Single Variable Calculus

Early Transcendentals

Complete Solutions Manual

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Fourth Edition

Contents

Chapter 1

Functions

1.1 Functions and Graphs

1. $$f(-5) = (-5)^2 - 1 = 25 - 1 = 24$$
$$f(-\sqrt{3}) = (-\sqrt{3})^2 - 1 = 3 - 1 = 2$$
$$f(3) = (3)^2 - 1 = 9 - 1 = 8$$
$$f(6) = (6)^2 - 1 = 36 - 1 = 35$$

2. $$f(-5) = -2(-5)^2 + (-5) = -2(25) - 5 = -55$$
$$f(-\tfrac{1}{2}) = -2(-\tfrac{1}{2})^2 + (-\tfrac{1}{2}) = -2(\tfrac{1}{4}) - \tfrac{1}{2} = -1$$
$$f(2) = -2(2)^2 + (2) = -2(4) + 2 = -6$$
$$f(7) = -2(7)^2 + (7) = -2(49) + 7 = -91$$

3. $$f(-1) = \sqrt{-1+1} = \sqrt{0} = 0$$
$$f(0) = \sqrt{0+1} = \sqrt{1} = 1$$
$$f(3) = \sqrt{3+1} = \sqrt{4} = 2$$
$$f(5) = \sqrt{5+1} = \sqrt{6}$$

4. $$f(-\tfrac{1}{2}) = \sqrt{2(-\tfrac{1}{2}) + 4} = \sqrt{-1+4} = \sqrt{3}$$
$$f(\tfrac{1}{2}) = \sqrt{2(\tfrac{1}{2}) + 4} = \sqrt{1+4} = \sqrt{5}$$
$$f(\tfrac{5}{2}) = \sqrt{2(\tfrac{5}{2}) + 4} = \sqrt{5+4} = \sqrt{9} = 3$$
$$f(4) = \sqrt{2(4) + 4} = \sqrt{8+4} = \sqrt{12} = 2\sqrt{3}$$

5. $$f(-1) = \frac{3(-1)}{(-1)^2+1} = \frac{-3}{1+1} = -\frac{3}{2}$$
$$f(0) = \frac{3(0)}{(0)^2+1} = 0$$
$$f(1) = \frac{3(1)}{(1)^2+1} = \frac{3}{2}$$
$$f(\sqrt{2}) = \frac{3(\sqrt{2})}{(\sqrt{2})^2+1} = \frac{3\sqrt{2}}{2+1} = \sqrt{2}$$

6. $$f(-\sqrt{2}) = \frac{(-\sqrt{2})^2}{(-\sqrt{2})^3-2} = \frac{2}{-2\sqrt{2}-2} = \frac{1}{-\sqrt{2}-1} = -\frac{1}{\sqrt{2}+1}$$
$$f(-1) = \frac{(-1)^2}{(-1)^3-2} = \frac{1}{-1-2} = \frac{1}{-3} = -\frac{1}{3}$$
$$f(0) = \frac{(0)^2}{(0)^3-2} = \frac{0}{-2} = 0$$
$$f(\tfrac{1}{2}) = \frac{\left(\frac{1}{2}\right)^2}{\left(\frac{1}{2}\right)^3-2} = \frac{\frac{1}{4}}{\frac{1}{8}-2} = \frac{\frac{1}{4}}{\frac{1}{8}-2}\left(\frac{8}{8}\right) = \frac{2}{1-16} = \frac{2}{-15} = -\frac{2}{15}$$

7. $$\begin{aligned} f(x) &= -2x^2+3x \\ f(2a) &= -2(2a)^2+3(2a) = -2(4a^2)+6a = -8a^2+6a \\ f(a^2) &= -2(a^2)^2 = 3(a^2) = -2a^4+3a^2 \\ f(-5x) &= -2(-5x)^2+3(-5x) = -2(25x^2)-15x = -50x^2-15x \\ f(2a+1) &= -2(2a+1)^2+3(2a+1) = -2(4a^2+4a+1)+6a+3 = -8a^2-8a-2+6a+3 \\ &= -8a^2-2a+1 \\ f(x+h) &= -2(x+h)^2+3(x+h) = -2(x^2+2xh+h^2)+3x+3h \\ &= -2x^2-4xh-2h^2+3x+3h \end{aligned}$$

8. $$\begin{aligned} f(x) &= x^3-2x^2+20 \\ f(2a) &= (2a)^3-2(2a)^2+20 = 8a^3-2(4a^2)+20 = 8a^3-8a^2+20 \\ f(a^2) &= (a^2)^3-2(a^2)^2+20 = a^6-2a^4+20 \\ f(-5x) &= (-5x)^3-2(-5x)^2+20 = -125x^3-2(25x^2)+20 = -125x^3-50x^2+20 \\ f(2a+1) &= (2a+1)^3-2(2a+1)^2+20 = 8a^3+3(2a)^2+3(2a)+1-2(4a^2+4a+1)+20 \\ &= 8a^3+12a^2+6a+1-8a^2-8a-2+20 = 8a^3+4a^2-2a+19 \\ f(x+h) &= (x+h)^3-2(x+h)^2+20 = x^3+3x^2h+3xh^2+h^3-2(x^2+2xh+h^2)+20 \\ &= x^3+3x^2h+3xh^2+h^3-2x^2-4xh-2h^2+20 \end{aligned}$$

9. Setting $f(x) = 23$ and solving for x, we find

$$\begin{aligned} 6x^2 - 1 &= 23 \\ 6x^2 &= 24 \\ x^2 &= 4 \\ x &= \pm 2. \end{aligned}$$

When we compute $f(-2)$ and $f(2)$ we obtain 23 in both cases, so $x = \pm 2$ is the answer.

10. We solve $f(x) = 4$:

$$\begin{aligned} \sqrt{x-4} &= 4 \\ x - 4 &= 4^2 = 16 \\ x &= 16 + 4 = 20. \end{aligned}$$

11. We need $4x - 2 \geq 0$:

$$\begin{aligned} 4x &\geq 2 \\ x &\geq \frac{1}{2}. \end{aligned}$$

The domain is $[\frac{1}{2}, \infty)$.

12. The domain of $f(x) = \sqrt{15 - 5x}$ is the set of all x for which $15 - 5x \geq 0$. This is equivalent to:

$$\begin{aligned} 15 &\geq 5x \\ 3 &\geq x \\ x &\leq 3. \end{aligned}$$

The domain of $f(x)$ is $(-\infty, 3]$.

13. We need $1 - x > 0$. This implies $x < 1$, so the domain is $(-\infty, 1)$.

14. We need $3x - 1 > 0$. This implies $x > \frac{1}{3}$, so the domain is $(\frac{1}{3}, \infty)$.

15. The domain of $f(x) = (2x-5)/x(x-3)$ is the set of all x for which $x(x-3) \neq 0$. Since $x(x-3) = 0$ when $x = 0$ or $x = 3$, the domain of $f(x)$ is $\{x \mid x \neq 0, x \neq 3\}$.

16. We need $x^2 - 1 \neq 0$ or $x^2 \neq 1$. Thus, $x \neq \pm 1$ and the domain is $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

17. We need $x^2 - 10x + 25 \neq 0$ or $(x-5)^2 \neq 0$. Thus, $x \neq 5$ and the domain is $(-\infty, 5) \cup (5, \infty)$.

18. The domain of $f(x) = (x+1)/(x^2 - 4x - 12)$ is the set of all x for which $x^2 - 4x - 12 \neq 0$. Since $x^2 - 4x - 12 = (x+4)(x-6) = 0$ when $x = -4$ or $x = 6$, the domain of $f(x)$ is $\{x | x \neq -4, x \neq 6\}$.

19. We need $x^2 - x + 1 \neq 0$. Applying the quadratic formula, we have

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)} = \frac{1 \pm \sqrt{-3}}{2}.$$

Neither value of x is real, so $x^2 - x + 1 \neq 0$ for all x, and the domain is $(-\infty, \infty)$.

20. We need $x^2 - 2x - 1 \neq 0$. Applying the quadratic formula, we have

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)} = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}.$$

Thus, the domain is $(-\infty, 1 - \sqrt{2}) \cup (1 - \sqrt{2}, 1 + \sqrt{2}) \cup (1 + \sqrt{2}, \infty)$.

21. The domain of $f(x) = \sqrt{25 - x^2}$ is the set of all x for which $25 - x^2 \geq 0$. Since $25 - x^2 = (5 + x)(5 - x)$, we have $25 - x^2 \geq 0$ when

$$5 + x \geq 0 \text{ and } 5 - x \geq 0 \qquad \text{or} \qquad 5 + x \leq 0 \text{ and } 5 - x \leq 0$$

Rewriting these inequalities for x, we get $x \geq -5$ and $x \leq 5$ for the first set of conditions or $x \leq -5$ and $x \geq 5$ for the second set of conditions. $x \leq -5$ and $x \geq 5$ at the same time can never happen, so the domain is determined by $-5 \leq x \leq 5$, or $[-5, 5]$.

22. We need $x(4 - x) \geq 0$, which is true when

$$x \geq 0 \text{ and } 4 - x \geq 0 \qquad \text{or} \qquad x \leq 0 \text{ and } 4 - x \leq 0$$

Rewriting these inequalities for x, we get $x \geq 0$ and $x \leq 4$ for the first set of conditions or $x \leq 0$ and $x \geq 4$ for the second set of conditions. $x \leq 0$ and $x \geq 4$ at the same time can never happen, so the domain is determined by $0 \leq x \leq 4$, or $[0, 4]$.

23. We need $x^2 - 5x = x(x - 5) \geq 0$, which is true when

$$x \geq 0 \text{ and } x - 5 \geq 0 \qquad \text{or} \qquad x \leq 0 \text{ and } x - 5 \leq 0$$

Rewriting these inequalities for x, we get $x \geq 0$ and $x \geq 5$ for the first set of conditions or $x \leq 0$ and $x \leq 5$ for the second set of conditions. The domain therefore requires that $x \geq 5$ or $x \leq 0$, so it is $(-\infty, 0] \cup [5, \infty)$.

24. The domain of $f(x) = \sqrt{x^2 - 3x - 10}$ is the set of all x for which $x^2 - 3x - 10 \geq 0$. Since $x^2 - 3x - 10 = (x + 2)(x - 5)$, we have $x^2 - 3x - 10 \geq 0$ when

$$x + 2 \geq 0 \text{ and } x - 5 \geq 0 \qquad \text{or} \qquad x + 2 \leq 0 \text{ and } x - 5 \leq 0$$

Rewriting these inequalities for x, we get $x \geq -2$ and $x \geq 5$ for the first set of conditions or $x \leq -2$ and $x \leq 5$ for the second set of conditions. The domain therefore requires that $x \geq 5$ or $x \leq -2$, so it is $(-\infty, -2] \cup [5, \infty)$.

25. We need $(3 - x)/(x + 2) \geq 0$ with $x \neq -2$, which is true when

$$3 - x \geq 0 \text{ and } x + 2 \geq 0 \qquad \text{or} \qquad 3 - x \leq 0 \text{ and } x + 2 \leq 0$$

Rewriting these inequalities for x, we get $x \leq 3$ and $x \geq -2$ for the first set of conditions or $x \geq 3$ and $x \leq -2$ for the second set of conditions. $x \geq 3$ and $x \leq -2$ at the same time can never happen, so the domain is determined by $-2 \leq x \leq 3$ and $x \neq -2$, or $(-2, 3]$.

26. We need $(5-x)/x \geq 0$ with $x \neq 0$, which is true when

$$5 - x \geq 0 \text{ and } x > 0 \qquad \text{or} \qquad 5 - x \leq 0 \text{ and } x < 0$$

Rewriting these inequalities for x, we get $x \leq 5$ and $x > 0$ for the first set of conditions or $x \geq 5$ and $x < 0$ for the second set of conditions. $x \geq 5$ and $x < 0$ at the same time can never happen, so the domain is determined by $0 < x \leq 5$, or $(0, 5]$.

27. Since the y-axis (a vertical line) intersects the graph in more than one point (three points in this case), the graph is not that of a function.

28. This is the graph of a function by the vertical line test.

29. This is the graph of a function by the vertical line test.

30. Since the y-axis (a vertical line) intersects the graph in more than one point (three points in this case), the graph is not that of a function.

31. Projecting the graph onto the x-axis, we see that the domain is $[-4, 4]$. Projecting the graph onto the y-axis, we see that the range is $[0, 5]$.

32. Projecting the graph onto the x-axis, we see that the domain is $[-1, 1]$. Projecting the graph onto the y-axis, we see that the range is $[-\pi/2, \pi/2]$.

33. Horizontally, the graph extends between $x = 1$ and $x = 9$ and terminates at both ends, as indicated by the solid dots. Thus, the domain is $[1, 9]$. Vertically, the graph extends between $y = 1$ and $y = 6$, so the range is $[1, 6]$.

34. Projecting the graph onto the x-axis, we see that the domain is $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$. Projecting the graph onto the y-axis, we see that the range is $(-\infty, 0) \cup (1, \infty)$.

35. **x-intercepts:** Solving $\frac{1}{2}x - 4 = 0$ we get $x = 8$. The x-intercept is $(8, 0)$.
y-intercept: Since $f(0) = \frac{1}{2}(0) - 4 = -4$, the y-intercept is $(0, -4)$.

36. **x-intercepts:** We solve $f(x) = x^2 - 6x + 5 = 0$:

$$\begin{aligned} x^2 - 6x + 5 &= 0 \\ (x-1)(x-5) &= 0 \\ x &= 1, 5. \end{aligned}$$

The x-intercepts are $(1, 0)$ and $(5, 0)$.
y-intercept: Since $f(0) = 0^2 - 6(0) + 5 = 5$, the y-intercept is $(0, 5)$.

37. **x-intercepts:** We solve $f(x) = 4(x-2)^2 - 1 = 0$:

$$\begin{aligned} 4(x-2)^2 - 1 &= 0 \\ 4(x-2)^2 &= 1 \\ (x-2)^2 &= \frac{1}{4} \\ x - 2 &= \pm\sqrt{\frac{1}{4}} = \pm\frac{1}{2} \\ x &= 2 \pm \frac{1}{2}. \end{aligned}$$

Both of these check with the original equation, so the x-intercepts are $(\frac{3}{2}, 0)$ and $(\frac{5}{2}, 0)$.

y-intercept: Since $f(0) = 4(0-2)^2 - 1 = 4(4) - 1 = 15$, the y-intercept is $(0, 15)$.

38. **x-intercepts:** We solve $f(x) = (2x-3)(x^2+8x+16) = 0$:

$$\begin{aligned} (2x-3)(x^2+8x+16) &= 0 \\ (2x-3)(x+4)^2 &= 0 \\ x &= \frac{3}{2}, -4. \end{aligned}$$

The x-intercepts are $(\frac{3}{2}, 0)$ and $(-4, 0)$.

y-intercept: Since $f(0) = [2(0)-3][(0)^2+8(0)+16] = -3(16) = -48$, the y-intercept is $(0, -48)$.

39. **x-intercepts:** We solve $f(x) = x^3 - x^2 - 2x = 0$:

$$\begin{aligned} x^3 - x^2 - 2x &= 0 \\ x(x^2 - x - 2) &= 0 \\ x(x+1)(x-2) &= 0 \\ x &= 0, -1, 2. \end{aligned}$$

The x-intercepts are $(0, 0)$, $(-1, 0)$, and $(2, 0)$.

y-intercept: Since $f(0) = 0$, the y-intercept is $(0, 0)$.

40. **x-intercepts:** We solve $f(x) = x^4 - 1 = 0$:

$$\begin{aligned} x^4 - 1 &= 0 \\ (x^2-1)(x^2+1) &= 0 \\ (x-1)(x+1)(x^2+1) &= 0 \\ x &= \pm 1. \end{aligned}$$

The x-intercepts are $(-1, 0)$ and $(1, 0)$.

y-intercept: Since $f(0) = (0)^4 - 1 = -1$, the y-intercept is $(0, -1)$.

41. **x-intercepts:** We solve $x^2 + 4 = 0$. Since $x^2 + 4$ is never 0, there are no x-intercepts.

y-intercept: Since $f(0) = [(0)^2 + 4]/[(0)^2 - 16] = 4/(-16) = -\frac{1}{4}$, the y-intercept is $(0, -\frac{1}{4})$.

42. **x-intercepts:** We solve $f(x) = x(x+1)(x-6)/(x+8) = 0$:

$$\begin{aligned} \frac{x(x+1)(x-6)}{x+8} &= 0 \\ x(x+1)(x-6) &= 0, \ x \neq 8. \end{aligned}$$

Thus, $x = 0$, $x = -1$, and $x = 6$. The x-intercepts are $(0, 0)$, $(-1, 0)$, and $(6, 0)$.

y-intercept: Since $f(0) = 0$, the y-intercept is $(0, 0)$.

43. **x-intercepts:** We solve $f(x) = \frac{3}{2}\sqrt{4-x^2} = 0$:

$$\begin{aligned} \frac{3}{2}\sqrt{4-x^2} &= 0 \\ 4-x^2 &= 0 \\ (2-x)(2+x) &= 0 \\ x &= \pm 2. \end{aligned}$$

The x-intercepts are $(-2, 0)$ and $(2, 0)$.

y-intercept: Since $f(0) = \frac{3}{2}\sqrt{4-(0)^2} = \frac{3}{2}(2) = 3$, the y-intercept is $(0, 3)$.

44. **x-intercepts:** We solve $f(x) = \frac{1}{2}\sqrt{x^2-2x-3} = 0$:

$$\begin{aligned} \frac{1}{2}\sqrt{x^2-2x-3} &= 0 \\ x^2-2x-3 &= 0 \\ (x-3)(x+1) &= 0 \\ x &= -1, 3. \end{aligned}$$

The x-intercepts are $(-1, 0)$ and $(3, 0)$.

y-intercept: Since $f(0) = \frac{1}{2}\sqrt{(0)^2-2(0)-3} = \frac{1}{2}\sqrt{-3}$, there is no y-intercept.

45. To find $f(a)$ for any number a, first locate a on the x-axis and then approximate the signed vertical distance to the graph from $(a, 0)$: $f(-3) \approx 0.5$ (because the graph is so steep at $x = -3$, $f(-3)$ could be reasonably approximated by any number from 0 to 1); $f(-2) \approx -3.4$; $f(-1) \approx 0.3$; $f(1) \approx 2$; $f(2) \approx 3.8$; $f(3) \approx 2.9$. The y-intercept is $(0, 2)$.

46. The function values are the directed distances from the x-axis at the given value of x.

$$f(-3) \approx 0;\ f(-2) \approx -3.5;\ f(-1) \approx 0.3;\ f(1) \approx 2;\ f(2) \approx 3.8;\ f(3) \approx 2.8.$$

47. To find $f(a)$ for any number a, first locate a on the x-axis and then approximate the signed vertical distance to the graph from $(a, 0)$:

$$f(-2) \approx 3.6;\ f(-1.5) \approx 2;\ f(0.5) \approx 3.3;\ f(1) \approx 4.1;\ f(2) \approx 2;\ f(3.2) \approx -4.1.$$

The x-intercepts are approximately $(-3.2, 0)$, $(2.3, 0)$, and $(3.8, 0)$.

48. To find $f(a)$ for any number a, first locate a on the x-axis and then approximate the signed vertical distance to the graph from $(a, 0)$:

$$f(-2) \approx 0;\ f(-1.5) \approx 1.6;\ f(0.5) \approx -2.3;\ f(1) \approx -3.8;\ f(2) \approx -2.2;\ f(3.2) \approx 0.$$

The x-intercepts are approximately $(-3, 0)$, $(-2, 0)$, $(0, 0)$, and $(3.2, 0)$.

49. Solving $x = y^2 - 5$ for y, we obtain

$$\begin{aligned} x &= y^2 - 5 \\ x + 5 &= y^2 \\ y^2 &= x + 5 \\ y &= \pm\sqrt{x+5} \end{aligned}$$

The two functions are $f_1(x) = -\sqrt{x+5}$ and $f_2(x) = \sqrt{x+5}$. The domains are both $[-5, \infty)$.

50. Solving for y^2, we have

$$\begin{aligned} x^2 - 4y^2 &= 16 \\ x^2 - 16 &= 4y^2 \\ y^2 &= \frac{1}{4}(x^2 - 16). \end{aligned}$$

The two functions are $f_1(x) = \frac{1}{2}\sqrt{x^2 - 16}$ and $f_2(x) = -\frac{1}{2}\sqrt{x^2 - 16}$. The domains are both $(-\infty, -4] \cup [4, \infty)$.

51. (a) $f(2) = 2! = 2 \cdot 1 = 2$
$f(3) = 3! = 3 \cdot 2 \cdot 1 = 6$
$f(5) = 5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$
$f(7) = 7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$
Note that we could have simplified the computation of 7! in this case by writing

$$7! = 7 \cdot 6 \cdot 5! = 7 \cdot 6 \cdot 120 = 5040.$$

(b) $f(n+1) = (n+1)! = (n+1)n! = n!(n+1) = f(n)(n+1)$

(c) Using the result from (b), we can simplify as follows:

$$\begin{aligned} \frac{f(5)}{f(4)} &= \frac{f(4) \cdot 5}{f(4)} = 5 \\ \frac{f(7)}{f(5)} &= \frac{f(6) \cdot 7}{f(5)} = \frac{f(5) \cdot 6 \cdot 7}{f(5)} = 42 \end{aligned}$$

(d) $\dfrac{f(n+3)}{f(n)} = \dfrac{(n+3)!}{n!} = \dfrac{(n+3)(n+2)(n+1)n!}{n!} = (n+1)(n+2)(n+3)$

52. (a) Identifying $n = 100$, we have

$$S(100) = \frac{1}{6}(100)(100+1)(200+1) = \frac{1}{3}(50)(101)(201) = 338,350.$$

(b) We can simply try various values of n:

$$S(8) = 204, S(9) = 285, S(10) = 385, S(11) = 506.$$

We see that $n = 10$ results in $300 < S(10) < 400$.

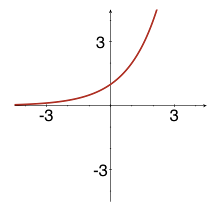
53. Generally, when the domain is a semi-infinite interval, the function can be one of $\sqrt{x-a}$, or $\sqrt{a-x}$. One of these should work when the number a is included in the interval. To exclude a number a, simply use the reciprocal of $\sqrt{x-a}$, or $\sqrt{a-x}$.

    (a) We try $f(x) = \sqrt{3-x}$. Since $x = 0$ is in the domain of this function, but not in the interval $[3, \infty)$, this is not the correct choice for $f(x)$. We then try $f(x) = \sqrt{x-3}$ and see that it does work.

    (b) Since 3 is not part of the interval, we let $f(x) = 1/\sqrt{x-3}$. This function has domain $(3, \infty)$.

54. (a) A function whose range is a semi-infinite closed interval will be $f(x) = x^2 + a$ when the range has the form $[a, \infty)$, or $f(x) = a - x^2$ when the range has the form $(\infty, a]$. Thus, a function with range $[3, \infty)$ is $f(x) = x^2 + 3$.

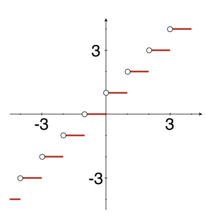
    (b) To find a function whose range is a semi-infinite open interval, we note the graph of $f(x) = 2^x$ shown to the right which as range $(0, \infty)$. The graph of $f(x) = 2^x + a$ is the graph of $f(x) = 2^x$ shifted upward by a units. Thus, the range of $f(x) = 2^x + 3$ is $(3, \infty)$.

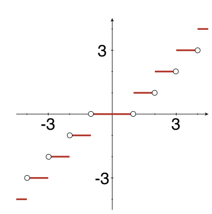


55. The graph indicates that $f(x) < 0$ when $x < -1$ or $x > 3$, so the domain of $g(x) = \sqrt{f(x)}$ must be $[-1, 3]$. Within this domain, $0 \le f(x) \le 4$, so the range of $g(x)$ must therefore be $[0, 2]$.

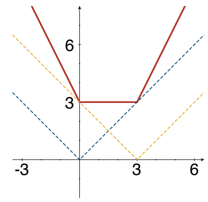
56. We can express M_1 as $(x, f(x)/2)$ and M_2 as $(x/2, f(x))$. To find $M_3(x)$, note that the points M_3 are the midpoints of line segments ST. We are told that $S = (0, f(x))$ and $T = (x, 0)$, so $M_3(x) = (\frac{0+x}{2}, \frac{f(x)+0}{2}) = (x/2, f(x)/2)$.

57. $$g(x) = \lceil x \rceil = \begin{cases} \vdots & \\ -2, & -3 < x \le -2 \\ -1, & -2 < x \le -1 \\ 0, & -1 < x \le 0 \\ 1, & 0 < x \le 1 \\ 2, & 1 < x \le 2 \\ 3, & 2 < x \le 3 \\ \vdots & \end{cases}$$

58. 

59. 

60. We discuss two methods for graphing $f(x) = |x| + |x-3|$. For the first method, graph both $y = |x|$ and $y = |x-3|$ on the same set of axes. Then, as illustrated on the right ($|x|$ and $|x-3|$ are rendered as dotted lines, while $|x|+|x-3|$ is the solid line), add the vertical distances from the x-axis up to each graph. This will give the point on the graph of $f(x) = |x|+|x-3|$. For the second method, since

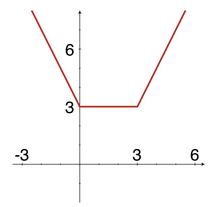


$$|x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases} \quad \text{and} \quad |x-3| = \begin{cases} -(x-3), & x-3 < 0 \\ x-3, & x-3 \geq 0, \end{cases} = \begin{cases} 3-x, & x < 3 \\ x-3, & x \geq 3, \end{cases}$$

then for:

$$\begin{aligned} x < 0, &\quad f(x) = (-x) + (3-x) = -2x+3 \\ 0 \leq x < 3, &\quad f(x) = (x) + (3-x) = 3 \\ x \geq 3, &\quad f(x) = (x) + (x-3) = 2x-3 \end{aligned} \qquad \text{and thus, } f(x) = \begin{cases} -2x+3, & x < 0 \\ 3, & 0 \leq x < 3 \\ 2x-3, & x \geq 3. \end{cases}$$

To graph $f(x)$, we note that $y = -2x+3$, where $x < 0$, is a line with slope -2 that approaches the point $(0,3)$ on the y-axis, while $y = 2x-3$, where $x \geq 3$, is a line with slope 2 that touches the point $(3,3)$ on the vertical line $x = 3$. These two partial lines are joined by the horizontal line segment from $(0,3)$ to $(3,3)$, shown on the right.



61. Since $\dfrac{x^2-9}{x-3} = \dfrac{(x+3)(x-3)}{x-3} = x+3, \quad x \neq 3,$

the graph of $f(x)$ is a line with a hole at $x = 3$. The graph of $g(x)$ is the same line with a hole at $x = 3$ and a dot at $(3,4)$. Since $x+3 = 6$ when $x = 3$, the graph of $h(x)$ is the same line with no holes.

62. Since $\dfrac{x^4-1}{x^2-1} = \dfrac{(x^2+1)(x^2-1)}{x^2-1} = x^2+1, \quad x \neq \pm 1,$

the graph of $f(x)$ is a parabola with holes at $x = \pm 1$. The graph of $g(x)$ is the same parabola with holes at $x = \pm 1$ and a dot at $(1,0)$. Since $x^2+1 = 2$ when $x = 1$, the graph of $h(x)$ is the same parabola with a hole only at $x = -1$.

1.2 Combining Functions

1. $(f+g)(x) = -2x+13 \qquad (f-g)(x) = 6x-3$

   $(fg)(x) = -8x^2-4x+40 \qquad (f/g)(x) = \dfrac{2x+5}{-4x+8},\ x \neq 2$

2. $(f+g)(x) = 5x^2+7x-9 \qquad (f-g)(x) = 5x^2-7x+9$

   $(fg)(x) = 35x^3-45x^2 \qquad (f/g)(x) = \dfrac{5x^2}{7x-9}, x \neq \dfrac{9}{7}$

3. $(f+g)(x) = \dfrac{x^2+x+1}{x^2+x} \qquad (f-g)(x) = \dfrac{x^2-x-1}{x^2+x}$

   $(fg)(x) = \dfrac{1}{x+1}, x \neq 0 \qquad (f/g)(x) = \dfrac{x^2}{x+1}, x \neq -1, 0$

4. $(f+g)(x) = \dfrac{9x^2-11}{4x^2+14x+6}$ $\quad (f-g)(x) = \dfrac{7x^2+7}{4x^2+14x+6}$

   $(fg)(x) = \dfrac{2x^2-7x+3}{4x^2+14x+6}$ $\quad (f/g)(x) = \dfrac{8x^2-2}{x^2-9}, x \neq \dfrac{1}{2}, -3, 3$

5. $(f+g)(x) = 2x^2+5x-7$ $\quad (f-g)(x) = -x+1$

   $(fg)(x) = x^4+5x^3-x^2-17x+12$ $\quad (f/g)(x) = \dfrac{x+3}{x+4}, x \neq -4, 1$

6. $(f+g)(x) = x^2+\sqrt{x}$ $\quad (f-g)(x) = x^2-\sqrt{x}$

   $(fg)(x) = x^{5/2}$ $\quad (f/g)(x) = x^{3/2}$

7. $(f+g)(x) = \sqrt{x-1}+\sqrt{2-x}$; the domain is $[1,2]$.

8. $(fg)(x) = \sqrt{(x-1)(2-x)}$; the domain is $[1,2]$.

9. $(f/g)(x) = \sqrt{x-1}/\sqrt{2-x}$; the domain is $[1,2)$.

10. $(g/f)(x) = \sqrt{2-x}/\sqrt{x-1}$; the domain is $(1,2]$.

11. $(f \circ g)(x) = 3x+16$; $\quad (g \circ f)(x) = 3x+4$

12. $(f \circ g)(x) = 4x^2+1$; $\quad (g \circ f)(x) = 16x^2+8x+1$

13. $(f \circ g)(x) = x^6+2x^5+x^4$; $\quad (g \circ f)(x) = x^6+x^4$

14. $(f \circ g)(x) = \dfrac{4x+9}{x+2}$; $\quad (g \circ f)(x) = \dfrac{1}{4x+12}$

15. $(f \circ g)(x) = \dfrac{3x+3}{x}$; $\quad (g \circ f)(x) = \dfrac{3}{3+x}$

16. $(f \circ g)(x) = (x^2)^2+\sqrt{x^2} = x^4+|x|$

    $(g \circ f)(x) = (x^2+\sqrt{x})^2 = x^4+2x^2\sqrt{x}+x$

    Note that the domain of $f \circ g$ is $\Re$, while the domain of $g \circ f$ is $[0,\infty)$.

17. $(f \circ g)(x) = f(x^2+2) = \sqrt{(x^2+2)-3} = \sqrt{x^2-1}$

    The domain of f, determined by $x \geq 3$, is $[3,\infty)$. Since the domain of g is all real numbers and $g(x) = x^2+2 \geq 3$ when either $x \leq -1$ or $x \geq 1$, the domain of $f \circ g$ is $(-\infty,-1] \cup [1,\infty)$.

18. $(g \circ f)(x) = g(\sqrt{x-3}) = (\sqrt{x-3})^2+2 = x-1$

    The domain of g is all real numbers and the domain of f, determined by $x-3 \geq 0$, is $[3,\infty)$. Thus, the domain of $g \circ f$ is $[3,\infty)$.

19. $(g \circ f)(x) = g(5-x^2) = 2-\sqrt{5-x^2}$

    The domain of g, determined by $x \geq 0$, is $[0,\infty)$. Since the domain of f is all real numbers and $f(x) = 5-x^2 \geq 0$ when $-\sqrt{5} \leq x \leq \sqrt{5}$, the domain of $g \circ f$ is $[-\sqrt{5},\sqrt{5}]$.

20. $(f \circ g)(x) = f(2-\sqrt{x}) = 5-(2-\sqrt{x})^2 = 5-(4-4\sqrt{x}+x) = 1+4\sqrt{x}-x$
    The domain of f is all real numbers and the domain of g, determined by $x \geq 0$, is $[0,\infty)$. Thus, the domain of $f \circ g$ is $[0,\infty)$.

21. $(f \circ (2f))(x) = 2(4x^3)^3 = 128x^9$; $\quad (f \circ (1/f))(x) = 2\left(\dfrac{1}{2x^3}\right)^3 = \dfrac{1}{4x^9}$

22. $(f \circ (2f))(x) = \dfrac{1}{\left(\dfrac{2}{x-1}\right)-1} = \dfrac{x-1}{3-x}$; $\quad (f \circ (1/f))(x) = \dfrac{1}{(x-1)-1} = \dfrac{1}{x-2}$

23. $(f \circ g \circ h)(x) = 36x^2 - 36x + 15$

24. $(f \circ g \circ h)(x) = \sqrt{2x-2}$

25. $2g(x) - 5 = -4x + 13$; $\quad g(x) = -2x + 9$

26. $\sqrt{2g(x)+6} = 4x^2$; $\quad g(x) = 8x^4 - 3$

27. $f(x) = 2x^2 - x$; $\quad g(x) = x^2$

28. $f(x) = 1/x$; $\quad g(x) = x^2 + 9$

29. The point (x,y) on the graph of f corresponds to the point $(x,y+2)$ on the shifted graph. Thus, $(-2,1)$ corresponds to $(-2,3)$ and $(3,-4)$ corresponds to $(3,-2)$.

30. The point (x,y) on the graph of f corresponds to the point $(x,y-5)$ on the shifted graph. Thus, $(-2,1)$ corresponds to $(2,-4)$ and $(3,-4)$ corresponds to $(3,-9)$.

31. The point (x,y) on the graph of f corresponds to the point $(x-6,y)$ on the shifted graph. Thus, $(-2,1)$ corresponds to $(-8,1)$ and $(3,-4)$ corresponds to $(-3,-4)$.

32. The point (x,y) on the graph of f corresponds to the point $(x+1,y)$ on the shifted graph. Thus, $(-2,1)$ corresponds to $(-1,1)$ and $(3,-4)$ corresponds to $(4,-4)$.

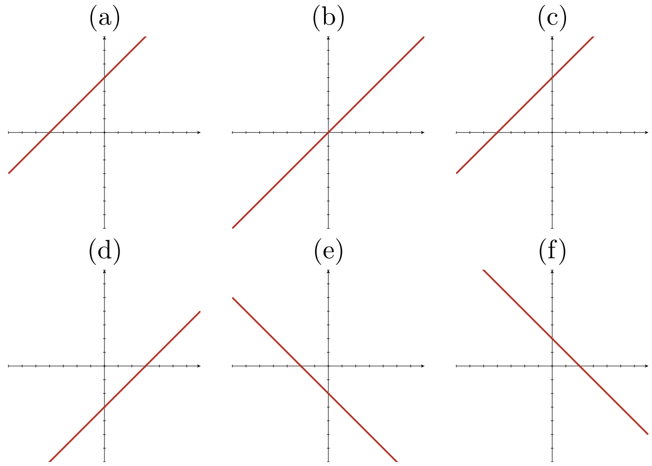
33. The point (x,y) on the graph of f corresponds to the point $(x-4,y+1)$ on the shifted graph. Thus, $(-2,1)$ corresponds to $(-6,2)$ and $(3,-4)$ corresponds to $(-1,-3)$.

34. The point (x,y) on the graph of f corresponds to the point $(x+5,y-3)$ on the shifted graph. Thus, $(-2,1)$ corresponds to $(3,-2)$ and $(3,-4)$ corresponds to $(8,-7)$.

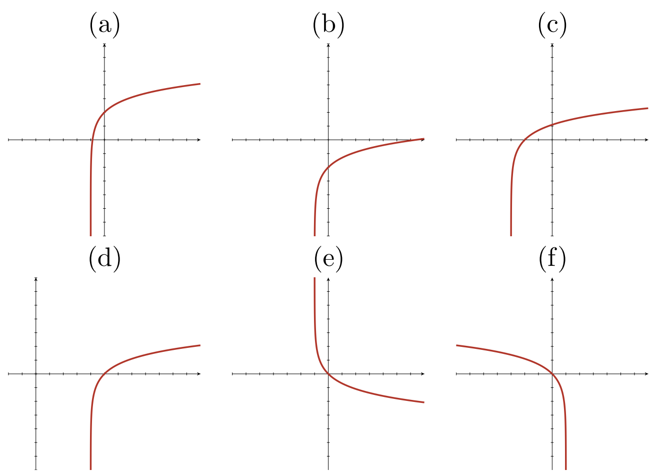
35. The point (x,y) on the graph of f corresponds to the point $(-x,y)$ on the shifted graph. Thus, $(-2,1)$ corresponds to $(2,1)$ and $(3,-4)$ corresponds to $(-3,-4)$.

36. The point (x,y) on the graph of f corresponds to the point $(x,-y)$ on the shifted graph. Thus, $(-2,-1)$ corresponds to $(-1,1)$ and $(3,-4)$ corresponds to $(3,4)$.

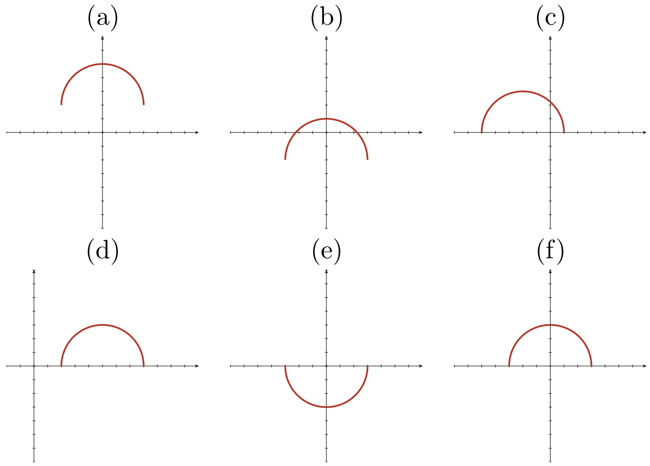
37. In (a) the graph is shifted up 2 units; in (b) it is shifted down 2 units; in (c) it is shifted left 2 units; in (d) it is shifted right 5 units; in (e) it is reflected in the x-axis; and in (f) it is reflected in the y-axis.



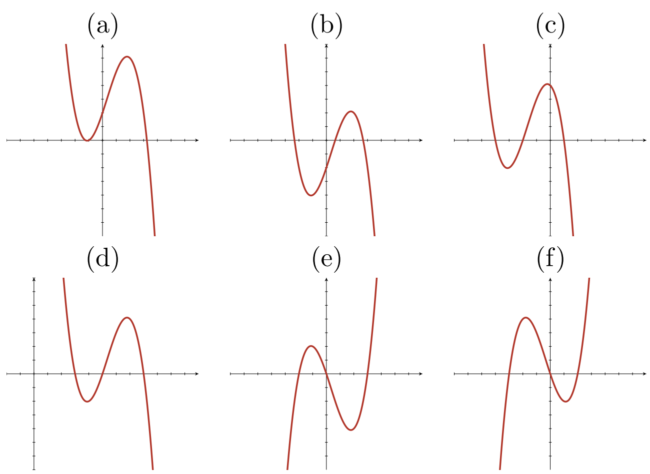
38. In (a) the graph is shifted up 2 units; in (b) it is shifted down 2 units; in (c) it is shifted left 2 units; in (d) it is shifted right 5 units; in (e) it is reflected in the x-axis; and in (f) it is reflected in the y-axis.



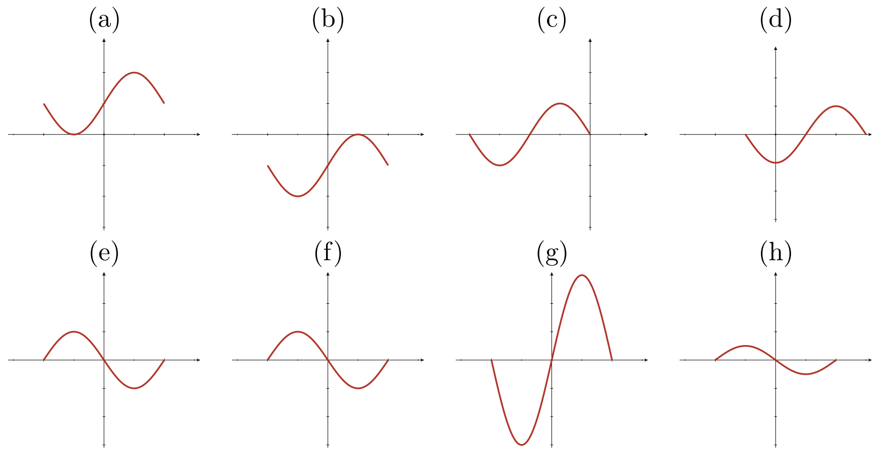
39. In (a) the graph is shifted up 2 units; in (b) it is shifted down 2 units; in (c) it is shifted left 2 units; in (d) it is shifted right 5 units; in (e) it is reflected in the x-axis; and in (f) it is reflected in the y-axis.



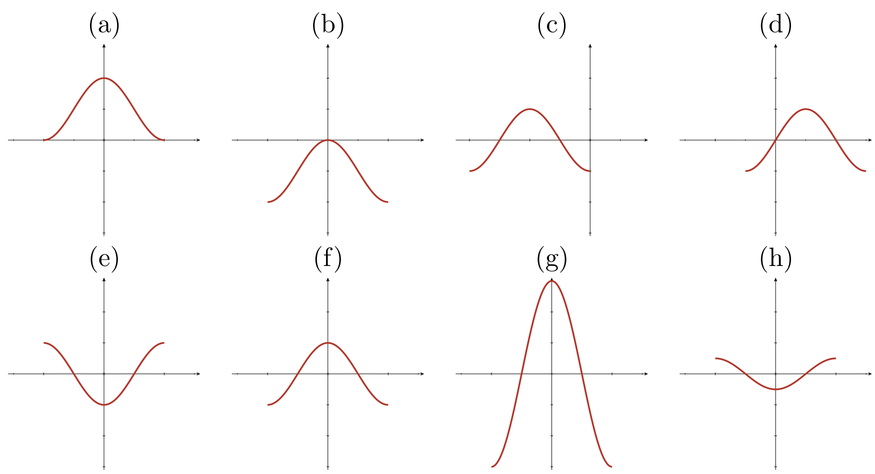
40. In (a) the graph is shifted up 2 units; in (b) it is shifted down 2 units; in (c) it is shifted left 2 units; in (d) it is shifted right 5 units; in (e) it is reflected in the x-axis; and in (f) it is reflected in the y-axis.



41. In (a) the graph is shifted up 1 unit; in (b) it is shifted down 1 unit; in (c) it is shifted left π units; in (d) it is shifted right $\pi/2$ units; in (e) it is reflected in the x-axis; in (f) it is reflected in the y-axis; in (g) it is stretched vertically by a factor of 3; and in (h) it is compressed vertically by a factor of $1/2$ and then reflected in the x-axis.



42. In (a) the graph is shifted up 1 unit; in (b) it is shifted down 1 unit; in (c) it is shifted left π units; in (d) it is shifted right $\pi/2$ units; in (e) it is reflected in the x-axis; in (f) it is reflected in the y-axis; in (g) it is stretched vertically by a factor of 3; and in (h) it is compressed vertically by a factor of 1/2 and then reflected in the x-axis.



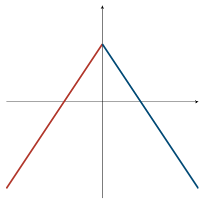
43. If $f(x)$ is shifted up 5 units and right 1 unit, the new function is $y = f(x-1)+5$. Since $f(x) = x^3$, this becomes $y = (x-1)^3 + 5$.

44. The function is first multiplied by 3 and then x is replaced by $x-2$. The equation of the new graph is thus $y = 3(x-2)^{2/3}$.

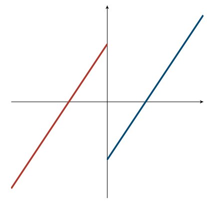
45. The function is first multiplied by -1 and then x is replaced by $x+7$. The equation of the new graph is thus $y = -(x+7)^4$.

46. If $f(x)$ is reflected in the y-axis, then shifted left 5 units and down 10 units, the new function is $y = -f(x+5) - 10$. Since $f(x) = 1/x$, this becomes $y = -1/(x+5) - 10$.

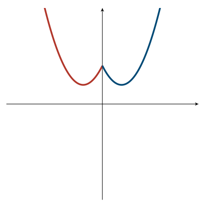
47. (a) An even function is symmetric with respect to the y-axis.



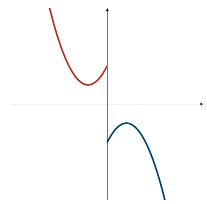
(b) An odd function is symmetric with respect to the origin.



48. (a) An even function is symmetric with respect to the y-axis.



(b) An odd function is symmetric with respect to the origin.



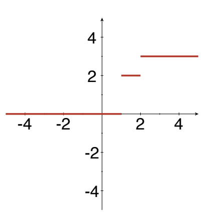
49. To fill in the bottom row, we use the fact that since f is an even function then $f(-x) = f(x)$, and so $(f \circ g)(-x) = (f \circ g)(x)$.

x	0	1	2	3	4
$f(x)$	-1	2	10	8	0
$g(x)$	2	-3	0	1	-4
$(f \circ g)(x)$	10	8	-1	2	0

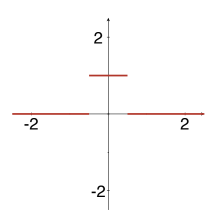
50. To fill in the bottom row, we use the fact that since g is an odd function then $g(-x) = -g(x)$, and so $(g \circ f)(-x) = -(g \circ f)(x)$.

x	0	1	2	3	4
$f(x)$	-2	-3	0	-1	-4
$g(x)$	9	7	-6	-5	13
$(g \circ f)(x)$	6	5	9	-7	-13

51.



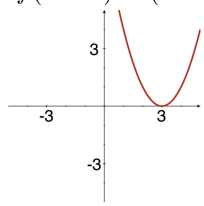
52.



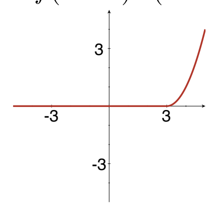
53. $2U(x-0) - 2U(x-2) + (-1)U(x-2) - (-1)U(x-3) = 2 - 3U(x-2) + U(x-3)$

54. When compared to the graph of $y = f(x-3)$, the graph of $y = f(x-3)U(x-3)$ is a horizontal line at zero for $x < 3$, then matches the graph of $y = f(x-3)$ after that.

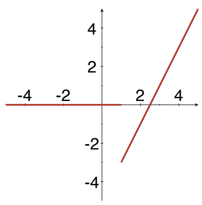
$y = f(x-3) = (x-3)^2$



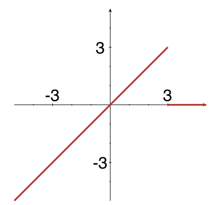
$y = f(x-3)U(x-3)$



55.



56.



57. False; let $f(x) = x^2$ and $g(x) = h(x) = 1$.

58. $[-3, -1]$

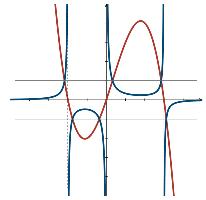
59. If a function is symmetric with respect to the x-axis, then, for any x, both (x, y) and $(x, -y)$ are on the graph. If $y \neq 0$ (as is the case for a function that is nonzero for at least one value of x), then the vertical line test fails and the graph is not the graph of a function.

60. A vertical stretch or compression by a factor of c units results in $y = cf(x)$. For a given x, this will result in the same y-value when $f(x) = 0$. Thus, the points on the graph of $y = f(x)$ are those that also lie on the x-axis.

When the graph is reflected in the x-axis, the point (x, y) becomes the point $(x, -y)$. Thus, the points that remain the same are those for which $y = 0$; that is, the x-intercepts.

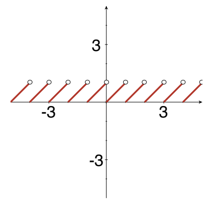
When the graph is reflected in the y-axis, the point (x, y) becomes the point $(-x, y)$. Thus, the only point that remains the same is the y-intercept.

61. Since $|x| = x$ when $x \geq 0$ and $|x| = -x$ when $x < 0$, the graph of $f(|x|)$ is the same as the graph of $f(x)$ when $x \geq 0$. When $x < 0$, $f(|x|) = f(-x)$, so the graph of $f(|x|)$ in this case is the reflection of the graph of $f(x), x > 0$, in the y-axis. To summarize: to obtain the graph of $f(|x|)$ from the graph of $f(x)$, simply ignore the portion of the graph of $f(x)$ to the left of the y-axis, and then reflect the portion of the graph of $f(x)$ to the right of the y-axis through the y-axis.

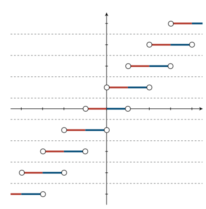


62. We see from Figure 1.2.1 in the text that when x is positive:
    (a) if $f(x)$ is near 0, then $1/x$ is very large;
    (b) if $f(x)$ is large, then $1/x$ is near 0;
    (c) if $f(x) = 1$, then $1/x = 1$.

    That is, for a fixed x, points (x, y) where $y > 1$ are "reflected" through the line $y = 1$ and compressed into the interval $0 < y < 1$. Also, for a fixed x, points (x, y) where $0 < y < 1$ are "reflected" through the line $y = 1$ and stretched into the interval $y > 1$. Similar statements hold for $x < 0$. The graph of $1/f(x)$ is shown in blue, and the graph of $f(x)$ is shown in red.



63. frac(x) is so named because its value is the non-integer part of x, i.e., the value that follows the decimal point. Its graph is shown on the right.



64. Based on the graph to the right, the graph of the ceiling function $g(x) = \lceil x \rceil$ (red) can be thought of as the reflection of the floor function $f(x) = \lfloor x \rfloor$ (blue) across the horizontal lines $y = f(x) + 1/2 = \lfloor x \rfloor + 1/2$ (gray, dotted) if x is not an integer. When x is an integer, $g(x) = f(x)$. Thus,

$$g(x) = \begin{cases} f(x) + 1, & x \text{ is not an integer} \\ f(x), & x \text{ is an integer.} \end{cases}$$

1.3 Polynomial and Rational Functions

1. The form of the equation of the line is $y = \frac{2}{3}x + b$. Letting $x = 1$ and $y = 2$, we have $2 = \frac{2}{3} + b$, so $b = 2 - \frac{2}{3} = \frac{4}{3}$. The equation of the line is $y = \frac{2}{3}x + \frac{4}{3}$.

2. Letting $m = \frac{1}{10}$, $x_1 = 1$, and $y_1 = 2$, we obtain from the point-slope form of the equation of a line that

$$\begin{aligned} y - 2 &= \frac{1}{10}(x - 1) = \frac{1}{10}x - \frac{1}{10} \\ y &= \frac{1}{10}x + \frac{19}{10}. \end{aligned}$$

3. A line with slope 0 is horizontal and has the form $y = b$. In this case, $b = 2$, so the equation if $y = 2$.

4. The form of the equation of the line is $y = -2x + b$. Letting $x = 1$ and $y = 2$, we have $2 = -2 + b$, so $b = 2 + 2 = 4$. The equation of the line is $y = -2x + 4$.

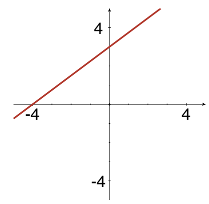
5. Letting $m = -1$, $x_1 = 1$, and $y_1 = 2$, we obtain from the point-slope form of the equation of a line that

$$y - 2 = -1(x - 1) = -x + 1$$
$$y = -x + 3.$$

6. When the slope is undefined, the line is vertical and has the form $x = a$. In this case $a = 1$, so the equation is $x = 1$.

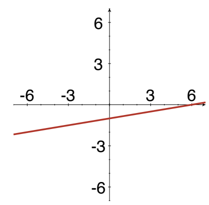
7. To find the x-intercept we set $y = 0$. This gives $3x + 12 = 0$ or $x = -4$. The x-intercept is $(-4, 0)$. Now, write the equation in slope-intercept form by solving for y:

$$3x - 4y + 12 = 0$$
$$-4y = -3x - 12$$
$$y = \frac{3}{4}x + 3.$$

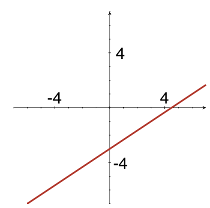


The slope of the line is $m = \frac{3}{4}$ and the y-intercept is $(0, 3)$.

8. Solving for y, we have $3y = \frac{1}{2}x - 3$ or $y = \frac{1}{6}x - 1$, so the slope is $\frac{1}{6}$ and the y-intercept is $(0, -1)$. Setting $y = 0$, we have $0 = \frac{1}{6}x - 1$ or $x = 6$, so the x-intercept is $(6, 0)$.

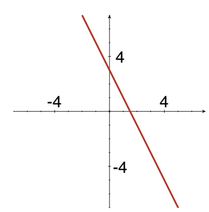


9. Solving for y, we have $3y = 2x - 9$ or $y = \frac{2}{3}x - 3$, so the slope is $\frac{2}{3}$ and the y-intercept is $(0, -3)$. Setting $y = 0$, we have $0 = \frac{2}{3}x - 3$ or $x = \frac{9}{2}$, so the x-intercept is $(\frac{9}{2}, 0)$.



10. To find the x-intercept we set $y = 0$. This gives $-4x + 6 = 0$ or $x = \frac{3}{2}$. The x-intercept is $(\frac{3}{2}, 0)$. Now, write the equation in slope-intercept form by solving for y:

$$4x - 2y + 6 = 0$$
$$-2y = 4x - 6$$
$$y = -2x + 3.$$



The slope of the line is $m = -2$ and the y-intercept is $(0, 3)$.

11. The slope of the line is $m = \dfrac{-5 - 3}{6 - 2} = \dfrac{-8}{4} = -2$. The form of the equation of the line is $y = -2x + b$. Letting $x = 2$ and $y = 3$, we have $3 = -2(2) + b$, so $b = 3 + 4 = 7$. The equation of the line is $y = -2x + 7$.

12. The slope of the line is $m = \dfrac{0-(-6)}{4-5} = \dfrac{6}{-1} = -6$, and we identify $x_1 = 5$ and $y_1 = -6$. Then, using the point-slope form of the equation of a line, we have $y-(-6) = -6(x-5) = -6x+30$, or $y = -6x+24$.

13. Solving $3x+y-5=0$ for y, we obtain $y=-3x+5$. The slope of this line is $m=-3$, so the form of the line through $(-2,4)$ is $y=-3x+b$. Letting $x=-2$ and $y=4$, we have $4=-3(-2)+b$, so $b=4-6=-2$ and the equation of the line is $y=-3x-2$.

14. Lines parallel to the y-axis are vertical, so the equation is $x=5$.

15. Solving for $x-4y+1=0$ for y, we obtain $-4y=-x-1$ and $y=\frac{1}{4}x+\frac{1}{4}$. Thus, the slope of a line perpendicular to this one is $m=-1/(1/4)=-4$. We identify $x_1=2$ and $y_1=3$ and use the point-slope form of the equation of a line:

$$\begin{aligned} y-3 &= -4(x-2) = -4x+8 \\ y &= -4x+11. \end{aligned}$$

16. The line through $(1,1)$ and $(3,11)$ has the slope $m=\dfrac{11-1}{3-1}=\dfrac{10}{2}=5$, so the desired line has slope $-\frac{1}{5}$. It then has the form $y=-\frac{1}{5}x+b$. Since it passes through $(-5,-4)$, we have $-4=-\frac{1}{5}(-5)+b$ or $b=-4-1=-5$. The equation of the line is then $y=-\frac{1}{5}x-5$.

17. Letting $x=-1$, $y=5$, and $x=1$, $y=6$ in $f(x)=ax+b$, we have

$$\begin{aligned} 5 &= a(-1)+b \\ 6 &= a(1)+b \end{aligned} \qquad \text{or} \qquad \begin{aligned} -a+b &= 5 \\ a+b &= 6. \end{aligned}$$

Adding these equations, we find $2b=11$, so $b=\frac{11}{2}$. Then $a=6-b=6-\frac{11}{2}=\frac{1}{2}$ and the function is $f(x)=\frac{1}{2}x+\frac{11}{2}$.

18. We are looking for a function having the form $f(x)=ax+b$. The first condition implies

$$\begin{aligned} a(-1)+b &= 1+a(2)+b \\ -a+b &= 2a+b+1 \\ -3a &= 1;\ a=-\frac{1}{3}. \end{aligned}$$

The second condition implies

$$\begin{aligned} a(3)+b &= 4[a(1)+b] \\ 3a+b &= 4a+4b;\ 0=a+3b. \end{aligned}$$

Since we have already determined that $a=-\dfrac{1}{3}$,

$$\begin{aligned} 0 &= -\frac{1}{3}+3b \\ \frac{1}{3} &= 3b;\ b=\frac{1}{9}. \end{aligned}$$

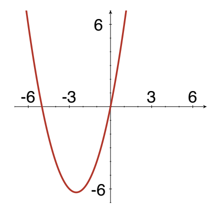
The linear function is $f(x)=-\dfrac{1}{3}x+\dfrac{1}{9}$.

19. When $x = -1$, the corresponding point on the blue curve has y-coordinate $y = (-1)^2 + 1 = 2$. When $x = 2$, the corresponding point on the blue curve has y-coordinate $y = 2^2 + 1 = 5$. The slope of the line through $(-1, 2)$ and $(2, 5)$ is $m = (5 - 2)/[2 - (-1)] = 3/3 = 1$. The form of the equation of the line is then $y = 1x + b = x + b$. Using $x = -1$ and $y = 2$ we have $2 = -1 + b$ or $b = 3$. Thus, the equation of the line is $y = x + 3$.

20. The center of the circle is at $(2, 3)$, and since the circle is tangent to the y-axis, its radius is 2. Thus, the equation of the circle is $(x - 2)^2 + (y - 3)^2 = 4$. The point on the circle has x-coordinate 3, so $(3 - 2)^2 + (y - 3)^2 = 4$ or $(y - 3)^2 = 4 - 1 = 3$ and $y = \sqrt{3} + 3$. (We use the positive square root of 3 because the point on the circle if above the center of the circle.) The point on the circle is then $(3, \sqrt{3} + 3)$. The slope of the line through the center of the circle and this point is $m = [(\sqrt{3} + 3) - 3]/(3 - 2) = \sqrt{3}/1 = \sqrt{3}$, so the slope of the tangent line, which is perpendicular to this line, is $-1/\sqrt{3}$. The equation of the tangent line then has the form $y = (-1/\sqrt{3})x + b$. Using $x = 3$ and $y = \sqrt{3} + 3$ we have $\sqrt{3} + 3 = (-1/\sqrt{3})(3) + b$ or $b = \sqrt{3} + 3 + \sqrt{3} = 2\sqrt{3} + 3$. Thus, the equation of the tangent line is:

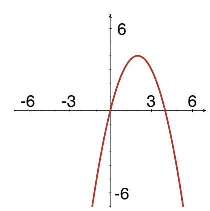
$$y = -\frac{1}{\sqrt{3}}x + 2\sqrt{3} + 3.$$

21. (a) **x-intercepts:** Solving $x(x + 5) = 0$ we get $x = 0, -5$, so the x-intercepts are $(0, 0)$ and $(-5, 0)$.
y-intercept: Since $f(0) = 0$, the y-intercept is $(0, 0)$.
(b) $f(x) = x(x + 5) = x^2 + 5x = \left[x^2 + 5x + \left(\frac{5}{2}\right)^2\right] - \left(\frac{5}{2}\right)^2 = (x + 5/2)^2 - (25/4)$
(c) Identifying $h = -\frac{5}{2}$ and $k = -\frac{25}{4}$ in part (b), we see that the vertex is $\left(-\frac{5}{2}, -\frac{25}{4}\right)$ and the axis of symmetry is $x = -\frac{5}{2}$.



(d) Since $a = 1 > 0$ in part (b), the parabola opens up.
(e) The range of $f(x)$ is $[-25/4, \infty)$.
(f) $f(x)$ is increasing on $[-5/2, \infty)$ and decreasing on $(-\infty, -5/2]$.

22. (a) **x-intercepts:** Solving $-x^2 + 4x = x(-x + 4) = 0$ we get $x = 0, 4$, so the x-intercepts are $(0, 0)$ and $(4, 0)$.
y-intercept: Since $f(0) = 0$, the y-intercept is $(0, 0)$.
(b) $f(x) = -x^2 + 4x = -[x^2 - 4x + (-2)^2] + (-2)^2 = -(x - 2)^2 + 4$
(c) Identifying $h = 2$ and $k = 4$ in part (b), we see that the vertex is $(2, 4)$ and the axis of symmetry is $x = 2$.



(d) Since $a = -1 < 0$ in part (b), the parabola opens down.
(e) The range of $f(x)$ is $(-\infty, 4]$.
(f) $f(x)$ is increasing on $(-\infty, 2]$ and decreasing on $[2, \infty)$.

23. (a) **x-intercepts:** Solving $(3-x)(x+1) = 0$ we get $x = 3, -1$, so the x-intercepts are $(-1, 0)$ and $(3, 0)$.

y-intercept: Since $f(0) = (3 - 0)(0 + 1) = 3$, the y-intercept is $(0, 3)$.

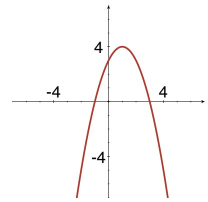
(b) $f(x) = (3 - x)(x + 1) = -x^2 + 2x + 3 = -(x^2 - 2x) + 3$
$= -(x^2 - 2x + 1) + 3 + 1 = -(x - 1)^2 + 4$

(c) Identifying $h = 1$ and $k = 4$ in part (b), we see that the vertex is $(1, 4)$ and the axis of symmetry is $x = 1$.

(d) Since $a = -1 < 0$ in part (b), the parabola opens down.

(e) The range of $f(x)$ is $(-\infty, 4]$.

(f) $f(x)$ is increasing on $(-\infty, 1]$ and decreasing on $[1, \infty)$.



24. (a) **x-intercepts:** Solving $(x - 2)(x - 6) = 0$ we get $x = 2, 6$, so the x-intercepts are $(2, 0)$ and $(6, 0)$.

y-intercept: Since $f(0) = 12$, the y-intercept is $(0, 12)$.

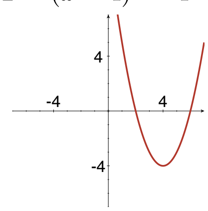
(b) $f(x) = (x - 2)(x - 6) = x^2 - 8x + 12 = [x^2 - 8x + (-4)^2] - (-4)^2 + 12 = (x - 4)^2 - 4$

(c) Identifying $h = 4$ and $k = -4$ in part (b), we see that the vertex is $(4, -4)$ and the axis of symmetry is $x = 4$.

(d) Since $a = 1 > 0$ in part (b), the parabola opens up.

(e) The range of $f(x)$ is $[-4, \infty)$.

(f) $f(x)$ is increasing on $[4, \infty)$ and decreasing on $(-\infty, 4]$.



25. (a) **x-intercepts:** Solving $x^2 - 3x + 2 = (x - 1)(x - 2) = 0$ we get $x = 1, 2$, so the x-intercepts are $(1, 0)$ and $(2, 0)$.

y-intercept: Since $f(0) = 2$, the y-intercept is $(0, 2)$.

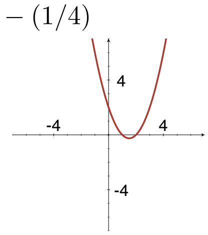
(b) $f(x) = x^2 - 3x + 2 = [x^2 - 3x + (-3/2)^2] - (-3/2)^2 + 2 = (x - 3/2)^2 - (1/4)$

(c) Identifying $h = 3/2$ and $k = -1/4$ in part (b), we see that the vertex is $(3/2, -1/4)$ and the axis of symmetry is $x = 3/2$.

(d) Since $a = 1 > 0$ in part (b), the parabola opens up.

(e) The range of $f(x)$ is $[-1/4, \infty)$.

(f) $f(x)$ is increasing on $[3/2, \infty)$ and decreasing on $(-\infty, 3/2]$.



26. (a) **x-intercepts:** Factoring, we obtain

$$f(x) = -x^2 + 6x - 5 = -(x^2 - 6x + 5) = -(x - 1)(x - 5).$$

Solving $-(x - 1)(x - 5) = 0$ we get $x = 1, 5$, so the x-intercepts are $(1, 0)$ and $(5, 0)$.

y-intercept: Since $f(0) = -5$, the y-intercept is $(0, -5)$.

(b) To complete the square and obtain the standard form, we start by factoring -1 from the two x-terms:

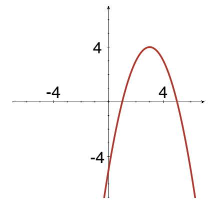
$$f(x) = -x^2 + 6x - 5 = -(x^2 - 6x) - 5 = -(x^2 - 6x + 9) - 5 + 9 = -(x - 3)^2 + 4$$

(c) Identifying $h = 3$ and $k = 4$ in part (b), we see that the vertex is $(3, 4)$ and the axis of symmetry is $x = 3$.

(d) Since $a = -1 < 0$ in part (b), the parabola opens down.

(e) The range of $f(x)$ is $(-\infty, 4]$.

(f) $f(x)$ is increasing on $(-\infty, 3]$ and decreasing on $[3, \infty)$.



27. The vertex of the graph is $(-10, 0)$, so the graph of $f(x)$ is the graph of $y = x^2$ shifted to the right by 10 units.

28. The vertex of the graph is $(-6, 0)$, so the graph of $f(x)$ is the graph of $y = x^2$ shifted to the left by 6 units.

29. The vertex of the graph is $(-4, 9)$, so the graph of $f(x)$ is the graph of $y = x^2$ reflected in the x-axis, compressed by a factor of $1/3$, shifted to the left by 4 units, and shifted up by 9 units.

30. The vertex of the graph is $(2, -1)$, so the graph of $f(x)$ is the graph of $y = x^2$ stretched by a factor of 10, shifted to the right by 2 units and shifted down by 1 unit.

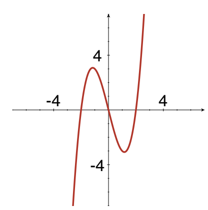
31. Since $f(x) = (-x - 6)^2 - 4 = (x + 6)^2 - 4$, the vertex of the graph is $(-6, -4)$, so the graph of $f(x)$ is the graph of $y = x^2$ shifted to the left by 6 units and shifted down by 4 units.

32. Since $f(x) = -(1 - x)^2 + 1 = -(x - 1)^2 + 1$, the vertex of the graph is $(1, -1)$, so the graph of $f(x)$ is the graph of $y = x^2$ reflected in the x-axis, shifted to the right by 1 unit, and shifted up by 1 unit.

33. **End behavior:** For large x, the graph is like that of $y = x^3$.

Symmetry: Since the powers of x are all odd, the graph is symmetric with respect to the origin.

Intercepts: Since $f(0) = 0$, the y-intercept is $(0, 0)$. Solving $f(x) = x^3 - 4x = x(x - 2)(x + 2) = 0$, we see that the x-intercepts are $(-2, 0)$, $(0, 0)$, and $(2, 0)$.

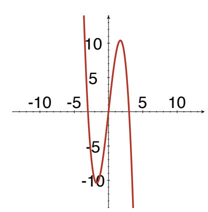


Graph: From $f(x) = x^1(x - 2)^1(x + 2)^1$ we see that 0, 2, and -2 are all simple zeros.

34. **End behavior:** For large x, the graph is like that of $y = -x^3$.

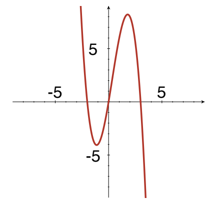
Symmetry: Since the powers of x are all odd, the graph is symmetric with respect to the origin.

Intercepts: Since $f(0) = 0$, the y-intercept is $(0, 0)$. Solving $f(x) = 9x - x^3 = x(3 + x)(3 - x) = 0$, we see that the x-intercepts are $(-3, 0)$, $(0, 0)$, and $(3, 0)$.



Graph: From $f(x) = -x^1(x + 3)^1(x - 3)^1$ we see that 0, -3, and 3 are all simple zeros.

35. **End behavior:** For large x, the graph is like that of $y = -x^3$.

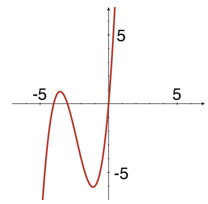


Symmetry: Since the powers of x are both even and odd, the graph has no symmetry with respect to the origin or y-axis.

Intercepts: Since $f(0) = 0$, the y-intercept is $(0, 0)$. Solving $f(x) = -x^3 + x^2 + 6x = -x(x+2)(x-3) = 0$, we see that the x-intercepts are $(0, 0)$, $(-2, 0)$, and $(3, 0)$.

Graph: From $f(x) = -x^1(x+2)^1(x-3)^1$ we see that 0, -2, and 3 are all simple zeros.

36. **End behavior:** For large x, the graph is like that of $y = x^3$.

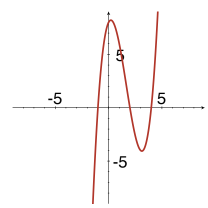


Symmetry: Since the powers of x are both even and odd, the graph has no symmetry with respect to the origin or y-axis.

Intercepts: Since $f(0) = 0$, the y-intercept is $(0, 0)$. Solving $f(x) = x^3 + 7x^2 + 12x = x(x+3)(x+4) = 0$, we see that the x-intercepts are $(0, 0)$, $(-3, 0)$, and $(-4, 0)$.

Graph: From $f(x) = x^1(x+3)^1(x+4)^1$ we see that 0, -3, and -4 are all simple zeros.

37. **End behavior:** For large x, the graph is like that of $y = x^3$.

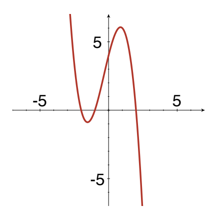


Symmetry: Since the powers of x in $f(x) = (x+1)(x-2)(x-4) = x^3 - 5x^2 + 2x + 8$ are both even and odd, the graph has no symmetry with respect to the origin or y-axis.

Intercepts: Since $f(0) = 8$, the y-intercept is $(0, 8)$. Solving $f(x) = (x+1)(x-2)(x-4) = 0$, we see that the x-intercepts are $(-1, 0)$, $(2, 0)$, and $(4, 0)$.

Graph: From $f(x) = (x+1)^1(x-2)^1(x-4)^1$ we see that -1, 2, and 4 are all simple zeros.

38. **End behavior:** For large x, the graph is like that of $y = -x^3$.

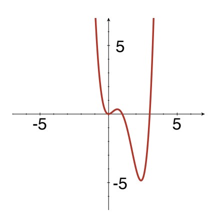


Symmetry: Since the powers of x in $f(x) = (2-x)(x+2)(x+1) = -x^3 - x^2 + 4x + 4$ are both even and odd, the graph has no symmetry with respect to the origin or y-axis.

Intercepts: Since $f(0) = 4$, the y-intercept is $(0, 4)$. Solving $f(x) = (2-x)(x+2)(x+1) = 0$, we see that the x-intercepts are $(2, 0)$, $(-2, 0)$, and $(-1, 0)$.

Graph: From $f(x) = -(x-2)^1(x+2)^1(x+1)^1$ we see that 2, -2, and -1 are all simple zeros.

39. **End behavior:** For large x, the graph is like that of $y = x^4$.



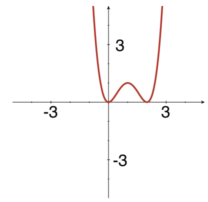
Symmetry: Since the powers of x are both even and odd, the graph has no symmetry with respect to the origin or y-axis.

Intercepts: Since $f(0) = 0$, the y-intercept is $(0, 0)$. Solving $f(x) = x^4 - 4x^3 + 3x^2 = x^2(x-1)(x-3) = 0$, we see that the x-intercepts are $(0, 0)$, $(1, 0)$, and $(3, 0)$.

Graph: From $f(x) = x^2(x-1)^1(x-3)^1$ we see that 1 and 3 are simple zeros and the graph is tangent to the x-axis at $x = 0$.

40. **End behavior:** For large x, the graph is like that of $y = x^4$.

Symmetry: Since the powers of x in $f(x) = x^2(x-2)^2 = x^4 - 4x^3 + 4x^2$ are both even and odd, the graph has no symmetry with respect to the origin or y-axis.

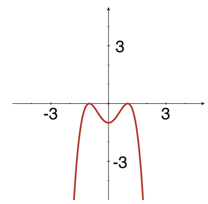


Intercepts: Since $f(0) = 0$, the y-intercept is $(0, 0)$. Solving $f(x) = x^2(x-2)^2 = 0$, we see that the x-intercepts are $(0, 0)$ and $(2, 0)$.

Graph: From $f(x) = x^2(x-2)^2$ we see that the graph is tangent to the x-axis at $x = 0$ and $x = 2$.

41. **End behavior:** For large x, the graph is like that of $y = -x^4$.

Symmetry: Since the powers of x are all even, the graph is symmetric with respect to the y-axis.

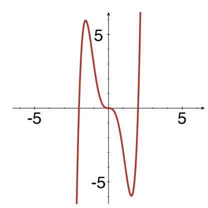


Intercepts: Since $f(0) = -1$, the y-intercept is $(0, -1)$. Solving $f(x) = -(x^4 - 2x^2 + 1) = -(x+1)^2(x-1)^2 = 0$, we see that the x-intercepts are $(-1, 0)$ and $(1, 0)$.

Graph: From $f(x) = -(x+1)^2(x-1)^2$ we see that the graph is tangent to the x-axis at $x = -1$ and $x = 1$.

42. **End behavior:** For large x, the graph is like that of $y = x^5$.

Symmetry: Since the powers of x are all odd, the graph is symmetric with respect to the origin.

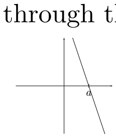


Intercepts: Since $f(0) = 0$, the y-intercept is $(0, 0)$. Solving $f(x) = x^5 - 4x^3 = x^3(x+2)(x-2) = 0$, we see that the x-intercepts are $(0, 0)$, $(-2, 0)$, and $(2, 0)$.

Graph: From $f(x) = x^3(x+2)^1(x-2)^1$ we see that -2 and 2 are simple zeros and the graph is tangent to but passes through, x-axis.

To solve Problems 43–48, first note that all of the functions have zeros at $x = 0$ and $x = 1$. Next, note whether the exponents of x and $x - 1$ are 1, even, or odd and greater than 1. Finally, use the facts that when the exponent of $x - a$ is

1: the graph passes directly through the x-axis at $x = a$;



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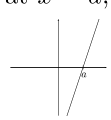
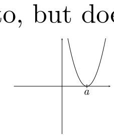


even: the graph is tangent to, but does not pass through the x-axis at $x = a$;



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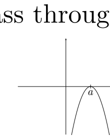
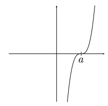


odd and greater than 1: the graph is tangent to, and passes through the x-axis.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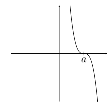


The forms of the functions in (a)-(f) are

(a) $x^{\text{even}}(x-1)^{\text{even}}$ (b) $x^{\text{odd}}(x-1)^1$
(c) $x^{\text{odd}}(x-1)^{\text{odd}}$ (d) $x^1(x-1)^{\text{odd}}$
(e) $x^{\text{even}}(x-1)^1$ (f) $x^{\text{odd}}(x-1)^{\text{even}}$

Because each of these is distinct, it is not necessary for this set of problems to consider whether the lead coefficient is positive or negative.

43. The form of the function must be $x^{\text{odd}}(x-1)^{\text{even}}$, so this graph corresponds to (f).

44. The form of the function must be $x^{\text{odd}}(x-1)^{\text{odd}}$, so this graph corresponds to (c).

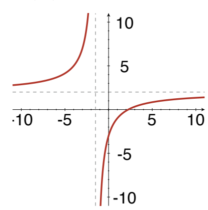
45. The form of the function must be $x^{\text{even}}(x-1)^1$, so this graph corresponds to (e).

46. The form of the function must be $x^{\text{even}}(x-1)^{\text{even}}$, so this graph corresponds to (a).

47. The form of the function must be $x^{\text{odd}}(x-1)^1$, so this graph corresponds to (b).

48. The form of the function must be $x^1(x-1)^{\text{odd}}$, so this graph corresponds to (d).

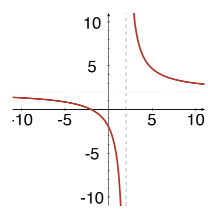
49. **Vertical asymptotes:** Setting $2x+3=0$ we see that $x=-3/2$ is a vertical asymptote.



Horizontal asymptote: The degree of the numerator equals the degree of the denominator, so $y=4/2=2$ is the horizontal asymptote.

Intercepts: Since $f(0)=-3$, the y-intercept is $(0,-3)$. Setting $4x-9=0$ we see that $x=9/4$, so $(9/4,0)$ is the x-intercept.

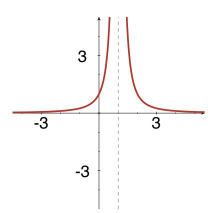
50. **Vertical asymptotes:** Setting $x-2=0$ we see that $x=2$ is a vertical asymptote.



Horizontal asymptote: The degree of the numerator equals the degree of the denominator, so $y=2/1=2$ is the horizontal asymptote.

Intercepts: Since $f(0)=-2$, the y-intercept is $(0,-2)$. Setting $2x+4=0$ we see that $x=-2$, so $(-2,0)$ is the x-intercept.

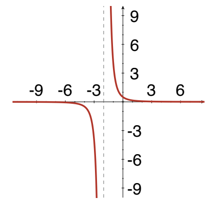
51. **Vertical asymptotes:** Setting $(x-1)^2=0$ we see that $x=1$ is a vertical asymptote.



Horizontal asymptote: The degree of the numerator is less than the degree of the denominator, so $y=0$ is the horizontal asymptote.

Intercepts: Since $f(0)=1$, the y-intercept is $(0,1)$. The numerator is never zero, so there are no x-intercepts.

52. **Vertical asymptotes:** Setting $(x+2)^3 = 0$ we see that $x = -2$ is a vertical asymptote.

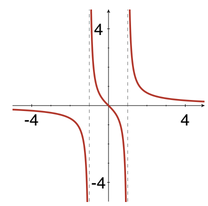


Horizontal asymptote: The degree of the numerator is less than the degree of the denominator, so $y = 0$ is the horizontal asymptote.

Intercepts: Since $f(0) = 4/(2^3) = 1/2$, the y-intercept is $(0, 1/2)$. The numerator is never 0, so there are no x-intercepts.

Graph: The left branch must lie entirely in the third quadrant since there are no x-intercepts, and $f(x) < 0$ for $x < -2$. The right branch must lie above the x-axis because it passes through $(0, 1/2)$ and there are no x-intercepts.

53. **Vertical asymptotes:** Setting $x^2 - 1 = (x+1)(x-1) = 0$ we see that $x = -1$ and $x = 1$ are vertical asymptotes.

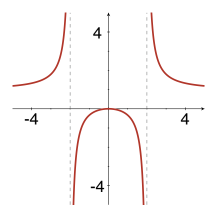


Horizontal asymptote: The degree of the numerator is less than the degree of the denominator, so $y = 0$ is the horizontal asymptote.

Intercepts: Since $f(0) = 0$, the y-intercept is $(0, 0)$. Since the numerator is simply x, $(0, 0)$ is also the only x-intercept.

Graph: We use the facts that the only x-intercept is $(0, 0)$ and the x-axis is a horizontal asymptote. For $x < -1$, $f(x) < 0$, so the left branch is below the x-axis. For $-1 < x < 0$, $f(x) > 0$, and for $0 < x < 1$, $f(x) < 0$, so the middle branch passes through the origin (as opposed to being tangent to the origin and lying strictly above or below the x-axis). For $x > 1$, $f(x) > 0$, so the right branch is above the x-axis.

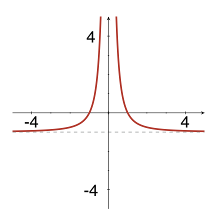
54. **Vertical asymptotes:** Setting $x^2 - 4 = (x+2)(x-1) = 2$ we see that $x = -2$ and $x = 2$ are vertical asymptotes.



Horizontal asymptote: The degree of the numerator equals the degree of the denominator, so $y = 1/1 = 1$ is the horizontal asymptote.

Intercepts: Since $f(0) = 0$, the y-intercept is $(0, 0)$. Setting $x^2 = 0$ we see that $x = 0$, so $(0, 0)$ is also the only x-intercept.

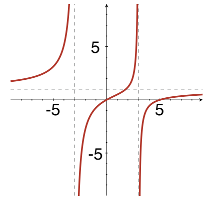
55. **Vertical asymptotes:** Setting $x^2 = 0$ we see that $x = 0$, or the y-axis, is a vertical asymptote.



Horizontal asymptote: The degree of the numerator equals the degree of the denominator, so $y = -1/1 = -1$ is the horizontal asymptote.

Intercepts: Since the y-axis is a vertical asymptote, there is no y-intercept. Setting $1 - x^2 = (1+x)(1-x) = 0$ we see that $x = -1$ and $x = 1$, so $(-1, 0)$ and $(1, 0)$ are x-intercepts.

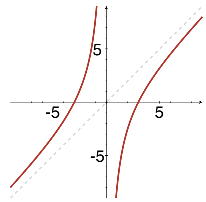
56. **Vertical asymptotes:** Setting $x^2 - 9 = (x+3)(x-3) = 0$ we see that $x = -3$ and $x = 3$ are vertical asymptotes.



Horizontal asymptote: The degree of the numerator equals the degree of the denominator, so $y = 1/1 = 1$ is the horizontal asymptote.

Intercepts: Since $f(0) = 0$, the y-intercept is $(0,0)$. Setting $x(x-5) = 0$ we see that $x = 0$ and $x = 5$, so $(0,0)$ and $(5,0)$ are x-intercepts.

57. **Vertical asymptotes:** Setting the denominator equal to zero, we see that $x = 0$, or the y-axis, is a vertical asymptote.



Slant asymptote: Since the degree of the numerator is one greater than the degree of the denominator, the graph of $f(x)$ possesses a slant asymptote. From $f(x) = (x^2 - 9)/x = x - 9/x$, we see that $y = x$ is a slant asymptote.

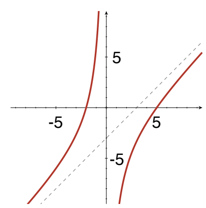
Intercepts: Since the y-axis is a vertical asymptote, the graph has no y-intercept. Setting $x^2 - 9 = (x+3)(x-3) = 0$ we see that $x = -3$ and $x = 3$, so $(-3,0)$ and $(3,0)$ are the x-intercepts.

Graph: We need to determine if the graph crosses the slant asymptote. To do this, we solve

$$\frac{x^2-9}{x} = x; \quad x^2 - 9 = x^2; \quad -9 = 0.$$

Since there is no solution, the graph does not cross its slant asymptote.

58. **Vertical asymptotes:** Setting the denominator equal to 0, we see that $x = 0$ or the y-axis is a vertical asymptote.



Slant asymptote: Since the degree of the numerator is one greater than the degree of the denominator, the graph of $f(x)$ possesses a slant asymptote. From $f(x) = (x^2 - 3x - 10)/x = x - 3 - 10/x$, we see that $y = x - 3$ is a slant asymptote.

Intercepts: Since the y-axis is a vertical asymptote, the graph has no y-intercept. Setting $(x+2)(x-5) = 0$ we see that $x = -2$ and $x = 5$ or $(-2,0)$ and $(5,0)$ are the x-intercepts.

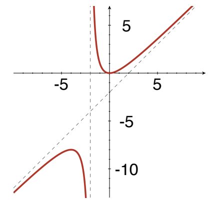
Graph: We can just about find the graph from the asymptotes and intercepts, but we need to determine if the graph crosses the slant asymptote. To do this, we solve

$$\frac{x^2-3x-10}{x} = x - 3; \quad x^2 - 3x - 10 = x^2 - 3x; \quad -10 = 0.$$

Since there is no solution, the graph does not cross its slant asymptote.

59. **Vertical asymptotes:** Setting $x + 2 = 0$, we see that $x = -2$ is a vertical asymptote.

Slant asymptote: Since the degree of the numerator is one greater than the degree of the denominator, the graph of $f(x)$ possesses a slant asymptote. Using synthetic division we see that $f(x) = x^2/(x+2) = x - 2 + 4/(x+2)$, and the slant asymptote is $y = x - 2$.



Intercepts: Since $f(0) = 0$, the y-intercept is $(0,0)$. Setting $x^2 = 0$ we see that $x = 0$, so $(0,0)$ is also the only x-intercept.

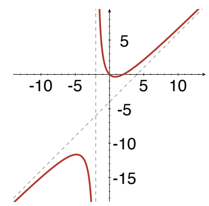
Graph: We need to determine if the graph crosses the slant asymptote. To do this, we solve

$$\frac{x^2}{x+2} = x - 2; \quad x^2 = x^2 - 4; \quad 0 = -4.$$

Since there is no solution, the graph does not cross its slant asymptote.

60. **Vertical asymptotes:** Setting $x + 2 = 0$, we see that $x = -2$ is a vertical asymptote.

Slant asymptote: Since the degree of the numerator is one greater than the degree of the denominator, the graph of $f(x)$ possesses a slant asymptote. Using synthetic division we see that $f(x) = (x^2 - 2x)/(x+2) = x - 4 + 8/(x+2)$, and the slant asymptote is $y = x - 4$.



Intercepts: Since $f(0) = 0$, the y-intercept is $(0,0)$. Setting $x^2 - 2x = x(x-2) = 0$ we see that $x = 0$ and $x = 2$, so $(0,0)$ and $(2,0)$ are the x-intercept.

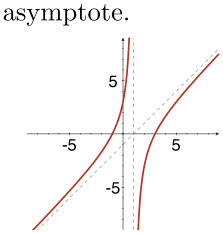
Graph: We need to determine if the graph crosses the slant asymptote. To do this, we solve

$$\frac{x^2 - 2x}{x+2} = x - 4; \quad x^2 - 2x = x^2 - 2x - 8; \quad 0 = -8.$$

Since there is no solution, the graph does not cross its slant asymptote.

61. **Vertical asymptotes:** Setting $x - 1 = 0$, we see that $x = 1$ is a vertical asymptote.

Slant asymptote: Since the degree of the numerator is one greater than the degree of the denominator, the graph of $f(x)$ possesses a slant asymptote. Using synthetic division we see that $f(x) = (x^2 - 2x - 3)/(x-1) = x - 1 + 4/(x-1)$, and the slant asymptote is $y = x - 1$.



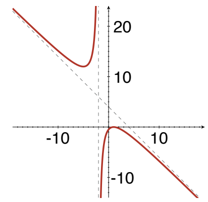
Intercepts: Since $f(0) = -3/(-1) = 3$, the y-intercept is $(0,3)$. Setting $(x+1)(x-3) = 0$ we see that $x = -1$ and $x = 3$ or $(-1,0)$ and $(3,0)$ are the x-intercepts.

Graph: We can just about find the graph from the asymptotes and intercepts, but we need to determine if the graph crosses the slant asymptote. To do this, we solve

$$\frac{x^2 - 2x - 3}{x-1} = x - 1; \quad x^2 - 2x - 3 = x^2 - 2x + 1; \quad -3 = 1.$$

Since there is no solution, the graph does not cross its slant asymptote.

62. **Vertical asymptotes:** Setting $x + 2 = 0$, we see that $x = -2$ is a vertical asymptote.



Slant asymptote: Since the degree of the numerator is one greater than the degree of the denominator, the graph of $f(x)$ possesses a slant asymptote. Using synthetic division we see that $f(x) = -(x-1)^2/(x+2) = (-x^2+2x-1)/(x+2) = -x+4-9/(x+2)$ and the slant asymptote is $y = -x+4$.

Intercepts: Since $f(0) = -1/2$, the y-intercept is $(0, -1/2)$. Setting $-(x-1)^2 = 0$ we see that $x = 1$, so $(1, 0)$ is also the only x-intercept.

Graph: We need to determine if the graph crosses the slant asymptote. To do this, we solve

$$\frac{-(x-1)^2}{x+2} = -x+4; \quad -x^2+2x-1 = -x^2+2x+8; \quad -1 = 8.$$

Since there is no solution, the graph does not cross its slant asymptote.

63. Set $f(x) = -1$: Set $f(x) = 2$:

$$\begin{aligned} \frac{2x-1}{x+4} &= -1 \\ 2x-1 &= -x-4 \\ x &= -1 \end{aligned} \qquad \begin{aligned} \frac{2x-1}{x+4} &= 2 \\ 2x-1 &= 2x+8 \\ -1 &= 8 \end{aligned}$$

Thus, -1 is in the range and 2 is not.

64. The degree of the numerator equals the degree of the denominator, so $y = 1$ is the horizontal asymptote. To determine the points where the graph of $f(x)$ crosses the horizontal asymptote $y = 1$ we solve

$$\begin{aligned} \frac{(x-3)^2}{x^2-5x} &= 1 \\ x^2-6x+9 &= x^2-5x \\ -6x+9 &= -5x; \quad 9 = x. \end{aligned}$$

Thus, $f(x)$ crosses its horizontal asymptote at $(9, 1)$.

65. Begin by calculating $\dfrac{\Delta T_F}{\Delta T_C} = \dfrac{140-32}{60-0} = \dfrac{9}{5}$. Then:

$$\begin{aligned} T_F - 32 &= \frac{9}{5}(T_C - 0) \\ T_C &= \frac{9}{5}T_C + 32 \end{aligned}$$

Try it out: when $T_C = 100$, $T_F = \dfrac{9}{5}(100) + 32 = 212$.

66. Begin by calculating $\dfrac{\Delta T_K}{\Delta T_C} = \dfrac{300-273}{27-0} = 1$. Then:

$$\begin{aligned} T_K - 273 &= T_C - 0 \\ T_K &= T_C + 273 \end{aligned}$$

Try it out: when $T_C = 100$, $T_K(100) = 100 + 273 = 373°$ K.

67. Identifying $t = 20$, $P = 1000$, and $r = 0.034$, we have

$$A = P + Prt = 1000 + 1000(0.034)(20) = 1000 + 680 = \$1680.$$

Assuming that P and r remain the same, we solve $220 = 1000 + 1000(0.034)t$ for t. This gives $t = 12/0.34 \approx 35.29$ years.

68. If x is the number of years, then $A(x) = ax + b$ for appropriate choices of a and b. Since $A = 20,000$ when $x = 0$, we have $20,000 = a(0) + b = b$. Since $A = 0$ when $x = 25$, we have $0 = a(25) + b = 25a + 20,000$, so $25a = -20,000$ and $a = -400$. Thus, $A(x) = -400x + 20,000$. For $x = 10$ years, the item will have a value of $A(10) = -400(10) + 20,000 = -4000 + 20,000 = 16,000$.

69. The ball is on the ground when $s(t) = 0$. Solving $-16t^2 + 96t = 0$, we find $t = 0$ seconds and 6 seconds.

70. The ball is 80 feet above the ground when $s(t) = 80$. Solving $-16t^2 + 96t = 80$, we find $t = 1$ second and 5 seconds. The ball reaches its highest point when $t = 3$ seconds. This is $s(3) = 144$ feet.

71. The slope of a line is its rate of change, which means the change in output for each (positive) unit change in input. In this case, the slope is $5/2$, so when x is changed by one unit, y will change by $5/2 = 2.5$ units. When x is changed by 2 units, y will change by $2(5/2) = 5$ units, and when x is changed by n units, y will change by $(5/2)n$ units.

72. Using $f(x) = ax + b$, we have

$$\begin{aligned} f\left(\frac{x_1 + x_2}{2}\right) &= a\left(\frac{x_1 + x_2}{2}\right) + b = \frac{1}{2}(ax_1 + ax_2) + b \\ &= \frac{1}{2}(ax_1 + b + ax_2 + b) = \frac{f(x_1) + f(x_2)}{2}. \end{aligned}$$

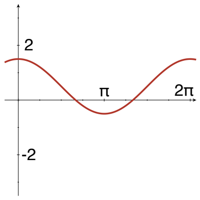
Geometrically, this says that for $a > 0$, the point on the line connecting $(x_1, f(x_1))$ and $(x_2, f(x_2))$ that lies directly above the midpoint of the interval $[x_1, x_2]$ lies level with the midpoint of the interval $[f(x_1), f(x_2)]$ on the y-axis.

73. First, find the slope of the line through $(\frac{1}{2}, 10)$ and $(\frac{3}{2}, 4)$; this is $m = -6$. The slope of a perpendicular line is then $-1/m$; in this case, $\frac{1}{6}$. To find the point the line passes through, find the midpoint of $(\frac{1}{2}, 10)$ and $(\frac{3}{2}, 4)$; this is $(1, 7)$. Now, find the equation of the line through the midpoint with slope $-1/m$; this is $y = \frac{1}{6}x + \frac{41}{6}$.

74. First, find the slopes of the lines through each pair of points; these are 2, $-\frac{1}{2}$, and 1. If the product of any two of these slopes is -1, then two of the sides are perpendicular and the triangle is a right triangle. In this case, $2\left(-\frac{1}{2}\right) = -1$, so $(2, 3)$, $(-1, -3)$, and $(4, 2)$ are vertices of a right triangle.

1.4 Transcendental Functions

1.



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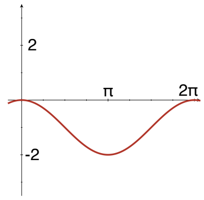

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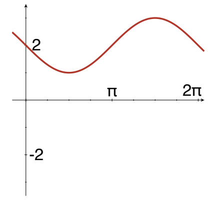

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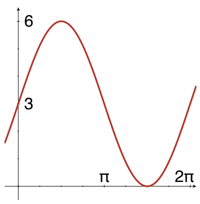

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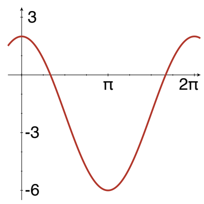

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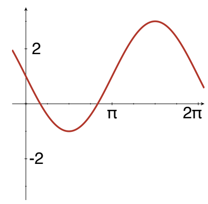
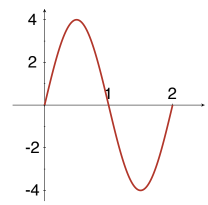
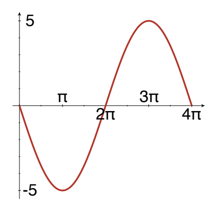


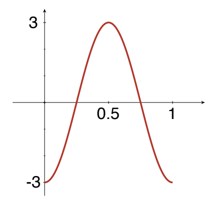
7. The amplitude of $y = 4\sin\pi x$ is $A = 4$ and the period is $\dfrac{2\pi}{\pi} = 2$.



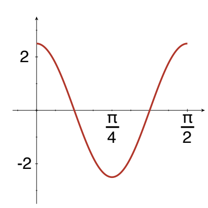
8. The amplitude of $y = -5\sin\dfrac{x}{2}$ is $A = |-5| = 5$ and the period is $\dfrac{2\pi}{\left(\frac{1}{2}\right)} = 4\pi$.



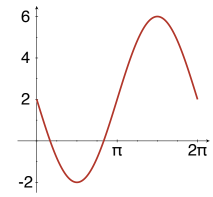
9. The amplitude of $y = -3\cos 2\pi x$ is $A = |-3| = 3$ and the period is $\dfrac{2\pi}{2\pi} = 1$.



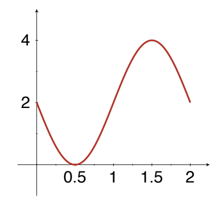
10. The amplitude of $y = \dfrac{5}{2}\cos 4x$ is $A = \dfrac{5}{2}$ and the period is $\dfrac{2\pi}{4} = \dfrac{\pi}{2}$.



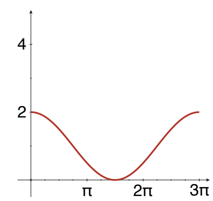
11. The amplitude of $y = 2 - 4\sin x$ is $A = |-4| = 4$ and the period is $\dfrac{2\pi}{1} = 2\pi$.



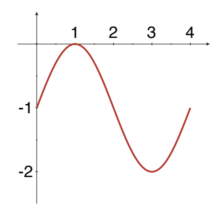
12. The amplitude of $y = 2 - 2\sin \pi x$ is $A = |-2| = 2$ and the period is $\dfrac{2\pi}{\pi} = 2$.



13. The amplitude of $y = 1 + \cos\dfrac{2x}{3}$ is $A = 1$ and the period is $\dfrac{2\pi}{\left(\frac{2}{3}\right)} = 3\pi$.



14. The amplitude of $y = -1 + \sin\dfrac{\pi x}{2}$ is $A = 1$ and the period is $\dfrac{2\pi}{\left(\frac{\pi}{2}\right)} = 4$.



15. We begin with $y = A\sin x$ since the graph flattens out at $\pi/2$ and $3\pi/2$. The amplitude is $\frac{1}{2}[3-(-3)] = 3$ and the graph has been reflected through the line $y = 0$, so $A = -3$. Thus, $y = -3\sin x$.

16. Since the graph flattens out at $x = 0$ we use $y = A\cos x$. The amplitude is $\frac{1}{2}\left[\frac{1}{4} - \left(\frac{1}{4}\right)\right] = \frac{1}{4}$. Thus $y = \frac{1}{4}\cos x$.

17. Since the graph flattens out at $x = 0$, we use $y = A\cos x + D$. The amplitude is $\frac{1}{2}[4-(-2)] = 3$ and the graph has been reflected through the line $y = 1$, so $A = -3$ and $D = 1$. Thus $y = -3\cos x + 1$.

18. We begin with $y = A\sin x + D$ since the graph flattens out at $\pi/2$ and $3\pi/2$. The amplitude is $\frac{1}{2}[0-(-1)] = \frac{1}{2}$ and the graph has been reflected through the line $y = -\frac{1}{2}$, so $A = -\frac{1}{2}$ and $D = -\frac{1}{2}$. Thus, $y = -\frac{1}{2}\sin x - \frac{1}{2}$.

19. Since the y-intercept is $(0, 0)$, the equation has the form $y = A\sin Bx$. The amplitude of the graph is $A = 3$ and the period is $\pi = 2\pi/B$, so $B = 2$ and $y = 3\sin 2x$.

20. Since the y-intercept is $(0, -2)$ and not $(0, 0)$, the equation has the form $y = A \cos Bx$. The amplitude is 2, and the graph has been reflected through the line $y = 0$, so $A = -2$. The period is $\pi = 2\pi/B$, so $B = 2$, and $y = -2\cos 2x$.

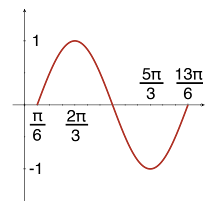
21. Since the y-intercept is $(0, \frac{1}{2})$ and not $(0, 0)$, the equation has the form $y = A \cos Bx$. The amplitude of the graph is $A = \frac{1}{2}$ and the period is $2 = 2\pi/B$, so $B = \pi$ and $y = \frac{1}{2}\cos \pi x$.

22. Since the y-intercept is $(0, 2)$ and not $(0, 0)$, the equation has the form $y = A \cos Bx$. The amplitude of the graph is $A = 2$ and the period is $4 = 2\pi/B$, so $B = \pi/2$ and $y = 2\cos(\pi x/2)$.

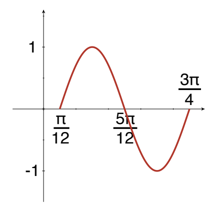
23. Since the y-intercept is $(0, 0)$, the equation has the form $y = A \sin Bx$. The amplitude of the graph is 1, and the graph has been reflected through the line $y = 0$, so $A = 1$. The period is $2 = 2\pi/B$, so $B = \pi$ and $y = -\sin \pi x$.

24. Since the y-intercept is $(0, 3)$ and not $(0, 0)$, the equation has the form $y = A \cos Bx$. The amplitude of the graph is $A = 3$ and the period is $8 = 2\pi/B$, so $B = \pi/4$ and $y = 3\cos(\pi x/4)$.

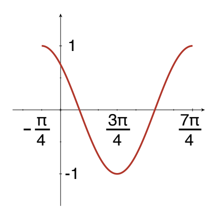
25. The amplitude of $y = \sin(x - \pi/6)$ is $A = 1$ and the period is $2\pi/1 = 2\pi$. The phase shift is $|-\pi/6|/1 = \pi/6$. Since $C = -\pi/6 < 0$, the shift is to the right.



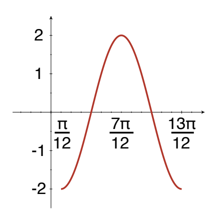
26. The amplitude of $y = \sin(3x - \pi/4)$ is $A = 1$ and the period is $2\pi/3$. The phase shift is $|-\pi/4|/3 = \pi/12$. Since $C = -\pi/4 < 0$, the shift is to the right.



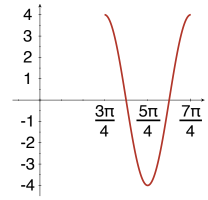
27. The amplitude of $y = \cos(x + \pi/4)$ is $A = 1$ and the period is $2\pi/1 = 2\pi$. The phase shift is $|\pi/4|/1 = \pi/4$. Since $C = \pi/4 > 0$, the shift is to the left.



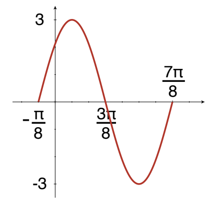
28. The amplitude of $y = -2\cos(2x - \pi/6)$ is $A = |-2| = 2$ and the period is $2\pi/2 = \pi$. The phase shift is $|-\pi/6|/2 = \pi/12$. Since $C = -\pi/6 < 0$, the shift is to the right.



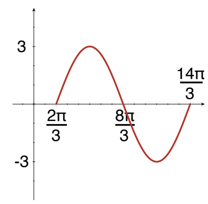
29. The amplitude of $y = 4\cos(2x - 3\pi/2)$ is $A = 4$ and the period is $2\pi/2 = \pi$. The phase shift is $|-3\pi/2|/2 = 3\pi/4$. Since $C = -3\pi/2 < 0$, the shift is to the right.



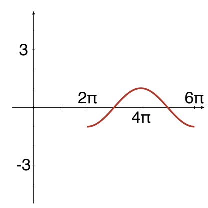
30. The amplitude of $y = 3\cos(2x + \pi/4)$ is $A = 3$ and the period is $2\pi/2 = \pi$. The phase shift is $|\pi/4|/2 = \pi/8$. Since $C = \pi/8 > 0$, the shift is to the left.



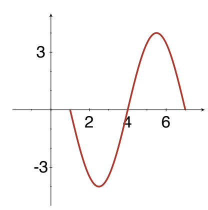
31. The amplitude of $y = 3\sin(x/2 - \pi/3)$ is $A = 3$ and the period is $2\pi/(1/2) = 4\pi$. The phase shift is $|-\pi/3|/(1/2) = 2\pi/3$. Since $C = -\pi/3 < 0$, the shift is to the right.



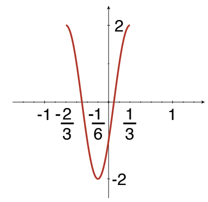
32. The amplitude of $y = -\cos(x/2 - \pi)$ is $A = |-1| = 1$ and the period is $2\pi/(1/2) = 4\pi$. The phase shift is $|-\pi|/(1/2) = 2\pi$. Since $C = -\pi < 0$, the shift is to the right.



33. The amplitude of $y = -4\sin(\pi x/3 - \pi/3)$ is $A = |-4| = 4$ and the period is $2\pi/(\pi/3) = 6$. The phase shift is $|-\pi/3|/(\pi/3) = 1$. Since $C = -\pi/3 < 0$, the shift is to the right.



34. The amplitude of $y = 2\cos(-2\pi x - 4\pi/3) = 2\cos(2\pi x + 4\pi/3)$ is $A = 2$ and the period is $2\pi/2\pi = 1$. The phase shift is $|4\pi/3|/2\pi = 2/3$. Since $C = 4\pi/3 > 0$, the shift is to the left.



35. $y = 5\sin\left[\pi\left(x - \dfrac{1}{2}\right)\right] = 5\sin\left(\pi x - \dfrac{\pi}{2}\right)$

36. $y = -8 + \cos\left[\dfrac{1}{2}\left(x - \dfrac{2\pi}{3}\right)\right] = -8 + \cos\left(\dfrac{x}{2} - \dfrac{\pi}{3}\right)$

37. Setting $-1 + \sin x = 0$ we have $\sin x = 1$, which is true for $x = \pi/2$ in $[0, 2\pi]$. By periodicity, the x-intercepts of $-1 + \sin x$ are $(\pi/2 + 2n\pi, 0)$, where n is an integer.

38. Setting $1 - 2\cos x = 0$ we have $2\cos x = 1$ or $\cos x = 1/2$. Since $\cos x$ is positive, x is in the first or fourth quadrant. Since $\cos x = 1/2$ for $x = \pi/3$, we see that $1 - 2\cos x = 0$ for $x = \pi/3$ and $x = 5\pi/3$ in $[0, 2\pi]$. By periodicity, then, the x-intercepts of $1 + 2\cos x$ are at
$$\frac{\pi}{3} + 2n\pi = \left(\frac{1}{3} + 2n\right)\pi \qquad \text{and} \qquad \frac{5\pi}{3} + 2n\pi = \left(\frac{5}{3} + 2n\right)\pi,$$
so the intercepts are
$$\left(\frac{\pi}{3} + 2n\pi, 0\right) \qquad \text{and} \qquad \left(\frac{5\pi}{3} + 2n\pi, 0\right), \text{ where } n \text{ is an integer.}$$

39. Setting $\sin \pi x = 0$, we have by (3) in this section of the text that the x-intercepts are determined by $\pi x = n\pi$, n an integer. Thus, $x = n$, and the x-intercepts are $(n, 0)$, where n is an integer.

40. Setting $-\cos 2x = 0$, we have by (4) in this section of the text that the x-intercepts are determined by $2x = (2n+1)\pi/2$, n an integer. Thus, $x = (2n+1)\pi/4$, so the intercepts are $(\pi/4 + n\pi/2, 0)$, where n is an integer.

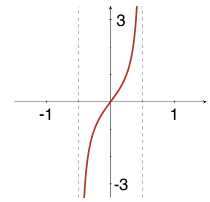
41. Setting $10\cos(x/2) = 0$, we have by (4) in this section of the text that the x-intercepts are determined by $x/2 = (2n+1)\pi/2$, n an integer. Thus, $x = (2n+1)\pi$, so the intercepts are $(\pi + 2n\pi, 0)$, where n is an integer.

42. Setting $3\sin(-5x) = 0$, we have by (3) in this section of the text that the x-intercepts are determined by $-5x = n\pi$, n an integer. Thus, $x = -n\pi/5$, and the x-intercepts are $(-n\pi/5, 0)$, where n is an integer.

43. Setting $\sin(x - \pi/4) = 0$, we have by (3) in this section of the text that the x-intercepts are determined by $x - \pi/4 = n\pi$, n an integer. Thus, $x = n\pi + \pi/4 = (n + 1/4)\pi$, and the x-intercepts are $(\pi/4 + n\pi, 0)$, where n is an integer.

44. Setting $\cos(2x - \pi) = 0$, we have by (4) in this section of the text that the x-intercepts are determined by $2x - \pi = (2n+1)\pi/2$, n an integer. Thus,
$$x = \frac{1}{2}\left[(2n+1)\frac{\pi}{2} + \pi\right] = \frac{3\pi}{4} + \frac{n\pi}{2},$$
so the intercepts are $(3\pi/4 + n\pi/2, 0)$, where n is an integer.

45. The period of $y = \tan \pi x$ is $\pi/\pi = 1$. Since

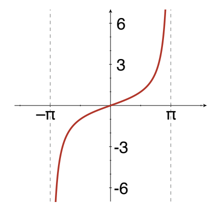


$$\tan \pi x = \frac{\sin \pi x}{\cos \pi x},$$

the x-intercepts of $\tan \pi x$ occur at the zeros of $\sin \pi x$; namely, at $\pi x = n\pi$ or $x = n$ for n an integer. The vertical asymptotes occur at the zeros of $\cos \pi x$; namely, at

$$\pi x = \frac{(2n+1)\pi}{2} \qquad \text{or} \qquad x = \frac{1}{2}(2n+1) \qquad \text{for } n \text{ an integer.}$$

46. The period of $y = \tan(x/2)$ is $\pi/(1/2) = 2\pi$. Since



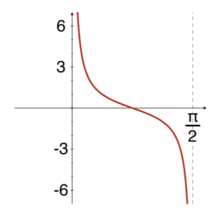
$$\tan \frac{x}{2} = \frac{\sin(x/2)}{\cos(x/2)},$$

the x-intercepts of $\tan(x/2)$ occur at the zeros of $\sin(x/2)$; namely, at $x/2 = n\pi$ or $x = 2n\pi$ for n an integer. The vertical asymptotes occur at the zeros of $\cos(x/2)$; namely, at

$$\frac{x}{2} = \frac{(2n+1)\pi}{2} \qquad \text{or} \qquad x = (2n+1)\pi = \pi + 2n\pi \qquad \text{for } n \text{ an integer.}$$

Since the graph has vertical asymptotes at $-\pi$ and π (using $n = -1$ and $n = 0$), we graph one cycle on the interval $(-\pi, \pi)$.

47. The period of $y = \cot 2x$ is $\pi/2$. Since



$$\cot 2x = \frac{\cos 2x}{\sin 2x},$$

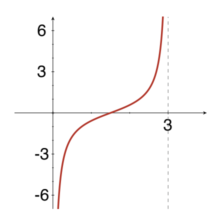
the x-intercepts of $\cot 2x$ occur at the zeros of $\cos 2x$; namely, at

$$2x = \frac{(2n+1)\pi}{2} \qquad \text{or} \qquad x = \frac{(2n+1)\pi}{4} \qquad \text{for } n \text{ an integer.}$$

The vertical asymptotes occur at the zeros of $\sin 2x$; namely, at

$$2x = n\pi \qquad \text{or} \qquad x = \frac{n\pi}{2} \qquad \text{for } n \text{ an integer.}$$

48. The period of $y = -\cot(\pi x/3)$ is $\pi/(\pi/3) = 3$. Since



$$-\cot \frac{\pi x}{3} = -\frac{\cos(\pi x/3)}{\sin(\pi x/3)},$$

the x-intercepts of $-\cot(\pi x/3)$ occur at the zeros of $\cos(\pi x/3)$; namely, at

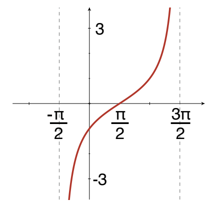
$$\frac{\pi x}{3} = \frac{(2n+1)\pi}{2} \qquad \text{or} \qquad x = \frac{3}{2}(2n+1) \qquad \text{for } n \text{ an integer.}$$

The vertical asymptotes occur at the zeros of $\sin(\pi x/3)$; namely, at

$$\frac{\pi x}{3} = n\pi \qquad \text{or} \qquad x = 3n \qquad \text{for } n \text{ an integer.}$$

Since $A = -1$, the graph of $y = -\cot(\pi x/3)$ is the graph of $y = \cot(\pi x/3)$ reflected through the x-axis.

49. The period of $y = \tan(x/2 - \pi/4)$ is $\pi/(1/2) = 2\pi$. Since



$$\tan\left(\frac{x}{2} - \frac{\pi}{4}\right) = \frac{\sin(x/2 - \pi/4)}{\cos(x/2 - \pi/4)},$$

the x-intercepts of $\tan(x/2 - \pi/4)$ occur at the zeros of $\sin(x/2 - \pi/4)$; namely, at

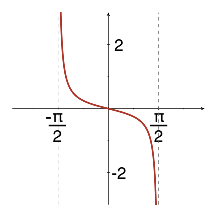
$$\frac{x}{2} - \frac{\pi}{4} = n\pi \qquad \text{or} \qquad x = 2n\pi + \frac{\pi}{2} \qquad \text{for } n \text{ an integer.}$$

The vertical asymptotes occur at the zeros of $\cos(x/2 - \pi/4)$; namely, at

$$\frac{x}{2} - \frac{\pi}{4} = \frac{(2n+1)\pi}{2} \qquad \text{or} \qquad x = (2n+1)\pi + \frac{\pi}{2} = \frac{3\pi}{2} + 2n\pi \qquad \text{for } n \text{ an integer.}$$

Since the graph has vertical asymptotes at $-\pi/2$ and $3\pi/2$ (using $n = -1$ and $n = 0$), we graph one cycle on the interval $(-\pi/2, 3\pi/2)$.

50. The period of $y = \frac{1}{4}\cot(x - \pi/2)$ is $\pi/1 = \pi$. Since



$$\frac{1}{4}\cot\left(x - \frac{\pi}{2}\right) = \frac{1}{4} \cdot \frac{\cos(x - \pi/2)}{\sin(x - \pi/2)},$$

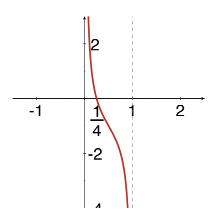
the x-intercepts of $\frac{1}{4}\cot(x - \pi/2)$ occur at the zeros of $\cos(x - \pi/2)$; namely, at

$$x - \frac{\pi}{2} = \frac{(2n+1)\pi}{2} \qquad \text{or} \qquad x = (n+1)\pi \qquad \text{for } n \text{ an integer.}$$

The vertical asymptotes occur at the zeros of $\sin(x - \pi/2)$; namely, at

$$x - \frac{\pi}{2} = n\pi \qquad \text{or} \qquad x = n\pi + \frac{\pi}{2} \qquad \text{for } n \text{ an integer.}$$

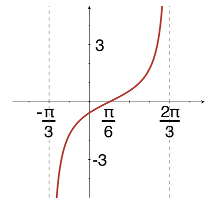
51. The period of $y = -1 + \cot \pi x$ is $\pi/\pi = 1$. To find the x-intercepts, we solve $-1 + \cot \pi x = 0$, or $\cot \pi x = \cos \pi x / \sin \pi x = 1$, which is equivalent to solving for $\cos \pi x = \sin \pi x$. This occurs when



$$\pi x = (4n+1)\frac{\pi}{4} \qquad \text{or} \qquad x = n + \frac{1}{4} \qquad \text{for } n \text{ an integer.}$$

The vertical asymptotes occur at the zeros of $\sin \pi x$; namely, $\pi x = n\pi$ or $x = n$, for n an integer.

52. The period of $y = \tan(x + 5\pi/6)$ is π. Since



$$\tan(x + 5\pi/6) = \frac{\sin(x + 5\pi/6)}{\cos(x + 5\pi/6)},$$

the x-intercepts of $\tan(x+5\pi/6)$ occur at the zeros of $\sin(x+5\pi/6)$; namely, at

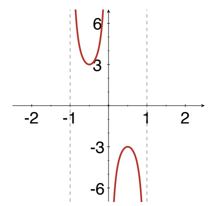
$$x + \frac{5\pi}{6} = n\pi \qquad \text{or} \qquad x = n\pi - \frac{5\pi}{6} \qquad \text{for } n \text{ an integer.}$$

The vertical asymptotes occur at the zeros of $\cos(x + 5\pi/6)$; namely, at

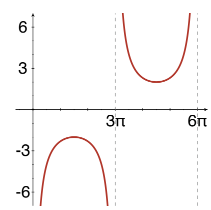
$$x + \frac{5\pi}{6} = (2n+1)\frac{\pi}{2} \qquad \text{or} \qquad x = (2n+1)\frac{\pi}{2} - \frac{5\pi}{6} = -\frac{\pi}{3} + n\pi \qquad \text{for } n \text{ an integer.}$$

Since the graph has vertical asymptotes at $-\pi/3$ and $2\pi/3$ (using $n = 0$ and $n = 1$), we graph one cycle on the interval $(-\pi/3, 2\pi/3)$.

53. The period of $y = 3\csc \pi x$ is $2\pi/\pi = 2$. Since $3\csc \pi x = 3/\sin \pi x$, the vertical asymptotes occur at the zeros of $\sin \pi x$; namely, at $\pi x = n\pi$ or $x = n$, for n an integer. We plot one cycle of the graph on $(-1, 1)$, since the period of the function is $2 = 1 - (-1)$ and vertical asymptotes occur at $x = -1$ and $x = 1$ (taking $n = -1$ and 1).



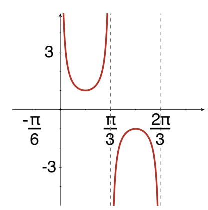
54. The period of $y = -2\csc(x/3)$ is $2\pi/(1/3) = 6\pi$. Since



$$-2\csc\frac{x}{3} = -\frac{2}{\sin(x/3)},$$

the vertical asymptotes occur at the zeros of $\sin(x/3)$; namely, at $x/3 = n\pi$ or $x = 3\pi n$, for n an integer.

55. The period of $y = \sec(3x - \pi/2)$ is $2\pi/3$. Since

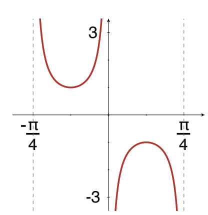


$$\sec\left(3x - \frac{\pi}{2}\right) = \frac{1}{\cos(3x - \pi/2)},$$

the vertical asymptotes occur at the zeros of $\cos(3x - \pi/2)$; namely, at

$$3x - \frac{\pi}{2} = (2n+1)\frac{\pi}{2} \qquad \text{or} \qquad x = (n+1)\frac{\pi}{3} \qquad \text{for } n \text{ an integer.}$$

56. The period of $y = \csc(4x + \pi)$ is $2\pi/4 = \pi/2$. Since



$$\csc(4x + \pi) = \frac{1}{\sin(4x + \pi)},$$

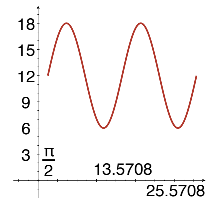
the vertical asymptotes occur at the zeros of $\sin(4x + \pi)$; namely, at

$$4x + \pi = n\pi \qquad \text{or} \qquad x = (n-1)\frac{\pi}{4} = -\frac{\pi}{4} + \frac{n\pi}{4} \qquad \text{for } n \text{ an integer.}$$

We plot one cycle of the graph on $(-\pi/4, \pi/4)$ since the period of the function is $\pi/2 = \pi/4 - (-\pi/4)$ and vertical asymptotes occur at $x = -\pi/4$ and $x = \pi/4$ (taking $n = 0$ and 2).

57. The amplitude is $A = \frac{1}{2}(18 - 6) = 6$. The tidal period is $2\pi/B = 12$, so $B = \pi/6$, and the average depth $D = (18 + 6)/2 = 12$. Thus, the function is

$$d(t) = 12 + 6 \sin \frac{\pi}{6} \left(t - \frac{\pi}{2}\right).$$



58. (a) Since 8 A.M. is 8 hours after midnight,

$$T(8) = 50 + 10 \sin \frac{\pi}{12}(8 - 8) = 50$$

and the temperature at 8 A.M. is 50° F.

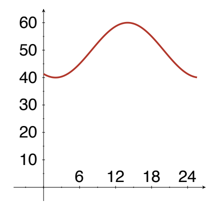
(b) We solve

$$\begin{aligned} 50 + 10 \sin \frac{\pi}{12}(t - 8) &= 60 \\ \sin \frac{\pi}{12}(t - 8) &= 1 \\ \frac{\pi}{12}(t - 8) &= \frac{\pi}{2} \\ t - 8 = 6;\quad t &= 14. \end{aligned}$$

Since 14 hours after midnight is noon plus 2 hours, the temperature will be at 60° F at 2 P.M.

(c) The amplitude of the graph is $A = 10$ and the graph is vertically centered at $T = 50$. The period is $2\pi/(\pi/12) = 24$ and the phase shift is $|C|/B = (8\pi/12)/(\pi/12) = 8$. Since $C = 8\pi/12 > 0$, the graph of $y = 50 + 10 \sin \pi t/12$ is shifted 8 units to the left to obtain the graph of

$$T = 50 + 10 \sin \left(\frac{\pi}{12} t - \frac{8\pi}{12}\right).$$



(d) The minimum temperature will be 40° F and this will occur when $\pi(t - 8)/12 = 3\pi/2$, since $x = 3\pi/2$ where $y = \sin x$ first reaches its minimum. Solving for t, we obtain $t = 26$. Since the period is 24, the temperature is first minimum when $t = 2$ or at 2 A.M. The maximum temperature will be 60° F and this will occur when $\pi(t - 8)/12 = \pi/2$, since $x = \pi/2$ where $y = \sin x$ first reaches its maximum. Solving for t, we obtain $t = 14$. Thus, the temperature is first maximum when $t = 14$ or at 2 P.M. (**Note:** We could have reached the same conclusion about the first occurrence of the maximum temperature from the information in parts (b) and (c).)

59. (a) When $\theta = 0°$, $\sin \theta = 0$ and $\sin 2\theta = 0$, so $g = 978.0309$ cm/s^2.

(b) At the north pole, $\theta = 90°$, so $\sin \theta = 1$ and $\sin 2\theta = \sin 180° = 0$, so $g = 978.0309 + 5.18552 = 983.2164$ cm/s^2.

(c) When $\theta = 45°$, $\sin \theta = \sqrt{2}/2$ and $\sin 2\theta = \sin 90° = 1$, so

$$\begin{aligned} g &= 978.0309 + 5.18552(\sqrt{2}/2)^2 - 0.00570(1)^2 \\ &= 978.0309 + 2.59276 - 0.00570 = 980.618 \text{ cm/s}^2. \end{aligned}$$

60. (a) The range of a shot put released from a height of 2.0 m above the ground is

$$R = \frac{13.7\cos 40^\circ}{9.81}\left[13.7\sin 40^\circ + \sqrt{(13.7^2\sin^2 40^\circ + 2(9.81)(2)}\right] = 20.98 \text{ m},$$

while the range of a shot put released from a height of 2.4 m above the ground is

$$R = \frac{13.7\cos 40^\circ}{9.81}\left[13.7\sin 40^\circ + \sqrt{(13.7^2\sin^2 40^\circ + 2(9.81)(2.4)}\right] = 21.36 \text{ m}.$$

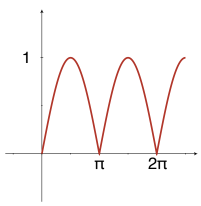
Thus, an additional 0.4 m above the ground results in an additional $21.36 - 20.98 = 0.38$ m in range.

(b) If h increases while the other parameters remain fixed, the $2gh$ term increases and, as a result, the product increases and the range R increases.

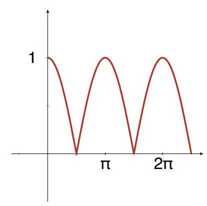
(c) This demonstrates that a taller shot-putter will achieve a longer range.

61. The period of $\sin\frac{1}{2}x$ is $2\pi/(1/2) = 4\pi$. The period of $\sin 2x$ is $2\pi/2 = \pi$. The period of a sum of periodic functions is equal to the least common multiple of the individual periods, so the period of $f(x)$ is 4π.

62. The graph of the absolute value of a function is the graph of the function with any portions of the graph that lie below the x-axis reflected through the x-axis.

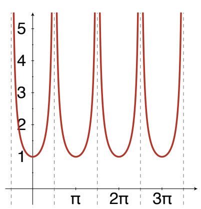


$y = |\sin x|$

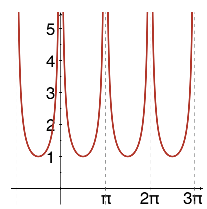


$y = |\cos x|$

63. The graph of the absolute value of a function is the graph of the function with any portions of the graph that lie below the x-axis reflected through the x-axis.



$y = |\sec x|$



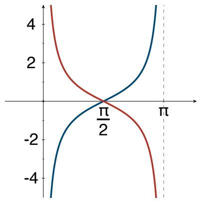
$y = |\csc x|$

64. (a) We have $9\csc x = 9/\sin x = 1$, or $\sin x = 9$, which is not possible for any real number x since $-1 \le \sin x \le 1$.

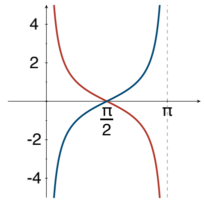
(b) We have $10\sec x = 10/\cos x = -7$, or $\cos x = -10/7$, which is not possible for any real number x since $-1 \le \cos x \le 1$.

(c) We have $\sec x = 1/\cos x = -10.5 = -11/2$, or $\cos x = -2/11$, which is possible for some real number x since $-1 \le \cos x \le 1$.

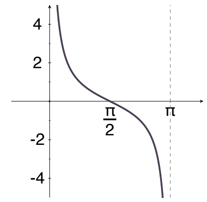
65. The graph of $y = \cot x$ is shown below as a red curve, while the graphs of $y = A\tan(x + C)$, for various choices of A and C, are shown as blue curves.



$A = 1, C = \pi/2$



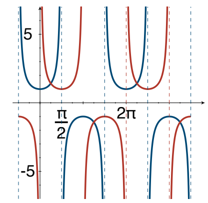
$A = 1, C = -\pi/2$



$A = -1, C = -\pi/2$

We see from the third graph that $\cot x = -\tan(x - \pi/2)$.

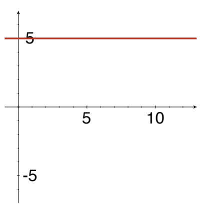
66. Comparing the graphs of $y = \sec x$, shown in blue, and $y = \csc x$, shown in red, we see that the graph of the cosecant is equivalent to the graph of the secant shifted $\pi/2$ units to the right. Thus, $\csc x = \sec(x - \pi/2)$, and we identify $A = 1$ and $C = -\pi/2$.



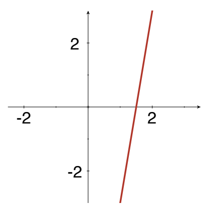
1.5 Inverse Functions

1. For $f(x) = 1 + x(x - 5)$, the value $y = 1$ in the range of f occurs at either $x = 0$ or $x = 5$ in the domain of f. Thus, $f(x)$ is not one-to-one.

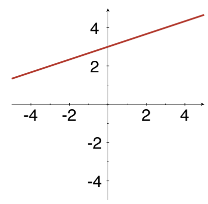
2. For $f(x) = x^4 + 2x^2$, the value $y = 3$ in the range of f occurs at either $x = -1$ or $x = 1$ in the domain of f. Thus, $f(x)$ is not one-to-one.

3. 

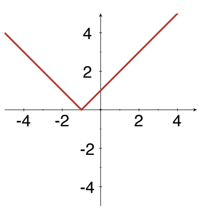
Not one-to-one.

4. 

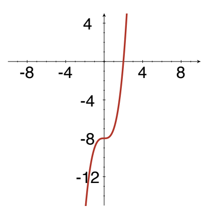
One-to-one.

5. 

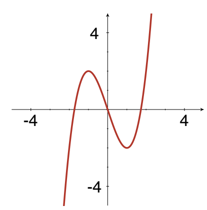
One-to-one.

6. 

Not one-to-one.

7. 

One-to-one.

8. 

Not one-to-one.

9. $y = 3x^3 + 7, \quad x = \left(\dfrac{y-7}{3}\right)^{1/3}; \quad f^{-1}(x) = \left(\dfrac{x-7}{3}\right)^{1/3}$

10. $y = \sqrt[3]{2x - 4}, \quad x = \dfrac{1}{2}y^3 + 2; \quad f^{-1}(x) = \dfrac{1}{2}x^3 + 2$

11. $y = \dfrac{2-x}{1-x}, \quad x = \dfrac{y-2}{y-1}; \quad f^{-1}(x) = \dfrac{x-2}{x-1}$

12. $y = 5 - \dfrac{2}{x}, \quad x = \dfrac{2}{5-y}; \quad f^{-1}(x) = \dfrac{2}{5-x}$

13. $f(f^{-1}(x)) = f\left(\dfrac{1}{5}x + 2\right) = 5\left(\dfrac{1}{5}x+2\right) - 10 = x + 10 - 10 = x$

$f^{-1}(f(x)) = f^{-1}(5x-10) = \dfrac{1}{5}(5x-10) + 2 = x - 2 + 2 = x$

14. $f(f^{-1}(x)) = f\left(\dfrac{1-x}{x}\right) = \dfrac{1}{\dfrac{1-x}{x}+1} = \dfrac{1}{\dfrac{1-x}{x}+1}\cdot\left(\dfrac{x}{x}\right) = \dfrac{x}{1-x+x} = x$

$f^{-1}(f(x)) = f^{-1}\left(\dfrac{1}{x+1}\right) = \dfrac{1-\dfrac{1}{x+1}}{\dfrac{1}{x+1}} = \dfrac{1-\dfrac{1}{x+1}}{\dfrac{1}{x+1}}\cdot\left(\dfrac{x+1}{x+1}\right) = \dfrac{x+1-1}{1} = x$

15. Domain: $[0, \infty)$; Range: $[-2, \infty)$

16. Domain: $[3, \infty)$; Range: $[1/2, \infty)$

17. Domain: $(-\infty, 0) \cup (0, \infty)$; Range: $(-\infty, -3) \cup (-3, \infty)$

18. Domain: $(-\infty, 1) \cup (1, \infty)$; Range: $(-\infty, 4) \cup (4, \infty)$

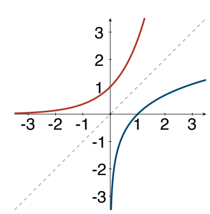
19. When $x = 2$, $y = f(2) = 2(2)^3 + 2(2) = 2(8) + 4 = 20$, so the point on the graph of f is $(2, 20)$. The corresponding point on the graph of f^{-1} is then $(20, 2)$.

20. When $x = 5$, $y = f(5) = 37$, so the point on the graph of f is $(5, 37)$. The corresponding point on the graph of f^{-1} is then $(37, 5)$.

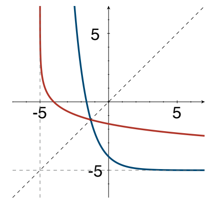
21. Since $f(9) = 12$, $(9, 12)$ is a point on the graph of f. The corresponding point on the graph of f^{-1} is then $(12, 9)$, and so $f^{-1}(12) = 9$.

22. Since $f\left(\dfrac{1}{2}\right) = \dfrac{4(1/2)}{1/2+1} = \dfrac{2}{3/2} = \dfrac{4}{3}$, $(1/2, 4/3)$ is a point on the graph of f. The corresponding point on the graph of f^{-1} is then $(4/3, 1/2)$, and therefore $f^{-1}(4/3) = 1/2$.

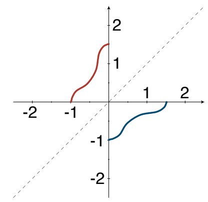
23. The graph of f^{-1}, shown in red, is obtained from the graph of f, shown in blue, by reflection through the line $y = x$.



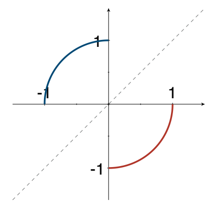
24. The graph of f^{-1}, shown in red, is obtained from the graph of f, shown in blue, by reflection through the line $y = x$.



25. To graph f, shown in blue, we use the fact that f is the inverse of f^{-1}, shown in red, and that the graph of the inverse of a function is the reflection of the graph of the original function reflected through the line $y = x$.



26. To graph f, shown in blue, we use the fact that f is the inverse of f^{-1}, shown in red, and that the graph of the inverse of a function is the reflection of the graph of the original function reflected through the line $y = x$.



27. By restricting f's domain to $x \geq 5/2$, we get $f^{-1}(x) = \dfrac{5 - \sqrt{x}}{2}, x \geq 0$.

28. By restricting f's domain to $x \geq 0$, we get $f^{-1}(x) = \left(\dfrac{x - 9}{3}\right)^{1/2}, x \geq 9$.

29. $f(x)$ can be rewritten as $(x + 1)^2 + 3$. Thus, by restricting the domain of f to $x \geq -1$, we get $f^{-1}(x) = \sqrt{x - 3} - 1$.

30. $f(x)$ can be rewritten as $-(x - 4)^2 + 16$. By restricting the domain of f to $x \geq 4$, we get $f^{-1}(x) = \sqrt{16 - x} + 4$.

31. For $f(x) = x^3$ and $g(x) = 4x + 5$, we get $(f \circ g)(x) = (4x + 5)^3$ and thus $(f \circ g)^{-1}(x) = \frac{1}{4}(x^{1/3} - 5)$. $f^{-1}(x) = x^{1/3}$ and $g^{-1}(x) = \frac{1}{4}(x - 5)$, so $(g^{-1} \circ f^{-1})(x) = \frac{1}{4}(x^{1/3} - 5)$, which is the same as $(f \circ g)^{-1}$.

32. Solving $y = \sqrt[3]{x} - \sqrt[3]{y}$ for x, we get $x = (y + \sqrt[3]{y})^3$. Thus, $f^{-1}(x) = (x + \sqrt[3]{x})^3$.

33. $3\pi/4$

34. $\pi/3$

35. $\pi/4$

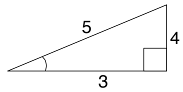
36. $\pi/3$

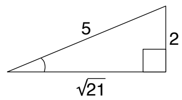
37. $3\pi/4$

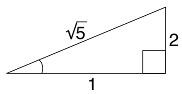
38. π

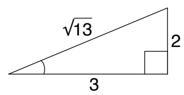
39. $-\pi/3$

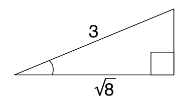
40. $5\pi/6$

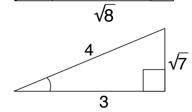
41. $\sin\left(\arctan\frac{4}{3}\right) = \frac{4}{5}$ 

42. $\cos\left(\sin^{-1}\frac{2}{5}\right) = \frac{\sqrt{21}}{5}$ 

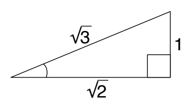
43. $\tan\left(\cot^{-1}\frac{1}{2}\right) = 2$ 

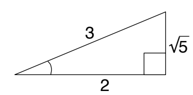
44. $\csc\left(\tan^{-1}\frac{2}{3}\right) = \frac{\sqrt{13}}{2}$ 

45. $\sin\left(2\sin^{-1}\frac{1}{3}\right) = 2\sin\left(\sin^{-1}\frac{1}{3}\right)\cos\left(\sin^{-1}\frac{1}{3}\right) = 2\left(\frac{1}{3}\right)\left(\frac{\sqrt{8}}{3}\right) = \frac{4\sqrt{2}}{9}$ 

46. $\cos\left(2\cos^{-1}\frac{3}{4}\right) = \cos^2\left(\cos^{-1}\frac{3}{4}\right) - \sin^2\left(\cos^{-1}\frac{3}{4}\right)$ 

$$= \left(\frac{3}{4}\right)^2 - \left(\frac{\sqrt{7}}{4}\right)^2 = \frac{1}{8}$$

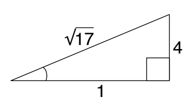
47. $\sin\left(\arcsin\frac{\sqrt{3}}{3} + \arccos\frac{2}{3}\right)$ 

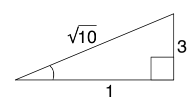


$$= \sin\left(\arcsin\frac{\sqrt{3}}{3}\right)\cos\left(\arccos\frac{2}{3}\right) + \cos\left(\arcsin\frac{\sqrt{3}}{3}\right)\sin\left(\arccos\frac{2}{3}\right)$$

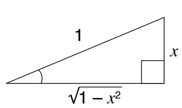
$$= \left(\frac{\sqrt{3}}{3}\right)\left(\frac{2}{3}\right) + \left(\frac{\sqrt{6}}{3}\right)\left(\frac{\sqrt{5}}{3}\right) = \frac{2\sqrt{3}}{9} + \frac{\sqrt{30}}{9} = \frac{(2+\sqrt{10})\sqrt{3}}{9}$$

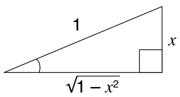
48. $\cos(\tan^{-1}4 - \tan^{-1}3)$

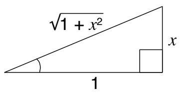
$$= \cos(\tan^{-1}4)\cos(\tan^{-1}3) + \sin(\tan^{-1}4)\sin(\tan^{-1}3)$$ 

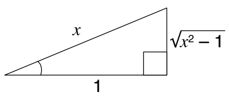


$$= \left(\frac{1}{\sqrt{17}}\right)\left(\frac{1}{\sqrt{10}}\right) + \left(\frac{4}{\sqrt{17}}\right)\left(\frac{3}{\sqrt{10}}\right) = \frac{1}{\sqrt{170}} + \frac{12}{\sqrt{170}} = \frac{13}{\sqrt{170}}$$

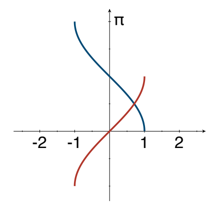
49. $\sqrt{1-x^2}$ 

50. $\dfrac{x}{\sqrt{1-x^2}}$ 

51. $\sqrt{1+x^2}$ 

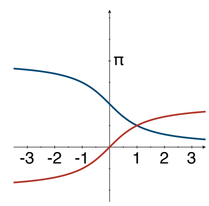
52. $\dfrac{\sqrt{x^2-1}}{x}$ 

53. From the graph on the right, it can be seen that $\sin^{-1} x$, shown in red, when reflected through the x-axis (thus multiplying by -1) then shifted up by $\pi/2$ units (thus adding $\pi/2$), is the graph of $\cos^{-1} x$, shown in blue. Thus,



$$\cos^{-1} x = -\sin^{-1} x + \frac{\pi}{2} \qquad \text{or} \qquad \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

54. From the graph on the right, it can be seen that $\arctan x$, shown in red, when reflected through the x-axis (thus multiplying by -1) then shifted up by $\pi/2$ units (thus adding $\pi/2$), is the graph of arccot x, shown in blue. Thus,

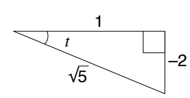


$$\text{arccot } x = -\arctan x + \frac{\pi}{2} \qquad \text{or} \qquad \text{arccot } x + \arctan x = \frac{\pi}{2}$$

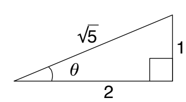
55. Let $y = \sec^{-1} x$. Then $x = \sec y = 1/\cos y$ and $\cos y = 1/x$. This implies that $y = \cos^{-1}(1/x)$. Thus, $\sec^{-1} x = \cos^{-1}(1/x)$. The domain of both $\sec^{-1} x$ and $\cos^{-1}(1/x)$ is $|x| \geq 1$.

56. Let $y = \sin^{-1}(1/x)$, for $|x| \geq 1$. Then, $\sin y = 1/x$, where $-\pi/2 \leq y \leq \pi/2$, and $\csc y = x$. It therefore follows that $\csc^{-1} x = y = \sin^{-1}(1/x)$.

57. Since $t - \sin^{-1}(-2/\sqrt{5})$ and $-\pi/2 < t < 0$, t is in the fourth quadrant. $\cos t = 1/\sqrt{5}$, $\tan t = -2$, $\cot t = -1/2$, $\sec t = \sqrt{5}$, $\csc t = -\sqrt{5}/2$.



58. Since $\theta = \arctan 1/2$ and $0 < \theta < \pi/2$, θ is in the first quadrant. $\sin\theta = 1/\sqrt{5}$, $\cos\theta = 2/\sqrt{5}$, $\cot\theta = 2$, $\sec\theta = \sqrt{5}/2$, $\csc\theta = \sqrt{5}$.



59. (a) $\sec^{-1}(-\sqrt{2}) = \cos^{-1}(-\sqrt{2}/2) \approx 2.3562$
   (b) $\csc^{-1} 2 = \sin^{-1}(1/2) \approx 0.5236$

60. (a) $\sec^{-1}(3.5) = \cos^{-1}(2/7) \approx 1.2810$
   (b) $\csc^{-1}(-1.25) = \sin^{-1}(-4/5) \approx -0.9273$

61. (b) The range of the arctangent function is $(-\pi/2, \pi/2)$, and 5 is not in this interval.

62. $\sin^{-1}(\sin x) \approx 1.4416$; $\sin(\sin^{-1} x)$ does not exist since the domain of the arcsine function is $[-1, 1]$ and 1.7 is not in this interval.

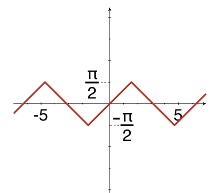
63. $\dfrac{x(1+c^2)}{Lc} - c = \tan\beta;\quad \beta = \tan^{-1}\left[\dfrac{(1+c^2)x}{cL} - c\right]$

(a) $\beta = \tan^{-1}\left[\dfrac{2L}{3L} - 1\right] = \tan^{-1}(1) = \dfrac{\pi}{4}$

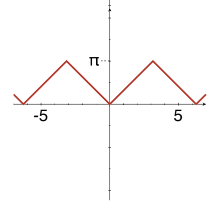
(b) $\beta = \tan^{-1}\left[\dfrac{(1.25)(\frac{3}{4}L)}{0.5L} - 0.5\right] = \tan^{-1}(1.375) \approx 0.942$ radian $\approx 53.97°$

64. $\theta = \tan^{-1}(60/300) \approx 0.1974$ radian $\approx 11.31°$

65. Theorem 1.5.2(i) is not violated because the range of the arcsine function is $[-\pi/2, \pi/2]$, while the range of $f(x) = x$ is $(-\infty, \infty)$.



66. Theorem 1.5.2(iii) is not violated because the range of the arcsine function is $[0, \pi]$, while the range of $f(x) = x$ is $(-\infty, \infty)$.

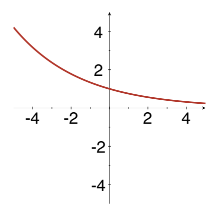


67. A periodic function cannot be one-to-one, since periodic functions have repeated y-values over regular intervals of x- values. For a function to be one-to-one, every y in its range must correspond to a single x in its domain.

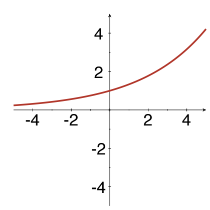
68. The functions' symmetry across the line $y = x$ means that $f^{-1}(x) = f(x)$. Functions that have this property include $f(x) = x$, $f(x) = -x$, and $f(x) = 1/x$, among many others.

1.6 Exponential and Logarithmic Functions

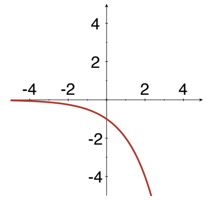
1. Since $f(0) = (3/4)^0 = 1$, the y-intercept is $(0, 1)$. The x-axis is a horizontal asymptote.



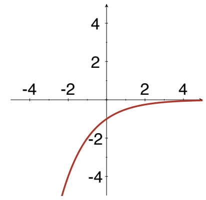
2. Since $f(0) = (4/3)^0 = 1$, the y-intercept is $(0, 1)$. The x-axis is a horizontal asymptote.



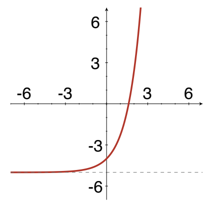
3. Since $f(0) = -2^0 = -1$, the y-intercept is $(0, -1)$. The x-axis is a horizontal asymptote and the graph of $f(x) = -2^x$ is the graph of $y = 2^x$ reflected in the x-axis.



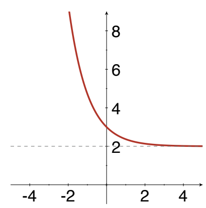
4. Since $f(0) = -2^0 = -1$, the y-intercept is $(0, -1)$. The x-axis is a horizontal asymptote.



5. Since $f(0) = -5 + e^0 = -5 + 1 = -4$, the y-intercept is $(0, -4)$. The line $y = -5$ is a horizontal asymptote.



6. Since $f(0) = 2 + e^0 = 2 + 1 = 3$, the y-intercept is $(0, 3)$. The line $y = 2$ is a horizontal asymptote.



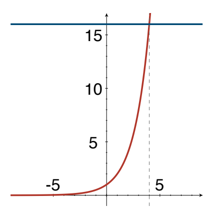
7. Letting $x = 3$ and $f(3) = 216$, we have $f(3) = 216 = b^3$, so $b = 6$ and $f(x) = 6^x$.

8. Letting $x = -1$ and $f(-1) = 5$, we have $f(-1) = 5 = b^{-1}$, so $b = \dfrac{1}{5}$ and $f(x) = \left(\dfrac{1}{5}\right)^x$.

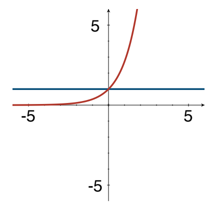
9. Letting $x = -1$ and $f(-1) = e^2$, we have $f(-1) = e^2 = b^{-1}$, so $b = e^{-2}$ and $f(x) = (e^{-2})^x = e^{-2x}$.

10. Letting $x = 2$ and $f(2) = e$, we have $f(2) = e = b^2$, so $b = e^{1/2}$ and $f(x) = (e^{1/2})^x = e^{x/2}$.

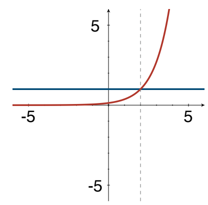
11. Graphing $y = 2^x$ and $y = 16$, we see that $2^x > 16$ for $x > 4$.



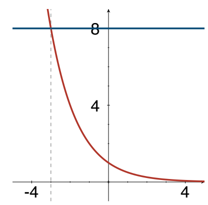
12. Graphing $y = e^x$ and $y = 1$, we see that $e^x \leq 1$ for $x \leq 0$.



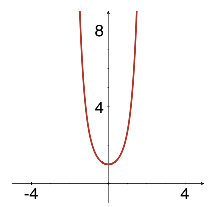
13. Graphing $y = e^{x-2}$ and $y = 1$, we see that $e^{x-2} < 1$ for $x < 2$.



14. Graphing $y = \left(\frac{1}{2}\right)^x$ and $y = 8$, we see that $\left(\frac{1}{2}\right)^x \geq 8$ for $x \leq -3$.

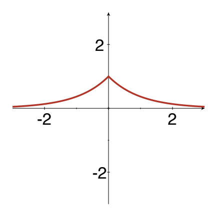


15. Since $f(-x) = e^{(-x)^2} = e^{x^2} = f(x)$, we see that $f(x)$ is even.

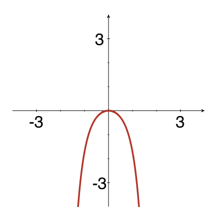


16. Since $f(-x) = e^{-|-x|} = e^{-|x|} = f(x)$, we see that $f(x)$ is even. To sketch the graph, we note that

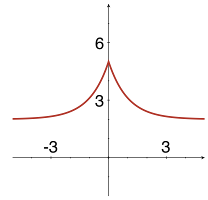
$$e^{-|x|} = \begin{cases} e^{-x}, & x \geq 0 \\ e^{x}, & x < 0. \end{cases}$$



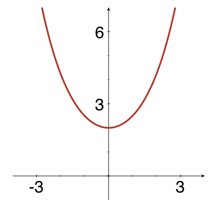
17. The graph of $f(x) = 1 - e^{x^2}$ is the graph of $y = e^{x^2}$ reflected through the x-axis and shifted up 1 unit.



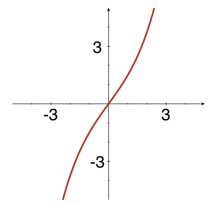
18. The graph of $f(x) = 2+3e^{-|x|}$ is the graph of $y = e^{-|x|}$ vertically stretched by a factor of 3 and shifted up 2 units.



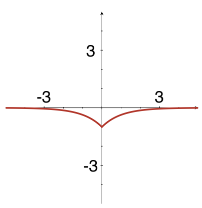
19. Since $f(-x) = 2^{-x} + 2^{-(-x)} = 2^{-x} + 2^{x} = f(x)$, we see that $f(x)$ is even.



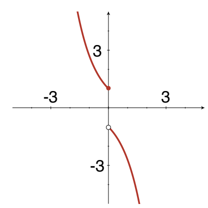
20. Since $f(-x) = 2^{-x} - 2^{-(-x)} = 2^{-x} - 2^{x} = -(2^{x} - 2^{-x}) = -f(x)$, we see that $f(x)$ is odd.



21.



22.



23. $4^{-1/2} = \frac{1}{2}$ is equivalent to $\log_4 \frac{1}{2} = -\frac{1}{2}$.

24. $9^0 = 1$ is equivalent to $\log_9 1 = 0$.

25. $10^4 = 10,000$ is equivalent to $\log_{10} 10,000 = 4$.

26. $10^{0.3010} = 2$ is equivalent to $\log_{10} 2 = 0.3010$.

27. $\log_2 128 = 7$ is equivalent to $2^7 = 128$.

28. $\log_5 \frac{1}{25} = -2$ is equivalent to $5^{-2} = \frac{1}{25}$.

29. $\log_{\sqrt{3}} 81 = 8$ is equivalent to $(\sqrt{3})^8 = 81$.

30. $\log_{16} 2 = \frac{1}{4}$ is equivalent to $16^{1/4} = 2$.

31. We solve $2 = \log_b 49$ or $b^2 = 49$, so $b = 7$ and $f(x) = \log_7 x$.

32. We solve $\frac{1}{3} = \log_b 4$ or $b^{1/3} = 4$. Cubing both sides, we have $b = 4^3 = 64$ and $f(x) = \log_{64} x$.

33. $\ln e^e = e \ln e = e$

34. $\ln(e^4 e^9) = \ln e^{4+9} = \ln e^{13} = 13$

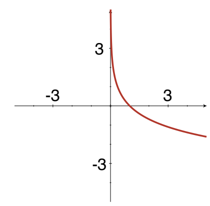
35. $10^{\log_{10} 6^2} = 6^2 = 36$

36. $25^{\log_5 8} = (5^2)^{\log_5 8} = 5^{2 \log_5 8} = 5^{\log_5 8^2} = 8^2 = 64$

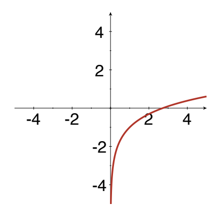
37. $e^{-\ln 7} = e^{\ln 7^{-1}} = 7^{-1} = \dfrac{1}{7}$

38. $e^{\frac{1}{2} \ln \pi} = e^{\ln \sqrt{\pi}} = \sqrt{\pi}$

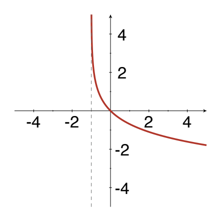
39. The domain of $\ln x$ is determined by $x > 0$, so the domain is $(0, \infty)$. The x-intercept is the solution of $-\ln x = 0$. This is equivalent to $e^0 = x$, so $x = 1$ and the x-intercept is $(1, 0)$. The vertical asymptote is $x = 0$ or the y-axis.



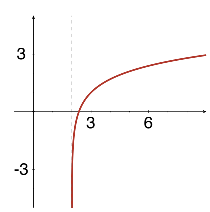
40. The domain of $\ln x$ is determined by $x > 0$, so the domain is $(0, \infty)$. The x-intercept is the solution of $-1 + \ln x = 0$ or $\ln x = 1$, so $x = e$ and the x-intercept is $(e, 0)$. The vertical asymptote is $x = 0$ or the y-axis.



41. The domain of $\ln x$ is determined by $x > 0$, so the domain of $-\ln(x+1)$ is determined by $x + 1 > 0$ or $x > -1$. Thus, the domain is $(-1, \infty)$. The x-intercept is the solution of $-\ln(x+1) = 0$. This is equivalent to $e^0 = x + 1$, or $1 = x + 1$, so $x = 0$ and the x-intercept is $(0, 0)$. The vertical asymptote is $x + 1 = 0$ or $x = -1$.



42. The domain of $\ln x$ is determined by $x > 0$, so the domain of $\ln(x-2)$ is determined by $x - 2 > 0$ or $x > 2$. Thus, the domain is $(2, \infty)$. The x-intercept is the solution of



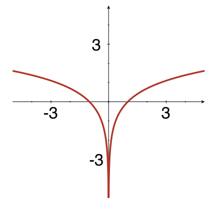
$$\begin{aligned} 1 + \ln(x-2) &= 0 \\ \ln(x-2) &= -1 \\ x - 2 &= e^{-1} \\ x &= 2 + e^{-1} \approx 2.37, \end{aligned}$$

so the x-intercept is $(2 + e^{-1}, 0)$. The vertical asymptote is $x - 2 = 0$ or $x = 2$.

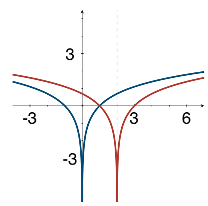
43. The domain of $\ln(9 - x^2)$ is determined by $9 - x^2 > 0$. This is equivalent to $(3+x)(3-x) > 0$. Thus, $3 + x$ and $3 - x$ must be both > 0 or both < 0, so the domain is $(-3, 3)$.

44. The domain of $\ln(x^2 - 2x)$ is determined by $x^2 - 2x > 0$. This is equivalent to $x(x-2) > 0$. Thus, x and $x - 2$ must be both > 0 or both < 0, so the domain is $(-\infty, 0) \cup (2, \infty)$.

45. Since $f(-x) = \ln|-x| = \ln|x| = f(x)$, we see that $f(x)$ is even. The x-intercepts are the solutions of $\ln|x| = 0$, so $x = 1$ or $x = -1$, and the x-intercepts are $(1, 0)$ and $(-1, 0)$. The vertical asymptote is $x = 0$ or the y-axis.



46. The graph of $y = \ln|x-2|$, shown in red, is the graph of $y = \ln|x|$, shown in blue, shifted 2 units to the right. The x-intercepts occur where $\ln|x-2| = 0$ or $|x-2| = 1$; they are $(1, 0)$ and $(3, 0)$. The vertical asymptote is $x = 2$.



47. Using (ii) of the laws of logarithms (Theorem 1.6.1) in the text, we have

$$\ln(x^4 - 4) - \ln(x^2 + 2) = \ln \frac{x^4 - 4}{x^2 + 2} = \ln \frac{(x^2 - 2)(x^2 + 2)}{x^2 + 2} = \ln(x^2 - 2).$$

48. Using the laws of logarithms (Theorem 1.6.1) in the text, we have

$$\begin{aligned} \ln\left(\frac{x}{y}\right) - 2\ln x^3 - 4\ln y &= \ln\left(\frac{x}{y}\right) - \ln x^6 - \ln y^4 = \ln\left(\frac{x}{y}\right) - \ln(x^6 y^4) \\ &= \ln\left(\frac{x}{y} \cdot \frac{1}{x^6 y^4}\right) = \ln\left(\frac{1}{x^5 y^5}\right) = -\ln(x^5 y^5). \end{aligned}$$

49. Using (i) of the laws of logarithms (Theorem 1.6.1) in the text, we have

$$\ln 5 + \ln 5^2 + \ln 5^3 - \ln 5^6 = \ln(5 \cdot 5^2 \cdot 5^3) - \ln 5^6 = \ln 5^6 - \ln 5^6 = 0 = \ln 1.$$

50. Using the laws of logarithms (Theorem 1.6.1) in the text, we have

$$5\ln 2 + 2\ln 3 - 3\ln 4 = \ln 2^5 + \ln 3^2 - \ln 4^3 = \ln \frac{2^5 \cdot 3^2}{4^3} = \ln \frac{32 \cdot 9}{64} = \ln \frac{9}{2}.$$

51. Using the laws of logarithms (Theorem 1.6.1) in the text, we have

$$\begin{aligned} \ln y = \ln \frac{x^{10}\sqrt{x^2 + 5}}{\sqrt[3]{8x^3 + 2}} &= \ln x^{10} + \ln(x^2 + 5)^{1/2} - \ln(8x^3 + 2)^{1/3} \\ &= 10\ln x + \frac{1}{2}\ln(x^2 + 5) - \frac{1}{3}\ln(8x^3 + 2). \end{aligned}$$

52. Using the laws of logarithms (Theorem 1.6.1) in the text, we have

$$\begin{aligned}\ln y &= \ln\sqrt{\frac{(2x+1)(3x+2)}{4x+3}} = \ln\left[\frac{(2x+1)(3x+2)}{4x+3}\right]^{1/2}\\ &= \frac{1}{2}\left[\ln(2x+1)+\ln(3x+2)-\ln(4x+3)\right].\end{aligned}$$

53. Using the laws of logarithms (Theorem 1.6.1) in the text, we have

$$\begin{aligned}\ln y &= \ln\frac{(x^3-3)^5(x^4+3x^2+1)^8}{\sqrt{x}(7x+5)^9} = \ln(x^3-3)^5+\ln(x^4+3x^2+1)^8-\ln x^{1/2}-\ln(7x+5)^9\\ &= 5\ln(x^3-3)+8\ln(x^4+3x^2+1)-\frac{1}{2}\ln x-9\ln(7x+5).\end{aligned}$$

54. Using the laws of logarithms (Theorem 1.6.1) in the text, we have

$$\begin{aligned}\ln y &= \ln(64x^6\sqrt{x+1}\,\sqrt[3]{x^2+2}) = \ln 64+\ln x^6+\ln(x+1)^{1/2}+\ln(x^2+2)^{1/3}\\ &= 6\ln 2+6\ln x+\frac{1}{2}\ln(x+1)+\frac{1}{3}\ln(x^2+2).\end{aligned}$$

55. We want to solve $6^x = 51$. This is equivalent to

$$\ln 6^x = \ln 51, \quad x\ln 6 = \ln 51$$
$$x = \frac{\ln 51}{\ln 6} \approx 2.1944.$$

56. We want to solve $\left(\frac{1}{2}\right)^x = 7$. This is equivalent to

$$\frac{1}{2^x} = 7, \quad 2^x = \frac{1}{7}$$
$$\ln 2^x = \ln\frac{1}{7} = \ln 7^{-1} = -\ln 7$$
$$x\ln 2 = -\ln 7, \quad x = -\frac{\ln 7}{\ln 2} \approx -2.8074.$$

57. Taking the natural logarithm of both sides, we have

$$\ln 2^{x+5} = (x+5)\ln 2 = \ln 9$$
$$x+5 = \frac{\ln 9}{\ln 2}, \quad x = -5+\frac{\ln 9}{\ln 2} \approx -1.8301.$$

58. Taking the natural logarithm of both sides, we have

$$\begin{aligned}\ln(4\cdot 7^{2x}) = \ln 4+\ln(7^{2x}) = \ln 4+2x\ln 7 &= \ln 9\\ 2x\ln 7 &= \ln 9-\ln 4\\ x = \frac{\ln 9-\ln 4}{2\ln 7} &= \frac{\ln(9/4)}{\ln 49} \approx 0.2084.\end{aligned}$$

59. Taking the natural logarithm of both sides, we have

$$\begin{aligned}
\ln 5^x &= \ln(2e^{x+1}) = \ln 2 + \ln e^{x+1} = \ln 2 + x + 1 \\
x\ln 5 &= \ln 2 + x + 1 \\
x\ln 5 - x = x(\ln 5 - 1) &= 1 + \ln 2 \\
x &= \frac{1+\ln 2}{\ln 5 - 1} \approx 2.7782.
\end{aligned}$$

60. Taking the natural logarithm of both sides, we have

$$\begin{aligned}
\ln(3^{2x-2}) &= \ln(2^{x-3}) \\
(2x-2)\ln 3 &= (x-3)\ln 2 \\
2x\ln 3 - 2\ln 3 &= x\ln 2 - 3\ln 2 \\
2x\ln 3 - x\ln 2 = x(2\ln 3 - \ln 2) &= 2\ln 3 - 3\ln 2 \\
x &= \frac{2\ln 3 - 3\ln 2}{2\ln 3 - \ln 2} = \frac{\ln(9/8)}{\ln(9/2)} \approx 0.0783.
\end{aligned}$$

In Problems 61–62, it is necessary to check that the solutions obtained actually satisfy the original equation. This is because equations involving logarithms may lead to extraneous solutions. An extraneous solution, in fact, occurs in Problem 61.

61. We use the laws of logarithms and the fact that $\ln x$ is one-to-one:

$$\begin{aligned}
\ln x + \ln(x-2) = \ln[x(x-2)] &= \ln 3 \\
x^2 - 2x &= 3 \\
x^2 - 2x - 3 &= 0 \\
(x-3)(x+1) &= 0.
\end{aligned}$$

Thus, $x = 3$ and $x = -1$. We disregard $x = -1$ since it is outside the domains of both $\ln x$ and $\ln(x-2)$. We see that $x = 3$ checks.

62. We use the laws of logarithms and the fact that $\ln x$ is one-to-one:

$$\begin{aligned}
\ln 3 + \ln(2x-1) &= \ln 4 + \ln(x+1) \\
\ln[3(2x-1)] &= \ln[4(x+1)] \\
3(2x-1) &= 4(x+1) \\
6x - 3 &= 4x + 4 \\
2x = 7, \quad x &= \frac{7}{2}.
\end{aligned}$$

The solution checks.

63. (a) Since the population doubles after 2 hours, we write $P(2) = P_0e^{2k} = 2P_0$. Solving for k gives

$$e^{2k} = 2, \quad 2k = \ln 2, \quad k = \frac{\ln 2}{2} \approx 0.3466.$$

Thus, $P(t) = P_0e^{0.3466t}$.

(b) In 5 hours, $P(5) = P_0 e^{(0.3446)(5)} \approx 5.66P_0$.

(c) Solving $P(t) = P_0 e^{0.3466t} = 20P_0$ for t, we have

$$e^{0.3466t} = 20, \quad 0.3466t = \ln 20, \quad t = \frac{\ln 20}{0.3466} \approx 8.64 \text{ hours.}$$

64. (a) We are given $A_0 = 200$, so $A(t) = 200e^{kt}$. Since 3% of 200 is 6, $A(6) = 200e^{6k} = 194$ and

$$e^{6k} = \frac{194}{200}, \quad 6k = \ln\frac{194}{200}$$
$$k = \frac{1}{6}\ln\frac{194}{200} \approx -0.0051.$$

Thus, $A(t) = 200e^{-0.0051t}$.

(b) $A(24) \approx 177.1$ mg.

(c) Solving $A(t) = 200e^{-0.0051t} = 100$ for t, we have

$$e^{-0.0051t} = \frac{100}{200} = \frac{1}{2}$$
$$-0.0051t = \ln\frac{1}{2}, \quad t = \frac{\ln(1/2)}{-0.0051} \approx 137 \text{ hours.}$$

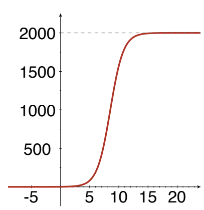
65. (a) $P(5) = \dfrac{2000}{1 + 1999e^{-0.8905(5)}} \approx 82$ students

(b) Solving $P(t) = \dfrac{2000}{1 + 1999e^{-0.8905t}} = 1000$ for t, we have

$$1 + 1999e^{-0.8905t} = 2, \quad 1999e^{-0.8905t} = 1$$
$$e^{-0.8905t} = \frac{1}{1999}$$
$$-0.8905t = \ln\left(\frac{1}{1999}\right) = -\ln 1999$$
$$t = \frac{-\ln 1999}{-0.8905} \approx 8.53 \text{ days.}$$

(c) As $t \to \infty$, $e^{-0.8905t} \to 0$, so $P(t) \to \dfrac{2000}{1+0} = 2000$.

(d) We note that as $t \to \infty$, $e^{-0.8905t} \to 0$ so the graph has a horizontal asymptote at $P = 2000$.



66. (a) When the cake is removed from the oven, its temperature is also 350° F, that is, $T_0 = 350$. The ambient temperature is the temperature of the kitchen, $T_m = 75$. Thus we have $T(t) = 75 + 275e^{kt}$. The measurement that $T(1) = 300$ is the condition that determines k. From $T(1) = 75 + 275e^k = 300$ we find

$$e^k = \frac{225}{275} = \frac{9}{11} \qquad \text{or} \qquad k = \ln\frac{9}{11} \approx -0.2007.$$

From the model $T(t) = 75 + 275e^{-0.2007t}$ we then find

$$T(6) = 75 + 275e^{-0.2007(6)} \approx 157.5° \text{ F}.$$

(b) To determine when the temperature of the cake will be 80° F, we solve the equation $T(t) = 80$ for t. Rewriting $T(t) = 75 + 275e^{-0.2007t} = 80$ as

$$e^{-0.2007t} = \frac{5}{275} = \frac{1}{55} \qquad \text{we find} \qquad t = \frac{\ln(1/55)}{-0.2007} \approx 20 \text{ min}.$$

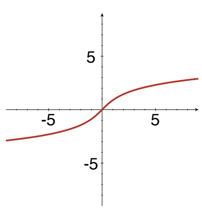
67. (a) Since $y = \ln 5x = \ln 5 + \ln x$, we can obtain the graph of $y = \ln 5x$ by shifting the graph of $y = \ln x$ up $\ln 5$ units.

(b) Since $y = \ln(x/4) = \ln x - \ln 4$, we can obtain the graph of $y = \ln(x/4)$ by shifting the graph of $y = \ln x$ down $\ln 4$ units.

(c) Since $y = \ln x^{-1} = -\ln x$, we can obtain the graph of $y = \ln x^{-1}$ by reflecting the graph of $y = \ln x$ in the x-axis.

(d) The graph of $y = \ln(-x)$ is the reflection of $y = \ln x$ in the y-axis.

68. (a)



(b) $$f(-x) = \ln(-x + \sqrt{x^2+1}) = \ln\left(\frac{-x + \sqrt{x^2+1}}{1} \cdot \frac{-x - \sqrt{x^2+1}}{-x - \sqrt{x^2+1}}\right)$$
$$= \ln\left(\frac{x^2 - (x^2+1)}{-x - \sqrt{x^2+1}}\right) = \ln\left(\frac{1}{x + \sqrt{x^2+1}}\right) = \ln 1 - \ln(x + \sqrt{x^2+1})$$
$$= -\ln(x + \sqrt{x^2+1}) = -f(x)$$

1.7 From Words to Functions

1. Let x and y be the positive numbers. Then $xy = 50$ and their sum is $S = x + y$. From $xy = 50$ we have $y = 50/x$, so

$$S(x) = x + \frac{50}{x} = \frac{x^2 + 50}{x}.$$

Since x is positive, the domain of S is $(0, \infty)$.

2. Let x be the nonzero number. Then the sum of x and its reciprocal is

$$S(x) = x + \frac{1}{x} = \frac{x^2+1}{x}.$$

Since x is nonzero, the domain of S is $\{x \mid x \neq 0\}$.

3. Let x and y be the nonnegative numbers. Then $x + y = 1$. Now, the sum of the square of x and twice the square of y is $x^2 + 2y^2$. From $x + y = 1$ we have $y = 1 - x$, so the function in this case is

$$s(x) = x^2 + 2(1-x)^2 = x^2 + 2(1 - 2x + x^2) = 3x^2 - 4x + 2.$$

Since x and y are both nonnegative, we must have $0 \leq x \leq 1$ and $0 \leq y \leq 1$. (If, say, $y > 1$, then we would have $x < 0$.) Thus

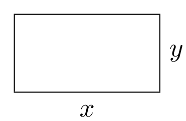
$$s(x) = 3x^2 - 4x + 2, \quad 0 \leq x \leq 1.$$

Alternatively, if we choose the independent variable of s to be y, we have $x = 1 - y$ and

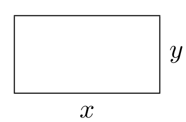
$$s(y) = (1-y)^2 + 2y^2 = 1 - 2y + y^2 + 2y^2 = 3y^2 - 2y + 1, \quad 0 \leq y \leq 1.$$

4. Let x and y be the two nonnegative numbers. Then $S = x + y$, and the product of the powers of x and y is $P = x^m y^n$. From $S = x + y$ we have $y = S - x$, so $P(x) = x^m (S - x)^n$. Since x is nonnegative, the domain of P is $(0, \infty)$.

5. Let x and y be the sides of the rectangle. Then the perimeter is $2x + 2y = 200$ and the area is $A = xy$. Solving $2x + 2y = 200$ for y, we have $y = 100 - x$, so $A(x) = x(100 - x) = 100x - x^2$. The domain of A is $[0, 100]$.



6. Let the sides of the rectangle be x and y as shown in the figure. Then $A = xy = 400$, $x,\ y > 0$. The perimeter of the rectangle is $P = 2x + 2y$. To express P in terms of just x, we use $y = 400/x$. Then



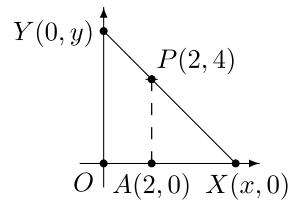
$$P(x) = 2x + 2\left(\frac{400}{x}\right) = 2x + \frac{800}{x} = \frac{2x^2 + 800}{x}, \quad x > 0.$$

Alternatively, from $xy = 400$ we have $x = 400/y$ and

$$P(x) = 2y + 2\left(\frac{400}{y}\right) = 2y + \frac{800}{y} = \frac{2y^2 + 300}{y}, \quad y > 0.$$

7. The lengths of the sides of the rectangle are x and y, so its area is $A = xy$. Since the lengths are related by $x + 2y = 4$, we have $x = 4 - 2y$ and $A(y) = (4 - 2y)y$. The y-intercept of the line is $(0, 2)$, so the domain of A is $[0, 2]$.

8. The triangle OXY is similar to the triangle AXP, so



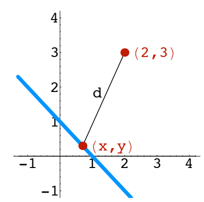
$$\frac{y}{x} = \frac{4}{x-2} \qquad \text{and} \qquad y = \frac{4x}{x-2}.$$

The line segment is the hypotenuse of the right triangle OXY, so its length is

$$L = \sqrt{x^2+y^2} = \sqrt{x^2 + \left(\frac{4x}{x-2}\right)^2} = \frac{x}{x-2}\sqrt{(x-2)^2+16x^2}$$
$$= \frac{x}{x-2}\sqrt{x^2-4x+4+16x^2} = \frac{x}{x-2}\sqrt{17x^2-4x+4}.$$

From the figure, we see that the domain of L is $(2,\infty)$.

9. The distance between (x,y) and $(2,3)$ is given by



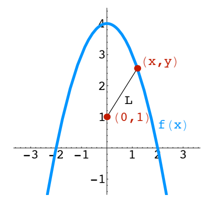
$$d = \sqrt{(x-2)^2+(y-3)^2}.$$

Since $x+y=1$, we have $y = 1-x$, so

$$d = \sqrt{(x-2)^2+[(1-x)-3]^2} = \sqrt{(x-2)^2+(-x-2)^2}$$
$$= \sqrt{x^2-4x+4+x^2+4x+4} = \sqrt{2x^2+8}.$$

The domain of d is $(-\infty,\infty)$.

10. The distance between $(0,1)$ to (x,y) on the graph of the parabola is given by the distance formula:



$$d = \sqrt{(x-0)^2+(y-1)^2} = \sqrt{x^2+(y-1)^2}.$$

Since $y = 4-x^2$,

$$d = \sqrt{x^2+(4-x^2-1)^2} = \sqrt{x^2+(3-x^2)^2} = \sqrt{x^2+(9-6x^2+x^4)} = \sqrt{x^4-5x^2+9}.$$

From the figure, we see that the domain of d is $(-\infty,\infty)$.

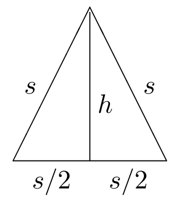
11. If the side of a square is x, then its area is $A = x^2$ and its perimeter is $P = 4x$. Solving $A = x^2$ for x, we have $x = \sqrt{A}$, so $P = 4x = 4\sqrt{A}$. The domain of P is $(0,\infty)$.

12. The diameter of a circle is twice its radius; that is, $d = 2r$ so $r = \frac{1}{2}d$. The area of a circle is then given by $A = \pi r^2 = \pi\left(\frac{1}{2}d\right)^2 = \frac{1}{4}\pi d^2$.

13. If the diameter of a circle is d, then its circumference is $C = \pi d$. Solving for d, we have $d = C/\pi$. The domain of $d(C)$ is $(0,\infty)$.

14. If the side of a cube is x, then its base has area $A = x^2$ and its volume is $V = x^3$. Solving $A = x^2$ for x, we have $x = A^{1/2}$, so $V = x^3 = (A^{1/2})^3 = A^{3/2}$. The domain of $V(A)$ is $(0,\infty)$.

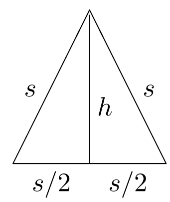
15. Let the sides of the equilateral triangle each be of length s. Then, referring to the figure and using the Pythagorean theorem, we have



$$\begin{aligned}\left(\frac{s}{2}\right)^2 + h^2 &= s^2\\ h^2 &= s^2 - \frac{s^2}{4} = \frac{3}{4}s^2\\ s^2 &= \frac{4}{3}h^2, \quad s = \frac{2}{\sqrt{3}}h.\end{aligned}$$

The area of the triangle is $A = \frac{1}{2}sh = \frac{1}{2}\left(\frac{2}{\sqrt{3}}h\right)h = \frac{1}{\sqrt{3}}h^2 = \frac{\sqrt{3}}{3}h^2$. The domain of $A(h)$ is $(0, \infty)$.

16. Referring to the diagram on the right, we see that the side s of the equilateral triangle is the hypotenuse of a right triangle whose legs have lengths $s/2$ and h. By the Pythagorean theorem,

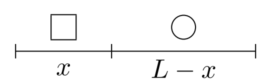


$$\left(\frac{s}{2}\right)^2 + h^2 = s^2, \qquad \text{so} \qquad h^2 = s^2 - \frac{s^2}{4} = \frac{3s^2}{4} \qquad \text{and} \qquad h = \frac{\sqrt{3}s}{2}.$$

The area of the equilateral triangle is $A = \frac{1}{2}sh = \frac{1}{2}s\left(\frac{\sqrt{3}s}{2}\right) = \frac{\sqrt{3}}{4}s^2$. The domain of $A(s)$ is $(0, \infty)$.

17. If r is the radius of a circle, then its circumference, x in this case, is $x = 2\pi r$, and its area is $A = \pi r^2$. Solving $x = 2\pi r$ for r, we have $r = x/2\pi$. Then $A = \pi r^2 = \pi\left(\frac{x}{2\pi}\right)^2 = \frac{x^2}{4\pi}$. The domain of $A(x)$ is $(0, \infty)$.

18. As shown in the diagram on the right, we will bend the portion of the wire of length x into a square and the portion of length $L - x$ into a circle. Then the sides of the square are each $x/4$ and the circumference of the circle is $C = L - x$. Since the circumference of a circle is related to its radius by $C = 2\pi r$, we have $r = C/2\pi = (L - x)/2\pi$. The sum of the areas is



$$\begin{aligned}A &= \text{area of square} + \text{area of circle}\\ &= \left(\frac{x}{4}\right)^2 + \pi\left(\frac{L-x}{2\pi}\right)^2 = \frac{x^2}{16} + \pi\left(\frac{(L-x)^2}{4\pi^2}\right) = \frac{x^2}{16} + \frac{(L-x)^2}{4\pi}.\end{aligned}$$

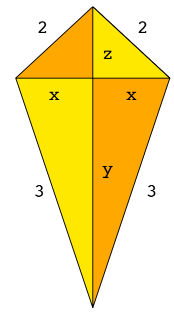
In this problem, $0 < x < L$.

19. Let x be the length of one \$4-per-foot side of the fence. The length of one \$1.60-per-foot side is therefore $1000/x$. Thus, the total cost $C(x)$ to enclose the corral is:

$$C(x) = 2(4x) + 2(1.6)\left(\frac{1000}{x}\right) = 8x + \frac{3200}{x}$$

The domain of $C(x)$ is $(0, \infty)$ — though increasingly higher values of x imply quite a narrow corral!

20. Let y and z be the lengths of the two parts of the vertical cross bar, as shown in the figure. Using the Pythagorean theorem on the lower yellow triangle and again on the upper orange rectangle, we have



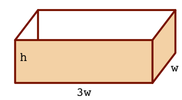
$$x^2 + y^2 = 9 \qquad \text{and} \qquad x^2 + z^2 = 4.$$

Thus, $y = \sqrt{9 - x^2}$ and $z = \sqrt{4 - x^2}$. The area of the kite is

$$\begin{aligned} A &= \frac{1}{2}xz + \frac{1}{2}xz + \frac{1}{2}xy + \frac{1}{2}xy = xz + xy = x(z + y) \\ &= x(\sqrt{4 - x^2} + \sqrt{9 - x^2}). \end{aligned}$$

The domain of $A(x)$ is $[0, 2]$. *Note:* When $x = 2$, the upper part of the kite completely collapses and the shape of the kite is triangular.

21. Let w be the width of the box, and h the height of the box. Then the length of the box is $3w$ and the volume of the box is $V = (3w)(w)(h) = 3w^2h = 450$.

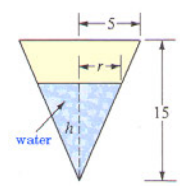


Since the box is open, its surface area is $S = 2wh + 2(3w)h + 3w(w) = 8wh + 3w^2$. From $3w^2h = 450$ we have $h = 150/w^2$, so

$$S = 8w\left(\frac{150}{w^2}\right) + 3w^2 = \frac{1200}{w} + 3w^2 = \frac{1200 + 3w^2}{w}.$$

In this problem, $w > 0$.

22. A cross section of the conical tank is shown in the figure. Using similar right triangles, we have

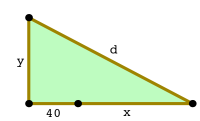


$$\frac{r}{h} = \frac{5}{15} = \frac{1}{3} \qquad \text{or} \qquad r = \frac{h}{3}.$$

The volume of the water when its depth is h is $V = \pi r^2 h = \pi\left(\dfrac{h}{3}\right)^2 h = \dfrac{1}{9}\pi h^3$.

The domain of $V(h)$ is $[0, 15]$.

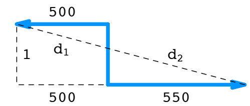
23. After 1 hour, car A is 40 miles from point O. Let $t = 0$ correspond to this point in time. After t hours, car B has travelled $y = 60t$ miles from point O and car A has travelled $40 + x = 40 + 40t$ miles from point O. By the Pythagorean theorem,



$$\begin{aligned} d &= \sqrt{(40 + x)^2 + y^2} = \sqrt{(40 + 40t)^2 + (60t)^2} = \sqrt{20^2(2 + 2t)^2 + 20^2(3t)^2} \\ &= 20\sqrt{4 + 8t + 4t^2 + 9t^2} = 20\sqrt{13t^2 + 8t + 4}. \end{aligned}$$

The domain of $d(t)$ is $(0, \infty)$.

24. (a) As shown in the figure, suppose the lower of the two airliners is traveling at 550 mi/h and the higher airliner is traveling at 500 mi/h. Then, after time t, the lower plane has traveled $550t$ miles and the higher plane has traveled $500t$ miles. The horizontal distance h is therefore $h = 500t + 550t = 1050t$.

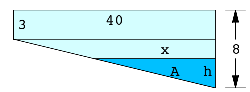


(b) For the diagonal distance d, we see from the figure that, using the Pythagorean theorem,

$$d = d_1 + d_2 = \sqrt{(500t + 550t)^2 + 1^2} = \sqrt{(1050t)^2 + 1} = \sqrt{1,102,500t^2 + 1}.$$

In this problem, $t > 0$.

25. A cross section of the pool as viewed from the side is shown. When the level of the water is h feet above the bottom of the deep end of the pool, where $0 \le h \le 5$, the volume of the water in the pool is the area A of the darker triangle times the width of the pool, 30 feet.



To find A, we let x be the distance shown in the figure and use similar triangles:

$$\frac{x}{h} = \frac{40}{5} = 8 \qquad \text{so} \qquad x = 8h.$$

Then $A = \dfrac{1}{2}hx = \dfrac{1}{2}h(8h) = 4h^2$. Thus, $V(h) = 30A = 30(4h^2) = 120h^2$, $0 \le h \le 5$.

When $h = 5$, the volume of water in the pool is $120(5^2) = 3000$ ft^3. For each foot y of water above this level, there is an additional $40 \times 30 \times y = 1200y$ ft^3 of water in the pool. Thus, using the fact that $y = h - 5$ for $h > 5$,

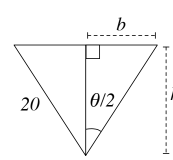
$$V(h) = \begin{cases} 120h^2, & 0 \le h < 5 \\ 3000 + 1200(h-5), & 5 \le h \le 8 \end{cases} = \begin{cases} 120h^2, & 0 \le h < 5 \\ 1200h - 3000, & 5 \le h \le 8. \end{cases}$$

The domain of $V(h)$ is $[0, 8]$.

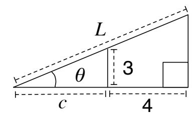
26. We will assume that for different values of x, y is always chosen so that the length plus the girth of the package equals 108 inches. This means $y + 4x = 108$ or $y = 108 - 4x$. The volume of the package is the area of the end of the package times the length of the package. Thus, $V = x^2 y = x^2(108 - 4x) = 108x^2 - 4x^3$. The domain of $V(x)$ is $[0, \sqrt{108}]$.

27. If θ is the angle of elevation, then $h(\theta) = 300 \tan\theta$, $\theta \ge 0$.

28. Start by labeling some key lengths in Figure 1.7.16 as shown. From this figure, we see that $\sin\frac{\theta}{2} = \frac{b}{20}$ and $\cos\frac{\theta}{2} = \frac{h}{20}$. Then area $A(\theta) = bh = \left(20\sin\frac{\theta}{2}\right)\left(20\cos\frac{\theta}{2}\right) = 200\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right) = 200\sin\theta$. The domain of $A(\theta)$ is $[0, 180]$, though $\theta = 0$ or $\theta = 180$ would turn the triangle into a vertical or horizontal line, respectively.



29. Start by labeling Figure 1.7.17 such that the distance between the end of the plank that rests on the ground to the center of the sawhorse is c. Thus, we see that $\cot\theta = c/3$ and $\sec\theta = L/(4+c)$. Thus:



$$L(\theta) = (4+c)\sec\theta = (4+3\cot\theta)\sec\theta = 4\sec\theta + 3\csc\theta$$

The domain of $L(\theta)$ is theoretically $(0, 90)$ degrees, but as stated in the problem, θ cannot get too close to 0 or 90 degrees because the sawhorse would no longer fit beneath the plank.

30. We see from Figure 1.7.18 that $\cot\theta = x/y$ and $\csc\theta = z/y$. Then, from $x + y + z = 2000$ we obtain:

$$\frac{x}{y} + 1 + \frac{z}{y} = \cot\theta + 1 + \csc\theta = \frac{2000}{y}, \quad y = \frac{2000}{1 + \cot\theta + \csc\theta}$$

$$\text{and} \quad A = \frac{1}{2}xy = \frac{1}{2}y^2\left(\frac{x}{y}\right) = \frac{1}{2}y^2\cot\theta = \frac{1}{2}\cot\theta\left(\frac{2000}{1+\cot\theta+\csc\theta}\right)^2.$$

The domain of $A(\theta)$ is theoretically $(0, 90]$ degrees, but as stated in the problem, θ cannot get too close to 0 or 90 degrees because the pasture would no longer be able to accommodate the farmer's livestock.

31. See Figure 1.7.19 in the text. Let θ_1 denote the angle of elevation from eye level to the top of the pedestal, just below the base of the statue. Then

$$\tan\theta_1 = \frac{1/2}{x} \quad \text{so} \quad \theta_1 = \arctan\left(\frac{1}{2x}\right).$$

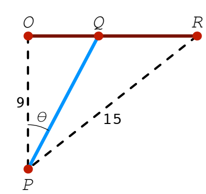
Let θ_2 denote the angle of elevation from eye level to the top of the statue. Then

$$\tan\theta_2 = \frac{1}{x} \quad \text{so} \quad \theta_2 = \arctan\left(\frac{1}{x}\right).$$

Thus, we see that $\theta = \theta_2 - \theta_1 = \arctan\left(\frac{1}{x}\right) - \arctan\left(\frac{1}{2x}\right)$, where x is measured in meters. The domain of $\theta(x)$ is $(0, \infty)$.

32. Referring to the figure, we see that $\cos\theta = \dfrac{9}{PQ}$ so $PQ = \dfrac{9}{\cos\theta} = 9\sec\theta$.

Also from the figure, we see that $\tan\theta = \dfrac{OQ}{9}$ so $OQ = 9\tan\theta$.



Using the Pythagorean theorem, we have $9^2 + (OR)^2 = 15^2$ so $OR = 12$, and the distance between points Q and R is $12 - 9\tan\theta$. The time it takes to row from point P to point Q is $(9\sec\theta)/3 = 3\sec\theta$. The time it takes to walk from point Q to point R is $(12 - 9\tan\theta)/5$. Thus, the total time t is given by

$$t(\theta) = 3\sec\theta + \frac{12 - 9\tan\theta}{5}.$$

Assuming that Q will lie strictly within line segment OR, the domain of $t(\theta)$ is therefore $[0, \arctan(12/9)]$, since the $\tan\theta = 12/9$ when $Q = R$.

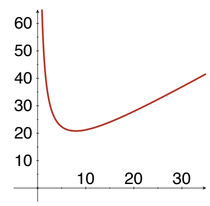
33. If the building is 60 ft high, then from $y = \dfrac{10(x+5)}{x}$, we get $y = 60$ when $x = 1$. Since we want the ladder to be no higher than the building, then:

$$\begin{aligned}\frac{10(x+5)}{x} &\le 60\\ x+5 &\le 6x\\ 5x \ge 5, &\quad x \ge 1\end{aligned}$$

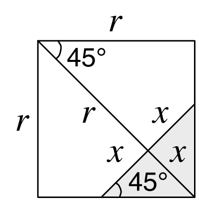
Thus, for $0 < x < 1$, we get $y > 60$, which is higher than the building (for example, if $x = 1/2$, then $y = 110$). The domain of the function

$$L(x) = \frac{x+5}{x}\sqrt{x^2+100}$$

must therefore be $[1, \infty)$. L decreases until it attains its absolute minimum at $x = \sqrt[3]{500} \approx 7.94$ ft.



34. Extend the horizontal and vertical sides of the octagon to form a square. The figure shows the lower right-hand corner of this square. Since the diagonal of the square in the figure is $r + x = r\sqrt{2}$, we have $x = r(\sqrt{2}-1)$. The area of the shaded region is $x^2 = r^2(3-2\sqrt{2})$ and thus the area of the octagon is:



$$\begin{aligned}A(r) &= 4r^2 - 4x^2 = 4r^2 - 4r^2(3-2\sqrt{2})\\ &= 8r^2(\sqrt{2}-1) \approx 3.3137r^2.\end{aligned}$$

Chapter 1 in Review

A. True/False

1. False; consider $f(x) = x^2$ with $x = -1$ and $x = 1$.
2. False; $f(-x) = -x^5 + 4x^3 + 2$.
3. True
4. False; it is the graph shifted 3 units left.
5. False; $f(3/2) = 0$.
6. False; consider $\dfrac{\sin x}{x}$.
7. True; one for $x \to -\infty$ and another for $x \to +\infty$.
8. False; consider $f(x) = (x-1)/(x-1)$.
9. False; amplitude is not defined for $\sec x$.
10. True

11. True

12. False; the range of the inverse tangent function is $(-\pi/2, \pi/2)$ and $5\pi/4$ is not in this interval.

13. True; if f is even, then $f(-x) = f(x)$ for all x, so f cannot be one-to-one (unless the domain of f is $\{0\}$, which is precluded by the fact that $a > 0$).

14. True; the graphs of f and f^{-1} are symmetric with respect to the line $y = x$, so any point of intersection of the graphs must lie on this line.

15. True; the range of $\sec x$ is $(-\infty, -1] \cup [1, \infty)$.

16. True; it is a one-to-one function.

17. True; since $y = 10^{-x} = (10^{-1})^x = (1/10)^x = (0.1)^x$.

18. True; since $\ln(e + e) = \ln(2e) = \ln e + \ln 2 = 1 + \ln 2$.

19. True; since $ln\dfrac{e^b}{e^a} = lne^b - \ln e^a = b\ln e - a\ln e = b - a$.

20. True; since $log_b b = 1$.

B. Fill in the Blanks

1. $[-2, 0) \cup (0, \infty)$

2. 107, 25, 491

3. $(-8, 6)$

4. -7 and 5

5. The graph is tangent to the x-axis at $(1, 0)$ since it is a zero of multiplicity 2. The graph passes through the x-axis at $(0, 0)$ (multiplicity 3) and $(5, 0)$ (simple zero).

6. $(0, 10]$

7. $-4/5$

8. For the horizontal asymptote to be $y = 1$, we want both the numerator and denominator to be polynomials of the same degree with leading coefficient 1. We also need the x-intercept to be $(3, 0)$, so $x - 3$ must be a factor of the numerator. One such function would be

$$f(x) = \frac{x-3}{x-1}.$$

9. The period is $\dfrac{2\pi}{B} = \dfrac{2\pi}{\pi/3} = 6$.

10. $\dfrac{\pi}{12}$ units to the right.

11. Since $\sin\pi = 0$, $\sin^{-1}(\sin\pi) = \sin^{-1} 0$. Letting $y = \sin^{-1} 0$, we must find y such that $\sin y = 0$ and $-\pi/2 \le y \le \pi/2$. Within this interval, we have $\sin 0 = 0$, so $\sin^{-1}(\sin\pi) = 0$.

12. $(1, 3)$

13. $(3, 5)$, since the graph of $y = 4 + e^{x-3}$ is obtained by shifting the graph of $y = e^x$ up four units and to the right three units.

14. $e^{3\ln 10} = e^{\ln 10^3} = 10^3 = 1000$

15. $\log_3 5$

16. Dividing both sides of $3e^x = 4e^{-3x}$ by e^x:

$$3 = 4\frac{e^{-3x}}{e^x} = 4e^{-4x}; \quad e^{-4x} = \frac{3}{4}$$
$$-4x = \ln\frac{3}{4}, \; x = -\frac{1}{4}\ln\frac{3}{4} \approx 0.0719.$$

17. If $\log_3 x = -2$, then $x = 3^{-2} = 1/9$.

18. $9^{1.5} = 27$

19. $y = \ln x$

20. $-\dfrac{5}{2}$

C. Exercises

1. (a) $f(-4) = 3$ (b) $f(-3) = 0$ (c) $f(-2) = -2$ (d) $f(-1) = 0$
   (e) $f(0) = 2.5$ (f) $f(1) = 2$ (g) $f(1.5) = 1$ (h) $f(2) = 0$
   (i) $f(3.5) = 3$ (j) $f(4) = 4$

2. (a) $g(1 + a) = 2 + 2a$ (b) $g(1 - a) = (1 - a)^2$

   (c) $g(1.5 - a) = \begin{cases} 3 - 2a, & 0 < a < 1/2 \\ (1.5 - a)^2, & 1/2 \le a < 1 \end{cases}$ (d) $g(a) = a^2$

   (e) $g(-a) = a^2$ (f) $g(2a) = \begin{cases} 4a^2, & 0 < a \le 1/2 \\ 4a, & 1/2 < a < 1 \end{cases}$

3. 1: in range ($f(1/2) = 1$); 5: not in range; 8: in range ($f(4) = 8$)

4. (a) $(f \circ h)(x) = \sqrt{x^2 + 4}$; the domain is $(-\infty, \infty)$.
   (b) $(g \circ h)(x) = \sqrt{5 - x^2}$; the domain is $[-\sqrt{5}, \sqrt{5}]$.
   (c) $(f \circ f)(x) = \sqrt{\sqrt{x + 4} + 4}$; the domain is $[-4, \infty)$.
   (d) $(g \circ g)(x) = \sqrt{5 - \sqrt{5 - x}}$; the domain is $[-20, 5]$.
   (e) $(f + g)(x) = \sqrt{x + 4} + \sqrt{5 - x}$; the domain is $[-4, 5]$.

(f) $(f/g)(x) = \dfrac{\sqrt{x+4}}{\sqrt{5-x}}$; the domain is $[-4, 5)$.

5. $$\frac{f(x+h)-f(x)}{h} = \frac{[-(x+h)^3+2(x+h)^2-(x+h)+5]-(-x^3+2x^2-x+5)}{h}$$
$$= \frac{(-x^3-3x^2h-3xh^2-h^3+2x^2+4xh+2h^2-x-h+5)-(-x^3+2x^2-x+5)}{h}$$
$$= \frac{-3x^2h-3xh^2-h^3+4xh+2h^2-h}{h} = \frac{h(-3x^2-3xh-h^2+4x+2h-1)}{h}$$
$$= -3x^2-3xh-h^2+4x+2h-1$$

6. $$\frac{f(x+h)-f(x)}{h} = \frac{\left[1+2(x+h)-\dfrac{3}{x+h}\right]-\left(1+2x-\dfrac{3}{x}\right)}{h}$$
$$= \frac{\left(1+2x+2h-\dfrac{3}{x+h}\right)-\left(1+2x-\dfrac{3}{x}\right)}{h} = \frac{\left(2h-\dfrac{3}{x+h}\right)+\dfrac{3}{x}}{h}$$
$$= \frac{1}{h}\cdot\frac{2x^2h+2xh^2-3x+3(x+h)}{x(x+h)} = \frac{1}{h}\cdot\frac{2x^2h+2xh^2-3x+3x+3h}{x(x+h)}$$
$$= \frac{1}{h}\cdot\frac{h(2x^2+2xh+3)}{x(x+h)} = \frac{2x^2+2xh+3}{x(x+h)}$$

7. Since
$$f(x) = \frac{2x}{x^2+1}$$
has no vertical asymptotes and its horizontal asymptote is $y = 0$, its graph must be either (a) or (f). Because $f(x)$ is negative for $x < 0$, it must be (f).

8. Since
$$f(x) = \frac{x^2-1}{x^2+1}$$
has no vertical asymptotes and its horizontal asymptote is $y = 1$, its graph must be (i).

9. Since
$$f(x) = \frac{2x}{x-2}$$
has vertical asymptote $x = 2$ and its horizontal asymptote is $y = 2$, its graph must be (d).

10. Since
$$f(x) = 2 - \frac{1}{x^2} = \frac{2x^2-1}{x^2}$$
has vertical asymptote $x = 0$ (the y-axis), its graph must be (g).

11. Since

$$f(x) = \frac{x}{(x-2)^2}$$

has vertical asymptote $x = 2$ and its horizontal asymptote is $y = 0$, its graph must be (h).

12. Since

$$f(x) = \frac{(x-1)^2}{x-2} = \frac{x^2 - 2x + 1}{x-2} = x + \frac{1}{x-2}$$

using synthetic division, we see that the graph of $f(x)$ has vertical asymptote $x = 2$ and slant asymptote $y = x$. Thus, the graph of f must be (e).

13. Since

$$f(x) = \frac{x^2 - 10}{2x - 4} = \frac{1}{2} \cdot \frac{x^2 - 10}{x - 2} = \frac{1}{2}\left(x + 2 - \frac{6}{x-2}\right)$$

using synthetic division, we see that the graph of $f(x)$ has vertical asymptote $x = 2$ and slant asymptote $y = \frac{1}{2}x + 1$. Thus, the graph of f must be (c).

14. Since

$$f(x) = \frac{-x^2 + 5x - 5}{x - 2} = -x + 3 + \frac{1}{x-2}$$

using synthetic division, we see that the graph of $f(x)$ has vertical asymptote $x = 2$ and slant asymptote $y = -x + 3$. Thus, the graph of f must be (j).

15. Since

$$f(x) = \frac{2x}{x^3 + 1} = \frac{2x}{(x+1)(x^2 - x + 1)}$$

has vertical asymptote $x = -1$ and horizontal asymptote $y = 0$, its graph must be (b).

16. Since

$$f(x) = \frac{3}{x^2 + 1}$$

has no vertical asymptotes and its horizontal asymptote is $y = 0$, the graph must be either (a) or (f). Because $f(x)$ is always positive, it must be (a).

17. Since $f(-2 + h) = 3^{-(-2+h+1)} = 3^{1-h}$, the line passes through $(-2 + h, 3^{1-h})$ and $(-2, 3)$. Its slope is

$$m = \frac{3^{1-h} - 3}{-2 + h - (-2)} = \frac{3^{1-h} - 3}{h}.$$

18. Since $\ln 1 = 0$ and $\ln e^2 = 2$, the line passes through $(1, 0)$ and $(e^2, 2)$. Its slope is

$$m = \frac{2-0}{e^2-1} = \frac{2}{e^2-1}.$$

19. (a) $12^t = (2 \cdot 6)^t = 2^t \cdot 6^t = 5 \cdot 2 = 10$

(b) $3^t = \left(\frac{6}{2}\right)^t = \frac{6^t}{2^t} = \frac{2}{5}$

(c) $6^{-t} = \frac{1}{6^t} = \frac{1}{2}$

20. (a) $6^{3t} = (6^t)^3 = 2^3 = 8$

(b) $2^{-3t}2^{7t} = 2^{4t} = (2^t)^4 = 5^4 = 625$

(c) $6^{t^2} = (6^t)^t = (2)^t = 5$

21. Since $(0, 5)$ is on the graph, $5 = Ae^{k \cdot 0} = A$ and the function is $f(x) = 5e^{kx}$. Since the graph passes through $(6, 1)$, $1 = 5e^{6k}$ and $k = \frac{1}{6}\ln\frac{1}{5} = -\frac{1}{6}\ln 5$. Thus,

$$f(x) = 5e^{(-\frac{1}{6}\ln 5)x} = 5e^{-0.2682x}.$$

22. Since $f(0) = 1/2$, we see that $1/2 = A10^{k \cdot 0} = A$ and the function is $f(x) = \frac{1}{2}(10^{kx})$. Since $f(3) = \frac{1}{2}(10^{3k}) = 8$, we have $10^{3k} = 16$. Taking the logarithm base 10 of both sides, we have

$$\begin{aligned} \log_{10} 10^{3k} &= \log_{10} 16 \\ 3k &= \log_{10} 16, \quad k = \frac{\log_{10} 16}{3} \end{aligned}$$

and

$$f(x) = \frac{1}{2}\left(10^{[(\log_{10} 16)/3]x}\right) \approx \frac{1}{2}(10^{0.4014x}).$$

23. The graph of $f(x) = b^x$, where $0 < b < 1$, has a horizontal asymptote at $y = 0$, so the graph of $f(x) = 5 + b^x$ has horizontal asymptote $y = 5$. The graph passes through $(1, 5.5)$ so $5.5 = 5 + b$, and $b = 0.5$. Thus, $f(x) = 5 + (1/2)^x$.

24. The graph of $y = \log_3 x$ has a vertical asymptote at $x = 0$, so the graph of $y = a + \log_3(x-2)$ has a vertical asymptote at $x = 2$. The graph passes through $(11, 10)$, so

$$10 = a + \log_3(11-2) = a + \log_3 9 = a + \log_3 3^2 = a + 2$$

and $a = 8$. Thus, $f(x) = 8 + \log_3(x-2)$.

25. This looks like the graph of $y = \ln x$ revolved around the x-axis (giving $-\ln x$) and shifted up 2 units. Thus, this is the graph of (b) $y = 2 - \ln x$.

26. This looks like the graph of $y = \ln x$ shifted right 2 units. Thus, this the graph of (a) $y = \ln(x - 2)$.

27. This looks like the graph of $y = \ln x$ revolved around the x-axis (giving $-\ln x$) and shifted left 2 units (giving $-\ln(x+2)$). But then the graph should pass through $(-1, 0)$. Instead, it appears to pass through $(-1, -2)$. Thus, this is the graph of (d) $y = -2 - \ln(x + 2)$.

28. This looks like the graph of $y = \ln x$ revolved around the x-axis (giving $-\ln x$). But then the graph should pass through $(-1, 0)$. Instead it appears to pass through $(1/2, 0)$. Thus, this is the graph of (e) $y = -\ln(2x)$.

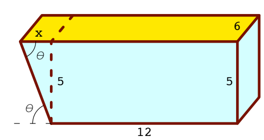
29. This looks like the graph of $y = \ln x$ shifted to the left 2 units (giving $\ln(x+2)$). But then the graph should pass through $(-1, 0)$. Instead, it appears to pass through $(-1, 2)$. Thus, this is the graph of (c) $y = 2 + \ln(x + 2)$.

30. This looks like the graph of $y = \ln(-x)$ shifted right by 2 units. This would be $y = \ln[-(x - 2)] = \ln(-x + 2)$. But then the graph should pass through $(1, 0)$. Instead, it appears to pass through $(1, 2)$. Thus, this is the graph of (f) $y = 2 + \ln(-x + 2)$.

31. (a) $V(l) = l(2l)(3l) = 6l^3$
    (b) $V(w) = w\left(\frac{1}{3}w\right)\left(\frac{2}{3}w\right) = \frac{2}{9}w^3$
    (c) $V(h) = h\left(\frac{1}{2}h\right)\left(\frac{3}{2}h\right) = \frac{3}{4}h^3$

32. With a length of x per side, the area of one face is x^2. 4 faces cost 1 cent per square centimeter, while 2 faces cost 2.5 cents per square centimeter. Thus, the total cost is $C(x) = 1(4x^2) + 2.5(2x^2) = 9x^2$ cents/cm.

33. To find the volume of the box, we begin by finding the area of the side of the box shown in the figure at the right. We see that $\tan\theta = 5/x$, so $x = 5\cot\theta$. Then the area of the figure is given by



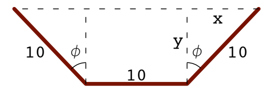
$$A = (12)(5) + \frac{1}{2}(5\cot\theta)(5) = 60 + \frac{25}{2}\cot\theta.$$

The volume of the box is $V = 6\left(60 + \frac{25}{2}\cot\theta\right) = 360 + 75\cot\theta$.

34. See Figure 1.R.21 in the text. The area of the shaded region is the area of the square with opposite corners at $(0, 0)$ and (h, h) minus the area of the quarter circle centered at (h, h) with radius h. That is, the area of the shaded region is

$$A = h^2 - \frac{1}{4}\pi h^2 = \left(1 - \frac{\pi}{4}\right)h^2.$$

35. Let x and y be as shown in the figure to the right. The cross-section is a trapezoid with parallel sides having lengths 10 and $10+2x$. The area of the trapezoid is

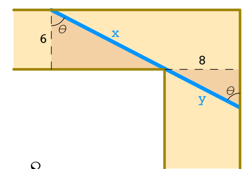


$$A = \frac{1}{2}[10 + (2x + 10)]y = \frac{1}{2}(20 + 2x)y = (10 + x)y,$$

so we need to express x and y in terms of ϕ. Using $\cos\phi = y/10$ and $\sin\phi = x/10$ we have $y = 10\cos\phi$ and $x = 10\sin\phi$, so

$$\begin{aligned} A &= (10 + x)y = (10 + 10\sin\phi)(10\cos\phi) = 100(1 + \sin\phi)(\cos\phi) \\ &= 100\cos\phi + 50\sin 2\phi. \end{aligned}$$

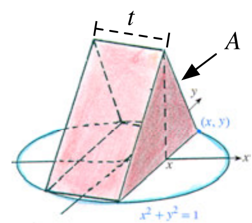
36. Let x denote the length of the section of pipe that stretches across the 6-foot-wise hallway to the corner, and let y denote the length of the section of pipe that stretches from the corner across the 8-foot-wide hallway. From the figure, we see that



$$\cos\theta = \frac{6}{x} \quad \text{or} \quad x = \frac{6}{\cos\theta} = 6\sec\theta, \qquad \text{and} \qquad \sin\theta = \frac{8}{y} \quad \text{or} \quad y = \frac{8}{\sin\theta} = 8\csc\theta.$$

Thus, we have $L = x + y = 6\sec\theta + 8\csc\theta$.

37. Let A be the area of a triangular side and t be the thickness of the prism (i.e., the distance between the two parallel faces). The prism's triangular sides are equilateral and its rectangular base is inscribed within the circle $x^2 + y^2 = 1$, so $A = \frac{1}{2}(2y)(\sqrt{3} \cdot y)$ and $t = 2x$, resulting in the volume



$$V = At = \left(\frac{1}{2}(2y)(\sqrt{3} \cdot y)\right)(2x)$$

Since $x^2 + y^2 = 1$, then $y = \sqrt{1 - x^2}$, and we can substitute y above:

$$\begin{aligned} V(x) &= \left[\frac{1}{2}(2\sqrt{1 - x^2})(\sqrt{3} \cdot \sqrt{1 - x^2})\right](2x) \\ &= 2\sqrt{3}(1 - x^2) \end{aligned}$$

38. Given that the lateral surface area of a cone is $\pi R\sqrt{R^2 + h^2}$, we need to express R and h in terms of θ. R is fixed, while $h = R\cot\theta$, as can be seen from Figure 1.R.25 in the text. Letting H be the height of the cylinder, the total surface area $S(\theta)$ of the container is the lateral surface area of the cone plus the surface area of the cylinder, which is $2\pi RH$:

$$\begin{aligned} S(\theta) &= \pi R\sqrt{R^2 + R^2\cot^2\theta} + 2\pi RH \\ &= \pi R^2\sqrt{1 + \cot^2\theta} + 2\pi RH = \pi R^2\csc\theta + 2\pi RH. \end{aligned}$$

Now, the volume of the cone is

$$\frac{\pi}{3}R^2 h = \frac{\pi}{3}R^2(R\cot\theta) = \frac{\pi}{3}R^3\cot\theta.$$

The volume of the cylinder is $\pi R^2 H$. Thus, the volume of the container is the sum of these two volumes:

$$V = \frac{\pi}{3} R^3 \cot\theta + \pi R^2 H.$$

Solving for H, we obtain $H = \dfrac{3V - \pi R^3 \cot\theta}{3\pi R^2}$ and substituting this in $S(\theta)$, we get:

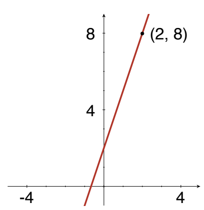
$$S(\theta) = \pi R^2 \csc\theta + 2\pi R \left(\frac{3V - \pi R^3 \cot\theta}{3\pi R^2} \right) = \pi R^2 \csc\theta - \frac{2}{3}\pi R^2 \cot\theta + \frac{2V}{R}.$$

Chapter 2

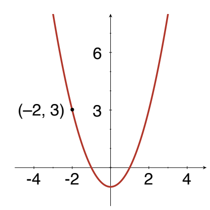
Limit of a Function

2.1 Limits — An Informal Approach

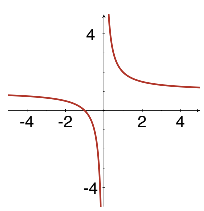
1. $\lim\limits_{x \to 2} (3x + 2) = 8$



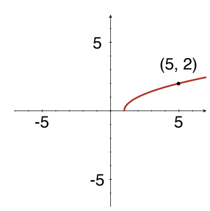
2. $\lim\limits_{x \to 2} (x^2 - 1) = 3$



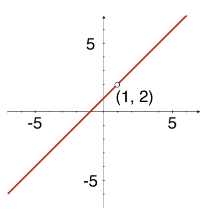
3. No limit as $x \to 0$.



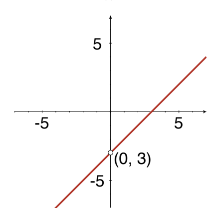
4. $\lim\limits_{x \to 5} \sqrt{x - 1} = 2$



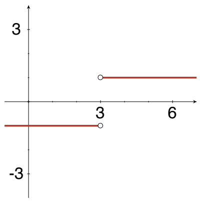
5. $\lim\limits_{x \to 1} \dfrac{x^2 - 1}{x - 1} = \lim\limits_{x \to 1} (x + 1) = 2$



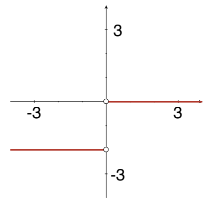
6. $\lim\limits_{x \to 0} \dfrac{x^2 - 3x}{x} = \lim\limits_{x \to 0} (x - 3) = -3$



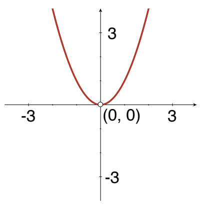
7. No limit as $x \to 3$.



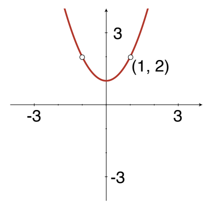
8. No limit as $x \to 0$.



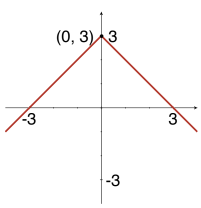
9. $\lim_{x \to 0} \dfrac{x^3}{x} = 0$



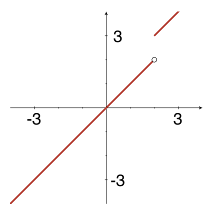
10. $\lim_{x \to 1} \dfrac{x^4 - 1}{x^2 - 1} = 2$



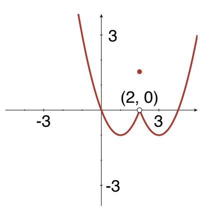
11. $\lim_{x \to 0} f(x) = 3$



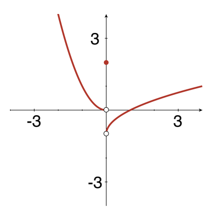
12. No limit as $x \to 2$.



13. $\lim_{x \to 2} f(x) = 0$



14. No limit as $x \to 0$.



15. (a) 1 (b) -1 (c) 2 (d) doesn't exist

16. (a) 0 (b) 3 (c) 3 (d) 3

17. (a) 2 (b) -1 (c) -1 (d) -1

18. (a) doesn't exist (b) 3 (c) -2 (d) doesn't exist

19. Correct

20. Incorrect; $\lim_{x \to 0^+} \sqrt[4]{x} = 0$

21. Incorrect; $\lim_{x\to 1^-} \sqrt{1-x} = 0$

22. Correct

23. Incorrect; $\lim_{x\to 0^+} \lfloor x \rfloor = 0$

24. Correct

25. Correct

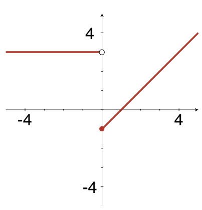
26. Incorrect; $\lim_{x\to 0^-} \cos^{-1} x = 0$

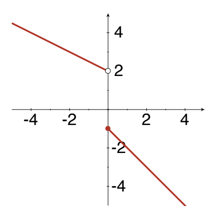
27. Incorrect; $\lim_{x\to 3^-} \sqrt{9-x^2} = 0$

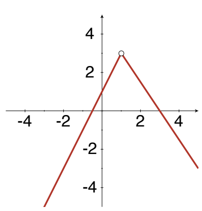
28. Correct

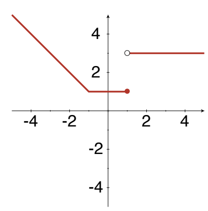
29. (a) Does not exist (b) 0 (c) 3 (d) -2 (e) 0 (f) 1

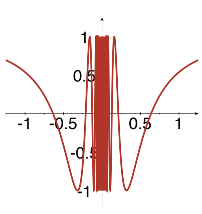
30. (a) ≈ 2.5 (b) 1 (c) -1 (d) Does not exist (e) 0 (f) 0

31.


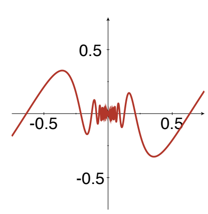
32.


33.


34.


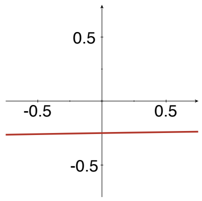
35.


The limit does not exist.

36.


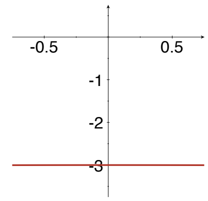
The limit is 0.

37.



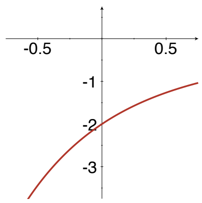
The limit is -0.25.

38.



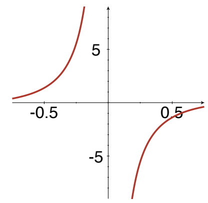
The limit is -3.

39.



The limit is -2.

40.



The limit does not exist.

41.

$x \to 1^-$	0.9	0.99	0.999	0.9999
$f(x)$	-3.25536642	-3.02276607	-3.00225263	-3.00022503
$x \to 1^+$	1.1	1.01	1.001	1.0001
$f(x)$	-2.79817601	-2.97775903	-2.99775260	-2.99977503

$\lim_{x\to1} f(x) = -3$

42.

$x \to 1^-$	0.9	0.99	0.999	0.9999
$f(x)$	1.05360516	1.00503359	1.00050033	1.00005000
$x \to 1^+$	1.1	1.01	1.001	1.0001
$f(x)$	0.95310180	0.99503309	0.99950033	0.99995000

$\lim_{x\to1} f(x) = 1$

43.

$x \to 0^-$	-0.1	-0.01	-0.001	-0.0001
$f(x)$	-0.04995835	-0.00499996	-0.00050000	-0.00005000
$x \to 0^+$	0.1	0.01	0.001	0.0001
$f(x)$	0.04995835	0.00499996	0.00050000	0.00005000

$\lim_{x\to0} f(x) = 0$

44. Since $\dfrac{1-\cos x}{x^2}$ is an even function, it suffices to consider only $x \to 0^+$.

$x \to 0^+$	0.1	0.01	0.001	0.0001
$f(x)$	0.49958347	0.49999583	0.49999996	0.50000000

$\lim_{x\to0} f(x) = 0.5$

45. Since $\dfrac{x}{\sin 3x}$ is an even function, it suffices to consider only $x \to 0^+$.

$x \to 0^+$	0.1	0.01	0.001	0.0001
$f(x)$	0.33838634	0.33338334	0.33333383	0.33333334

$\lim_{x\to 0} f(x) = 0.33333333$

46. Since $\dfrac{\tan x}{x}$ is an even function, it suffices to consider only $x \to 0^+$.

$x \to 0^+$	0.1	0.01	0.001	0.0001
$f(x)$	1.00334672	1.00003333	1.00000033	1.00000000

$\lim_{x\to 0} f(x) = 1$

47.

$x \to 4^-$	3.9	3.99	3.999	3.9999
$f(x)$	0.25158234	0.25015645	0.25001563	0.25000156
$x \to 4^+$	4.1	4.01	4.001	4.0001
$f(x)$	0.24845673	0.24984395	0.24998438	0.24999844

$\lim_{x\to 4} f(x) = 0.25$

48.

$x \to 3^-$	2.9	2.99	2.999	2.9999
$f(x)$	-0.52186477	-0.50209311	-0.50020843	-0.50002083
$x \to 3^+$	3.1	3.01	3.001	3.0001
$f(x)$	-0.48008703	-0.49792633	-0.49979176	-0.49997917

$\lim_{x\to 3} f(x) = -0.5$

49.

$x \to 1^-$	0.9	0.99	0.999	0.9999
$f(x)$	4.43900000	4.94039900	4.99400400	4.99940004
$x \to 1^+$	1.1	1.01	1.001	1.0001
$f(x)$	5.64100000	5.06040010	5.00600400	5.00060004

$\lim_{x\to 1} f(x) = 5$

50.

$x \to -2^-$	-2.1	-2.01	-2.001	-2.0001
$f(x)$	12.61000000	12.06010000	12.00600100	12.00060001
$x \to -2^+$	-1.9	-1.99	-1.999	-1.9999
$f(x)$	11.41000000	11.94010000	11.99400100	11.99940001

$\lim_{x\to -2} f(x) = 12$

2.2 Limit Theorems

1. 15
2. $\cos \pi = -1$
3. -12
4. -3
5. 4

6. -125

7. 4

8. -136

9. $-8/5$

10. does not exist

11. 14

12. 4

13. $28/9$

14. $\lim_{x\to 6} \dfrac{x^2-6x}{x^2-7x+6} = \lim_{x\to 6} \dfrac{x(x-6)}{(x-1)(x-6)} = \lim_{x\to 6} \dfrac{x}{x-1} = \dfrac{6}{5}$

15. -1

16. 16

17. $\sqrt{7}$

18. 3

19. does not exist

20. 16

21. $\lim_{y\to -5} \dfrac{y^2-25}{y+5} = \lim_{y\to -5}(y-5) = -10$

22. $\lim_{u\to 8} \dfrac{u^2-5u-24}{u-8} = \lim_{u\to 8}(u+3) = 11$

23. $\lim_{x\to 1} \dfrac{x^3-1}{x-1} = \lim_{x\to 1} \dfrac{(x-1)(x^2+x+1)}{x-1} = \lim_{x\to 1}(x^2+x+1) = 3$

24. $\lim_{t\to -1} \dfrac{t^3+1}{t^2-1} = \lim_{t\to -1} \dfrac{(t+1)(t^2-t+1)}{(t+1)(t-1)} = \lim_{t\to -1} \dfrac{t^2-t+1}{t-1} = -\dfrac{3}{2}$

25. $\lim_{x\to 10} \dfrac{(x-2)(x+5)}{x-8} = \dfrac{8(15)}{2} = 60$

26. $\lim_{x\to -3} \dfrac{2x+6}{4x^2-36} = \lim_{x\to -3} \dfrac{2(x+3)}{4(x+3)(x-3)} = \lim_{x\to -3} \dfrac{1}{2(x-3)} = -\dfrac{1}{12}$

27. $\lim_{x\to 2} \dfrac{x^3+3x^2-10x}{x-2} = \lim_{x\to 2} \dfrac{x(x+5)(x-2)}{x-2} = \lim_{x\to 2} x(x+5) = 14$

28. $\lim_{x\to 1.5} \dfrac{2x^2+3x-9}{x-1.5} = \lim_{x\to 1.5} \dfrac{(2x-3)(x+3)}{x-1.5} = \lim_{x\to 1.5} \dfrac{2(x-1.5)(x+3)}{x-1.5} = \lim_{x\to 1.5} 2(x+3) = 9$

29. $\lim_{t\to 1} \dfrac{t^3-2t+1}{t^3+t^2-2} = \lim_{t\to 1} \dfrac{(t-1)(t^2+t-1)}{(t-1)(t^2+2t+2)} = \lim_{t\to 1} \dfrac{t^2+t-1}{t^2+2t+2} = \dfrac{1}{5}$

30. $\lim_{x\to 0} x^3(x^4+2x^3)^{-1} = \lim_{x\to 0} \dfrac{x^3}{x^4+2x^3} = \lim_{x\to 0} \dfrac{1}{x+2} = \dfrac{1}{2}$

31. $\lim_{x\to 0^+} \dfrac{(x+2)(x^5-1)^3}{(\sqrt{x}+4)^2} = \dfrac{2(-1)}{16} = -\dfrac{1}{8}$

32. $\lim_{x\to -2} x\sqrt{x+4}\sqrt[3]{x-6} = -2\sqrt{2}\sqrt[3]{-8} = 4\sqrt{2}$

33. $\lim_{x\to 0} \left[\dfrac{x^2+3x-1}{x} + \dfrac{1}{x}\right] = \lim_{x\to 0} \dfrac{x^2+3x}{x} = \lim_{x\to 0}(x+3) = 3$

34. $$\begin{aligned} \lim_{x\to 2} \left[\frac{1}{x-2} - \frac{6}{x^2+2x-8}\right] &= \lim_{x\to 2} \left[\frac{1}{x-2} - \frac{6}{(x-2)(x+4)}\right] \\ &= \lim_{x\to 2} \left[\frac{x+4}{(x-2)(x+4)} - \frac{6}{(x-2)(x+4)}\right] \\ &= \lim_{x\to 2} \frac{x-2}{(x-2)(x+4)} = \lim_{x\to 2} \frac{1}{x+4} = \frac{1}{6} \end{aligned}$$

35. does not exist

36. -2^{10} or -1024

37. 2

38. $\dfrac{2\sqrt{2}}{\sqrt[3]{4}}$

39. $\lim_{h\to 4} \sqrt{\dfrac{h}{h+5}} \left[\dfrac{h^2-16}{h-4}\right]^2 = \lim_{h\to 4} \sqrt{\dfrac{h}{h+5}}(h^2+8h+16) = \dfrac{128}{3}$

40. 16

41. $\lim_{x\to 0^-} \sqrt[5]{\dfrac{x^3-64x}{x^2+2x}} = \lim_{x\to 0^-} \sqrt[5]{\dfrac{x^2-64}{x+2}} = -2$

42. $-100,000$

43. $a^2-2ab+b^2$

44. $\lim_{x\to -1} \sqrt{u^2x^2+2xu+1} = \lim_{x\to -1} \sqrt{u^2-2u+1} = \lim_{x\to -1} \sqrt{(u-1)^2} = |u-1|$

45. $\lim_{h\to 0} \dfrac{(8+h)^2-64}{h} = \lim_{h\to 0} \dfrac{16h+h^2}{h} = \lim_{h\to 0}(16+h) = 16$

46. $\lim_{h\to 0} \dfrac{1}{h}[(1+h)^3-1] = \lim_{h\to 0}(h^2+3h+3) = 3$

47. $\lim_{h\to 0}\frac{1}{h}\left(\frac{1}{x+h}-\frac{1}{x}\right)=\lim_{h\to 0}\frac{1}{h}\left(\frac{x-(x+h)}{(x+h)x}\right)=\lim_{h\to 0}\frac{-h}{hx(x+h)}$
$$=\lim_{h\to 0}-\frac{1}{x^2+hx}=-\frac{1}{x^2}$$

48. $\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$
$$=\lim_{h\to 0}\frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})}=\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}=\frac{1}{2\sqrt{x}}$$

49. $\lim_{t\to 1}\frac{\sqrt{t}-1}{t-1}=\lim_{t\to 1}\frac{\sqrt{t}-1}{t-1}\frac{\sqrt{t}+1}{\sqrt{t}+1}=\lim_{t\to 1}\frac{1}{\sqrt{t}+1}=\frac{1}{2}$

50. $\lim_{u\to 5}\frac{\sqrt{u+4}-3}{u-5}=\lim_{u\to 5}\frac{\sqrt{u+4}-3}{u-5}\frac{\sqrt{u+4}+3}{\sqrt{u+4}+3}$
$$=\lim_{u\to 5}\frac{u-5}{(u-5)(\sqrt{u+4}+3)}=\lim_{u\to 5}\frac{1}{\sqrt{u+4}+3}=\frac{1}{6}$$

51. $\lim_{v\to 0}\frac{\sqrt{25+v}-5}{\sqrt{1+v}-1}=\lim_{v\to 0}\left[\frac{\sqrt{25+v}-5}{\sqrt{1+v}-1}\frac{\sqrt{25+v}+5}{\sqrt{1+v}+1}\right]\frac{\sqrt{1+v}+1}{\sqrt{25+v}+5}$
$$=\lim_{v\to 0}\frac{v}{v}\frac{\sqrt{1+v}+1}{\sqrt{25+v}+5}=\frac{1}{5}$$

52. $\lim_{x\to 1}\frac{4-\sqrt{x+15}}{x^2-1}=\lim_{x\to 1}\frac{4-\sqrt{x+15}}{x^2-1}\frac{4+\sqrt{x+15}}{4+\sqrt{x+15}}$
$$=\lim_{x\to 1}\frac{1-x}{(x+1)(x-1)(4+\sqrt{x+15})}$$
$$=\lim_{x\to 1}\frac{-(x-1)}{(x+1)(x-1)(4+\sqrt{x+15})}$$
$$=\lim_{x\to 1}\frac{-1}{(x+1)(4+\sqrt{x+15})}=-\frac{1}{16}$$

53. 32

54. 64

55. $\frac{1}{2}$

56. $\sqrt{\frac{4}{2}}=\sqrt{2}$

57. does not exist

58. 8

59. $8a$

60. $\frac{3}{2}$

61. (a) $$\lim_{x\to1}\frac{x^{100}-1}{x^2-1}=\lim_{x\to1}\frac{x^{100}-1}{(x+1)(x-1)}$$
$$=\lim_{x\to1}\frac{1}{x+1}\cdot\frac{x^{100}-1}{x-1}=\frac{1}{2}\cdot100=50$$
(b) $$\lim_{x\to1}\frac{x^{50}-1}{x-1}=\lim_{x\to1}\frac{x^{50}-1}{x-1}\cdot\frac{x^{50}+1}{x^{50}+1}$$
$$=\lim_{x\to1}\frac{x^{100}-1}{x-1}\cdot\frac{1}{x^{50}+1}=100\cdot\frac{1}{2}=50$$
(c) $$\lim_{x\to1}\frac{(x^{100}-1)^2}{(x-1)^2}=\lim_{x\to1}\frac{x^{100}-1}{x-1}\cdot\frac{x^{100}-1}{x-1}=100\cdot100=10,000$$

62. (a) $$\lim_{x\to0}\frac{2x}{\sin x}=2\lim_{x\to0}\frac{1}{\left(\frac{\sin x}{x}\right)}=2\cdot\frac{\lim_{x\to0}1}{\lim_{x\to0}\frac{\sin x}{x}}=2$$
(b) $$\lim_{x\to0}\frac{1-\cos^2 x}{x^2}=\lim_{x\to0}\frac{\sin^2 x}{x^2}=\lim_{x\to0}\frac{\sin x}{x}\cdot\frac{\sin x}{x}=1\cdot1=1$$
(c) $$\lim_{x\to0}\frac{8x^2-\sin x}{x}=\lim_{x\to0}\left(8x-\frac{\sin x}{x}\right)=\lim_{x\to0}8x-\lim_{x\to0}\frac{\sin x}{x}=-1$$

63. $$\lim_{x\to0}\sin x=\lim_{x\to0}\left(x\cdot\frac{\sin x}{x}\right)=\lim_{x\to0}x\cdot\lim_{x\to0}\frac{\sin x}{x}=0\cdot1=0$$

64. $$\lim_{x\to2}[2f(x)-5]=\lim_{x\to2}(x+3)\left(\frac{2f(x)-5}{x+3}\right)=5\cdot4=20$$
$$2\lim_{x\to2}f(x)-\lim_{x\to2}5=20$$
$$\lim_{x\to2}f(x)=\frac{20+5}{2}=12.5$$

2.3 Continuity

1. Continuous everywhere
2. Continuous everywhere
3. Discontinuous at 3 and 6
4. Discontinuous at -1 and 1
5. Discontinuous at $\frac{n\pi}{2}$, for $n=0,1,2,\ldots$
6. Discontinuous at -3 and $\frac{\pi}{2}+n\pi$, for n an integer
7. Discontinuous at 2
8. Discontinuous at 0

9. Continuous everywhere

10. Discontinuous at $x < 0$ and $\dfrac{1}{2}$

11. Discontinuous at e^{-2}

12. Discontinuous at 0

13. (a) yes (b) yes

14. (a) no (b) yes

15. (a) yes (b) yes

16. (a) yes (b) yes

17. (a) no (b) no

18. (a) yes (b) yes

19. (a) yes (b) no

20. (a) no (b) no

21. Solving $2 + \sec x = 0$, we obtain $\cos x = -\frac{1}{2}$, so $x = \frac{2\pi}{3} + 2n\pi$ or $x = \frac{4\pi}{3} + 2n\pi$. Thus, $f(x)$ is discontinuous on $(-\infty, \infty)$ and on $[\frac{\pi}{2}, \frac{3\pi}{2}]$.

22. Since $\sin \frac{1}{x}$ is discontinuous only at $x = 0$, it is continuous on $[\frac{1}{\pi}, \infty)$ and discontinuous on $[\frac{-2}{\pi}, \frac{2}{\pi}]$.

23. Since $f(x)$ is discontinuous only at $x = 2$, it is discontinuous on $[-1, 3]$ and continuous on $(2, 4]$.

24. Since $f(x)$ is defined and continuous exactly on $(1, 5]$, it is continuous on $[2, 4]$ and discontinuous on $[1, 5]$.

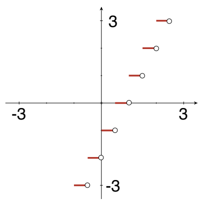
25. Since $\displaystyle\lim_{x\to 4^-} f(x) = 4m$ and $\displaystyle\lim_{x\to 4^+} f(x) = 16$, we have $4m = 16$ and $m = 4$.

26. Since $\displaystyle\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} = 4$ we have $f(2) = m$ and $m = 4$.

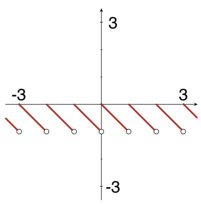
27. Since $\displaystyle\lim_{x\to 3^-} f(x) = 3m$, $\displaystyle\lim_{x\to 3^+} f(x) = 3$, and $f(3) = n$, we have $3m = 3 = n$, so $m = 1$ and $n = 3$.

28. Since $\displaystyle\lim_{x\to 1^-} f(x) = m - n$, $\displaystyle\lim_{x\to 1^+} f(x) = 2m + n$, and $f(1) = 5$, we have $m - n = 5$ and $2m + n = 5$. Adding, we obtain $3m = 10$, so $m = 10/3$ and $n = -5/3$.

29. Discontinuous at $\frac{n}{2}$, n an integer



30. Discontinuous at every integer



31. Since $\lim_{x\to 9} \frac{x-9}{\sqrt{x}-3} = \lim_{x\to 9} \frac{(\sqrt{x}+3)(\sqrt{x}-3)}{\sqrt{x}-3} = \lim_{x\to 9}(\sqrt{x}+3) = 6$, define $f(9) = 6$.

32. Since $\lim_{x\to 1} \frac{x^4-1}{x^2-1} = \lim_{x\to 1} \frac{(x^2+1)(x^2-1)}{x^2-1} = \lim_{x\to 1}(x^2+1) = 2$, define $f(1) = 2$.

33. $\lim_{x\to\pi/6} \sin(2x+\frac{\pi}{3}) = \sin\left[\lim_{x\to\pi/6}(2x+\frac{\pi}{3})\right] = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$

34. $\lim_{x\to\pi^2} \cos\sqrt{x} = \cos\left(\lim_{x\to\pi^2}\sqrt{x}\right) = \cos\pi = -1$

35. $\lim_{x\to\pi/2} \sin(\cos x) = \sin\left(\lim_{x\to\pi/2}\cos x\right) = \sin(\cos\frac{\pi}{2}) = \sin 0 = 0$

36. $\lim_{x\to\pi/2}[1+\cos(\cos x)] = 1+\cos\left(\lim_{x\to\pi/2}\cos x\right) = 1+\cos(\cos\frac{\pi}{2}) = 1+\cos 0 = 2$

37. $\lim_{t\to\pi} \cos\left(\frac{t^2-\pi^2}{t-\pi}\right) = \cos\left[\lim_{t\to\pi}\frac{(t-\pi)(t+\pi)}{t-\pi}\right] = \cos 2\pi = 1$

38. $\lim_{t\to 0} \tan\left(\frac{\pi t}{t^2+3t}\right) = \tan\left[\lim_{t\to 0}\frac{\pi t}{t(t+3)}\right] = \tan\left(\lim_{t\to 0}\frac{\pi}{t+3}\right) = \tan\frac{\pi}{3} = \sqrt{3}$

39. $\lim_{t\to\pi} \sqrt{t-\pi+\cos^2 t} = \sqrt{\cos^2\pi} = \sqrt{1} = 1$

40. $\lim_{t\to 1}(4t+\sin 2\pi t)^3 = \left[\lim_{t\to 1}(4t+\sin 2\pi t)\right]^3 = (4+\sin 2\pi)^3 = 4^3 = 64$

41. $$\lim_{x\to -3} \sin^{-1}\left(\frac{x+3}{x^2+4x+3}\right) = \sin^{-1}\left(\lim_{x\to -3} \frac{x+3}{x^2+4x+3}\right)$$
$$= \sin^{-1}\left[\lim_{x\to -3} \frac{x+3}{(x+3)(x+1)}\right]$$
$$= \sin^{-1}\left(\lim_{x\to -3} \frac{1}{x+1}\right) = \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

42. $\lim_{x\to\pi} e^{\cos 3x} = e^{\lim_{x\to\pi} \cos 3x} = e^{\cos 3\pi} = e^{-1}$

43. Since $(f \circ g)(x) = \dfrac{1}{\sqrt{x+3}}$, $f \circ g$ is continuous for $x + 3 > 0$ or on $(-3, \infty)$.

44. Since $(f\circ g)(x) = \dfrac{5(x-2)^2}{(x-2)^2-1} = \dfrac{5(x-2)^2}{x^2-4x+3} = \dfrac{5(x-2)^2}{(x-1)(x-3)}$, we see that $f\circ g$ is continuous for $x \neq 1$ and $x \neq 3$ or on $(\infty, 1) \cup (1, 3) \cup (3, \infty)$.

45. $f(1) = -1$, $f(5) = 15$. By the Intermediate Value Theorem, since $-1 \le 8 \le 15$, there exists $c \in [1, 5]$ such that $c^2 - 2c = 8$. Setting $c^2 - 2c - 8 = 0$ gives us $(c-4)(c+2) = 0$ or $c = -2, 4$. On $[1, 5]$, $c = 4$.

46. $f(-2) = 3$, $f(3) = 13$. By the Intermediate Value Theorem, since $3 \le 6 \le 13$, there exists $c \in [-2, 3]$ such that $c^2+c+1 = 6$. Setting $c^2+c-5 = 0$ gives us $c = \dfrac{-1 \pm \sqrt{1-4(1)(-5)}}{2} = \dfrac{-1 \pm \sqrt{21}}{2}$. On $[-2, 3]$, $c = \dfrac{-1+\sqrt{21}}{2}$.

47. $f(-2) = -3$, $f(2) = 5$. By the Intermediate Value Theorem, since $-3 \le 1 \le 5$, there exists $c \in [-2, 2]$ such that $c^3 - 2x + 1 = 1$. Setting $c^3 - 2c = 0$ gives us $c(c^2 - 2) = 0$. On $[-2, 2]$, $c = 0, \pm\sqrt{2}$.

48. $f(0) = 10$, $f(1) = 5$. By the Intermediate Value Theorem, since $5 \le 8 \le 10$, there exists $c \in [0, 1]$ such that $\dfrac{10}{c^2+1} = 8$. Setting $c^2 + 1 = \dfrac{5}{4}$ or $c^2 - \dfrac{1}{4} = 0$ gives us $(c + \frac{1}{2})(c - \frac{1}{2}) = 0$ or $c = \pm\frac{1}{2}$. On $[0, 1]$, $c = \frac{1}{2}$.

49. Since $f(0) = -7$, $f(3) = 242$, and $-7 \le 50 \le 242$, then by the Intermediate Value Theorem there exists $c \in [0, 3]$ such that $f(c) = 50$.

50. Since $f(a) > g(a)$, then $(f - g)(a) > 0$. Since $f(b) < g(b)$, then $(f - g)(b) < 0$. By the corollary to the Intermediate Value Theorem, there exists $c \in (a, b)$ such that $(f - g)(c) = 0$. Then $f(c) = g(c)$.

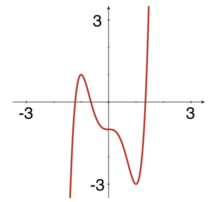
51. The equation will have a solution on $(0, 1)$ if $f(x) = 2x^7 + x - 1$ is 0 on $(0, 1)$. Since $f(0) = -1$ and $f(1) = 2$, then by the Intermediate Value Theorem $f(c) = 0$ for some $c \in (0, 1)$.

52. Let $f(x) = \dfrac{x^2+1}{x+3} + \dfrac{x^4+1}{x-4}$. Then $f(0) = \dfrac{1}{3} - \dfrac{1}{4} > 0$ and $f(1) = \dfrac{1}{2} - \dfrac{2}{3} < 0$. Thus, by the corollary to the Intermediate Value Theorem, $f(c) = 0$ for some c between 0 and 1, and hence between -3 and 4.

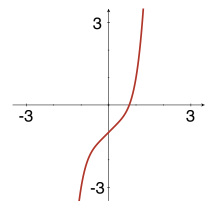
53. Let $f(x) = e^{-x} - \ln x$. Then $f(1) = e^{-1} - \ln 1 = e^{-1} > 0$ and $f(2) = e^{-2} - ln2 < 0$. Thus, by the corollary to the Intermediate Value Theorem, $f(c) = 0$ for some $c \in (1, 2)$.

54. Since $\dfrac{\left(\sin\frac{\pi}{2}\right)}{\left(\frac{\pi}{2}\right)} = \dfrac{2}{\pi}$, $\dfrac{\sin\pi}{\pi} = 0$, and $0 \le \dfrac{1}{2} \le \dfrac{2}{\pi}$, then by the Intermediate Value Theorem, there exists c between $\dfrac{\pi}{2}$ and π such that $\dfrac{\sin x}{x} = \dfrac{1}{2}$.

55. In $[-2, -1]$ the zero is approximately -1.21. In $[-1, 0]$ the zero is approximately -0.64. In $[1, 2]$ the zero is approximately 1.34.



56. In $[0, 1]$ the zero is approximately 0.75.



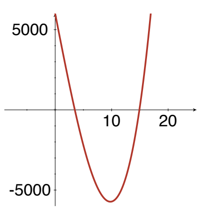
57. We want to solve $f(x) = x^5 + 2x - 7 = 50$ or $x^5 + 2x - 57 = 0$. It is easily seen that the expression on the left side of this equation is negative when $x = 2$ and positive when $x = 3$. Applying the bisection method on $[2, 3]$, we find $c \approx 2.21$.

58. Applying the bisection method to $f(x) = 2x^7 + x - 1$ on $[0, 1]$, we find $c \approx 0.75$.

59. In the solution of Problem 52 we saw that there is a zero in $[0, 1]$. Applying the bisection method on this interval, we find $c \approx 0.78$.

60. (a) If h is the height of the cylinder, then the volume is given by $V = \pi r^2 h$ and the surface area is $S = 2\pi r^2 + 2\pi rh$. Solving the latter equation for h, we obtain $h = \dfrac{S}{2\pi r} - r$. Substituting into the formula for V, we find $V = \dfrac{1}{2}Sr - \pi r^3$ or $2\pi r^3 - Sr + 2V = 0$.

(b)



(c) From the graph, we observe zeros in $[3, 4]$ and $[14, 15]$. The bisection method gives $r \approx 3.48$ ft and $r \approx 14.91$ ft. The corresponding values of h are $h = \dfrac{1800}{2\pi r} - r \approx 78.84$ ft and $h = \dfrac{1800}{2\pi r} - r \approx 4.29$ ft.

61. Since f and g are continuous at a, then $\lim_{x\to a} f(x) = f(a)$ and $\lim_{x\to a} g(x) = g(a)$. From this, we get:

$$\begin{aligned}\lim_{x\to a}(f+g)(x) &= \lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)\\ &= f(a)+g(a) = (f+g)(a)\end{aligned}$$

Thus, $f+g$ is continuous at a.

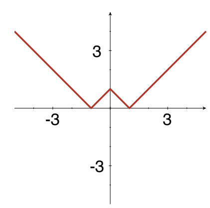
62. Since f and g are continuous at a, then $\lim_{x\to a} f(x) = f(a)$ and $\lim_{x\to a} g(x) = g(a)$. From this we get:

$$\begin{aligned}\lim_{x\to a}(f/g)(x) &= \lim_{x\to a}[f(x)/g(x)] = \lim_{x\to a} f(x)/\lim_{x\to a} g(x)\\ &= f(a)/g(a) = (f/g)(a) \text{ (since } g(a)\neq 0)\end{aligned}$$

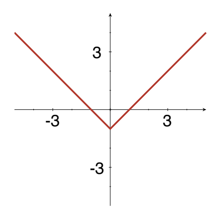
Thus, f/g is continuous at a.

63. $f\circ g$ will be discontinuous whenever $\cos x$ is an integer. In the interval $[0, 2\pi)$, this will be the case whenever $x = 0$, $\pi/2$, π, or $3\pi/2$. Thus, $f\circ g$ will be discontinuous for $x = n\pi/2$, n an integer.

64. $(f\circ g)(x) = \begin{cases} |x+1|, & x<0\\ |x-1|, & x\geq 0\end{cases}$ is continuous at $x=0$.



$(g\circ f)(x) = |x| - 1$ is continuous at $x = 0$.



65. (a) For any real a, $\lim_{x\to a} f(x)$ does not exist since f takes on the values 0 and 1 arbitrarily close to any real number. Therefore, the Dirichlet function is discontinuous at every real number.

(b) The graph consists of infinitely many points on each of the lines $y = 0$ and $y = 1$. In fact, between any two real numbers, there are infinitely many points of the graph on the line $y = 1$ and infinitely many points of the graph on the line $y = 0$.

(c) Let r be a positive rational number. If x is rational, then $x + r$ is rational so that $f(x+r) = 1 = f(x)$. If x is irrational, then $x+r$ is irrational so that $f(x+r) = 0 = f(x)$.

2.4 Trigonometric Limits

1. $\lim_{t\to 0}\dfrac{\sin 3t}{2t} = \dfrac{1}{2}\lim_{t\to 0}\dfrac{\sin 3t}{t} = \dfrac{3}{2}$

2. $\lim\limits_{t\to 0} \dfrac{\sin(-4t)}{t} = -4$

3. $\lim\limits_{x\to 0} \dfrac{\sin x}{4+\cos x} = \dfrac{0}{4+1} = 0$

4. $\lim\limits_{x\to 0} \dfrac{1+\sin x}{1+\cos x} = \dfrac{1+0}{1+1} = \dfrac{1}{2}$

5. $\lim\limits_{x\to 0} \dfrac{\cos 2x}{\cos 3x} = 1$

6. $\lim\limits_{x\to 0} \dfrac{\tan x}{3x} = \dfrac{1}{3}\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\cdot\dfrac{1}{\cos x}\right) = \dfrac{1}{3}(1\cdot 1) = \dfrac{1}{3}$

7. $\lim\limits_{t\to 0} \dfrac{1}{t\sec t\csc 4t} = \lim\limits_{t\to 0}\left(\dfrac{\sin 4t}{t}\cdot\cos t\right) = 4\cdot 1 = 4$

8. $$\begin{aligned}\lim_{t\to 0}(5t\cot 2t) &= 5\lim_{t\to 0}\frac{t\cos 2t}{\sin 2t} = 5\lim_{t\to 0}\left(\cos 2t\cdot\frac{1}{(\sin 2t)/t}\right)\\ &= 5\left(\lim_{t\to 0}\cos 2t\right)\left[\lim_{t\to 0}\frac{1}{(\sin 2t)/t}\right] = 5\cdot 1\cdot\frac{1}{2} = \frac{5}{2}\end{aligned}$$

9. $\lim\limits_{t\to 0} \dfrac{2\sin^2 t}{t\cos^2 t} = 2\lim\limits_{t\to 0}\left(\dfrac{\sin t}{t}\cdot\dfrac{\sin t}{\cos^2 t}\right) = 2\cdot 1\cdot 0 = 0$

10. $$\begin{aligned}\lim_{t\to 0}\frac{\sin^2(t/2)}{\sin t} &= \lim_{t\to 0}\left[\frac{\sin(t/2)}{t}\cdot\frac{t\sin(t/2)}{\sin t}\right]\\ &= \left[\lim_{t\to 0}\frac{\sin(t/2)}{t}\right]\left[\lim_{t\to 0}\frac{\sin(t/2)}{(\sin t)/t}\right] = \frac{1}{2}\cdot\frac{0}{1} = 0\end{aligned}$$

11. $\lim\limits_{t\to 0} \dfrac{\sin^2 6t}{t^2} = \lim\limits_{t\to 0}\left(\dfrac{\sin 6t}{t}\right)^2 = 6^2 = 36$

12. $\lim\limits_{t\to 0} \dfrac{t^3}{\sin^2 3t} = \lim\limits_{t\to 0}\left(t\cdot\dfrac{t^2}{\sin^2 3t}\right) = \left(\lim\limits_{t\to 0} t\right)\left(\lim\limits_{t\to 0}\dfrac{1}{[(\sin 3t)/t]^2}\right) = 0\cdot\dfrac{1}{3^2} = 0$

13. $\lim\limits_{x\to 1} \dfrac{\sin(x-1)}{2x-2} = \dfrac{1}{2}\lim\limits_{x\to 1}\dfrac{\sin(x-1)}{x-1} = \dfrac{1}{2}$

14. $\lim\limits_{x\to 2\pi} \dfrac{x-2\pi}{\sin x} = \lim\limits_{x\to 2\pi}\dfrac{x-2\pi}{\sin(x-2\pi)} = \lim\limits_{x\to 2\pi}\dfrac{1}{\sin(x-2\pi)/(x-2\pi)} = 1$

15. $\lim\limits_{x\to 0} \dfrac{\cos x}{x}$ does not exist.

16. $\lim\limits_{\theta\to \pi/2} \dfrac{1+\sin\theta}{\cos\theta}$ does not exist.

17. $\lim\limits_{x\to 0} \dfrac{\cos(3x-\pi/2)}{x} = \lim\limits_{x\to 0}\dfrac{\sin 3x}{x} = 3$

18. $\displaystyle\lim_{x\to-2}\frac{\sin(5x+10)}{4x+8}=\frac{5}{5}\cdot\frac{1}{4}\cdot\lim_{x\to-2}\frac{\sin(5x+10)}{x+2}=\frac{5}{4}\lim_{x\to-2}\frac{\sin(5x+10)}{5x+10}=\frac{5}{4}$

19. $\displaystyle\lim_{t\to0}\frac{\sin 3t}{\sin 7t}=\lim_{t\to0}\left(\frac{\sin 3t}{t}\cdot\frac{t}{\sin 7t}\right)=\left(\lim_{t\to0}\frac{\sin 3t}{t}\right)\left[\lim_{t\to0}\frac{1}{(\sin 7t)/t}\right]=3\cdot\frac{1}{7}=\frac{3}{7}$

20. $$\begin{aligned}\lim_{t\to0}\sin 2t\csc 3t&=\lim_{t\to0}\frac{\sin 2t}{\sin 3t}=\lim_{t\to0}\left(\frac{\sin 2t}{t}\cdot\frac{t}{\sin 3t}\right)\\&=\left(\lim_{t\to0}\frac{\sin 2t}{t}\right)\left[\lim_{t\to0}\frac{1}{(\sin 3t)/t}\right]=2\cdot\frac{1}{3}=\frac{2}{3}\end{aligned}$$

21. $\displaystyle\lim_{t\to0^+}\frac{\sin t}{\sqrt{t}}=\lim_{t\to0^+}\left(\sqrt{t}\cdot\frac{\sin t}{t}\right)=\left(\lim_{t\to0^+}\sqrt{t}\right)\left(\lim_{t\to0^+}\frac{\sin t}{t}\right)=0\cdot1=0$

22. Letting $u=\sqrt{t}$, we have $\displaystyle\lim_{t\to0}\frac{1-\cos\sqrt{t}}{\sqrt{t}}=\lim_{u\to0}\frac{1-\cos u}{u}=0.$

23. $\displaystyle\lim_{t\to0}\frac{t^2-5t\sin t}{t^2}=\lim_{t\to0}\left[1-5\left(\frac{\sin t}{t}\right)\right]=1-5=-4$

24. $\displaystyle\lim_{t\to0}\frac{\cos 4t}{\cos 8t}=\frac{1}{1}=1$

25. $$\begin{aligned}\lim_{x\to0^+}\frac{(x+2\sqrt{\sin x})^2}{x}&=\lim_{x\to0^+}\frac{x^2+4x\sqrt{\sin x}+4\sin x}{x}\\&=\lim_{x\to0^+}\left(x+4\sqrt{\sin x}+\frac{4\sin x}{x}\right)=0+0+4=4\end{aligned}$$

26. $\displaystyle\lim_{x\to0}\frac{(1-\cos x)^2}{x}=\left[\lim_{x\to0}(1-\cos x)\right]\left(\lim_{x\to0}\frac{1-\cos x}{x}\right)=0\cdot0=0$

27. $\displaystyle\lim_{x\to0}\frac{\cos x-1}{\cos^2 x-1}=\left(\lim_{x\to0}\frac{\cos x-1}{\cos x-1}\right)\left(\lim_{x\to0}\frac{1}{\cos x+1}\right)=1\cdot\frac{1}{2}=\frac{1}{2}$

28. $\displaystyle\lim_{x\to0}\frac{\sin x+\tan x}{x}=\lim_{x\to0}\left[\frac{\sin x}{x}+\left(\frac{\sin x}{x}\cdot\frac{1}{\cos x}\right)\right]=1+(1\cdot1)=2$

29. Letting $u=x^2$, we have $\displaystyle\lim_{x\to0}\frac{\sin 5x^2}{x^2}=\lim_{u\to0}\frac{\sin 5u}{u}=5.$

30. $$\begin{aligned}\lim_{t\to0}\frac{t^2}{1-\cos t}&=\lim_{t\to0}\left(\frac{t^2}{1-\cos t}\cdot\frac{1+\cos t}{1+\cos t}\right)=\lim_{t\to0}\left[\frac{t^2}{\sin^2 t}\cdot(1+\cos t)\right]\\&=\left[\lim_{t\to0}\frac{1}{(\sin t)/t}\right]^2\left[\lim_{t\to0}(1+\cos t)\right]=1^2\cdot2=2\end{aligned}$$

31. First, rewrite $\displaystyle\lim_{x\to2}\frac{\sin(x-2)}{x^2+2x-8}$ as $\displaystyle\lim_{x\to2}\frac{\sin(x-2)}{(x-2)(x+4)}$.

Letting $u=x-2$, we get $\displaystyle\lim_{u\to0}\left(\frac{\sin u}{u}\cdot\frac{1}{u+6}\right)=1\cdot\frac{1}{6}=\frac{1}{6}.$

32. First, rewrite $\lim\limits_{x\to 3} \dfrac{x^2-9}{\sin(x-3)}$ as $\lim\limits_{x\to 3} \dfrac{(x-3)(x+3)}{\sin(x-3)}$. Letting $u = x-3$:

$$\lim_{u\to 0} \left[\frac{u}{\sin u} \cdot (u+6)\right] = \lim_{u\to 0} \left[\frac{1}{(\sin u)/u} \cdot (u+6)\right] = \frac{1}{1} \cdot 6 = 6$$

33. $\lim\limits_{x\to 0} \dfrac{2\sin 4x + 1 - \cos x}{x} = \lim\limits_{x\to 0} \left(\dfrac{2\sin 4x}{x} + \dfrac{1-\cos x}{x}\right) = 8 + 0 = 8$

34. $\lim\limits_{x\to 0} \dfrac{4x^2 - 2\sin x}{x} = \lim\limits_{x\to 0} \left(4x - \dfrac{2\sin x}{x}\right) = 0 - 2 = -2$

35. Start by multiplying the function by $\dfrac{1+\tan x}{1+\tan x}$, producing:

$$\begin{aligned}\lim_{x\to\pi/4} \frac{1-\tan x}{\cos x - \sin x} &= \lim_{x\to\pi/4} \left(\frac{1-\tan x}{\cos x - \sin x} \cdot \frac{1+\tan x}{1+\tan x}\right) \\ &= \lim_{x\to\pi/4} \frac{1-\tan^2 x}{(\cos x - \sin x)(1+\tan x)}\end{aligned}$$

Focusing first on the denominator, we multiply out and simplify:

$$\begin{aligned}(\cos x - \sin x)(1+\tan x) &= \cos x + \cos x \tan x - \sin x - \sin x \tan x \\ &= \cos x + \cos x \left(\frac{\sin x}{\cos x}\right) - \sin x \left(\frac{\cos x}{\cos x}\right) - \sin x \left(\frac{\sin x}{\cos x}\right) \\ &= \cos x - \frac{\sin^2 x}{\cos x} = \frac{\cos^2 x - \sin^2 x}{\cos x}\end{aligned}$$

Substituting this result back into the function, we get:

$$\begin{aligned}\frac{1-\tan^2 x}{(\cos x - \sin x)(1+\tan x)} &= (1-\tan^2 x)\left(\frac{\cos x}{\cos^2 x - \sin^2 x}\right) \\ &= \frac{\cos x - \cos x\left(\dfrac{\sin^2 x}{\cos^2 x}\right)}{\cos^2 x - \sin^2 x} = \frac{\cos x - \left(\dfrac{\sin^2 x}{\cos x}\right)}{\cos^2 x - \sin^2 x} \\ &= \left(\frac{\cos^2 x - \sin^2 x}{\cos x}\right)\left(\frac{1}{\cos^2 x - \sin^2 x}\right) = \frac{1}{\cos x}\end{aligned}$$

Finally, returning to the limit, we have:

$$\lim_{x\to\pi/4} \frac{1-\tan x}{\cos x - \sin x} = \lim_{x\to\pi/4} \frac{1}{\cos x} = \frac{1}{\sqrt{2}/2} = \sqrt{2}$$

36. Using the trigonometric identity $\cos 2x = \cos^2 x - \sin^2 x$, we have:

$$\begin{aligned}\lim_{x\to\pi/4} \frac{\cos 2x}{\cos x - \sin x} &= \lim_{x\to\pi/4} \frac{\cos^2 x - \sin^2 x}{\cos x - \sin x} \\ &= \lim_{x\to\pi/4} \frac{(\cos x + \sin x)(\cos x - \sin x)}{\cos x - \sin x} \\ &= \lim_{x\to\pi/4} (\cos x + \sin x) = \sqrt{2}\end{aligned}$$

37. $$\lim_{h\to 0}\frac{f\left(\frac{\pi}{4}+h\right)-f\left(\frac{\pi}{4}\right)}{h}=\lim_{h\to 0}\frac{\sin\left(\frac{\pi}{4}+h\right)-\sin\left(\frac{\pi}{4}\right)}{h}$$
$$=\lim_{h\to 0}\frac{\sin(\pi/4)\cos h+\cos(\pi/4)\sin h-\sin(\pi/4)}{h}$$
$$=\lim_{h\to 0}\frac{(\sqrt{2}/2)\cos h+(\sqrt{2}/2)\sin h-(\sqrt{2}/2)}{h}$$
$$=\frac{\sqrt{2}}{2}\lim_{h\to 0}\frac{\cos h+\sin h-1}{h}=\frac{\sqrt{2}}{2}\lim_{h\to 0}\left(\frac{\cos h-1}{h}+\frac{\sin h}{h}\right)=\frac{\sqrt{2}}{2}$$

38. $$\lim_{h\to 0}\frac{f\left(\frac{\pi}{6}+h\right)-f\left(\frac{\pi}{6}\right)}{h}=\lim_{h\to 0}\frac{\cos\left(\frac{\pi}{6}+h\right)-\cos\left(\frac{\pi}{6}\right)}{h}$$
$$=\lim_{h\to 0}\frac{\cos(\pi/6)\cos h-\sin(\pi/6)\sin h-\cos(\pi/6)}{h}$$
$$=\lim_{h\to 0}\frac{(\sqrt{3}/2)\cos h-(1/2)\sin h-(\sqrt{3}/2)}{h}$$
$$=\lim_{h\to 0}\left[\frac{\sqrt{3}}{2}\left(\frac{\cos h-1}{h}\right)-\frac{1}{2}\left(\frac{\sin h}{h}\right)\right]=\frac{\sqrt{3}}{2}\cdot 0-\frac{1}{2}\cdot 1=-\frac{1}{2}$$

39. Since $-1\le\sin\frac{1}{x}\le 1$, then $-|x|\le x\sin\frac{1}{x}\le|x|$. Since $\lim_{x\to 0}(-|x|)=0$ and $\lim_{x\to 0}|x|=0$, then by the Squeeze Theorem, $\lim_{x\to 0}x\sin\frac{1}{x}=0$.

40. Since $-1\le\cos\frac{\pi}{x}\le 1$, then $-x^2\le x^2\cos\frac{\pi}{x}\le x^2$. Since $\lim_{x\to 0}-x^2=0$ and $\lim_{x\to 0}x^2=0$, then by the Squeeze Theorem, $\lim_{x\to 0}x^2\cos\frac{\pi}{x}=0$.

41. For both limits, we use the result from Problem 39, $\lim_{x\to 0}x\sin\frac{1}{x}=0$:

(a) $\lim_{x\to 0}x^3\sin\frac{1}{x}=\lim_{x\to 0}\left(x^2\cdot x\sin\frac{1}{x}\right)=\lim_{x\to 0}x^2\cdot\lim_{x\to 0}x\sin\frac{1}{x}=0\cdot 0=0$

(b) $\lim_{x\to 0}x^2\sin^2\frac{1}{x}=\left(\lim_{x\to 0}x\sin\frac{1}{x}\right)\left(\lim_{x\to 0}x\sin\frac{1}{x}\right)=0\cdot 0=0$

42. $|f(x)|\le B$ means that $B\ge 0$ and therefore $-B\le f(x)\le B$. Thus, $-Bx^2\le x^2f(x)\le Bx^2$ in that interval. Since $\lim_{x\to 0}(-Bx^2)=0$ and $\lim_{x\to 0}Bx^2=0$, then by the Squeeze Theorem, $\lim_{x\to 0}x^2f(x)=0$.

43. Since $\lim_{x\to 2}(2x-1)=3$ and $\lim_{x\to 2}(x^2-2x+3)=3$, then by the Squeeze Theorem, $\lim_{x\to 2}f(x)=3$.

44. Since $|f(x)-1|\le x^2$, then $f(x)-1\le x^2$, or $f(x)\le x^2+1$ when $f(x)-1>0$. However, $f(x)\le x^2+1$ is in fact true for all x, since $x^2\ge 0$ for all x. Similarly, we have $-x^2\le f(x)-1$, or $-x^2+1\le f(x)$ for all x. Since $\lim_{x\to 0}(-x^2+1)=1$ and $\lim_{x\to 0}(x^2+1)=1$, then by the Squeeze Theorem, $\lim_{x\to 0}f(x)=1$.

45. Let $t = x - \frac{\pi}{4}$. Thus, $x = t + \frac{\pi}{4}$ and we have the following substitutions:

$$\begin{aligned}
\sin x &= \sin(t + \tfrac{\pi}{4}) = \sin t \cos\frac{\pi}{4} + \cos t \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}\sin t + \frac{\sqrt{2}}{2}\cos t \\
\cos x &= \cos(t + \tfrac{\pi}{4}) = \cos t \cos\frac{\pi}{4} - \sin t \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}\cos t - \frac{\sqrt{2}}{2}\sin t \\
\sin x - \cos x &= \left(\frac{\sqrt{2}}{2}\sin t + \frac{\sqrt{2}}{2}\cos t\right) - \left(\frac{\sqrt{2}}{2}\cos t - \frac{\sqrt{2}}{2}\sin t\right) = \sqrt{2}\sin t
\end{aligned}$$

With these substitutions, $\lim\limits_{x\to\pi/4} \frac{\sin x - \cos x}{x - (\pi/4)} = \lim\limits_{t\to 0} \frac{\sqrt{2}\sin t}{t} = \sqrt{2}$.

46. Let $t = x - \pi$. Thus, $x = t + \pi$. Substituting, we get:

$$\begin{aligned}
\lim_{x\to\pi} \frac{x-\pi}{\tan 2x} &= \lim_{t\to 0} \frac{t}{\tan(2t + 2\pi)} = \lim_{t\to 0} \frac{t}{\tan 2t} \\
&= \lim_{t\to 0} \frac{1}{\left(\frac{\tan 2t}{t}\right)} = \lim_{t\to 0} \frac{1}{\frac{1}{\cos 2t} \cdot \frac{\sin 2t}{t}} = \frac{1}{1\cdot 2} = \frac{1}{2}
\end{aligned}$$

47. Let $t = \pi - (\pi/x)$. Therefore $\pi/x = \pi - t$ and $\sin(\pi/x) = \sin(\pi - t) = \sin t$. In addition, we can derive $x - 1 = \frac{t}{\pi - t}$, giving us:

$$\lim_{x\to 1} \frac{\sin(\pi/x)}{x-1} = \lim_{t\to 0} \frac{(\sin t)(\pi - t)}{t} = \lim_{t\to 0} \frac{\sin t}{t} \cdot \lim_{t\to 0}(\pi - t) = 1\cdot\pi = \pi$$

48. Let $t = \frac{\pi}{2} - \frac{\pi}{x}$. Substituting in the same way as in Problem 47, we get:

$$\lim_{x\to 2} \frac{\cos(\pi/x)}{x-2} = \lim_{t\to 0} \frac{(\sin t)(\pi - 2t)}{4t} = \lim_{t\to 0} \frac{\sin t}{t} \cdot \lim_{t\to 0} \frac{\pi - 2t}{4} = \frac{\pi}{4}$$

49. f is continuous at $x = 0$ because $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1 = f(0)$.

50. Since $|x| = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases}$, knowing that $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$ means:

$$\begin{aligned}
&\lim_{x\to 0^+} \frac{\sin|x|}{x} = \lim_{x\to 0^+} \frac{\sin x}{x} = 1 \\
&\lim_{x\to 0^-} \frac{\sin|x|}{x} = \lim_{x\to 0^-} \frac{\sin(-x)}{x} = \lim_{x\to 0^-} \frac{-\sin x}{x} = -\lim_{x\to 0^-} \frac{\sin x}{x} = -1
\end{aligned}$$

Since $\lim\limits_{x\to 0^+} \frac{\sin|x|}{x} \neq \lim\limits_{x\to 0^-} \frac{\sin|x|}{x}$, then $\lim\limits_{x\to 0} \frac{\sin|x|}{x}$ does not exist.

2.5 Limits that Involve Infinity

1. $-\infty$
2. ∞
3. ∞
4. $-\infty$
5. ∞
6. $-\infty$
7. ∞
8. $-\infty$

9. $\displaystyle\lim_{x\to\infty}\frac{x^2-3x}{4x^2+5}=\lim_{x\to\infty}\frac{1-3/x}{4+5/x^2}=\frac{1}{4}$

10. $\displaystyle\lim_{x\to\infty}\frac{x^2}{1+x^{-2}}=\lim_{x\to\infty}\frac{1}{1/x^2+1/x^4}=\infty$

11. 5

12. $\displaystyle\lim_{x\to-\infty}\left(\frac{6}{\sqrt[3]{x}}+\frac{1}{\sqrt[5]{x}}\right)=0$

13. $\displaystyle\lim_{x\to\infty}\frac{8-\sqrt{x}}{1+4\sqrt{x}}=\lim_{x\to\infty}\frac{(8/\sqrt{x})-1}{(1/\sqrt{x})+4}=-\frac{1}{4}$

14. $\displaystyle\lim_{x\to-\infty}\frac{1+7\sqrt[3]{x}}{2\sqrt[3]{x}}=\lim_{x\to-\infty}\frac{1/\sqrt[3]{x}+7}{2}=\frac{7}{2}$

15. $\displaystyle\lim_{x\to\infty}\left(\frac{3x}{x+2}-\frac{x-1}{2x+6}\right)=\lim_{x\to\infty}\left(\frac{3}{1+2/x}-\frac{1-1/x}{2+6/x}\right)=3-\frac{1}{2}=\frac{5}{2}$

16. $\displaystyle\lim_{x\to\infty}\left(\frac{x}{3x+1}\right)\left(\frac{4x^2+1}{2x^2+x}\right)^3=\lim_{x\to\infty}\left(\frac{1}{3+1/x}\right)\left(\frac{4+1/x^2}{2+1/x}\right)^3=\frac{1}{3}\cdot 2^3=\frac{8}{3}$

17. $\displaystyle\lim_{x\to\infty}\sqrt{\frac{3x+2}{6x-8}}=\lim_{x\to\infty}\sqrt{\frac{3+2/x}{6-8/x}}=\sqrt{\frac{1}{2}}=\frac{\sqrt{2}}{2}$

18. $\displaystyle\lim_{x\to-\infty}\sqrt[3]{\frac{2x-1}{7-16x}}=\lim_{x\to-\infty}\sqrt[3]{\frac{2-1/x}{7/x-16}}=\sqrt[3]{-\frac{2}{16}}=-\frac{1}{2}$

19. $\displaystyle\lim_{x\to\infty}\left(x-\sqrt{x^2+1}\right)=\lim_{x\to\infty}\left(x-\sqrt{x^2+1}\right)\cdot\frac{x+\sqrt{x^2+1}}{x+\sqrt{x^2+1}}=\lim_{x\to\infty}\frac{-1}{x+\sqrt{x^2+1}}=0$

20. $$\begin{aligned}\lim_{x\to\infty}\left(\sqrt{x^2+5x}-x\right)&=\lim_{x\to\infty}\left(\sqrt{x^2+5x}-x\right)\cdot\frac{\sqrt{x^2+5x}+x}{\sqrt{x^2+5x}+x}\\&=\lim_{x\to\infty}\frac{5x}{\sqrt{x^2+5x}+x}=\lim_{x\to\infty}\frac{5}{\sqrt{1+5/x}+1}=\frac{5}{2}\end{aligned}$$

21. $\displaystyle\lim_{x\to\infty}\cos\left(\frac{5}{x}\right)=\cos\left[\lim_{x\to\infty}\left(\frac{5}{x}\right)\right]=1$

22. $\displaystyle\lim_{x\to-\infty}\sin\left(\frac{\pi x}{3-6x}\right)=\lim_{x\to-\infty}\sin\left(\frac{\pi}{3/x-6}\right)=\sin\left[\lim_{x\to-\infty}\left(\frac{\pi}{3/x-6}\right)\right]=-\frac{1}{2}$

23. $$\lim_{x\to-\infty}\sin^{-1}\left(\frac{x}{\sqrt{4x^2+1}}\right)=\lim_{x\to-\infty}\sin^{-1}\left[\frac{\left(\frac{x}{|x|}\right)}{\sqrt{4+1/x^2}}\right]=\lim_{x\to-\infty}\sin^{-1}\left[\frac{\left(\frac{x}{-x}\right)}{\sqrt{4+1/x^2}}\right]$$
$$=\sin^{-1}\left[\lim_{x\to-\infty}\left(\frac{-1}{\sqrt{4+1/x^2}}\right)\right]=\sin^{-1}\left(-\frac{1}{2}\right)=-\frac{\pi}{6}$$

24. $\lim_{x\to\infty}\ln\left(\frac{x}{x+8}\right)=\lim_{x\to\infty}\ln\left(\frac{1}{1+8/x}\right)=\ln\left[\lim_{x\to\infty}\left(\frac{1}{1+8/x}\right)\right]=\ln 1=0$

25. Start with $\frac{4x+1}{\sqrt{x^2+1}}=\frac{\left(\frac{4x}{|x|}+\frac{1}{|x|}\right)}{\sqrt{1+1/x^2}}$. From this, $\lim_{x\to-\infty}f(x)=\lim_{x\to-\infty}\frac{-4-1/x}{\sqrt{1+1/x^2}}=-4$ and $\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\frac{4+1/x}{\sqrt{1+1/x^2}}=4$.

26. Start with $\frac{\sqrt{9x^2+6}}{5x-1}=\frac{\sqrt{9+6/x^2}}{\left(\frac{5x}{|x|}-\frac{1}{|x|}\right)}$. From this, $\lim_{x\to-\infty}f(x)=\lim_{x\to-\infty}\frac{\sqrt{9+6/x^2}}{-5+1/x}=\frac{\sqrt{9}}{-5}=-\frac{3}{5}$ and $\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\frac{\sqrt{9+6/x^2}}{5-1/x}=\frac{\sqrt{9}}{5}=\frac{3}{5}$.

27. Start with $\frac{2x+1}{\sqrt{3x^2+1}}=\frac{\left(\frac{2x}{|x|}+\frac{1}{|x|}\right)}{\sqrt{3+1/x^2}}$. From this, $\lim_{x\to-\infty}f(x)=\lim_{x\to-\infty}\frac{-2-1/x}{\sqrt{3+1/x^2}}=-\frac{2}{\sqrt{3}}=-\frac{2\sqrt{3}}{3}$ and $\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\frac{2+1/x}{\sqrt{3+1/x^2}}=\frac{2}{\sqrt{3}}=\frac{2\sqrt{3}}{3}$.

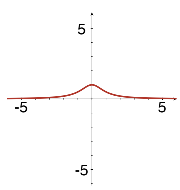
28. Start with $\frac{-5x^2+6x+3}{\sqrt{x^4+x^2+1}}=\frac{\left(-5+\frac{6}{|x|}+\frac{3}{x^2}\right)}{\sqrt{1+1/x^2+1/x^4}}$. From this, $\lim_{x\to-\infty}f(x)=\lim_{x\to-\infty}\frac{-5-6/x+3/x^2}{\sqrt{1+1/x^2+1/x^4}}=\frac{-5}{\sqrt{1}}=-5$ and $\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\frac{-5+6/x+3/x^2}{\sqrt{1+1/x^2+1/x^4}}=\frac{-5}{\sqrt{1}}=-5$.

29. $$\lim_{x\to-\infty}\frac{e^x-e^{-x}}{e^x+e^{-x}}=\frac{\left(\lim_{x\to-\infty}e^x\right)-\left(\lim_{x\to-\infty}e^{-x}\right)}{\left(\lim_{x\to-\infty}e^x\right)+\left(\lim_{x\to-\infty}e^{-x}\right)}=\frac{0-\left(\lim_{x\to-\infty}e^{-x}\right)}{0+\left(\lim_{x\to-\infty}e^{-x}\right)}$$
$$=\lim_{x\to-\infty}\frac{-e^{-x}}{e^{-x}}=\lim_{x\to-\infty}-1=-1$$
$$\lim_{x\to\infty}\frac{e^x-e^{-x}}{e^x+e^{-x}}=\frac{\left(\lim_{x\to\infty}e^x\right)-\left(\lim_{x\to\infty}e^{-x}\right)}{\left(\lim_{x\to\infty}e^x\right)+\left(\lim_{x\to\infty}e^{-x}\right)}=\frac{\left(\lim_{x\to\infty}e^x\right)-0}{\left(\lim_{x\to\infty}e^x\right)+0}$$
$$=\lim_{x\to\infty}\frac{e^x}{e^x}=\lim_{x\to\infty}1=1$$

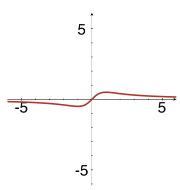
30. $$\lim_{x\to-\infty}\left(1+\frac{2e^{-x}}{e^x+e^{-x}}\right)=1+\frac{\lim\limits_{x\to-\infty}2e^{-x}}{\left(\lim\limits_{x\to-\infty}e^x\right)+\left(\lim\limits_{x\to-\infty}e^{-x}\right)}$$
$$=1+\frac{\lim\limits_{x\to-\infty}2e^{-x}}{0+\left(\lim\limits_{x\to-\infty}e^{-x}\right)}=1+\lim_{x\to-\infty}\frac{2e^{-x}}{e^{-x}}=1+\lim_{x\to-\infty}2=3$$
$$\lim_{x\to\infty}\left(1+\frac{2e^{-x}}{e^x+e^{-x}}\right)=1+\frac{\lim\limits_{x\to\infty}2e^{-x}}{\left(\lim\limits_{x\to\infty}e^x\right)+\left(\lim\limits_{x\to\infty}e^{-x}\right)}=1+\lim_{x\to\infty}\frac{0}{e^x}=1$$

31. $$\lim_{x\to-\infty}\frac{|x-5|}{x-5}=\lim_{x\to-\infty}\frac{-x+5}{x-5}=\lim_{x\to-\infty}\frac{-1+5/x}{1-5/x}=-1$$
$$\lim_{x\to\infty}\frac{|x-5|}{x-5}=\lim_{x\to\infty}\frac{x-5}{x-5}=1$$

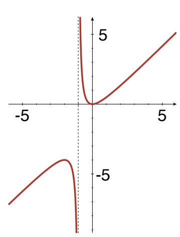
32. $$\lim_{x\to-\infty}\frac{|4x|+|x-1|}{x}=\lim_{x\to-\infty}\frac{-4x-(x-1)}{x}=\lim_{x\to-\infty}\frac{-5x+1}{x}$$
$$=\lim_{x\to-\infty}\frac{-5+1/x}{1}=-5$$
$$\lim_{x\to\infty}\frac{|4x|+|x-1|}{x}=\lim_{x\to\infty}\frac{4x+x-1}{x}=\lim_{x\to\infty}\frac{5x-1}{x}=\lim_{x\to\infty}\frac{5-1/x}{1}=5$$

33. 

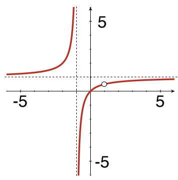
Vertical asymptote: none
Horizontal asymptote: $y=0$

34. 

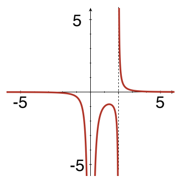
Vertical asymptote: none
Horizontal asymptote: $y=0$

35. 

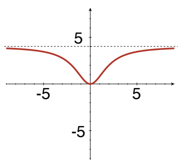
Vertical asymptote: $x=-1$
Horizontal asymptote: none

36. 

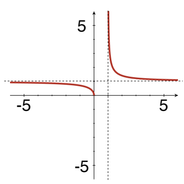
Vertical asymptote: $x = -1$
Horizontal asymptote: $y = 1$

37. 

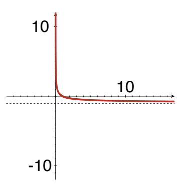
Vertical asymptote: $x = 0,\ x = 2$
Horizontal asymptote: $y = 0$

38. 

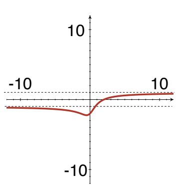
Vertical asymptote: none
Horizontal asymptote: $y = 4$

39. 

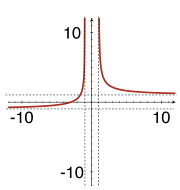
Vertical asymptote: $x = 1$
Horizontal asymptote: $y = 1$

40. 

Vertical asymptote: $x = 0$
Horizontal asymptote: $y = -1$

41. 

Vertical asymptote: none
Horizontal asymptote: $y = -1,\ y = 1$

42. 

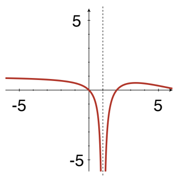
Vertical asymptote: $x = -1,\ x = 1$
Horizontal asymptote: $y = -1,\ y = 1$

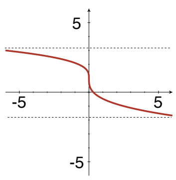
43. (a) 2 (b) $-\infty$ (c) 0 (d) 2

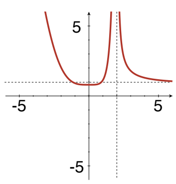
44. (a) ∞ (b) ∞ (c) 1 (d) 3

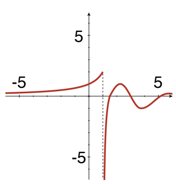
45. (a) $-\infty$ (b) $-3/2$ (c) ∞ (d) 0

46. (a) ∞ (b) $-\infty$ (c) 0 (d) 0

47. 

48. 

49. 

50. 

51. $\lim\limits_{x\to\infty} x\sin\dfrac{3}{x} = \lim\limits_{x\to\infty}\left(x\sin\dfrac{3}{x}\right)\left(\dfrac{3/x}{3/x}\right) = \lim\limits_{x\to\infty} x(3/x)\left(\dfrac{\sin 3/x}{3/x}\right)$
$= \left(\lim\limits_{x\to\infty} x\cdot\dfrac{3}{x}\right)\left(\lim\limits_{x\to\infty}\dfrac{\sin 3/x}{3/x}\right) = \left(\lim\limits_{x\to\infty} 3\right)\left(\lim\limits_{x\to\infty}\dfrac{\sin 3/x}{3/x}\right)$

At this point, we substitute $t = 3/x$, resulting in:

$$\left(\lim_{x\to\infty} 3\right)\left(\lim_{x\to\infty}\frac{\sin 3/x}{3/x}\right) = 3\lim_{t\to 0}\frac{\sin t}{t} = 3$$

52. $\lim\limits_{v\to c^-}\dfrac{m_0}{\sqrt{1-v^2/c^2}} = \lim\limits_{v\to c^-}\dfrac{m_0}{\sqrt{1-1}} = \lim\limits_{v\to c^-}\dfrac{m_0}{0}$; so as $v\to c^-$, $m\to\infty$.

53.

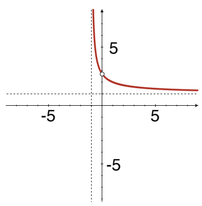
$x \to \infty$	10	100	1000	10000
$f(x)$	1.99986667	1.99999999	2.00000000	2.00000000

$\lim_{x \to \infty} x^2 \sin \frac{2}{x^2} = 2$

54.

$x \to \infty$	10	100	1000	10000
$f(x)$	0.95114995	0.99501240	0.99950012	0.99995000

$\lim_{x \to \infty} \left(\cos \frac{1}{x}\right)^x = 1$

55. 

(a) $\lim_{x \to -1^+} f(x) = \infty$ (b) $\lim_{x \to 0} f(x) \approx 2.7$ (c) $\lim_{x \to \infty} f(x) = 1$

56. (a) The area of the right triangle shown in Figure 2.5.18 is $\frac{1}{2}r^2 \sin \frac{\pi}{n} \cos \frac{\pi}{n}$. Since there are $2n$ such right triangles, the area of the polygon is:

$$A(n) = 2n\left(\frac{1}{2}r^2 \sin \frac{\pi}{n} \cos \frac{\pi}{n}\right) = nr^2\left(\frac{1}{2} \sin \frac{2\pi}{n}\right) = \frac{n}{2}r^2 \sin \frac{2\pi}{n}$$

(b) $A(100) \approx 3.1395r^2$; $A(1000) \approx 3.1416r^2$

(c) Letting $x = 2\pi/n$ (while noting that $n = 2\pi/x$) and substituting into $A(n)$ above, we obtain:

$$A(n) = \frac{\pi}{x}r^2 \sin x = \pi r^2\left(\frac{\sin x}{x}\right)$$

From (10) of Section 2.4, we see that:

$$\lim_{n \to \infty} A(n) = \pi r^2\left(\lim_{x \to 0} \frac{\sin x}{x}\right) = \pi r^2$$

57. (a) $\lim_{x \to \pm\infty} [f(x) - g(x)] = \lim_{x \to \pm\infty} \left[\frac{x^2}{x+1} - (x-1)\right]$

$$= \lim_{x \to \pm\infty} \left[\frac{x^2}{x+1} - \frac{(x-1)(x+1)}{x+1}\right] = \lim_{x \to \pm\infty} \frac{x^2 - (x^2-1)}{x+1}$$

$$= \lim_{x \to \pm\infty} \frac{1}{x+1} = 0$$

(b) The graphs of f and g get closer and closer to each other when $|x|$ is large.

(c) g is a *slant asymptote* to f.

58. All points P are of the form (x, x^2+1) while all points Q are of the form (x, x^2). When the y coordinates of P and Q are the same, we have $x_P^2 + 1 = x_Q^2$ or $x_Q = \sqrt{x_P^2+1}$, and thus the horizontal distance between P and Q is $|x_Q - x_P| = |\sqrt{x_P^2+1} - x_P|$. Thus:

$$\lim_{x\to\infty} |\sqrt{x^2+1} - x| = \lim_{x\to\infty} \left| \left(\sqrt{x^2+1} - x\right) \left(\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}\right) \right|$$
$$= \lim_{x\to\infty} \left| \frac{x^2+1-x^2}{\sqrt{x^2+1}+x} \right| = \lim_{x\to\infty} \left| \frac{1}{\sqrt{x^2+1}+x} \right| = 0.$$

2.6 Limits — A Formal Approach

1. $|10-10| = 0 < \epsilon$ for any choice of δ.

2. $|\pi - \pi| = 0 < \epsilon$ for any choice of δ.

3. $|x-3| < \epsilon$ whenever $0 < |x-3| < \epsilon$. Choose $\delta = \epsilon$.

4. $|2x-8| = 2|x-4| < \epsilon$ whenever $0 < |x-4| < \epsilon/2$. Choose $\delta = \epsilon/2$.

5. $|x+6-5| = |x+1| < \epsilon$ whenever $0 < |x-(-1)| < \epsilon$. Choose $\delta = \epsilon$.

6. $|x-4-(-4)| = |x-0| < \epsilon$ whenever $0 < |x-0| < \epsilon$. Choose $\delta = \epsilon$.

7. $|3x+7-7| = 3|x-0| < \epsilon$ whenever $0 < |x-0| < \epsilon/3$. Choose $\delta = \epsilon/3$.

8. $|9-6x-3| = |6-6x| = 6|x-1| < \epsilon$ whenever $0 < |x-1| < \epsilon/6$. Choose $\delta = \epsilon/6$.

9. $\left|\frac{2x-3}{4} - \frac{1}{4}\right| = \frac{1}{4}|2x-4| = \frac{1}{2}|x-2| < \epsilon$ whenever $0 < |x-2| < 2\epsilon$. Choose $\delta = 2\epsilon$.

10. $|8(2x+5)-48| = |16x-8| = 16\left|x - \frac{1}{2}\right| < \epsilon$ whenever $0 < |x-2| < \epsilon/16$. Chose $\delta = \epsilon/16$.

11. $\left|\frac{x^2-25}{x+5} - (-10)\right| = |x-5+10| = |x-(-5)| < \epsilon$ whenever $0 < |x-(-5)| < \epsilon$. Choose $\delta = \epsilon$.

12. $\left|\frac{x^2-7x+12}{2x-6} - \left(\frac{1}{2}\right)\right| = \left|\frac{(x-3)(x-4)}{2(x-3)} + \frac{1}{2}\right| = \frac{1}{2}|x-4+1| = \frac{1}{2}|x-3| < \epsilon$ whenever $0 < |x-3| < 2\epsilon$. Choose $\delta = 2\epsilon$.

13. $\left|\frac{8x^5+12x^4}{x^4} - 12\right| = |8x+12-12| = 8|x-0| < \epsilon$ whenever $0 < |x-0| < \epsilon/8$. Choose $\delta = \epsilon/8$.

14. $\left|\frac{2x^3+5x^2-2x-5}{x^2-1} - 7\right| = \left|\frac{(2x+5)(x^2-1)}{x^2-1} - 7\right| = |2x+5-7| = |2x-2| = 2|x-1| < \epsilon$ whenever $0 < |x-1| < \epsilon/2$. Choose $\delta = \epsilon/2$.

15. $|x^2-0| = |x-0|^2 < \epsilon$ whenever $0 < |x-0| < \sqrt{\epsilon}$. Choose $\delta = \sqrt{\epsilon}$.

16. $|8x^3-0| = 8|x-0|^3 < \epsilon$ whenever $0 < |x-0| < \sqrt[3]{\epsilon}/2$. Choose $\delta = \sqrt[3]{\epsilon}/2$.

17. $|\sqrt{5x} - 0| = \sqrt{5}|x - 0|^{1/2} < \epsilon$ whenever $0 < x < \epsilon^2/5$. Choose $\delta = \epsilon^2/5$.

18. $|\sqrt{2x-1} - 0| = \sqrt{2}|x - 1/2|^{1/2} < \epsilon$ whenever $1/2 < x < 1/2 + \epsilon^2/2$. Choose $\delta = \epsilon^2/2$.

19. $|2x - 1 - (-1)| = |2x| = 2|x - 0| < \epsilon$ whenever $0 - \epsilon/2 < x < 0$. Choose $\delta = \epsilon/2$.

20. $|3 - 3| = 0 < \epsilon$ whenever $x > 1$, for any choice of δ.

21. Note that $|x^2 - 9| = |x - 3||x + 3|$ and consider only values of x for which $|x - 3| < 1$. Then $2 < x < 4$ and $5 < x + 3 < 7$, so $|x + 3| < 7$. Thus, $|x^2 - 9| = |x - 3||x + 3| < 7|x - 3| < \epsilon$ whenever $|x - 3| < \epsilon/7$. Choose $\delta = \min\{1, \epsilon/7\}$.

22. Note that $|2x^2 + 4 - 12| = 2|x^2 - 4| = 2|x - 2||x + 2|$ and consider only values of x for which $|x - 2| < 1$. Then $1 < x < 3$ and $3 < x + 2 < 5$, so $|x + 2| < 5$. Thus $|2x^2 + 4 - 12| = 2|x - 2||x + 2| < 10|x - 2| < \epsilon$ whenever $|x - 2| < \epsilon/10$. Choose $\delta = \min\{1, \epsilon/10\}$.

23. Note that $|x^2 - 2x + 4 - 3| = |x - 1|^2 < \epsilon$ whenever $|x - 1| < \sqrt{\epsilon}$. Choose $\delta = \sqrt{\epsilon}$.

24. Note that $|x^2+2x-35| = |x-5||x+7|$ and consider only values of x for which $|x-5| < 1$. Then $4 < x < 6$ and $11 < x+7 < 13$, so $|x+7| < 13$. Thus $|x^2+2x-35| = |x-5||x+7| < 13|x-5|\epsilon$ whenever $|x - 5| < \epsilon/13$. Choose $\delta = \min\{1, \epsilon/13\}$.

25. We need to show $|\sqrt{x} - \sqrt{a}| < \epsilon$ whenever $0 < |x - a| < \delta$ for an appropriate choice of δ. For $\delta = \sqrt{a}\epsilon$, we have

$$|\sqrt{x} - \sqrt{a}| = |\sqrt{x} - \sqrt{a}| \cdot \frac{\sqrt{x} + \sqrt{a}}{\sqrt{x} + \sqrt{a}} = \frac{|x - a|}{\sqrt{x} + \sqrt{a}} < \frac{|x - a|}{\sqrt{a}} < \frac{\sqrt{a}\epsilon}{\sqrt{a}} = \epsilon$$

whenever $0 < |x - a| < \delta$. Thus, $\lim\limits_{x \to a} \sqrt{x} = \sqrt{a}$.

26. We need to show that $|1/x - 1/2| < \epsilon$, whenever $0 < |x - 2| < \delta$, for an appropriate choice of δ. Without loss of generality, we may assume that $\delta < 1$. Then $|x - 2| < 1$ or $1 < x < 3$. For these values of x, $1/3 < 1/x < 1$. Then, for $\delta = 2\epsilon$, we have

$$\left|\frac{1}{x} - \frac{1}{2}\right| = \frac{1}{2}\left(\frac{1}{x}\right)|2 - x| < \frac{1}{2}(1)|x - 2| < \frac{1}{2}(2\epsilon) = \epsilon$$

whenever $0 < |x - 2| < \delta$. Thus, $\lim\limits_{x \to 2} 1/x = 1/2$.

27. Assume $\lim\limits_{x \to 1} f(x) = L$. Take $\epsilon = 1$. Then there exists $\delta > 0$ such that $|f(x) - L| < 1$ whenever $0 < |x - 1| < \delta$. To the right of 1, choose $x = 1 + \delta/2$.

$$\begin{array}{rc} \text{Since} & 0 < |1 + \delta/2 - 1| = |\delta/2| < \delta, \\ \text{we must have} & |f(1 + \delta/2) - L| = |0 - L| = |L| < 1, \\ \text{or} & -1 < L < 1. \end{array}$$

To the left of 1, choose $x = 1 - \delta/2$.

$$\begin{array}{rc} \text{Since} & 0 < |1 - \delta/2 - 1| = |-\delta/2| < \delta, \\ \text{we must have} & |f(1 - \delta/2) - L| = |2 - L| < 1, \\ \text{or} & 1 < L < 3. \end{array}$$

Since no L can satisfy the conditions that $-1 < L < 1$ *and* $1 < L < 3$, we conclude that $\lim_{x\to 1} f(x)$ does not exist.

28. Assume $\lim_{x\to 3} f(x) = L$. Take $\epsilon = 1$. Then there exists $\delta > 0$ such that $|f(x)-L| < 1$ whenever $0 < |x-3| < \delta$. To the right of 3, choose $x = 3+\delta/2$.

$$\begin{aligned} \text{Since} \quad & 0 < |3+\delta/2-3| = |\delta/2| < \delta, \\ \text{we must have} \quad & |f(3+\delta/2)-L| = |-1-L| = |L+1| < 1, \\ \text{or} \quad & -2 < L < 0. \end{aligned}$$

To the left of 3, choose $x = 3-\delta/2$.

$$\begin{aligned} \text{Since} \quad & 0 < |3-\delta/2-3| = |-\delta/2| < \delta, \\ \text{we must have} \quad & |f(3-\delta/2)-L| = |1-L| = |L-1| < 1, \\ \text{or} \quad & 0 < L < 2. \end{aligned}$$

Since no L can satisfy the conditions that $-2 < L < 0$ *and* $0 < L < 2$, we conclude that $\lim_{x\to 3} f(x)$ does not exist.

29. Assume $\lim_{x\to 0} f(x) = L$. Take $\epsilon = 1$. Then there exists $\delta > 0$ such that $|f(x)-L| < 1$ whenever $0 < |x-0| < \delta$. To the right of 0, choose $x = \delta/2$.

$$\begin{aligned} \text{Since} \quad & 0 < |\delta/2-0| = |\delta/2| < \delta, \\ \text{we must have} \quad & |f(\delta/2)-L| = |2-\delta/2-L| < 1, \\ \text{or} \quad & 1-\delta/2 < L < 3-\delta/2. \end{aligned}$$

To the left of 0, choose $x = -\delta/2$.

$$\begin{aligned} \text{Since} \quad & 0 < |-\delta/2-0| = |-\delta/2| < \delta, \\ \text{we must have} \quad & |f(-\delta/2)-L| = |-\delta/2-L| < 1, \\ \text{or} \quad & -1-\delta/2 < L < 1-\delta/2. \end{aligned}$$

Since no L can satisfy the conditions that $1-\delta/2 < L < 3-\delta/2$ *and* $-1-\delta/2 < L < 1-\delta/2$, we conclude that $\lim_{x\to 0} f(x)$ does not exist.

30. Assume $\lim_{x\to 0} f(x) = L$. Take $\epsilon = 1$. Then there exists $\delta > 0$ such that $|f(x)-L| < 1$ whenever $0 < |x-0| < \delta$. Since $|f(x)-L| < 1$ for all x such that $0 < |x| < \delta$, we may assume that $\delta < 2$. To the right of 0, choose $x = \delta/2$.

$$\begin{aligned} \text{Since} \quad & 0 < |\delta/2-0| = |\delta/2| < \delta, \\ \text{we must have} \quad & |f(\delta/2)-L| = |2/\delta-L| = |L-2\delta| < 1, \\ \text{or} \quad & 2/\delta-1 < L < \delta/2+1. \end{aligned}$$

To the left of 0, choose $x = -\delta/2$.

$$\begin{aligned} \text{Since} \quad & 0 < |-\delta/2-0| = |\delta/2| < \delta, \\ \text{we must have} \quad & |f(-\delta/2)-L| = |-2/\delta-L| = |L+2/\delta| < 1, \\ \text{or} \quad & -2/\delta-1 < L < -2/\delta+1. \end{aligned}$$

Since we assumed $\delta < 2$, we have

$$\begin{array}{lccc} & 1 < 2/\delta & \text{or} & 0 < 2/\delta - 1 \\ \text{and} & -1 > -2/\delta & \text{or} & 0 > -2/\delta + 1. \end{array}$$

Having established $2/\delta - 1 < L < \delta/2 + 1$ and $-2/\delta - 1 < L < -2/\delta + 1$, these imply

$$0 < L < 2\delta + 1 \qquad \text{and} \qquad -2/\delta - 1 < L < 0.$$

Since it is impossible for L to satisfy both of these inequalities, $\lim_{x\to 0} f(x)$ does not exist.

31. By Definition 2.6.5(i), for any $\epsilon > 0$ we must find an $N > 0$ such that

$$\left|\frac{5x-1}{2x+1} - \frac{5}{2}\right| < \epsilon \text{ whenever } x > N.$$

Now by considering $x > 0$,

$$\left|\frac{5x-1}{2x+1} - \frac{5}{2}\right| = \left|\frac{-7}{4x+2}\right| = \frac{7}{4x+2} < \frac{7}{4x} < \epsilon$$

whenever $x > 7/4\epsilon$. Hence, choose $N = 7/4\epsilon$.

32. By Definition 2.6.5(i), for any $\epsilon > 0$ we must find an $N > 0$ such that

$$\left|\frac{2x}{3x+8} - \frac{2}{3}\right| < \epsilon \text{ whenever } x < N.$$

Now by considering $x > 0$,

$$\left|\frac{2x}{3x+8} - \frac{2}{3}\right| = \left|\frac{-16}{9x+24}\right| = \frac{16}{9x+24} < \frac{16}{9x} < \epsilon$$

whenever $x > 16/9\epsilon$. Hence, choose $N = 16/9\epsilon$.

33. By Definition 2.6.5(ii), for any $\epsilon > 0$ we must find an $N < 0$ such that

$$\left|\frac{10x}{x-3} - 10\right| < \epsilon \text{ whenever } x < N.$$

Now by considering $x < 0$,

$$\left|\frac{10x}{x-3} - 10\right| = \left|\frac{30}{x-3}\right| = \left|\frac{30}{-(-x+3)}\right| = \frac{30}{-x+3} < \frac{30}{x} < \epsilon$$

whenever $x < -30/\epsilon$. Hence, choose $N = -30/\epsilon$.

34. By Definition 2.6.5(ii), for any $\epsilon > 0$ we must find an $N < 0$ such that

$$\left|\frac{x^2}{x^2+3} - 1\right| < \epsilon \text{ whenever } x < N.$$

Now by considering $x < 0$,

$$\left|\frac{x^2}{x^2+3} - 1\right| = \left|\frac{-3}{x^2+3}\right| = \frac{3}{x^2+3} < \frac{3}{x^2} < \epsilon$$

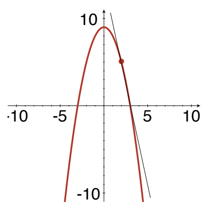
whenever $x^2 > 3/\epsilon$ or $x < -\sqrt{3/\epsilon}$. Hence, choose $N = -\sqrt{3/\epsilon}$.

35. We need to show $|f(x)-0| = |f(x)| < \epsilon$ whenever $0 < |x-0| = |x| < \delta$ for an appropriate choice of δ. For $\delta = \epsilon$,

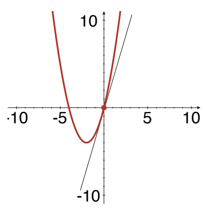
$$|f(x)| = \begin{cases} |x|, & x \text{ rational} \\ 0, & x \text{ irrational} \end{cases} < \epsilon \text{ whenever } 0 < |x| < \delta.$$

Thus, $\lim_{x\to 0} f(x) = 0$.

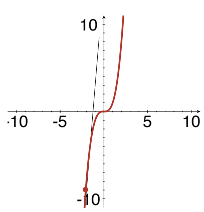
2.7 The Tangent Line Problem

1. 

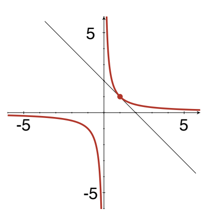
change in $x = h = 2.5 - 2 = 0.5$
change in $y = f(2+0.5) - f(2) = 2.75 - 5 = -2.25$
$$m_{\text{sec}} = \frac{\text{change in } y}{\text{change in } x} = \frac{-2.25}{0.5} = -4.5$$

2. 

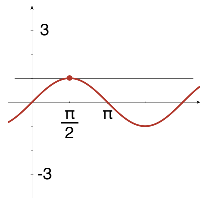
change in $x = h = 0 - (-1/4) = 1/4$
change in $y = f(0+1/4) - f(0) = 17/16 - 0 = 17/16$
$$m_{\text{sec}} = \frac{\text{change in } y}{\text{change in } x} = \frac{17/16}{1/4} = \frac{17}{4}$$

3. 

change in $x = h = -1 - (-2) = 1$
change in $y = f(-2+1) - f(-2) = -1 - (-8) = 7$
$$m_{\text{sec}} = \frac{\text{change in } y}{\text{change in } x} = \frac{7}{1} = 7$$

4. 

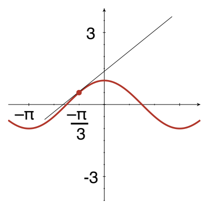
change in $x = h = 1 - 0.9 = 0.1$
change in $y = f(1+0.1) - f(1) = 10/11 - 1 = -1/11$
$$m_{\text{sec}} = \frac{\text{change in } y}{\text{change in } x} = \frac{-1/11}{1/10} = -\frac{10}{11}$$

5. 

$$\text{change in } x = h = \frac{2\pi}{3} - \frac{\pi}{2} = \frac{\pi}{6}$$

$$\text{change in } y = f\left(\frac{\pi}{2} + \frac{\pi}{6}\right) - f\left(\frac{\pi}{2}\right) = \sin\frac{2}{3}\pi - 1 = \sqrt{3}/2 - 1$$

$$m_{\text{sec}} = \frac{\text{change in } y}{\text{change in } x} = \frac{\sqrt{3}/2 - 1}{\pi/6} = \frac{3\sqrt{3} - 6}{\pi}$$

6. 

$$\text{change in } x = h = -\frac{\pi}{3} - \left(-\frac{\pi}{2}\right) = \frac{\pi}{6}$$

$$\text{change in } y = f\left(-\frac{\pi}{3} + \frac{\pi}{6}\right) - f\left(-\frac{\pi}{3}\right) = \cos\left(-\frac{\pi}{6}\right) - \frac{1}{2}$$

$$= \frac{\sqrt{3}}{2} - \frac{1}{2} = \frac{\sqrt{3} - 1}{2}$$

$$m_{\text{sec}} = \frac{\text{change in } y}{\text{change in } x} = \frac{(\sqrt{3} - 1)/2}{\pi/6} = \frac{3\sqrt{3} - 3}{\pi}$$

7. $f(a) = f(3) = 3;\ f(a+h) = f(3+h) = (h+3)^2 - 6$

$$f(a+h) - f(a) = [(h+3)^2 - 6] - 3 = [(h^2 + 6h + 9) - 6] - 3 = h^2 + 6h = h(h+6)$$

$$m_{\tan} = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{h(h+6)}{h} = \lim_{h\to 0}(h+6) = 6$$

With point of tangency $(3,3)$, we have $y - 3 = 6(x-3)$ or $y = 6x - 15$.

8. $f(a) = f(-1) = 7;\ f(a+h) = f(-1+h) = -3(h-1)^2 + 10$

$$f(a+h) - f(a) = [-3(h-1)^2 + 10] - 7 = [(-3h^2 + 6h - 3) + 10] - 7 = -3h^2 + 6h = h(6 - 3h)$$

$$m_{\tan} = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{h(6-3h)}{h} = \lim_{h\to 0}(6 - 3h) = 6$$

With point of tangency $(-1,7)$, we have $y - 7 = 6(x+1)$ or $y = 6x + 13$.

9. $f(a) = f(1) = -2;\ f(a+h) = f(1+h) = (h+1)^2 - 3(h+1)$

$$f(a+h) - f(a) = [(h+1)^2 - 3(h+1)] - (-2) = (h^2 - h - 2) - (-2) = h^2 - h = h(h-1)$$

$$m_{\tan} = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{h(h-1)}{h} = \lim_{h\to 0}(h-1) = -1$$

With point of tangency $(1,-2)$, we have $y + 2 = -(x-1)$ or $y = -x - 1$.

10. $f(a) = f(-2) = -17;\ f(a+h) = f(-2+h) = -(h-2)^2 + 5(h-2) - 3$

$$f(a+h) - f(a) = [-(h-2)^2 + 5(h-2) - 3] - (-17)$$

$$= (-h^2 + 9h - 17) - (-17) = -h^2 + 9h = h(9-h)$$

$$m_{\tan} = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{h(9-h)}{h} = \lim_{h\to 0}(9-h) = 9$$

With point of tangency $(-2,-17)$, we have $y + 17 = 9(x+2)$ or $y = 9x + 1$.

11. $$f(a) = f(2) = -14;\; f(a+h) = f(2+h) = -2(h+2)^3 + (h+2)$$
$$\begin{aligned} f(a+h) - f(a) &= [-2(h+2)^3 + (h+2)] - (-14) \\ &= (-2h^3 - 12h^2 - 23h - 14) - (-14) = h(-2h^2 - 12h - 23) \\ m_{\tan} &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{h(-2h^2 - 12h - 23)}{h} \\ &= \lim_{h\to 0} (-2h^2 - 12h - 23) = -23 \end{aligned}$$
With point of tangency $(2, -14)$, we have $y + 14 = -23(x - 2)$ or $y = -23x + 32$.

12. $$f(a) = f(1/2) = -3;\; f(a+h) = f(1/2+h) = 8(h+1/2)^3 - 4$$
$$\begin{aligned} f(a+h) - f(a) &= [8(h+1/2)^3 - 4] - (-3) \\ &= (8h^3 + 12h^2 + 6h - 3) - (-3) = 2h(4h^2 + 6h + 3) \\ m_{\tan} &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{2h(4h^2 + 6h + 3)}{h} \\ &= \lim_{h\to 0} 2(4h^2 + 6h + 3) = 6 \end{aligned}$$
With point of tangency $(1/2, -3)$, we have $y + 3 = 6(x - 1/2)$ or $y = 6x - 6$.

13. $$f(a) = f(-1) = -1/2;\; f(a+h) = f(-1+h) = \frac{1}{2(h-1)}$$
$$\begin{aligned} f(a+h) - f(a) &= \frac{1}{2(h-1)} - \left(-\frac{1}{2}\right) = \frac{1 + h - 1}{2(h-1)} = \frac{h}{2(h-1)} \\ m_{\tan} &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \left[\frac{h}{2(h-1)} \cdot \frac{1}{h}\right] \\ &= \lim_{h\to 0} \frac{1}{2(h-1)} = -\frac{1}{2} \end{aligned}$$
With point of tangency $(-1, -1/2)$, we have $y + \dfrac{1}{2} = -\dfrac{1}{2}(x+1)$ or $y = -\dfrac{x}{2}$.

14. $$f(a) = f(2) = 4;\; f(a+h) = f(2+h) = \frac{4}{(h+2) - 1}$$
$$\begin{aligned} f(a+h) - f(a) &= \frac{4}{(h+2) - 1} - 4 = \frac{4 - 4h - 4}{h+1} = \frac{-4h}{h+1} \\ m_{\tan} &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \left[\frac{-4h}{h+1} \cdot \frac{1}{h}\right] = \lim_{h\to 0} \frac{-4}{h+1} = -4 \end{aligned}$$
With point of tangency $(2, 4)$, we have $y - 4 = -4(x - 2)$ or $y = -4x + 12$.

15. $$f(a) = f(0) = 1;\; f(a+h) = f(h) = \frac{1}{(h-1)^2}$$
$$\begin{aligned} f(a+h) - f(a) &= \frac{1}{(h-1)^2} - 1 = \frac{-h^2 + 2h}{(h-1)^2} = \frac{h(2-h)}{(h-1)^2} \\ m_{\tan} &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \left[\frac{h(2-h)}{(h-1)^2} \cdot \frac{1}{h}\right] = \lim_{h\to 0} \frac{2-h}{(h-1)^2} = 2 \end{aligned}$$
With point of tangency $(0, 1)$, we have $y - 1 = 2(x - 0)$ or $y = 2x + 1$.

16. $f(a) = f(-1) = 12;\ f(a+h) = f(-1+h) = 4 - \dfrac{8}{-1+h}$

$$f(a+h) - f(a) = \left(4 - \frac{8}{-1+h}\right) - 12 = \frac{-8}{-1+h} - 8 = \frac{-8-8h+8}{h-1} = \frac{-8h}{h-1}$$

$$m_{\tan} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0}\left(\frac{-8h}{h-1}\cdot\frac{1}{h}\right) = \lim_{h\to 0}\frac{-8}{h-1} = 8$$

With point of tangency $(-1, 12)$, we have $y - 12 = 8(x+1)$ or $y = 8x + 20$.

17. $f(a) = f(4) = 2;\ f(a+h) = f(4+h) = \sqrt{4+h}$

$$f(a+h) - f(a) = \sqrt{4+h} - 2 = (\sqrt{4+h} - 2)\frac{\sqrt{4+h}+2}{\sqrt{4+h}+2} = \frac{4+h-4}{\sqrt{4+h}+2} = \frac{h}{\sqrt{4+h}+2}$$

$$m_{\tan} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0}\left(\frac{h}{\sqrt{4+h}+2}\cdot\frac{1}{h}\right)$$

$$= \lim_{h\to 0}\frac{1}{\sqrt{4+h}+2} = \frac{1}{4}$$

With point of tangency $(4, 2)$, we have $y - 2 = \dfrac{1}{4}(x-4)$ or $y = \dfrac{1}{4}x + 1$.

18. $f(a) = f(1) = 1;\ f(a+h) = f(1+h) = \dfrac{1}{\sqrt{h+1}}$

$$f(a+h) - f(a) = \frac{1}{\sqrt{h+1}} - 1 = \frac{1-\sqrt{h+1}}{\sqrt{h+1}} = \frac{1-\sqrt{h+1}}{\sqrt{h+1}}\cdot\frac{1+\sqrt{h+1}}{1+\sqrt{h+1}}$$

$$= \frac{1-h-1}{\sqrt{h+1}+h+1} = \frac{-h}{\sqrt{h+1}+h+1}$$

$$m_{\tan} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0}\left(\frac{-h}{\sqrt{h+1}+h+1}\cdot\frac{1}{h}\right)$$

$$= \lim_{h\to 0}\frac{-1}{\sqrt{h+1}+h+1} = -\frac{1}{2}$$

With point of tangency $(1, 1)$, we have $y - 1 = -\dfrac{1}{2}(x-1)$ or $y = -\dfrac{1}{2}x + \dfrac{3}{2}$.

19. $f(a) = f(\pi/6) = 1/2;\ f(a+h) = f(\pi/6+h) = \sin(\pi/6+h)$

$$f(a+h) - f(a) = \sin\left(\frac{\pi}{6}+h\right) - \frac{1}{2} = \sin\frac{\pi}{6}\cos h + \cos\frac{\pi}{6}\sin h - \frac{1}{2}$$

$$= \frac{1}{2}\cos h + \frac{\sqrt{3}}{2}\sin h - \frac{1}{2} = \frac{1}{2}(\cos h - 1) + \frac{\sqrt{3}}{2}\sin h$$

$$m_{\tan} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0}\left(\frac{1}{2}\cdot\frac{\cos h - 1}{h} + \frac{\sqrt{3}}{2}\cdot\frac{\sin h}{h}\right)$$

$$= (1/2)(0) + (\sqrt{3}/2)(1) = \sqrt{3}/2$$

With point of tangency $(\pi/6, 1/2)$, we have $y - \dfrac{1}{2} = \dfrac{\sqrt{3}}{2}\left(x - \dfrac{\pi}{6}\right)$ or $y = \dfrac{\sqrt{3}}{2}x - \dfrac{\sqrt{3}\pi}{12} + \dfrac{1}{2}$.

20. $f(a) = f(\pi/4) = \sqrt{2}/2$; $f(a+h) = f(\pi/4+h) = \cos(\pi/4+h)$

$$\begin{aligned}
f(a+h) - f(a) &= \cos\left(\frac{\pi}{4}+h\right) - \frac{\sqrt{2}}{2} = \cos\frac{\pi}{4}\cos h - \sin\frac{\pi}{4}\sin h - \frac{\sqrt{2}}{2} \\
&= \frac{\sqrt{2}}{2}\cos h - \frac{\sqrt{2}}{2}\sin h - \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}(\cos h - \sin h - 1) \\
m_{\tan} &= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{h\to 0}\frac{\sqrt{2}}{2}\left(\frac{\cos h - 1}{h} - \frac{\sin h}{h}\right) \\
&= (\sqrt{2}/2)(0-1) = -\sqrt{2}/2
\end{aligned}$$

With point of tangency $(\pi/4, \sqrt{2}/2)$, we have $y - \frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right)$ or $y = -\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}\pi}{8} + \frac{\sqrt{2}}{2}$.

21. $f(a) = f(1) = 1$; $f(a+h) = f(1+h) = (h+1)^2$

$$\begin{aligned}
f(a+h) - f(a) &= [(h+1)^2] - 1 = (h^2+2h+1) - 1 = h(h+2) \\
m_{\tan} &= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{h\to 0}\frac{h(h+2)}{h} = \lim_{h\to 0}(h+2) = 2
\end{aligned}$$

The slope of the tangent at the blue point $(1,1)$ is 2. The slope of the line through $(1,1)$ and $(4,6)$ is $m = (6-1)/(4-1) = 5/3$. Since the slopes are not equal, then this line is not tangent to the graph.

22. Since there is more than one line, we first find the slope of the tangent line at the "general point" $(a, f(a))$.

$$\begin{aligned}
f(a) &= a^2;\ f(a+h) = (h+a)^2 \\
f(a+h) - f(a) &= [(h+a)^2] - (a^2) = (h^2+2ha+a^2) - a^2 = h(h+2a) \\
m_{\tan} &= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{h\to 0}\frac{h(h+2a)}{h} = \lim_{h\to 0}(h+2a) = 2a
\end{aligned}$$

Now that we have determined that $m_{\tan} = 2a$, then the slope of the tangent at the blue point $(-1,1)$ is $m_{\tan}(-1) = 2(-1) = -2$. The slope of the line through $(-1,1)$ and $(1,-3)$ is $m = (-3-1)/(1+1) = -2$. Since the slopes are equal, then this line is tangent to the graph.

The slope of the tangent at the blue point $(3,9)$ is $m_{\tan}(3) = 2(3) = 6$. The slope of the line through $(3,9)$ and $(1,-3)$ is $m = (9+3)/(3-1) = 6$. Since the slopes are equal, then this line is tangent to the graph.

23. We know that the points $(2,0)$ and $(6,4)$ are on the tangent line, so its equation is

$$y - 0 = \frac{0-4}{2-6}(x-2) \qquad \text{or} \qquad y = x - 2$$

The line's y-intercept is $(0,-2)$.

24. We know that the points $(0,4)$ and $(7,0)$ are on the tangent line, so its equation is

$$y - 0 = \frac{4-0}{0-7}(s-7) \qquad \text{or} \qquad y = -\frac{4}{7}x + 4$$

Since the point of tangency $(-5, f(-5))$ is on this tangent line, then

$$f(-5) = -\frac{4}{7}(-5) + 4 = \frac{48}{7}$$

25.
$$\begin{aligned} f(a) &= -a^2 + 6a + 1;\ f(a+h) = -(h+a)^2 + 6(h+a) + 1 \\ f(a+h) - f(a) &= [-(h+a)^2 + 6(h+a) + 1] - (-a^2 + 6a + 1) \\ &= -h^2 - 2ha - a^2 + 6h + 6a + 1 - (-a^2) - 6a - 1 \\ &= -h^2 - 2ha + 6h = h(-h - 2a + 6) \\ m_{\tan} &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{h(-h - 2a + 6)}{h} \\ &= \lim_{h\to 0} (-h - 2a + 6) = -2a + 6 \end{aligned}$$

The tangent line is horizontal when $m_{\tan} = 0$, so we substitute and solve $m_{\tan} = 0 = -2a+6$, yielding $2a = 6$ and $a = 3$. Thus, the tangent line is horizontal at $(3, f(3)) = (3, 10)$.

26.
$$\begin{aligned} f(a) &= 2a^2 + 24a - 22;\ f(a+h) = 2(h+a)^2 + 24(h+a) - 22 \\ f(a+h) - f(a) &= [2(h+a)^2 + 24(h+a) - 22] - (2a^2 + 24a - 22) \\ &= 2h^2 + 4ha + 2a^2 + 24h + 24a - 22 - 2a^2 - 24a - (-22) \\ &= 2h^2 + 4ha + 24h = h(2h + 4a + 24) \\ m_{\tan} &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{h(2h + 4a + 24)}{h} \\ &= \lim_{h\to 0} (2h + 4a + 24) = 4a + 24 \end{aligned}$$

The tangent line is horizontal when $m_{\tan} = 0$, so we substitute and solve $m_{\tan} = 0 = 4a+24$, yielding $4a = -24$ and $a = -6$. Thus, the tangent line is horizontal at $(-6, f(-6)) = (-6, -94)$.

27.
$$\begin{aligned} f(a) &= a^3 - 3a;\ f(a+h) = (h+a)^3 - 3(h+a) \\ f(a+h) - f(a) &= [(h+a)^3 - 3(h+a)] - (a^3 - 3a) \\ &= h^3 + 3h^2a + 3ha^2 + a^3 - 3h - 3a - a^3 - (-3a) \\ &= h^3 + 3h^2a + 3ha^2 - 3h = h(h^2 + 3ah + 3a^2 - 3) \\ m_{\tan} &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{h(h^2 + 3ah + 3a^2 - 3)}{h} \\ &= \lim_{h\to 0} (h^2 + 3ah + 3a^2 - 3) = 3a^2 - 3 \end{aligned}$$

The tangent line is horizontal when $m_{\tan} = 0$, so we substitute and solve $m_{\tan} = 0 = 3a^2 - 3$, yielding $3a^2 = 3$ and $a = \pm 1$. Thus, the tangent line is horizontal at $(-1, f(-1)) = (-1, 2)$ and $(1, f(1)) = (1, -2)$.

28.
$$\begin{aligned}
f(a) &= -a^3 + a^2;\ f(a+h) = -(h+a)^3 + (h+a)^2 \\
f(a+h) - f(a) &= [-(h+a)^3 + (h+a)^2] - (-a^3 + a^2) \\
&= -h^3 - 3h^2a - 3ha^2 - a^3 + h^2 + 2ah + a^2 - (-a^3) - a^2 \\
&= -h^3 - 3h^2a - 3ha^2 + h^2 + 2ah = h(-h^2 - 3ah - 3a^2 + h + 2a) \\
m_{\tan} &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{h(-h^2 - 3ah - 3a^2 + h + 2a)}{h} \\
&= \lim_{h\to 0} (-h^2 - 3ah - 3a^2 + h + 2a) = -3a^2 + 2a
\end{aligned}$$

The tangent line is horizontal when $m_{\tan} = 0$, so we substitute and solve $m_{\tan} = 0 = -3a^2 + 2a$, yielding $a(3a-2) = 0$ and $a = 0, 2/3$. Thus, the tangent line is horizontal at $(0, f(0)) = (0, 0)$ and $(2/3, f(2/3)) = (2/3, 4/27)$.

29. $v_{\text{ave}} = \dfrac{\text{change of distance}}{\text{change in time}} = \dfrac{290 \text{ mi}}{5 \text{ h}} = 58 \text{ mi/h}$

30. $v_{\text{ave}} = \dfrac{\text{change of distance}}{\text{change in time}} = \dfrac{1/2 \text{ mi}}{40 \text{ s}} = \dfrac{(1/2 \text{ mi})}{(40 \text{ s})/(3600 \text{ s/h})} = \dfrac{1/2 \text{ mi}}{1/90 \text{ h}} = 45 \text{ mi/h}$

The car will not be stopped for speeding.

31. $v_{\text{ave}} = \dfrac{\text{change of distance}}{\text{change in time}};\ 920 \text{ km/h} = \dfrac{3500 \text{ km}}{t};\ t \approx 3.8 \text{ h} = 3 \text{ h } 48 \text{ min}$

32. $v_{\text{ave}} = \dfrac{\text{change of distance}}{\text{change in time}} = \dfrac{20 \text{ mi} - 10 \text{ mi}}{3\frac{1}{6} \text{ h} - 1\frac{1}{2} \text{ h}} = \dfrac{20 \text{ mi} - 10 \text{ mi}}{19/6 \text{ h} - 3/2 \text{ h}} = \dfrac{10 \text{ mi}}{5/3 \text{ h}} = 6 \text{ mi/h}$

33. $\Delta s = s(t_0 + \Delta t) - s(t_0) = f(3 + \Delta t) - f(3) = [-4(3+\Delta t)^2 + 10(3+\Delta t) + 6] - 0 = -14\Delta t - 4\Delta t^2$

The instantaneous velocity at $t = 3$ is

$$v(3) = \lim_{\Delta t\to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t\to 0} \frac{-14\Delta t - 4\Delta t^2}{\Delta t} = \lim_{\Delta t\to 0} (-14 - 4\Delta t) = -14.$$

34. $\Delta s = s(t_0 + \Delta t) - s(t_0) = f(\Delta t) - f(0) = \Delta t^2 + \dfrac{1}{5\Delta t + 1} - 1 = \dfrac{5\Delta t^3 + \Delta t^2 - 5\Delta t}{5\Delta t + 1} = -5$

The instantaneous velocity at $t = 0$ is

$$v(0) = \lim_{\Delta t\to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t\to 0} \frac{5\Delta t^3 + \Delta t^2 - 5\Delta t}{(5\Delta t + 1)\Delta t} = \lim_{\Delta t\to 0} \frac{5\Delta t^2 + \Delta t - 5}{5\Delta t + 1} = -5.$$

35. (a) $\Delta s = s(t_0 + \Delta t) - s(t_0) = f(1/2 + \Delta t) - f(1/2) = -4.9(1/2 + \Delta t)^2 + 122.5 - 121.275$
$= -4.9\Delta t^2 - 4.9\Delta t$

The instantaneous velocity at $t = 1/2$ is

$$v(1/2) = \lim_{\Delta t\to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t\to 0} \frac{-4.9\Delta t^2 - 4.9\Delta t}{\Delta t} = \lim_{\Delta t\to 0} (-4.9\Delta t - 4.9) = -4.9 \text{ m/s}.$$

(b) The ball hits the ground when $s(t) = 0$:

$$-4.9t^2 + 122.5 = 0;\ t^2 = 122.5/4.9;\ t = 5 \text{ s.}$$

(c) Since the ball impacts at $t = 5$,

$$\begin{aligned}\Delta s &= s(t_0 + \Delta t) - s(t_0) = f(5 + \Delta t) - f(5) = [-4.9(5 + \Delta t)^2 + 122.5] - [-4.9(5)^2 + 122.5]\\ &= -49\Delta t^2 - 49\Delta t\end{aligned}$$

The impact velocity at $t = 5$ is

$$v(5) = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} \frac{-49\Delta t^2 - 49\Delta t}{\Delta t} = \lim_{\Delta t \to 0} (-49\Delta t - 49) = -49 \text{ m/s.}$$

36. (a) Setting $-\frac{1}{2}gt^2 + h = 0$ and solving for $t > 0$, we obtain $t = \sqrt{2h/g}$.

(b) Earth: $t_{\text{impact}} = \sqrt{2(100)/32} = 2.5$ s

Mars: $t_{\text{impact}} = \sqrt{2(100)/12} \approx 4.08$ s

Moon: $t_{\text{impact}} = \sqrt{2(100)/5.5} \approx 6.03$ s

(c) $\Delta s = s(t_0 + \Delta t) - s(t_0) = -\dfrac{1}{2}g(t_0 + \Delta t)^2 + h - (-\dfrac{1}{2}gt_0^2 + h) = -\dfrac{1}{2}g\Delta t^2 - gt_0\Delta t$

The instantaneous velocity at t_{impact} is

$$v(t_{\text{impact}}) = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} \frac{-\frac{1}{2}g\Delta t^2 - gt_0\Delta t}{\Delta t} = \lim_{\Delta t \to 0} \left(-\frac{1}{2}g\Delta t - gt_0\right) = -gt_0.$$

(d) The impact velocities are

$$\begin{aligned}v_{\text{Earth}} &= -(32)(2.5) = -80 \text{ ft/s}\\ v_{\text{Mars}} &\approx -(12)(4.08) = -48.96 \text{ ft/s}\\ v_{\text{Moon}} &\approx -(5.5)(6.03) = -33.165 \text{ ft/s.}\end{aligned}$$

37. (a)

$$\begin{aligned}s(t) &= -16t^2 + 256t\\ s(2) &= -16(2^2) + 256(2) = 448 \text{ ft}\\ s(6) &= -16(6^2) + 256(6) = 960 \text{ ft}\\ s(9) &= -16(9^2) + 256(9) = 1008 \text{ ft}\\ s(10) &= -16(10^2) + 256(10) = 960 \text{ ft}\end{aligned}$$

(b) $s(5) = -16(5^2) + 256(5) = 880$ ft

$s(2) = 448$ ft [from (a)]

$$v_{\text{ave}} = \frac{\text{change of distance}}{\text{change in time}} = \frac{880 \text{ ft} - 448 \text{ ft}}{5 \text{ s} - 2 \text{ s}} = 144 \text{ ft/s}$$

(c) $s(7) = -16(7^2) + 256(7) = 1008$ ft
$s(9) = 1008$ ft [from (a)]
$v_{\text{ave}} = \dfrac{\text{change of distance}}{\text{change in time}} = \dfrac{1008 \text{ ft} - 1008 \text{ ft}}{9 \text{ s} - 7 \text{ s}} = \dfrac{0}{2} = 0$ ft/s
At $t = 7$ s, the projectile is at a height of 1008 ft on its way upward. After it reaches a maximum height, it begins to fall downward and, at $t = 9$ s, the height is once again 1008 ft. Since distance upward is positive and distance downward is negative, the net distance is zero.

(d) The projectile hits the ground when $s(t) = 0$:

$$-16t^2 + 256t = 0;\ 16t^2 = 256t;\ t = 256/16 = 16 \text{ s}$$

(e) For some general time t:
$\Delta s = s(t + \Delta t) - s(t) = [-16(t + \Delta t)^2 + 256(t + \Delta t)] - (-16t^2 + 256t)$
$\quad = -16\Delta t^2 + 256\Delta t - 32t\Delta t = \Delta t(-16\Delta t + 256 - 32t)$
The instantaneous velocity at a general time t is

$$\begin{aligned} v(t) &= \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta t(-16\Delta t + 256 - 32t)}{\Delta t} \\ &= \lim_{\Delta t \to 0} (-16\Delta t + 256 - 32t) = (256 - 32t) \text{ ft/s.} \end{aligned}$$

(f) From (d), the projectile impacts at $t = 16$ s. From (e), $v(t) = 256 - 32t$ so $v(16) = 256 - 32(16) = -256$ ft/s.

(g) The maximum height is reached when $v(t) = 0$: $256 - 32t = 0$ gives us $t = 8$ s. Since $s(t) = -16t^2 + 256t$, we have $s(8) = -16(8^2) + 256(8) = 1024$ ft.

38. (a) $s(4) \approx 1.3$ ft; $s(6) \approx 2.7$ ft

(b) $v_{\text{ave}} \approx \dfrac{s(6) - s(4)}{6 - 4} = \dfrac{2.7 - 1.3}{2} = 0.7$ ft/s

(c) The instantaneous velocity at $t = 0$ is the slope of the tangent line to the graph at $t = 0$. In this case, $v_0 \approx 1$ ft/s.

(d) $t \approx 3$ s

(e) The velocity is decreasing where the slopes of the tangent lines are decreasing; in this case, for $0 < t < 3$.

(f) The velocity is increasing where the slopes of the tangent lines are increasing; in this case, for $3 < t < 7$.

39. The slopes m of a tangent line at $(a, f(a))$ and m' of a tangent line at $(-a, f(-a))$ are:

$$m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h};\ m' = \lim_{h' \to 0} \frac{f(-a + h') - f(-a)}{h'}$$

As defined in Section 1.2, an even function is a function which is symmetric with respect to the y-axis: $f(-x) = f(x)$ for all x. Since f is even, then $f(-a) = f(a)$ and $f(-a + h') = f(-[-a + h']) = f(a - h')$, resulting in:

$$m' = \lim_{h' \to 0} \frac{f(a - h') - f(a)}{h'} = \lim_{h' \to 0} \frac{f(a + [-h']) - f(a)}{h'}$$

Without loss of generality, we apply the substitution $h' = -h$ to obtain:

$$m' = \lim_{h' \to 0} \frac{f(a + [-h']) - f(a)}{h'} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{-h} = -m$$

40. The slopes m of a tangent line at $(a, f(a))$ and m' of a tangent line at $(-a, f(-a))$ are:

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}; \; m' = \lim_{h' \to 0} \frac{f(-a+h') - f(-a)}{h'}$$

As defined in Section 1.2, an odd function is a function which is symmetric with respect to the origin: $f(-x) = -f(x)$ for all x. Since f is odd, then $f(-a) = -f(a)$ and $f(-a+h') = -f(-[-a+h']) = -f(a-h')$, resulting in:

$$m' = \lim_{h' \to 0} \frac{-f(a-h') - [-f(a)]}{h'} = -\lim_{h' \to 0} \frac{-[f(a-h') + f(a)]}{h'}$$

Without loss of generality, we apply the substitution $h' = -h$ to obtain:

$$m' = \lim_{h' \to 0} \frac{-[f(a-h') + f(a)]}{h'} = \lim_{h \to 0} \frac{-[f(a+h) - f(a)]}{-h} = m$$

41. To show that the graph of $f(x) = x^2 + |x|$ does not possess a tangent line at $(0, 0)$, we examine

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{[(0+h)^2 + |0+h|] - 0}{h} = \lim_{h \to 0} \frac{h^2 + |h|}{h}$$

From the definition of absolute value, we see that

$$\lim_{h \to 0^+} \frac{h^2 + |h|}{h} = \frac{h^2 + h}{h} = h + 1 = 1$$

whereas

$$\lim_{h \to 0^-} \frac{h^2 + |h|}{h} = \frac{h^2 - h}{h} = h - 1 = -1$$

Since the right-hand and left-hand limits are not equal, we conclude that $\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{h^2 + |h|}{h}$ does not exist, and that therefore f has no tangent line at $(0, 0)$.

Chapter 2 in Review

A. True/False

1. True
2. False; $\lim_{x \to 5^+} \sqrt{x-5} = 0$.
3. False; $\lim_{x \to 0^-} \frac{|x|}{x} = -1$.

4. False; $\lim_{x\to\infty} e^{2x-x^2} = 0$.

5. False; $\lim_{x\to 0^+}\left(\tan^{-1}\dfrac{1}{x}\right) = \dfrac{\pi}{2}$.

6. True

7. True

8. False; let $f(x) = 0$.

9. False; consider $f(x) = \dfrac{1}{x^2}$, $g(x) = \dfrac{1}{x^4}$, and $a = 0$.

10. False; consider $f(x) = \dfrac{1}{x^2}$, $g(x) = \dfrac{1}{\tan^2 x}$, and $a = 0$.

11. False; consider $f(x) = -x$.

12. True

13. True; since $f(-1) < 0$ and $f(1) > 0$.

14. False; consider $f(x) = 1$ and $g(x) = x - 2$.

15. True

16. False; consider $f(x) = \begin{cases} -1, & x < 0 \\ 1, & x > 0 \end{cases}$ and $a = 0$.

17. False; consider $f(x) = \begin{cases} 1, & x \le 3 \\ 2, & x > 3 \end{cases}$.

18. True; since $\lim_{x\to a}[(x-a)f(x)] = [\lim_{x\to a}(x-a)][\lim_{x\to a} f(x)] = 0\cdot f(a) = 0$.

19. True

20. False; $\lim_{x\to 5} f(x) = 4 = f(5)$.

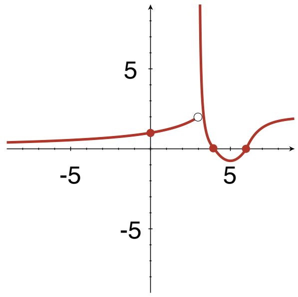
21. False; since $\dfrac{\sqrt{x}}{x+1}$ is undefined for $x < 0$.

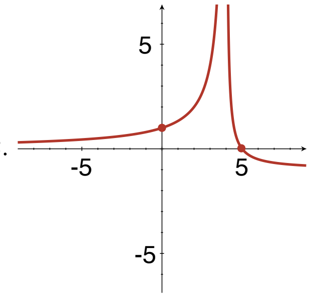
22. False; the slope m of the tangent line at $(3, f(3))$ is 1. There is not enough information to determine the value of $f(3)$.

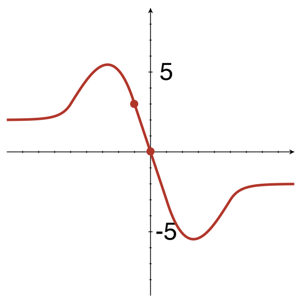
B. Fill in the Blanks

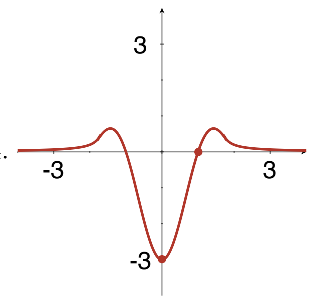
1. 4
2. 1
3. -1/5
4. -1/2
5. 0
6. 3/5
7. ∞
8. 0
9. 1
10. 1/4
11. 3^-
12. 4
13. $-\infty$
14. 0^+
15. -2
16. Dividing by x^2 we have $1 - \dfrac{x^2}{3} \leq \dfrac{f(x)}{x^2} \leq 1$. Since $\lim\limits_{x\to 0}\left(1 - \dfrac{x^2}{3}\right) = 1 = \lim\limits_{x\to 0} 1$, by the Squeeze Theorem we have $\lim\limits_{x\to 0} \dfrac{f(x)}{x^2} = 1$.
17. 10
18. 8
19. continuous
20. 2
21. 9
22. Since $f(x) = x^2$ is continuous, $\lim\limits_{x\to -5} f(g(x)) = f(\lim\limits_{x\to -5} g(x)) = f(-9) = (-9)^2 = 81$.

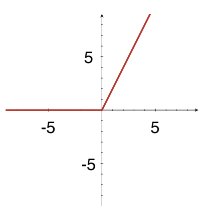
C. Exercises

1. 

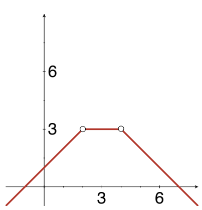
2. 

3. 

4. 

5. (a), (e), (f), (h)
6. (b), (e), (h)
7. (c), (h)
8. (b), (c), (d), (e), (f), (i)
9. (b), (c), (d), (e), (f)
10. (a), (g), (j)
11. 

The function is continuous everywhere.

12. 

The function is discontinuous at $x = 2$ and $x = 4$.

13. $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, and $(1, \infty)$

14. $[-2, 1)$ and $(1, 2]$

15. $(-\infty, -\sqrt{5})$ and $(\sqrt{5}, \infty)$

16. $(n\pi, n\pi + \pi)$ for $n = 0, 1, 2, \ldots$

17. For $f(x)$ to be continuous at the number 3, we must have $f(3) = 3k + 1 = \lim_{x \to 3^+} (2 - kx)$. Thus, we must solve for k in the equation $3k + 1 = 2 - 3k$, resulting in $k = 1/6$. Therefore:

$$f(x) = \begin{cases} \dfrac{x}{6} + 1, & x \le 3 \\ 2 - \dfrac{x}{6}, & x > 3 \end{cases}$$

18. For $f(x)$ to be continuous everywhere, we must have $f(1) = 5 = \lim_{x \to 1^+} (ax + b)$ and $f(3) = 3a + b = \lim_{x \to 3^+} (3x - 8)$. Thus, we get two equations $5 = a + b$ and $1 = 3a + b$. Solving for a an b yields $a = -2$, $b = 7$. Therefore:

$$f(x) = \begin{cases} x + 4, & x \le 1 \\ -2x + 7, & 1 < x \le 3 \\ 3x - 8, & x > 3 \end{cases}$$

19.
$$\begin{aligned} f(a) = f(2) = 32;\ f(a+h) &= f(2+h) = -3(h+2)^2 + 16(h+2) + 12 \\ f(a+h) - f(a) &= [-3(h+2)^2 + 16(h+2) + 12] - 32 \\ &= -3h^2 - 12h - 12 + 16h + 32 + 12 - 32 = -3h^2 + 4h = h(-3h+4) \\ m_{\tan} &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{h(-3h+4)}{h} = 4 \end{aligned}$$

With point of tangency $(2, 32)$, we have $y - 32 = 4(x - 2)$ or $y = 4x + 24$.

20.
$$\begin{aligned} f(a) = f(-1) = -2;\ f(a+h) &= f(-1+h) = (h-1)^3 - (h-1)^2 \\ f(a+h) - f(a) &= [(h-1)^3 - (h-1)^2] - (-2) \\ &= h^3 - 3h^2 + 3h - 1 - h^2 + 2h - 1 - (-2) \\ &= h^3 - 4h^2 + 5h = h(h^2 - 4h + 5) \\ m_{\tan} &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{h(h^2 - 4h + 5)}{h} = 5 \end{aligned}$$

With point of tangency $(-1, -2)$, we have $y + 2 = 5(x + 1)$ or $y = 5x + 3$.

21.
$$\begin{aligned} f(a) = f(1/2) = -2;\ f(a+h) &= f(1/2 + h) = \frac{-1}{2(h+1/2)^2} \\ f(a+h) - f(a) &= \frac{-1}{2(h+1/2)^2} - (-2) = \frac{-1}{2h^2 + 2h + 1/2} - (-2) \\ &= \frac{-1 + 4h^2 + 4h + 1}{2h^2 + 2h + 1/2} = \frac{4h(h+1)}{2h^2 + 2h + 1/2} \\ m_{\tan} &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{4h(h+1)}{(2h^2 + 2h + 1/2)h} = 8 \end{aligned}$$

With point of tangency $(1/2, -2)$, we have $y + 2 = 8(x - 1/2)$ or $y = 8x - 6$.

22. $$f(a) = f(4) = 12;\; f(a+h) = f(4+h) = (h+4) + 4\sqrt{h+4}$$

$$\begin{aligned}
f(a+h) - f(a) &= [(h+4) + 4\sqrt{h+4}] - 12 = [(h-8) + 4\sqrt{h+4}] \cdot \frac{(h-8) - 4\sqrt{h+4}}{(h-8) - 4\sqrt{h+4}} \\
&= \frac{(h-8)^2 - 16(h+4)}{(h-8) - 4\sqrt{h+4}} = \frac{h^2 - 16h + 64 - 16h - 64}{(h-8) - 4\sqrt{h+4}} \\
&= \frac{h(h-32)}{(h-8) - 4\sqrt{h+4}} \\
m_{\tan} &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{h(h-32)}{[(h-8) - 4\sqrt{h+4}]h} = \frac{-32}{-16} = 2
\end{aligned}$$

With point of tangency $(4, 12)$, we have $y - 12 = 2(x - 4)$ or $y = 2x + 4$.

23. $$f(a) = f(1) = 2;\; f(a+h) = f(1+h) = -4(h+1)^2 + 6(h+1)$$

$$\begin{aligned}
f(a+h) - f(a) &= [-4(h+1)^2 + 6(h+1)] - 2 = -4h^2 - 8h - 4 + 6h + 6 - 2 \\
&= -4h^2 - 2h = h(-4h - 2) \\
m_{\tan} &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{h(-4h-2)}{h} = -2
\end{aligned}$$

With point of tangency $(1, 2)$, we have $y - 2 = -2(x - 1)$ or $y = -2x + 4$. Thus, the line that is perpendicular to this line would have a slope of $1/2$ and also passes through $(1, 2)$, resulting in the equation $y - 2 = (x - 1)/2$ or $y = (x + 3)/2$.

24. $|2x + 5 - 7| = |2x - 2| = 2|x - 1| < \epsilon$ whenever $|x - 1| < \epsilon/2$. Thus, we choose $\delta = \epsilon/2$ and so $\delta = 0.005$ when $\epsilon = 0.01$. Finding δ proves that $\lim_{x\to 1}(2x + 5) = 7$.

Chapter 3

The Derivative

3.1 The Derivative

1. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{10-10}{h} = 0$

2. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{[(x+h)-1]-(x-1)}{h} = \lim_{h\to 0} \dfrac{h}{h} = \lim_{h\to 0} 1 = 1$

3. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{[-3(x+h)+5]-(-3x+5)}{h} = \lim_{h\to 0} \dfrac{-3h}{h} = \lim_{h\to 0} -3 = -3$

4. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{[\pi(x+h)]-\pi x}{h} = \lim_{h\to 0} \dfrac{\pi h}{h} = \lim_{h\to 0} \pi = \pi$

5. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{[3(x+h)^2]-3x^2}{h} = \lim_{h\to 0} \dfrac{6xh+3h^2}{h} = \lim_{h\to 0}(6x+3h) = 6x$

6. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{[-(x+h)^2+1]-(-x^2+1)}{h} = \lim_{h\to 0} \dfrac{-2xh-h^2}{h}$
$= \lim_{h\to 0}(-2x-h) = -2x$

7. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{[-(x+h)^2+4(x+h)+1]-(-x^2+4x+1)}{h}$
$= \lim_{h\to 0} \dfrac{-2xh-h^2+4h}{h} = \lim_{h\to 0}(-2x-h+4) = -2x+4$

8. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{[(1/2)(x+h)^2+6(x+h)-7]-[(1/2)x^2+6x-7]}{h}$
$= \lim_{h\to 0} \dfrac{xh+(1/2)h^2+6h}{h} = \lim_{h\to 0}[x+(1/2)h+6] = x+6$

9. $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{[(x+h)+1]^2-(x+1)^2}{h}$$
$$= \lim_{h\to 0} \frac{2xh+h^2+2h}{h} = \lim_{h\to 0}(2x+h+2) = 2x+2 = 2(x+1)$$

10. $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{[2(x+h)-5]^2-(2x-5)^2}{h}$$
$$= \lim_{h\to 0} \frac{8xh+4h^2-20h}{h} = \lim_{h\to 0}(8x+4h-20) = 8x-20 = 4(2x-5)$$

11. $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{[(x+h)^3+(x+h)]-(x^3+x)}{h}$$
$$= \lim_{h\to 0} \frac{3x^2h+3xh^2+h^3+h}{h} = \lim_{h\to 0}(3x^2+3xh+h^2+1) = 3x^2+1$$

12. $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{[2(x+h)^3+(x+h)^2]-(2x^3+x^2)}{h}$$
$$= \lim_{h\to 0} \frac{6x^2h+6xh^2+2h^3+2xh+h^2}{h} = \lim_{h\to 0}(6x^2+6xh+2h^2+2x+h) = 6x^2+2x$$

13. $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{[-(x+h)^3+15(x+h)^2-(x+h)]-(-x^3+15x^2-x)}{h}$$
$$= \lim_{h\to 0} \frac{-3x^2h-3xh^2-h^3+30xh+15h^2-h}{h}$$
$$= \lim_{h\to 0}(-3x^2-3xh-h^2+30x+15h-1) = -3x^2+30x-1$$

14. $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{[3(x+h)^4]-3x^4}{h} = \lim_{h\to 0} \frac{12x^3h+18x^2h^2+12h^3+3h^4}{h}$$
$$= \lim_{h\to 0}(12x^3+18x^2h+12h^2+3h^3) = 12x^3$$

15. $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\dfrac{2}{(x+h)+1}-\dfrac{2}{x+1}}{h}$$
$$= \lim_{h\to 0} \frac{-2h}{h(x+1)(x+h+1)} = \lim_{h\to 0} \frac{-2}{(x+1)(x+h+1)} = -\frac{2}{(x+1)^2}$$

16. $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\dfrac{x+h}{(x+h)-1}-\dfrac{x}{x-1}}{h}$$
$$= \lim_{h\to 0} \frac{-h}{h(x-1)(x+h-1)} = \lim_{h\to 0} \frac{-1}{(x-1)(x+h-1)} = -\frac{1}{(x-1)^2}$$

17. $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\dfrac{2(x+h)+3}{(x+h)+4}-\dfrac{2x+3}{x+4}}{h}$$
$$= \lim_{h\to 0} \frac{5h}{h(x+4)(x+h+4)} = \lim_{h\to 0} \frac{5}{(x+4)(x+h+4)} = \frac{5}{(x+4)^2}$$

18. $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\left[\dfrac{1}{x+h} + \dfrac{1}{(x+h)^2}\right] - \left(\dfrac{1}{x} + \dfrac{1}{x^2}\right)}{h}$$
$$= \lim_{h\to 0} \frac{\dfrac{x-(x+h)}{x(x+h)} + \dfrac{x^2-(x+h)^2}{x^2(x+h)^2}}{h} = \lim_{h\to 0} \left[\frac{-h}{xh(x+h)} + \frac{-2xh-h^2}{x^2h(x+h)^2}\right]$$
$$= \lim_{h\to 0} \left[-\frac{1}{x(x+h)} - \frac{2x+h}{x^2(x+h)^2}\right] = -\frac{1}{x^2} - \frac{2}{x^3}$$

19. $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\dfrac{1}{\sqrt{x+h}} - \dfrac{1}{\sqrt{x}}}{h} = \lim_{h\to 0} \frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x^2+xh}}$$
$$= \lim_{h\to 0} \left(\frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x^2+xh}} \cdot \frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}}\right)$$
$$= \lim_{h\to 0} \frac{-h}{h\left(\sqrt{x^3+x^2h}+\sqrt{x^3+2x^2h+xh^2}\right)}$$
$$= \lim_{h\to 0} \frac{-1}{\sqrt{x^3+x^2h}+\sqrt{x^3+2x^2h+xh^2}} = -\frac{1}{2\sqrt{x^3}} = -\frac{1}{2x\sqrt{x}}$$

20. $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{2(x+h)+1}-\sqrt{2x+1}}{h}$$
$$= \lim_{h\to 0} \left[\frac{\sqrt{2(x+h)+1}-\sqrt{2x+1}}{h} \cdot \frac{\sqrt{2(x+h)+1}+\sqrt{2x+1}}{\sqrt{2(x+h)+1}+\sqrt{2x+1}}\right]$$
$$= \lim_{h\to 0} \frac{2h}{h\left(\sqrt{2x+2h+1}+\sqrt{2x+1}\right)}$$
$$= \lim_{h\to 0} \frac{2}{\sqrt{2x+2h+1}+\sqrt{2x+1}} = \frac{1}{\sqrt{2x+1}}$$

21. $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{[4(x+h)^2+7(x+h)]-(4x^2+7x)}{h}$$
$$= \lim_{h\to 0} \frac{8xh+4h^2+7h}{h} = \lim_{h\to 0}(8x+4h+7) = 8x+7$$
$$m_{\tan} = f'(-1) = 8(-1)+7 = -1$$

With point of tangency $(-1, f(-1))$ or $(-1,-3)$, we have $y+3=-1(x+1)$ or $y=-x-4$.

22. $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\left[\frac{1}{3}(x+h)^3+2(x+h)-4\right]-\left(\frac{1}{3}x^3+2x-4\right)}{h}$$
$$= \lim_{h\to 0} \frac{x^2h+xh^2+\frac{1}{3}h^3+2h}{h} = \lim_{h\to 0}\left(x^2+xh+\tfrac{1}{3}h^2+2\right) = x^2+2$$
$$m_{\tan} = f'(0) = 0^2+2 = 2$$

With point of tangency $(0, f(0))$ or $(0,-4)$, we have $y+4=2(x-0)$ or $y=2x-4$.

23. $$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\left[(x+h)-\dfrac{1}{x+h}\right]-\left(x-\dfrac{1}{x}\right)}{h}$$
$$= \lim_{h\to 0}\frac{x^2h+xh^2+h}{h(x^2+xh)} = \lim_{h\to 0}\frac{x^2+xh+1}{x^2+xh} = 1+\frac{1}{x^2}$$
$$m_{\tan} = f'(1) = 1+\frac{1}{1^2} = 2$$
With point of tangency $(1, f(1))$ or $(1, 0)$, we have $y-0 = 2(x-1)$ or $y = 2(x-1)$.

24. $$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\left[2(x+h)+1+\dfrac{6}{x+h}\right]-\left(2x+1+\dfrac{6}{x}\right)}{h}$$
$$= \lim_{h\to 0}\frac{2x^2h+2xh^2-6h}{h(x^2+xh)} = \lim_{h\to 0}\frac{2x^2+2xh-6}{x^2+xh} = \lim_{h\to 0}\frac{2x^2-6}{x^2} = 2-\frac{6}{x^2}$$
$$m_{\tan} = f'(2) = 2-\frac{6}{2^2} = \frac{1}{2}$$
With point of tangency $(2, f(2))$ or $(2, 8)$, we have $y-8 = \frac{1}{2}(x-2)$ or $y = \frac{1}{2}x+7$.

25. $$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{[(x+h)^2+8(x+h)+10]-(x^2+8x+10)}{h}$$
$$= \lim_{h\to 0}\frac{2xh+h^2+8h}{h} = \lim_{h\to 0}(2x+h+8) = 2x+8$$
The tangent is horizontal when $2x+8=0$ or $x=-4$. Since $f(-4)=-6$, the tangent is horizontal at $(-4,-6)$.

26. $$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)[(x+h)-5]-x(x-5)}{h}$$
$$= \lim_{h\to 0}\frac{2xh+h^2-5h}{h} = \lim_{h\to 0}(2x+h-5) = 2x-5$$
The tangent is horizontal when $2x-5=0$ or $x=5/2$. Since $f(5/2)=-25/4$, the tangent is horizontal at $(5/2,-25/4)$.

27. $$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{[(x+h)^3-3(x+h)]-(x^3-3x)}{h}$$
$$= \lim_{h\to 0}\frac{3x^2h+3xh^2+h^3-3h}{h} = \lim_{h\to 0}(3x^2+3xh+h^2-3) = 3x^2-3$$
The tangent is horizontal when $3x^2-3 = 3(x+1)(x-1) = 0$ or $x=\pm 1$. Since $f(-1)=2$ and $f(1)=-2$, the tangent is horizontal at $(-1,2)$ and $(1,-2)$.

28. $$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{[(x+h)^3-(x+h)^2+1]-(x^3-x^2+1)}{h}$$
$$= \lim_{h\to 0}\frac{3x^2h+h^3-2xh-h^2}{h} = \lim_{h\to 0}(3x^2+h^2-2x-h) = 3x^2-2x$$
The tangent is horizontal when $3x^2-2x = x(3x-2) = 0$ or $x = 0, 2/3$. Since $f(0)=1$ and $f(2/3)=23/27$, the tangent is horizontal at $(0,1)$ and $(2/3, 23/27)$.

29. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{\left[\frac{1}{2}(x+h)^2-1\right]-\left(\frac{1}{2}x^2-1\right)}{h}$

$= \lim_{h\to 0} \dfrac{xh+\frac{1}{2}h^2}{h} = \lim_{h\to 0}(x+\frac{1}{2}h) = x$

The given line $3x-y=1$ has a slope of 3, and so the tangent line is parallel to it when $x=3$, or at $(3, f(3)) = (3, 7/2)$.

30. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{[(x+h)^2-(x+h)]-(x^2-x)}{h}$

$= \lim_{h\to 0} \dfrac{2xh+h^2-h}{h} = \lim_{h\to 0}(2x+h-1) = 2x-1$

The given line $-2x+y=0$ has a slope of 2, and so the tangent line is parallel to it when $2x-1=2$, $x=3/2$, or at $(3/2, f(3/2)) = (3/2, 3/4)$.

31. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{[-(x+h)^3+4]-(-x^3+4)}{h}$

$= \lim_{h\to 0} \dfrac{-3x^2h-3xh^2-h^3}{h} = \lim_{h\to 0}(-3x^2-3xh-h^2) = -3x^2$

The given line $12x+y=4$ has a slope of -12, and so the tangent line is parallel to it when $-3x^2=-12$, $x=\pm 2$, or at $(-2, f(-2)) = (-2, 12)$ and $(2, f(2)) = (2, -4)$.

32. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{(6\sqrt{x+h}+2)-(6\sqrt{x}+2)}{h}$

$= \lim_{h\to 0} \dfrac{6\sqrt{x+h}-6\sqrt{x}}{h} = \lim_{h\to 0} \left[\dfrac{6(\sqrt{x+h}-\sqrt{x})}{h} \cdot \dfrac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right]$

$= \lim_{h\to 0} \dfrac{6(x+h-x)}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0} \dfrac{6h}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0} \dfrac{6}{\sqrt{x+h}+\sqrt{x}}$

$= \dfrac{6}{2\sqrt{x}} = \dfrac{3}{\sqrt{x}}$

The given line $-x+y=2$ has a slope of 1, and so the tangent line is parallel to it when $3/\sqrt{x}=1$, $x=9$, or at $(9, f(9)) = (9, 20)$.

33. $f'_+(2) = \lim_{h\to 0^+} \dfrac{f(2+h)-f(2)}{h} = \lim_{h\to 0^+} \dfrac{[2(2+h)-4]-[2(2)-4]}{h} = \lim_{h\to 0^+} \dfrac{2h}{h} = 2$

$f'_-(2) = \lim_{h\to 0^-} \dfrac{f(2+h)-f(2)}{h} = \lim_{h\to 0^-} \dfrac{[-(2+h)+2]-(-2+2)}{h} = \lim_{h\to 0^-} \dfrac{-h}{h} = -1$

Since $f'_+(2) \neq f'_-(2)$, f is not differentiable at 2.

34. $f'_+(0) = \lim_{h\to 0^+} \dfrac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \dfrac{-4h-[-4(0)]}{h} = \lim_{h\to 0^+} \dfrac{-4h}{h} = -4$

$f'_-(0) = \lim_{h\to 0^-} \dfrac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \dfrac{3h-[3(0)]}{h} = \lim_{h\to 0^-} \dfrac{3h}{h} = 3$

Since $f'_+(0) \neq f'_-(0)$, f is not differentiable at 0.

35. $$\begin{aligned} f'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{(10x^2-3)-(10a^2-3)}{x-a} \\ &= \lim_{x\to a} \frac{10(x+a)(x-a)}{x-a} = \lim_{x\to a} 10(x+a) = 20a \end{aligned}$$

36. $$\begin{aligned} f'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{(x^2-3x-1)-(a^2-3a-1)}{x-a} \\ &= \lim_{x\to a} \frac{(x+a)(x-a)-3(x-a)}{x-a} = \lim_{x\to a}(x+a-3) = 2a-3 \end{aligned}$$

37. $$\begin{aligned} f'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{(x^3-4x^2)-(a^3-4a^2)}{x-a} \\ &= \lim_{x\to a} \frac{(x^2+ax+a^2)(x-a)-4(x+a)(x-a)}{x-a} \\ &= \lim_{x\to a}(x^2+ax+a^2-4x-4a) = 3a^2-8a \end{aligned}$$

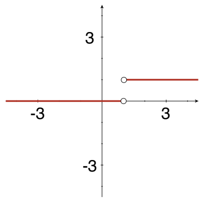
38. $$\begin{aligned} f'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{x^4-a^4}{x-a} = \lim_{x\to a} \frac{(x^2+a^2)(x+a)(x-a)}{x-a} \\ &= \lim_{x\to a}(x^2+a^2)(x+a) = 2a^2(2a) = 4a^3 \end{aligned}$$

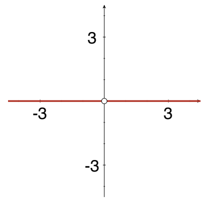
39. $$\begin{aligned} f'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{\dfrac{4}{3-x}-\dfrac{4}{3-a}}{x-a} \\ &= \lim_{x\to a} \frac{12-4a-12+4x}{(x-a)(3-x)(3-a)} = \lim_{x\to a} \frac{4(x-a)}{(x-a)(3-x)(3-a)} \\ &= \lim_{x\to a} \frac{4}{(3-x)(3-a)} = \frac{4}{(3-a)^2} \end{aligned}$$

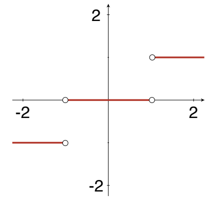
40. $$\begin{aligned} f'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{\sqrt{x}-\sqrt{a}}{x-a} = \lim_{x\to a} \left(\frac{\sqrt{x}-\sqrt{a}}{x-a} \cdot \frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}} \right) \\ &= \lim_{x\to a} \frac{x-a}{(x-a)(\sqrt{x}+\sqrt{a})} = \lim_{x\to a} \frac{1}{\sqrt{x}+\sqrt{a}} = \frac{1}{2\sqrt{a}} \end{aligned}$$

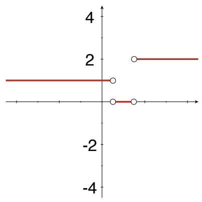
41. Since $(0,3)$ and $(-6,0)$ are points on the line, and the line is tangent to f at $x=-3$, then its slope is $f'(-3) = m = \dfrac{3-0}{0-(-6)} = \dfrac{1}{2}$. Using $(0,3)$, an equation of the tangent line is $y-3 = \dfrac{1}{2}x$ or $y = \dfrac{1}{2}x+3$. Thus, $f(-3) = \dfrac{3}{2}$.

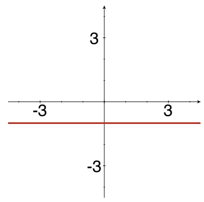
42. Since $(3,2)$ and $(9/2,0)$ are points on the line, and the line is tangent to f at $x=3$, then its slope is $f'(3) = m = \dfrac{2-0}{3-9/2} = -\dfrac{4}{3}$. Using $(3,2)$, an equation of the tangent line is $y-2 = -\dfrac{4}{3}(x-3)$ or $y = -\dfrac{4}{3}x+6$. Thus, the tangent line's y-intercept is $-\dfrac{4}{3}(0)+6 = 6$.

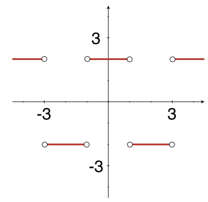
43.


44.


45.


46.


47.


48.


49. (e); consider $f(x) = \tan^{-1} x$.

50. (d); consider $f(x) = (x-1)^2$.

51. (b); consider $f(x) = \dfrac{1}{3}x^3 - 2x$.

52. (f); consider $f(x) = |x^2 - 2|$.

53. (a); consider $f(x) = 3\sqrt{|x|}$.

54. (c); consider $f(x) = 3\sqrt[3]{x}$.

55. $$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{x^{1/3}-a^{1/3}}{x-a} = \lim_{x\to a} \frac{x^{1/3}-a^{1/3}}{(x^{1/3})^3-(a^{1/3})^3}$$
$$= \lim_{x\to a} \frac{x^{1/3}-a^{1/3}}{(x^{1/3}-a^{1/3})(x^{2/3}+x^{1/3}a^{1/3}+a^{2/3})} = \lim_{x\to a} \frac{1}{x^{2/3}+x^{1/3}a^{1/3}+a^{2/3}} = \frac{1}{3a^{2/3}}$$

56. The vertical tangents for these roots occur when the value within the root is zero; based on this, the graphs of $y = (x-4)^{1/3}$ and $y = \sqrt{x+2}$ may have vertical tangents at $x = 4$ and $x = -2$, respectively.

57. $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{f(x)f(h)-f(x)}{h} = \lim_{h\to 0} \frac{f(x)[f(h)-1]}{h}$$
Since $f(0) = 1$ and $f'(0) = 1$, we can substitute $x = 0$ above to obtain:
$$f'(0) = \lim_{h\to 0} \frac{f(0)[f(h)-1]}{h} = \lim_{h\to 0} \frac{1[f(h)-1]}{h} = \lim_{h\to 0} \frac{f(h)-1}{h} = 1$$
Going back to the general expression of the limit for all x, we get:
$$f'(x) = \lim_{h\to 0} \frac{f(x)[f(h)-1]}{h} = \left[\lim_{h\to 0} f(x)\right] \cdot \left[\lim_{h\to 0} \frac{f(h)-1}{h}\right] = \left[\lim_{h\to 0} f(x)\right] \cdot [1]$$
$$= \lim_{h\to 0} f(x) = f(x)$$
Thus, $f'(x) = f(x)$ for all x.

58. (a) When the graph of f is reflected through the y-axis, tangent lines with slopes m, on reflection, have slopes $-m$. Thus $f'(-x) = -f'(x)$ and f' is an odd function.
    (b) When the graph of f is reflected across the origin, tangent lines with slopes m, on reflection, have slopes $-(-m) = m$ (i.e., reflection across the origin is equivalent to reflecting across the one axis first, thus negating the slope, then reflecting across the other axis, thus negating again). Thus $f'(-x) = f'(x)$ and f' is an even function.

59. The statement is true because, since $f(a) = f(b) = 0$, every "upward" or "downward" movement from $x = a$ must eventually have an equal but opposite amount of movement before reaching $x = b$. Since the function is differentiable on $[a, b]$, there must be a point c where the function "transitions" from going up or down to the opposite direction, and so the slope at that point must be horizontal, or $f'(c) = 0$.

60. The graphs of functions f for which $f'(x) > 0$ for all x in $[a, b]$ will have $f(x_1) > f(x_0)$ for all x_0, x_1 in $[a, b]$ where $x_1 > x_0$.

61. For $n = 1$, $f(x) = \begin{cases} x + x = 2x, & x > 0 \\ x + (-x) = 0, & x < 0 \end{cases}$

$$f'_+(x) = \lim_{h\to 0^+} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0^+} \frac{2(x+h) - 2x)}{h} = \lim_{h\to 0^+} \frac{2h}{h} = 2;\ f'_+(0) = 2$$
$$f'_-(x) = \lim_{h\to 0^-} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0^-} \frac{0 - 0)}{h} = 0;\ f'_-(0) = 0$$

Since $f'_+(0) \neq f'_-(0)$ then f is not differentiable at 0 for $n = 1$.

For $n = 2$, $f(x) = \begin{cases} x^2 + x, & x > 0 \\ x^2 - x, & x < 0 \end{cases}$

$$\begin{aligned} f'_+(x) &= \lim_{h\to 0^+} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0^+} \frac{[(x+h)^2 + (x+h)] - (x^2 + x)}{h} \\ &= \lim_{h\to 0^+} \frac{2xh + h^2 + h}{h} = 2x + 1;\ f'_+(0) = 1 \\ f'_-(x) &= \lim_{h\to 0^-} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0^-} \frac{[(x+h)^2 - (x+h)] - (x^2 - x)}{h} \\ &= \lim_{h\to 0^+} \frac{2xh + h^2 - h}{h} = 2x - 1;\ f'_-(0) = -1 \end{aligned}$$

Since $f'_+(0) \neq f'_-(0)$ then f is not differentiable at 0 for $n = 2$. Proceeding similarly for $n = 3, 4$, and 5, we get:

$$\begin{array}{lll} n = 3: & f'_+(x) = 3x^2 + 1 & f'_-(x) = 3x^2 - 1 \\ n = 4: & f'_+(x) = 4x^3 + 1 & f'_-(x) = 4x^3 - 1 \\ n = 5: & f'_+(x) = 5x^4 + 1 & f'_-(x) = 5x^4 - 1 \end{array}$$

The general case n yields $f'_+(x) = nx^{n-1} + 1$ and $f'_-(x) = nx^{n-1} - 1$. In all cases for $n > 1$, $f' + -(0) = 1$ and $f'_-(0) = -1$, and so f is not differentiable at 0 for any positive integer n.

3.2 Power and Sum Rules

1. $\dfrac{dy}{dx} = 0$

2. $\dfrac{dy}{dx} = 0$

3. $\dfrac{dy}{dx} = 9x^8$

4. $\dfrac{dy}{dx} = 48x^{11}$

5. $\dfrac{dy}{dx} = 14x - 4$

6. $\dfrac{dy}{dx} = 18x^2 + 6x$

7. $\dfrac{dy}{dx} = 2x^{-1/2} + 4x^{-5/3} = \dfrac{2}{\sqrt{x}} + \dfrac{4\sqrt[3]{x}}{x^2}$

8. $\dfrac{dy}{dx} = \dfrac{1}{2}x^{-1/2} - \dfrac{3}{2}x^{1/2} = \dfrac{1}{2\sqrt{x}} - \dfrac{3}{2}\sqrt{x}$

9. $f'(x) = x^4 - 12x^3 + 18x$

10. $f'(x) = -4x^5 + 20x^4 - 26x + 8$

11. $f(x) = 4x^5 - 5x^4 - 6x^3$
    $f'(x) = 20x^4 - 20x^3 - 18x^2$

12. $f(x) = 2x^3 + 3x^2 - x + \dfrac{2}{x^2}$
    $f'(x) = 6x^2 + 6x - 1 - \dfrac{4}{x^3}$

13. $f(x) = x^2(x^2+5)^2 = x^6 + 10x^4 + 25x^2;\quad f'(x) = 6x^5 + 40x^3 + 50x$

14. $f(x) = x^9 + 3x^8 + 3x^7 + x^6;\quad f'(x) = 9x^8 + 24x^7 + 21x^6 + 6x^5$

15. $f(x) = 16x + 8\sqrt{x} + 1;\quad f'(x) = 16 + \dfrac{4}{\sqrt{x}}$

16. $f(x) = 81 - x^2;\quad f'(x) = -2x$

17. $h(u) = 64u^3;\quad h'(u) = 192u^2$

18. $p(t) = \dfrac{1}{16}t^{-4} - 4t^{-2};\quad p'(t) = -\dfrac{1}{4}t^{-5} + 8t^{-3} = -\dfrac{1}{4t^5} + \dfrac{8}{t^3}$

19. $g(r) = r^{-1} + r^{-2} + r^{-3} + r^{-4};\quad g'(r) = -r^{-2} - 2r^{-3} - 3r^{-4} - 4r^{-5} = -\dfrac{1}{r^2} - \dfrac{2}{r^3} - \dfrac{3}{r^4} - \dfrac{4}{r^5}$

20. $Q'(t) = \dfrac{5t^4 + 8t}{6}$

21. $y' = 6x^2$. When $x = -1$, the slope of the tangent line is 6 and the point of tangency is $(-1, y(-1))$ or $(-1, -3)$. Hence, an equation of the tangent line is $y + 3 = 6(x + 1)$ or $y = 6x + 3$.

22. $y' = -1 - (8/x^2)$. When $x = 2$, the slope of the tangent line is -3 and the point of tangency is $(2, y(2))$ or $(2, 2)$. Hence, an equation of the tangent line is $y - 2 = -3(x - 2)$ or $y = -3x + 8$.

23. $y' = -2x^{-3/2} + x^{-1/2}$. When $x = 4$, the slope of the tangent line is $1/4$ and the point of tangency is $(4, y(4))$ or $(4, 6)$. Hence, an equation of the tangent line is $y - 6 = (x - 4)/4$ or $y = x/4 + 5$.

24. $y' = -3x^2 + 12x$. When $x = 1$, the slope of the tangent line is 9 and the point of tangency is $(1, y(1))$ or $(1, 5)$. Hence, an equation of the tangent line is $y - 5 = 9(x - 1)$ or $y = 9x - 4$.

25. $y' = 2x - 8$. The tangent is horizontal when $2x - 8 = 0$ or $x = 4$. Since $y(4) = -11$, the tangent is horizontal at $(4, -11)$.

26. $y' = x^2 - x$. The tangent is horizontal when $x^2 - x = 0$ or $x = 0, 1$. Since $y(0) = 0$ and $y(1) = -1/6$, the tangent is horizontal at $(0, 0)$ and $(1, -1/6)$.

27. $y' = 3x^2 - 6x - 9$. The tangent is horizontal when $3x^2 - 6x - 9 = 0$ or $x = 3, -1$. Since $y(3) = -25$ and $y(-1) = 7$, the tangent is horizontal at $(3, -25)$ and $(-1, 7)$.

28. $y' = 4x^3 - 12x^2$. The tangent is horizontal when $4x^3 - 12x^2 = 0$ or $x = 0, 3$. Since $y(0) = 0$ and $y(3) = -27$, the tangent is horizontal at $(0, 0)$ and $(3, -27)$.

29. $y' = -2x$, so $m_{\tan} = -4$ at $(2, -3)$. Thus, the slope of the normal line is $m = 1/4$ and its equation is $y + 3 = \dfrac{1}{4}(x - 2)$ or $y = \dfrac{1}{4}x - \dfrac{7}{2}$.

30. $y' = 3x^2$, so $m_{\tan} = 3$ at $(1, 1)$. Thus, the slope of the normal line is $m = -1/3$ and its equation is $y - 1 = -\dfrac{1}{3}(x - 1)$ or $y = -\dfrac{1}{3}x + \dfrac{4}{3}$.

31. $y' = x^2 - 4x$, so $m_{\tan} = 0$ at $(4, -32/3)$. Thus, tangent line is horizontal and the normal line is vertical. Its equation is $x = 4$.

32. $y' = 4x^3 - 1$, so $m_{\tan} = -5$ at $(-1, 2)$. Thus, the slope of the normal line is $m = 1/5$ and its equation is $y - 2 = \dfrac{1}{5}(x + 1)$ or $y = \dfrac{1}{5}x + \dfrac{11}{5}$.

33. $\dfrac{dy}{dx} = -2x + 3;\quad \dfrac{d^2y}{dx^2} = -2$

34. $\dfrac{dy}{dx} = 30x - \dfrac{12}{\sqrt{x}};\quad \dfrac{d^2y}{dx^2} = 30 + 6x^{-3/2}$

35. $\dfrac{dy}{dx} = 32x - 72;\quad \dfrac{d^2y}{dx^2} = 32$

36. $\dfrac{dy}{dx} = 10x^4 + 12x^2 - 12x$; $\dfrac{d^2y}{dx^2} = 40x^3 + 24x - 12$

37. $\dfrac{dy}{dx} = -20x^{-3}$; $\dfrac{d^2y}{dx^2} = 60x^{-4}$

38. $y = x + 8x^{-6}$; $\dfrac{dy}{dx} = 1 - 48x^{-7}$; $\dfrac{d^2y}{dx^2} = 336x^{-8}$

39. $f'(x) = 24x^5 + 5x^4 - 3x^2$; $f''(x) = 120x^4 + 20x^3 - 6x$;
$f'''(x) = 480x^3 + 60x^2 - 6$; $f^{(4)}(x) = 1440x^2 + 120x$

40. $\dfrac{dy}{dx} = 4x^3 + 10x^{-2}$; $\dfrac{d^2y}{dx^2} = 12x^2 - 20x^{-3}$; $\dfrac{d^3y}{dx^3} = 24x + 60x^{-4}$;
$\dfrac{d^4y}{dx^4} = 24 - 240x^{-5}$; $\dfrac{d^5y}{dx^5} = 1200x^{-6}$

41. $f'(x) = 2x + 8$. Solving $2x + 8 > 0$ we obtain $x > -4$. Thus $f'(x) > 0$ or $(-4, \infty)$. Solving $2x + 8 < 0$ we obtain $x < -4$. Thus $f'(x) < 0$ on $(-\infty, -4)$.

42. $f'(x) = 3x^2 - 6x - 9 = 3(x-3)(x+1)$. Checking the sign of $f'(x)$ on the intervals $(-\infty, -1)$, $(-1, 3)$, and$(3, \infty)$ we find that $f'(x) > 0$ on $(-\infty, -1)$ and $(3, \infty)$ and $f'(x) < 0$ on $(-1, 3)$.

43. $f'(x) = 3x^2 + 24x + 20$; $f''(x) = 6x + 24$. Solving $6x + 24 = 0$, we obtain $x = -4$. Thus the point on the graph is $(-4, f(-4))$ or $(-4, 48)$.

44. $f'(x) = 4x^3 - 6x^2$; $f''(x) = 12x^2 - 12x$. Solving $12x^2 - 12x = 0$, we obtain $x = 0$ and $x = 1$. Since $f(0) = 0$ and $f(1) = -1$, the points on the graph are $(0, 0)$ and $(1, -1)$.

45. $f'(x) = 3(x-1)^2$; $f''(x) = 6(x-1)$. $f''(x) > 0$ for $x > 1$ and $f''(x) < 0$ for $x < 1$. Thus, $f''(x) > 0$ on $(1, \infty)$ and $f''(x) < 0$ on $(-\infty, 1)$.

46. $f'(x) = 3x^2 + 2x$; $f''(x) = 6x + 2 = 6(x + 1/3)$. $f''(x) > 0$ for $x > -1/3$ and $f''(x) < 0$ for $x < -1/3$. Thus, $f''(x) > 0$ on $(-1/3, \infty)$ and $f''(x) < 0$ on $(-\infty, -1/3)$.

47. $y' = -x^{-2} + 4x^3$; $y'' = 2x^{-3} + 12x^2$. Substituting into the differential equation, we get:

$$\begin{aligned} x^2y'' - 2xy' - 4y &= x^2(2x^{-3} + 12x^2) - 2x(-x^{-2} + 4x^3) - 4(x^{-1} + x^4) \\ &= 2x^{-1} + 12x^4 + 2x^{-1} - 8x^4 - 4x^{-1} - 4x^4 = 0 \end{aligned}$$

Thus, the function satisfies $x^2y'' - 2xy' - 4y = 0$.

48. $y' = 1 + 3x^2$; $y'' = 6x$. Substituting into the differential equation, we get:

$$\begin{aligned} x^2y'' - 3xy' + 3y &= x^2(6x) - 3x(1 + 3x^2) + 3(x + x^3 + 4) \\ &= 6x^3 - 3x - 9x^3 + 3x + 3x^3 + 12 = 12 \end{aligned}$$

Thus, the function satisfies $x^2y'' - 3xy' + 3y = 12$.

49. $f'(x) = 4x - 3$. Since the slope of the tangent line is 5, $4x - 3 = 5$ and $x = 2$. Since $f(2) = 8$, the point on the graph is $(2, 8)$.

50. Writing $3x-9y-4=0$ as $y=\dfrac{1}{3}x-\dfrac{4}{9}$, we see that the slope of the tangent line is 1/3. Now, $f'(x)=2x-1=\dfrac{1}{3}$, so $x=\dfrac{2}{3}$. Since $f(2/3)=-2/9$, the point on the graph is $(2/3,-2/9)$.

51. Since the slope of the normal line is 2, the slope of the tangent line is $-1/2$. Thus $f'(x)=2x-1=-1/2$ and $x=1/4$. Since $f(1/4)=-3/16$, the point on the graph is $(1/4,-3/16)$.

52. Writing $3x-2y+1=0$ as $y=\dfrac{3}{2}x+\dfrac{1}{2}$, we see that the slope of the tangent line is 3/2. Now $f'(x)=\dfrac{1}{2}x-2=\dfrac{3}{2}$, so $x=7$. Since $f(7)=-7/4$, the point on the graph is $(7,-7/4)$.

53. $y'=3x^2+6x-4$; $y''=6x+6$. The second derivative is zero when $x=-1$. Since $y(-1)=7$ and $y'(-1)=-7$, the point on the graph is $(-1,7)$ and the slope of the tangent line is -7. Hence, an equation of the tangent line is $y-7=-7(x+1)$ or $y=-7x$.

54. $y'=4x^3$; $y''=12x^2$; $y'''=24x$. The third derivative is 12 when $x=1/2$. Since $y(1/2)=1/16$ and $y'(1/2)=1/2$, the point on the graph is $(1/2,1/16)$ and the slope of the tangent line is 1/2. Hence, an equation of the tangent line is $y-\dfrac{1}{16}=\dfrac{1}{2}\left(x-\dfrac{1}{2}\right)$ or $y=\dfrac{1}{2}x-\dfrac{3}{16}$.

55. $V(r)=\dfrac{4\pi}{3}r^3$; $S=V'(r)=4\pi r^2$

56. $v(r)=\dfrac{P}{4vl}(R^2-r^2)=\dfrac{PR^2}{4vl}-\dfrac{P}{4vl}r^2$. Since $v'(r)=-\dfrac{P}{2vl}r=0$, $r=0$ and $v(0)=\dfrac{PR^2}{4vl}$.

57. $U(x)=\dfrac{k}{2}x^2$; $F=-\dfrac{dU}{dx}=-\dfrac{k}{2}2x=-kx$

Given $k=30$ N/m and $x=\dfrac{m}{2}$, $F=-(30\text{ N/m})\dfrac{m}{2}=-15$ N

58. $s(t)=-\dfrac{1}{2}gt^2+v_0t+s_0$; $s'(t)=-gt+v_0$; $s'(4)=-4g+v_0$

59. When $n=1$, $\dfrac{d}{dx}x=1$; when $n=2$, $\dfrac{d^2}{dx^2}x^2=2$; when $n=3$, $\dfrac{d^3}{dx^3}=3\cdot 2=3!$; when $n=4$, $\dfrac{d^4}{dx^4}=4\cdot 3\cdot 2=4!$. We can verify by induction that $\dfrac{d^n}{dx^n}x^n=n!$:

$$\frac{d^n}{dx^n}x^n=\frac{d^{n-1}}{dx^{n-1}}\left(\frac{d}{dx}x^n\right)=\frac{d^{n-1}}{dx^{n-1}}(nx^{n-1})=n\left(\frac{d^{n-1}}{dx^{n-1}}x^{n-1}\right)=n(n-1)!=n!.$$

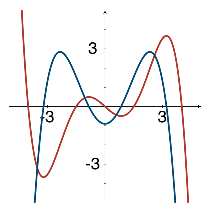
60. When $n=1$, $\dfrac{d}{dx}\left(\dfrac{1}{x}\right)=-\dfrac{1}{x^2}$; when $n=2$, $\dfrac{d^2}{dx^2}\left(\dfrac{1}{x}\right)=\dfrac{2}{x^3}$; when $n=3$, $\dfrac{d^3}{dx^3}\left(\dfrac{1}{x}\right)=-\dfrac{3\cdot 2}{x^4}=-\dfrac{3!}{x^4}$; when $n=4$, $\dfrac{d^4}{dx^4}\left(\dfrac{1}{x}\right)=\dfrac{4\cdot 3!}{x^5}=\dfrac{4!}{x^5}$. We can verify by induction that

$$\frac{d^n}{dx^n}\left(\frac{1}{x}\right) = (-1)^n \frac{n!}{x^{n+1}}:$$

$$\begin{aligned}\frac{d^n}{dx^n}\left(\frac{1}{x}\right) &= \frac{d}{dx}\left[\frac{d^{n-1}}{dx^{n-1}}\left(\frac{1}{x}\right)\right] = \frac{d}{dx}\left[(-1)^{n-1}\frac{(n-1)!}{x^{(n-1)+1}}\right] = (-1)^{n-1}(n-1)!\frac{d}{dx}\left(\frac{1}{x^n}\right)\\ &= (-1)^{n-1}(n-1)!\left(\frac{d}{dx}x^{-n}\right) = (-1)^{n-1}(n-1)!(-nx^{-n-1})\\ &= (-1)^{n-1}(n-1)!(-1)(n)(x^{-n-1}) = (-1)^n(n!)\left[x^{-(n+1)}\right] = (-1)^n\frac{n!}{x^{n+1}}.\end{aligned}$$

61. $f(x) = 0$ at 3 points in the figure, and the tangents to $g(x)$ are horizontal at the same locations. In contrast, $g(x) = 0$ at 4 points in the figure, while the tangents to $f(x)$ are horizontal at only two. This implies that $f(x)$ is the derivative of $g(x)$. Similarly, $f(x) < 0$ exactly at the intervals where $g(x)$ is moving downward and $f(x) > 0$ where $g(x)$ is moving upward, but not vice versa; this also implies that $f(x)$ is the derivative of $g(x)$.

62. The graph from Figure 3.2.7 is reproduced in red, while its derivative is shown in blue.



63. Let $f(x) = ax^2 + bx + c$. Then $f'(x) = 2ax + b$ and $f''(x) = 2a$. From $f''(-1) = 2a = -4$ we see that $a = -2$ and $f'(x) = -4x + b$. From $f'(-1) = -4(-1) + b = 7$ we see that $b = 3$ and $f(x) = -2x^2 + 3x + c$. From $f(-1) = -2 - 3 + c = -11$ we see that $c = -6$ and $f(x) = -2x^2 + 3x - 6$.

64. Solving $\frac{1}{8}x^2 = -\frac{1}{4}x^2 + 3$, we obtain $x = \pm 2\sqrt{2}$. From the derivatives $\frac{1}{4}x$ and $-\frac{1}{2}x$ we see that at $x = 2\sqrt{2}$, the slopes of the tangent lines are $m_1 = \sqrt{2}/2$ and $m_2 = -\sqrt{2}$. Then $m_1m_2 = -1$ and the tangent lines are perpendicular. Similarly, at $x = -2\sqrt{2}$, the slopes of the tangent lines are $m_1 = -\sqrt{2}/2$ and $m_2 = \sqrt{2}$. Again, $m_1m_2 = -1$ and the tangent lines are perpendicular. Thus, the graphs are orthogonal.

65. $f(-3) = 9 - 3b$, $f'(x) = 2x + b$, and $f'(-3) = -6 + b$. The equation of the tangent line at $(-3, 9 - 3b)$ is $y - (9 - 3b) = (-6 + b)(x + 3)$ or $y = (-6 + b)x - 9$. We are given that the tangent line is $y = 2x + c$. Thus, $-6 + b = 2$ or $b = 8$ and $c = -9$.

66. $f'(x) = 2x + 2$. Letting $(a, f(a))$ be the point of tangency on the graph, we find the equation of the line:

$$\begin{aligned} y - f(a) &= f'(a)(x - a)\\ y - (a^2 + 2a + 2) &= (2a + 2)(x - a)\\ y - a^2 - 2a - 2 &= (2a + 2)x - 2a^2 - 2a\\ y &= (2a + 2)x - a^2 + 2.\end{aligned}$$

Setting $x = 3/2$ and $y = 1$, we have

$$\begin{aligned} 1 &= (2a+2)\frac{3}{2} - a^2 + 2 \\ a^2 - 3a - 4 &= 0 \\ (a-4)(a+1) &= 0. \end{aligned}$$

Thus, $a = -1$ and $a = 4$. The equations of the tangent lines are $y = 1$ and $y = 10x - 14$.

67. Let $(a, f(a))$ or $(a, a^2 - 5)$ be the point on the graph. The slope of the line through $(a, a^2 - 5)$ and $(-3, 0)$ is $\dfrac{(a^2-5)-0}{a-(-3)}$ or $\dfrac{a^2-5}{a+3}$. Now we find the slope of the tangent line by finding $f'(a)$ and setting it equal to $\dfrac{a^2-5}{a+3}$. Since $f'(x) = 2x$, $m_{\text{tan}} = f'(a) = 2a$. Then $2a = \dfrac{a^2-5}{a+3}$ or $a^2 + 6a + 5 = 0$, and $a = -1, -5$. Since $f(-1) = -4$ and $f(-5) = 20$, the points on the graph are $(-1, -4)$ and $(-5, 20)$.

68. Let $(a, f(a))$ or (a, a^2) be the point on the graph. The slope of the line through (a, a^2) and $(0, -2)$ is $\dfrac{a^2-(-2)}{a-0}$ or $\dfrac{a^2+2}{a}$. Now we find the slope of the tangent line by finding $f'(a)$ and setting it equal to $\dfrac{a^2+2}{a}$. Since $f'(x) = 2x$, $m_{\text{tan}} = f'(a) = 2a$. Then $2a = \dfrac{a^2+2}{a}$ or $a^2 = 2$, and $a = \pm\sqrt{2}$. Since $f(\sqrt{2}) = 2 = f(-\sqrt{2})$, the points on the graph are $(\sqrt{2}, 2)$ and $(-\sqrt{2}, 2)$.

69. $f'(x) = x^4 + x^2$. Thus, $f'(x)$ is never negative, and therefore $f(x)$ cannot possibly have a tangent line with slope -1.

70. $y' = 2Ax + B$ and $y'' = 2A$. When $x = 0$, $y' = B$ and $y'' = 2A$. Substituting these into the differential equation, we get

$$2(2A) + 3(B) = 0 - 1, \quad 4A + 3B + 1 = 0.$$

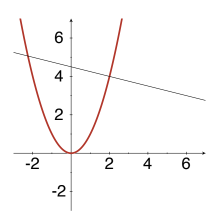
When $x = 1$, $y' = 2A + B$ and $y'' = 2A$. Substituting once more, we get

$$2(2A) + 3(2A + B) = 1 - 1, \quad 10A + 3B = 0.$$

Solving these two equations in two unknowns, we get $A = 1/6$ and $B = -5/9$.

71. $f(1) = a + b = 4$, $f'(x) = 2ax + b$, and $f'(1) = 2a + b = -5$. Solving for the two equations in two unknowns, we see $a = -9$ and $b = 13$.

72. Since $(2, 4)$ is on the graph of $f(x) = x^2$, there is only one normal line to the graph at this point. From $f'(x) = 2x$ we see that the slope of this normal line is $-1/4$. To find any other normal lines through $(2, 4)$ we consider (a, a^2) on the graph. The slope of the tangent line at this point is $2a$ and the slope of the normal line is $-1/2a$. Now, the slope of the line through $(2, 4)$ and (a, a^2) is $(a^2 - 4)/(a - 2) = a + 2$, for $a \neq 2$. Setting $-1/2a = a + 2$ we obtain $2a^2 + 4a + 1 = 0$ and $a = (-1 \pm \sqrt{2})/2$. The slopes of other two normal lines are then $-1/2a = (2 \pm \sqrt{2})/2$.



73. $f'(x) = 2x+1$ and $g'(x) = 4x+4$. At $(a, f(a))$ on the graph of f, the slope of the tangent line is $2a+1$. Solving $g'(x) = 4x+4 = 2a+1$ for x we obtain $x = a/2 - 3/4$. Thus, points on the graphs of f and g where the tangent lines are parallel are $(a, f(a))$ and $(a/2-3/4, g(a/2-3/4))$.

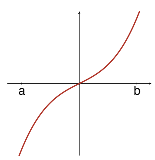
74. The slope of the tangent to $f(x)$ is smallest when $f'(x) = 15x^4 + 15x^2 + 2$ is smallest, and this is the case when $x = 0$. Thus, the slope is smallest at $(0, f(0)) = (0, 0)$.

75. $f'(x) = 3ax^2 + 2bx + c$. Thus, $f(x)$ has exactly one, two, or no horizontal tangents if and only if the equation $3ax^2 + 2bx + c = 0$ has exactly one, two, or no solutions, respectively. By the quadratic formula, the solutions to this equation are

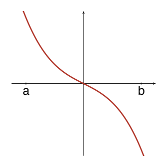
$$x = \frac{-(2b) \pm \sqrt{(2b)^2 - 4(3a)(c)}}{2(3a)} = \frac{-2b \pm \sqrt{4b^2 - 12ac}}{6a} = \frac{-b \pm \sqrt{b^2 - 3ac}}{3a}$$

Thus, $f(x)$ will have exactly one horizontal tangent when $b^2 - 3ac = 0$, and it will have no horizontal tangents when either $a = 0$ or $b^2 - 3ac < 0$. $f(x)$ will have two horizontal tangents when $a \neq 0$ and $b^2 - 3ac > 0$.

76.



When $f'(x) > 0$ for all x in an interval (a, b), the function increases and the graph rises.



When $f'(x) < 0$ for all x in an interval (a, b), the function decreases and the graph falls.

77. Since $f'(x) - f(x) = 0$, then $f'(x) = f(x)$. That means $f''(x) = (f')'(x) = f'(x) = f(x)$, $f'''(x) = (f'')'(x) = f'(x) = f(x)$, and so on. Thus, $f^{(100)}(x) = f(x)$.

78. Letting (a, a^2) be a point of tangency on $y = x^2$, the slope at that point is $2a$. Similarly, letting $(b, -b^2 + 2b - 3)$ be a point of tangency on $y = -x^2 + 2x - 3$, the slope at that point is $-2b + 2$. For L_1 and L_2, these slopes are equal:

$$2a = -2b + 2 \qquad \text{or} \qquad a + b = 1.$$

Expressing L_1 and L_2 in terms of (a, a^2) and $(b, -b^2 + 2b - 3)$ using slope $2a$, we have

$$a^2 - (-b^2 + 2b - 3) = 2a(a - b).$$

Simplifying this equation produces

$$a^2 - (-b^2 + 2b - 3) = 2a^2 - 2ab$$
$$-a^2 + 2ab + b^2 - 2b + 3 = 0.$$

Since $a + b = 1$, we can substitute $b = 1 - a$:

$$-a^2 + 2a(1 - a) + (1 - a)^2 - 2(1 - a) + 3 = 0$$
$$-a^2 + 2a - 2a^2 + 1 - 2a + a^2 - 2 + 2a + 3 = 0$$
$$-2a^2 + 2a + 2 = 0, \quad a^2 - a - 1 = 0.$$

Applying the quadratic formula, $a = \dfrac{1 \pm \sqrt{5}}{2}$ and $b = 1 - a = \dfrac{1 \mp \sqrt{5}}{2}$. Substituting these into (a, a^2) and $(b, -b^2+2b-3)$ and inspecting the resulting points' locations in Figure 3.2.8, we see that $(\frac{1+\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2})$ and $(\frac{1-\sqrt{5}}{2}, -\frac{7+\sqrt{5}}{2})$ lie on L_1 while $(\frac{1-\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2})$ and $(\frac{1+\sqrt{5}}{2}, -\frac{7-\sqrt{5}}{2})$ lie on L_2.

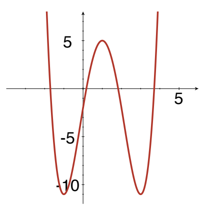
Therefore, an equation for L_1 is

$$y - \left(\frac{3+\sqrt{5}}{2}\right) = 2\left(\frac{1+\sqrt{5}}{2}\right)\left(x - \frac{1+\sqrt{5}}{2}\right)$$
$$y = (1+\sqrt{5})x - \left(\frac{6+2\sqrt{5}}{2}\right) + \left(\frac{3+\sqrt{5}}{2}\right) = (1+\sqrt{5})x - \frac{3+\sqrt{5}}{2}.$$

Similarly, an equation for L_2 is

$$y - \left(\frac{3-\sqrt{5}}{2}\right) = 2\left(\frac{1-\sqrt{5}}{2}\right)\left(x - \frac{1-\sqrt{5}}{2}\right)$$
$$y = (1-\sqrt{5})x - \frac{3-\sqrt{5}}{2}.$$

79. (a)

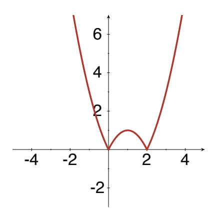


(b) $f'(x) = 4x^3 - 12x^2 - 4x + 12$; $f''(x) = 12x^2 - 24x - 4$

x	-2	-1	0	1	2	3	4
$f''(x)$	92	32	-4	-16	-4	32	92

(c) Where the graph is concave up, the second derivative is positive; where it is concave down the second derivative is negative.

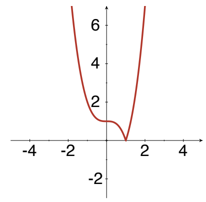
80. (a) By inspection, $f(x)$ is not differentiable at $x = 0, 2$.



$$f'(x) = \begin{cases} 2x-2, & x<0 \text{ or } x>2 \\ -2x+2, & 0<x<2 \end{cases}$$

(b) By inspection, $f(x)$ is not differentiable at $x = 1$.

$$f'(x) = \begin{cases} -3x^2, & x < 1 \\ 3x^2, & x > 1 \end{cases}$$



3.3 Product and Quotient Rules

1. $\dfrac{dy}{dx} = (x^2 - 7)(3x^2 + 4) + (x^3 + 4x + 2)(2x) = 5x^4 - 9x^2 + 4x - 28$

2. $\dfrac{dy}{dx} = (7x + 1)(4x^3 - 3x^2 - 9) + (x^4 - x^3 - 9x)(7) = 35x^4 - 24x^3 - 3x^2 - 126x - 9$

3. $$\begin{aligned} y &= (4x^{1/2} + x^{-1})(2x - 6x^{-1/3}) \\ \frac{dy}{dx} &= (4x^{1/2} + x^{-1})(2 + 2x^{-4/3}) + (2x - 6x^{-1/3})(2x^{-1/2} - x^{-2}) \\ &= 8x^{1/2} + 8x^{-5/6} + 2x^{-1} + 2x^{-7/3} + 4x^{1/2} - 2x^{-1} - 12x^{-5/6} + 6x^{-7/3} \\ &= 12x^{1/2} - 4x^{-5/6} + 8x^{-7/3} = 12\sqrt{x} - \frac{4}{\sqrt[6]{x^5}} + \frac{8}{\sqrt[3]{x^7}} \end{aligned}$$

4. $$\begin{aligned} y &= (x^2 - x^{-2})(x^3 + x^{-3}) \\ \frac{dy}{dx} &= (x^2 - x^{-2})(3x^2 - 3x^{-4}) + (x^3 + x^{-3})(2x + 2x^{-3}) \\ &= 3x^4 - 3x^{-2} - 3 + 3x^{-6} + 2x^4 + 2 + 2x^{-2} + 2x^{-6} \\ &= 5x^4 - x^{-2} - 1 + 5x^{-6} = 5x^4 - 1 - \frac{1}{x^2} + \frac{5}{x^6} \end{aligned}$$

5. $\dfrac{dy}{dx} = \dfrac{(x^2 + 1)(0) - (10)(2x)}{(x^2 + 1)^2} = -\dfrac{20x}{(x^2 + 1)^2}$

6. $\dfrac{dy}{dx} = \dfrac{(4x - 3)(0) - (5)(4)}{(4x - 3)^2} = -\dfrac{20}{(4x - 3)^2}$

7. $\dfrac{dy}{dx} = \dfrac{(2x - 5)(3) - (3x + 1)(2)}{(2x - 5)^2} = -\dfrac{17}{(2x - 5)^2}$

8. $\dfrac{dy}{dx} = \dfrac{(7 - x)(-3) - (2 - 3x)(-1)}{(7 - x)^2} = -\dfrac{19}{(7 - x)^2}$

9. $y = (6x - 1)(6x - 1)$; $\quad \dfrac{dy}{dx} = (6x - 1)(6) + (6x - 1)(6) = 12(6x - 1) = 72x - 12$

10. $y = (x^4 + 5x)(x^4 + 5x)$; $\quad \dfrac{dy}{dx} = (x^4 + 5x)(4x^3 + 5) + (x^4 + 5x)(4x^3 + 5) = 8x^7 + 50x^4 + 50x$

11. $f(x) = (x^{-1} - 4x^{-3})(x^3 - 5x - 1)$

$$f'(x) = (x^{-1} - 4x^{-3})(3x^2 - 5) + (x^3 - 5x - 1)(-x^{-2} + 12x^{-4})$$
$$= 2x - 40x^{-3} + x^{-2} - 12x^{-4} = 2x + \frac{1}{x^2} - \frac{40}{x^3} - \frac{12}{x^4}$$

12. $f(x) = (x^2 - 1)(x^2 - 10x + 2x^{-2})$

$$f'(x) = (x^2 - 1)(2x - 10 - 4x^{-3}) + (x^2 - 10x + 2x^{-2})(2x)$$
$$= 2x^3 - 10x^2 - 4x^{-1} - 2x + 10 + 4x^{-3} + 2x^3 - 20x^2 + 4x^{-1}$$
$$= 4x^3 - 30x^2 - 2x + 10 + 4x^{-3} = 4x^3 - 30x^2 - 2x + 10 + \frac{4}{x^3}$$

13. $f'(x) = \dfrac{(2x^2 + x + 1)(2x) - (x^2)(4x + 1)}{(2x^2 + x + 1)^2} = \dfrac{x^2 + 2x}{(2x^2 + x + 1)^2}$

14. $f(x) = \dfrac{x^2 - 10x + 2}{x^3 - x}$

$$f'(x) = \frac{(x^3 - x)(2x - 10) - (x^2 - 10x + 2)(3x^2 - 1)}{(x^3 - x)^2} = \frac{-x^4 + 20x^3 - 7x^2 + 2}{(x^3 - x)^2}$$

15. $f'(x) = [(x + 1)(2x + 1)](3) + (3x + 1)[(x + 1)(2) + (2x + 1)(1)] = 18x^2 + 22x + 6$

16. $f'(x) = [(x^2 + 1)(x^3 - x)](12x^3 + 2) + (3x^4 + 2x - 1)[(x^2 + 1)(3x^2 - 1) + (x^3 - x)(2x)]$

$$= (x^2 + 1)(x^3 - x)(12x^3 + 2) + (3x^4 + 2x - 1)(x^2 + 1)(3x^2 - 1)$$
$$+ (2x)(3x^4 + 2x - 1)(x^3 - x)$$

17. $f'(x) = \dfrac{(3x + 2)[(2x + 1)(1) + (x - 5)(2)] - [(2x + 1)(x - 5)](3)}{(3x + 2)^2} = \dfrac{6x^2 + 8x - 3}{(3x + 2)^2}$

18. $f'(x) = \dfrac{[(x^2 + 1)(x^3 + 4)](5x^4) - (x^5)[(x^2 + 1)(3x^2) + (x^3 + 4)(2x)]}{[(x^2 + 1)(x^3 + 4)]^2} = \dfrac{2x^7 + 12x^6 + 20x^4}{(x^2 + 1)^2(x^3 + 4)^2}$

19. $f'(x) = (x^2 - 2x - 1)\left[\dfrac{(x + 3)(1) - (x + 1)(1)}{(x + 3)^2}\right] + \left(\dfrac{x + 1}{x + 3}\right)(2x - 2)$

$$= \frac{2x^2 - 4x - 2}{(x + 3)^2} + \frac{2x^2 - 2}{x + 3} = \frac{2x^3 + 8x^2 - 6x - 8}{(x + 3)^2}$$

20. $f(x) = (x + 1)\left[(x + 1) - \dfrac{1}{x + 2}\right] = (x + 1)^2 - \dfrac{x + 1}{x + 2} = x^2 + 2x + 1 - \dfrac{x + 1}{x + 2}$

$$f'(x) = 2x + 2 - \frac{(x + 2)(1) - (x + 1)(1)}{(x + 2)^2} = 2x + 2 - \frac{1}{(x + 2)^2} = \frac{2x^3 + 10x^2 + 16x + 7}{(x + 2)^2}$$

21. $y' = \dfrac{(x - 1)(1) - (x)(1)}{(x - 1)^2} = -\dfrac{1}{(x - 1)^2}$

When $x = 1/2$, the slope of the tangent line is $-1/(1/2 - 1)^2$ or -4. The point of tangency is $(1/2, y(1/2))$ or $(1/2, -1)$. Hence, an equation of the tangent line is $y - (-1) = -4(x - 1/2)$ or $y = -4x + 1$.

22. $y' = \dfrac{(x^2+1)(5)-(5x)(2x)}{(x^2+1)^2} = \dfrac{-5x^2+5}{(x^2+1)^2}$

When $x=2$, the slope of the tangent line is $\dfrac{-5(2)^2+5}{(2^2+1)^2}$ or $-3/5$. The point of tangency is $(2, y(2))$ or $(2,2)$. Hence, an equation of the tangent line is $y-2 = -\dfrac{3}{5}(x-2)$ or $y = -\dfrac{3}{5}x+\dfrac{16}{5}$.

23. $y = (2x^{1/2}+x)(-2x^2+5x-1)$; $y' = (2x^{1/2}+x)(-4x+5)+(-2x^2+5x-1)(x^{-1/2}+1)$

When $x=1$, the slope of the tangent line is $(2+1)(-4+5)+(-2+5-1)(1+1)$ or 7. The point of tangency is $(1, y(1))$ or $(1,6)$. Hence, an equation of the tangent line is $y-6 = 7(x-1)$ or $y = 7x-1$.

24. $y' = (2x^2-4)(3x^2+5)+(x^3+5x+3)(4x)$

When $x=0$, the slope of the tangent line is $(-4)(5)+(3)(0)$ or -20. The point of tangency is $(0, y(0))$ or $(0,-12)$. Hence, an equation of the tangent line is $y+12 = -20(x-0)$ or $y = -20x-12$.

25. $y' = (x^2-4)(2x)+(x^2-6)(2x) = 4x^3-20x$

The tangent line is horizontal when $4x^3-20x = 4x(x^2-5) = 0$, or $x = 0, \pm\sqrt{5}$. Given $y = (x^2-4)(x^2-6)$ we see that for $x=0$, $y=24$, and for $x=\pm\sqrt{5}$, $y=-1$. Thus, the points on the graph are $(0,24)$, $(\sqrt{5},-1)$, and $(-\sqrt{5},-1)$.

26. $y = x(x^2-2x+1) = x^3-2x^2+x$; $y' = 3x^2-4x+1 = (x-1)(3x-1)$

The tangent is horizontal when $(x-1)(3x-1)=0$, or $x = 1, 1/3$. Given $y = x(x-1)^2$ we see that for $x=1$, $y=0$, and for $x=1/3$, $y=4/27$. Thus, the points on the graph are $(1,0)$ and $(1/3, 4/27)$.

27. $y' = \dfrac{(x^4+1)(2x)-(x^2)(4x^3)}{(x^4+1)^2} = \dfrac{2x-2x^5}{(x^4+1)^2}$

The tangent is horizontal when $y'=0$ or $2x-2x^5=0$. Then $2x(1-x^4)=0$ and $x = 0, \pm 1$. Given $y = \dfrac{x^2}{x^4+1}$, we see that for $x=0$, $y=0$, and for $x=\pm 1$, $y=1/2$. Thus, the points on the graph are $(0,0)$, $(1,1/2)$, and $(-1,1/2)$.

28. $y' = \dfrac{(x^2-6x)(0)-(1)(2x-6)}{(x^2-6x)^2} = \dfrac{-2x+6}{(x^2-6x)^2}$

The tangent is horizontal when $y'=0$, so $-2x+6=0$ and $x=3$. Given $y = \dfrac{1}{x^2-6x}$ we see that for $x=3$, $y=-1/9$. Thus, the point on the graph is $(3,-1/9)$.

29. $y' = \dfrac{(x+1)(1)-(x+3)(1)}{(x+1)^2} = \dfrac{-2}{(x+1)^2}$

Since the slope of the tangent line is $-1/8$, $-2/(x+1)^2 = -1/8$, so $x^2+2x-15=0$ and $x = 3, -5$. Given $y = \dfrac{x+3}{x+1}$, we see that for $x=3$, $y=3/2$, and for $x=-5$, $y=1/2$. Thus, the points on the graph are $(3,3/2)$ and $(-5,1/2)$.

30. $y' = (x+1)(2) + (2x+5)(1) = 4x+7$. Since the slope of the tangent line is -3, $4x+7 = -3$ and $x = -5/2$. Given $y = (x+1)(2x+5)$, we see that for $x = -5/2$, $y = 0$. Thus, the point on the graph is $(-5/2, 0)$.

31. $y' = \dfrac{(x+5)(1) - (x+4)(1)}{(x+5)^2} = \dfrac{1}{(x+5)^2}$. Since the tangent line is supposed to be perpendicular to $y = -x$, its slope is 1, so $\dfrac{1}{(x+5)^2} = 1$ and $x = -4, -6$. Given $y = \dfrac{x+4}{x+5}$, we see that for $x = -4$, $y = 0$, and for $x = -6$, $y = 2$. Thus, the points on the graph are $(-4, 0)$ and $(-6, 2)$.

32. $y' = \dfrac{(x+1)(1) - (x)(1)}{(x+1)^2} = \dfrac{1}{(x+1)^2}$. Since the slope of the tangent line is supposed to be parallel to $y = \dfrac{1}{4}x - 1$, its slope is $1/4$, so $\dfrac{1}{(x+1)^2} = \dfrac{1}{4}$ and $x = 1, -3$. Given $y = \dfrac{x}{x+1}$, we see that for $x = 1$, $y = 1/2$, and for $x = -3$, $y = 3/2$. Thus, the points on the graph are $(1, 1/2)$ and $(-3, 3/2)$.

33. $f'(x) = \dfrac{(x^2)(1) - (k+x)(2x)}{(x^2)^2} = \dfrac{-x-2k}{x^3}$. Solving $\dfrac{-2-2k}{2^3} = 5$, we obtain $k = -21$.

34. $f'(x) = \dfrac{(x^2+9)(2x) - (x^2+14)(2x)}{(x^2+9)^2} = \dfrac{-10x}{(x^2+9)^2}$; $\quad f'(1) = -\dfrac{1}{10}$

$g'(x) = (1+x^2)(2) + (1+2x)(2x) = 6x^2 + 2x + 2$; $\quad g'(1) = 10$

Since $f'(1)g'(1) = 1$, the tangent lines are perpendicular.

35. $F'(x) = 2[f(x)g'(x) + g(x)f'(x)]$; $F'(1) = 2[f(1)g'(1) + g(1)f'(1)] = 2[2(2) + 6(-3)] = -28$.

36. $F'(x) = (x^2)[f(x)g'(x) + g(x)f'(x)] + [f(x)g(x)](2x)$

$F'(1) = (1^2)[f(1)g'(1) + g(1)f'(1)] + [f(1)g(1)](2 \cdot 1) = 2(2) + 6(-3) + 2(6)(2) = 10$

37. $F'(x) = \dfrac{2[f(x)g'(x) - g(x)f'(x)]}{3[f(x)]^2}$; $F'(1) = \dfrac{2[f(1)g'(1) - g(1)f'(1)]}{3[f(1)]^2} = \dfrac{2[2(2) - 6(-3)]}{3(2^2)} = \dfrac{11}{3}$.

38. $F'(x) = \dfrac{[x - g(x)][2f'(x)] - [1 + 2f(x)][1 - g'(x)]}{[x - g(x)]^2}$

$$F'(1) = \frac{[1 - g(1)][2f'(1)] - [1 + 2f(1)][1 - g'(1)]}{[1 - g(1)]^2} = \frac{(1-6)[2(-3)] - [1 + 2(2)](1-2)}{(1-6)^2} = \frac{7}{5}$$

39. $F'(x) = [4x^{-1} + f(x)]g'(x) + g(x)[-4x^{-2} + f'(x)]$

$F'(1) = [4(1^{-1}) + f(1)]g'(1) + g(1)[-4(1^{-2}) + f'(1)] = (4+2)(2) + (6)(-4-3) = -30$

40. $F'(x) = \dfrac{g(x)[xf'(x) + f(x)] - [xf(x)]g'(x)}{[g(x)]^2}$

$$F'(1) = \frac{g(1)[f'(1) + f(1)] - f(1)g'(1)}{[g(1)]^2} = \frac{6(-3+2) - 2(2)}{6^2} = -\frac{5}{18}$$

41. $F'(x) = (x^{1/2})f'(x) + f(x)\left(\frac{1}{2}x^{-1/2}\right)$

$$F''(x) = (x^{1/2})f''(x) + f'(x)\left(\frac{1}{2}x^{-1/2}\right) + f(x)\left(-\frac{1}{4}x^{-3/2}\right) + \left(\frac{1}{2}x^{-1/2}\right)f'(x)$$

$$F''(4) = (4^{1/2})f''(4) + f'(4)\left(\frac{1}{2}\cdot 4^{-1/2}\right) + f(4)\left(-\frac{1}{4}\cdot 4^{-3/2}\right) + \left(\frac{1}{2}\cdot 4^{-1/2}\right)f'(4)$$

$$= 2(3) + 2\left(\frac{1}{4}\right) - 16\left(-\frac{1}{32}\right) + \frac{1}{4}(2) = 6 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{15}{2}$$

42. $F'(x) = xf'(x) + f(x) + xg'(x) + g(x)$

$$F''(x) = xf''(x) + f'(x) + f'(x) + xg''(x) + g'(x) + g'(x)$$
$$= 2f'(x) + 2g'(x) + xf''(x) + xg''(x)$$
$$F''(0) = 2f'(0) + 2g'(0) = 2(-1) + 2(6) = 10$$

43. $F'(x) = \dfrac{xf'(x) - f(x)}{x^2} = \dfrac{f'(x)}{x} - \dfrac{f(x)}{x^2}$

$$F''(x) = \frac{xf''(x) - f'(x)}{x^2} - \frac{x^2f'(x) - f(x)(2x)}{x^4} = \frac{f''(x)}{x} - \frac{2f'(x)}{x^2} + \frac{2f(x)}{x^3}$$

44. $F'(x) = x^3f'(x) + f(x)(3x^2) = x^3f'(x) + 3x^2f(x)$

$$F''(x) = [x^3f''(x) + f'(x)(3x^2)] + [3x^2f'(x) + f(x)(6x)] = x^3f''(x) + 6x^2f'(x) + 6xf(x)$$
$$F'''(x) = [x^3f'''(x) + f''(x)(3x^2)] + [6x^2f''(x) + f'(x)(12x)] + [6xf'(x) + f(x)(6)]$$
$$= x^3f'''(x) + 9x^2f''(x) + 18xf'(x) + 6f(x)$$

45. $f'(x) = \dfrac{(x^2-2x)(0) - 5(2x-2)}{(x^2-2x)^2} = \dfrac{10-10x}{x^2(x-2)^2}$

For $f'(x)$ to be positive, we need $10 - 10x > 0$ or $x < 1$, and $x \neq 0, 2$. So $f'(x) > 0$ on $(-\infty, 0) \cup (0, 1)$.

For $f'(x)$ to be negative, we need $10 - 10x < 0$ or $x > 1$, and $x \neq 0, 2$. So $f'(x) < 0$ on $(1, 2) \cup (2, \infty)$.

46. $f'(x) = \dfrac{(x+1)(2x) - (x^2+3)(1)}{(x+1)^2} = \dfrac{x^2+2x-3}{(x+1)^2} = \dfrac{(x+3)(x-1)}{(x+1)^2}$

For $f'(x)$ to be positive, $(x+3)$ and $(x-1)$ must both be positive or both be negative, and $x \neq -1$.

CASE I	$x+3>0$	and	$x-1>0;$	$x>-3$	and	$x>1$
CASE II	$x+3<0$	and	$x-1<0;$	$x<-3$	and	$x<1$

Thus, $x > 1$ or $x < -3$, so $f'(x) > 0$ on $(-\infty, -3) \cup (1, \infty)$.

For $f'(x)$ to be negative, one of $(x+3)$ or $(x-1)$ must be positive and the other must be negative, and $x \neq -1$.

CASE I	$x+3>0$	and	$x-1<0;$	$x>-3$	and	$x<1$
CASE II	$x+3<0$	and	$x-1>0;$	$x<-3$	and	$x>1$

Thus, $-3 < x < 1$, $x \neq -1$, so $f'(x) < 0$ on $(-3,-1) \cup (-1,1)$.

47. $f'(x) = (-2x+6)(4) + (4x+7)(-2) = 10 - 16x$

For $f'(x)$ to be positive, we need $10 - 16x > 0$ or $x < 5/8$, so $f'(x) > 0$ on $(-\infty, 5/8)$.

For $f'(x)$ to be negative, we need $10 - 16x < 0$ or $x > 5/8$, so $f'(x) < 0$ on $(5/8, \infty)$.

48. $f'(x) = (x-2)(8x+8) + (4x^2+8x+4)(1) = 8x^2+8x-16x-16+4x^2+8x+4 = 12x^2-12 = 12(x^2-1)$

For $f'(x)$ to be positive, we need $x^2 - 1 > 0$ or $-1 < x < 1$, so $f'(x) > 0$ on $(-1, 1)$.

For $f'(x)$ to be negative, we need $x^2 - 1 < 0$, which is true when $x < -1$ or $x > 1$. So $f'(x) < 0$ on $(-\infty,-1) \cup (1,\infty)$.

49. $F(r) = km_1m_2r^{-2}$; $\quad F'(r) = -2km_1m_2r^{-3}$; $\quad F'(1/2) = -2km_1m_2(1/2)^{-3} = -16km_1m_2$

50.
$$U(x) = q_1x^{-12} - q_2x^{-6}; \quad U'(x) = -12q_1x^{-13} + 6q_2x^{-7} = \frac{-12q_1 + 6q_2x^6}{x^{13}}$$
$$F(x) = -U'(x) = \frac{12q_1 - 6q_2x^6}{x^{13}}$$
$$F\left(\sqrt[6]{2q_1/q_2}\right) = F([2q_1/q_2]^{1/6}) = \frac{12q_1 - 6q_2[(2q_1/q_2)^{1/6}]^6}{[(2q_1/q_2)^{1/6}]^{13}} = \frac{12q_1 - 12q_1}{(2q_1/q_2)^{13/6}} = 0$$

51. Solving for P and differentiating we obtain

$$P = \frac{RT}{V-b} - \frac{a}{V^2} = \frac{RT}{V-b} - aV^{-2}$$
$$\frac{dP}{dV} = \frac{-RT}{(V-b)^2} - (-2aV^{-3}) = -\frac{RT}{(V-b)^2} + \frac{2a}{V^3}.$$

52. Solving for q and differentiating we obtain

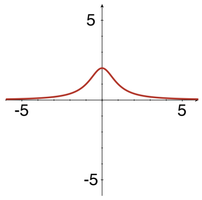
$$\frac{1}{q} = \frac{1}{f} - \frac{1}{p} = \frac{p-f}{fp}$$
$$q = \frac{fp}{p-f}$$
$$\frac{dq}{dp} = \frac{(p-f)(f) - (fp)(1)}{(p-f)^2} = \frac{-f^2}{(p-f)^2} = -\left(\frac{f}{p-f}\right)^2$$

Since we always want f, p, and q to be positive, we must have $p > f$. Now the presence of the negative sign in the derivative of q with respect to p guarantees that $\frac{dq}{dp} < 0$, for all values of $p > f$. Thus, q always decreases as p increases. Since

$$\lim_{p\to\infty} q = \lim_{p\to\infty} \frac{fp}{p-f} = \lim_{p\to\infty} \frac{f}{1 - f/p} = f,$$

we see that image distance q approaches the focal length f as the object distance p increases.

53. (a)



(b) $f'(x) = \dfrac{-2(2x)}{(x^2+1)^2} = -\dfrac{4x}{(x^2+1)^2}$. At $x = a$ the slope of the normal line is $\dfrac{(a^2+1)^2}{4a}$ and the equation of the normal line through $\left(a, \dfrac{2}{a^2+1}\right)$ is $y - \dfrac{2}{a^2+1} = \dfrac{(a^2+1)^2}{4a}(x-a)$. When the line passes through the origin,

$$-\frac{2}{a^2+1} = \frac{(a^2+1)^2}{4a}(-a)$$
$$8 = (a^2+1)^3, \quad 2 - a^2 + 1$$

and $a = \pm 1$. Thus, the points on the graph where the normal line passes through the origin are $(-1, 1)$ and $(1, 1)$. From the graph we see that the y-axis is also a normal line, so that another point is $(0, 2)$.

54. (a) $y = f(x)f(x)$; $\quad \dfrac{dy}{dx} = f(x)f'(x) + f(x)f'(x) = 2f(x)f'(x)$

(b) $y = [f(x)]^2 f(x)$

Using the result from part (a), $\dfrac{dy}{dx} = [f(x)]^2 f'(x) + f(x)[2f(x)f'(x)] = 3[f(x)]^2 f'(x)$.

(c) $\dfrac{dy}{dx}[f(x)]^n = n[f(x)]^{n-1} f'(x)$

(d) $\dfrac{dy}{dx}(x^2+2x-6)^{500} = 500(x^2+2x-6)^{499}(2x+2)$

55. For $y = u(x)y_1(x)$, $y' = u(x)y_1'(x) + y_1(x)u'(x)$. Substituting these into the differential equation, we get $u(x)y_1'(x) + u'(x)y_1(x) + P(x)u(x)y_1(x) = f(x)$.

Since $du/dx = f(x)/y_1(x)$, we can substitute $u'(x)$ above to obtain

$$u(x)y_1'(x) + \left[\frac{f(x)}{y_1(x)}\right] y_1(x) + P(x)u(x)y_1(x) = f(x)$$
$$u(x)y_1'(x) + f(x) + P(x)u(x)y_1(x) = f(x)$$
$$u(x)y_1'(x) + P(x)u(x)y_1(x) = u(x)[y_1'(x) + P(x)y_1(x)] = 0$$

Since $y_1(x)$ satisfies $y' + P(x)y = 0$, we have $u(x) \cdot 0 = 0$, and so $y = u(x)y_1(x)$ satisfies $y' + P(x)y = f(x)$.

3.4 Trigonometric Functions

1. $\dfrac{dy}{dx} = 2x + \sin x$

2. $\dfrac{dy}{dx} = 12x^2 + 1 + 5\cos x$

3. $\dfrac{dy}{dx} = 7\cos x - \sec^2 x$

4. $\dfrac{dy}{dx} = -3\sin x + 5\csc^2 x$

5. $\dfrac{dy}{dx} = (x)(\cos x) + (\sin x)(1) = x\cos x + \sin x$

6. $y = (4x^{1/2} - 3x^{1/3})(\cos x);$
$$\begin{aligned}\frac{dy}{dx} &= (4x^{1/2} - 3x^{1/3})(-\sin x) + (\cos x)(2x^{-1/2} - x^{-2/3}) \\ &= -4x^{1/2}\sin x + 3x^{1/3}\sin x + 2x^{-1/2}\cos x - x^{-2/3}\cos x \\ &= -4\sqrt{x}\sin x + 3\sqrt[3]{x}\sin x + \frac{2\cos x}{\sqrt{x}} - \frac{\cos x}{\sqrt[3]{x^2}}\end{aligned}$$

7. $\dfrac{dy}{dx} = (x^3 - 2)(\sec^2 x) + (\tan x)(3x^2) = (x^3 - 2)\sec^2 x + 3x^2\tan x$

8. $\dfrac{dy}{dx} = (\cos x)(-\csc^2 x) + (\cot x)(-\sin x) = -\cos x\csc^2 x - \cos x$

9. $y = x^2\sec x + \sin x\sec x = x^2\sec x + \tan x;\quad \dfrac{dy}{dx} = x^2\sec x\tan x + 2x\sec x + \sec^2 x$

10. $y = \csc x\tan x = \dfrac{1}{\sin x}\cdot\dfrac{\sin x}{\cos x} = \sec x;\quad \dfrac{dy}{dx} = \sec x\tan x$

11. $y = \cos^2 x + \sin^2 x = 1;\quad \dfrac{dy}{dx} = 0$

12. $y = x^3(\cos x - \sin x);$
$$\begin{aligned}\frac{dy}{dx} &= (x^3)(-\sin x - \cos x) + (\cos x - \sin x)(3x^2) \\ &= -x^3\sin x - x^3\cos x + 3x^2\cos x - 3x^2\sin x\end{aligned}$$

13. $f(x) = \sin x;\quad f'(x) = \cos x$

14. $f(x) = 2\tan x\sec x;\quad f'(x) = (2\tan x)(\sec x\tan x) + (\sec x)(2\sec^2 x) = 2\tan^2 x\sec x + 2\sec^3 x$

15. $f'(x) = \dfrac{(x+1)(-\csc^2 x) - (\cot x)(1)}{(x+1)^2} = \dfrac{-(x+1)\csc^2 x - \cot x}{(x+1)^2} = -\dfrac{x\csc^2 x + \csc^2 x + \cot x}{(x+1)^2}$

16. $f'(x) = \dfrac{(1+\cos x)(2x-6) - (x^2-6x)(-\sin x)}{(1+\cos x)^2} = \dfrac{(1+\cos x)(2x-6) + (\sin x)(x^2-6x)}{(1+\cos x)^2}$

17. $f'(x) = \dfrac{(1+2\tan x)(2x) - (x^2)(2\sec^2 x)}{(1+2\tan x)^2} = \dfrac{2x + 4x\tan x - 2x^2\sec^2 x}{(1+2\tan x)^2}$

18. $f'(x) = \dfrac{(x)(\cos x) - (2+\sin x)(1)}{x^2} = \dfrac{x\cos x - \sin x - 2}{x^2}$

19. $f'(x) = \dfrac{(1+\cos x)(\cos x) - (\sin x)(-\sin x)}{(1+\cos x)^2} = \dfrac{\cos x + 1}{(1+\cos x)^2} = \dfrac{1}{1+\cos x}$

20. $$f'(x) = \frac{(1+\sec x)(-\csc x \cot x) - (1+\csc x)(\sec x \tan x)}{(1+\sec x)^2}$$
$$= \frac{-\csc x \cot x - \csc^2 x - \sec x \tan x - \sec^2 x}{(1+\sec x)^2}$$

21. $$f'(x) = (x^4 \sin x)(\sec^2 x) + (\tan x)[(x^4)(\cos x) + (\sin x)(4x^3)][t]$$
$$= x^4 \sin x \sec^2 x + x^4 \sin x + 4x^3 \sin x \tan x$$

22. $$f'(x) = \frac{(x\cos x)(\cos x) - (1+\sin x)[(x)(-\sin x) + (\cos x)(1)]}{(x\cos x)^2}$$
$$= \frac{x\cos^2 x + x\sin x - \cos x + x\sin^2 x - \sin x \cos x}{x^2\cos^2 x} = \frac{x + x\sin x - \cos x - \sin x \cos x}{x^2 \cos^2 x}$$

23. $f'(x) = -\sin x$. When $x = \pi/3$, the slope of the tangent line is $f'(\pi/3) = -\sin \pi/3 = -\sqrt{3}/2$. The point of tangency is $(\pi/3, f(\pi/3))$ or $(\pi/3, 1/2)$. Hence, an equation of the tangent line is $y - \dfrac{1}{2} = -\dfrac{\sqrt{3}}{2}\left(x - \dfrac{\pi}{3}\right)$ or $y = -\dfrac{\sqrt{3}}{2}x + \dfrac{\pi\sqrt{3}+3}{6}$.

24. $f'(x) = \sec^2 x$. When $x = \pi$, the slope of the tangent line is $f'(\pi) = \sec^2 \pi = 1$. The point of tangency is $(\pi, f(\pi))$ or $(\pi, 0)$. Hence, an equation of the tangent line is $y - 0 = 1(x-\pi)$ or $y = x - \pi$.

25. $f'(x) = \sec x \tan x$. When $x = \pi/6$, the slope of the tangent line is $f'(\pi/6) = \sec(\pi/6)\tan(\pi/6) = 2/3$. The point of tangency is $(\pi/6, f(\pi/6))$ or $(\pi/6, 2\sqrt{3}/3)$. Hence, an equation of the tangent line is $y - \dfrac{2\sqrt{3}}{3} = \dfrac{2}{3}(x - \dfrac{\pi}{6})$ or $y = \dfrac{2}{3}x - \dfrac{6\sqrt{3}-\pi}{9}$.

26. $f'(x) = -\csc x \cot x$. When $x = \pi/2$, the slope of the tangent line is $f'(\pi/2) = -\csc(\pi/2)\cot(\pi/2) = 0$. The point of tangency is $(\pi/2, f(\pi/2))$ or $(\pi/2, 1)$. Hence, an equation of the tangent line is $y - 1 = 0(x - \pi/2)$ or $y = 1$.

27. The tangent is horizontal when $f'(x) = 0$. $f'(x) = 1 - 2\sin x = 0$; $\sin x = 1/2$; therefore $x = \pi/6, 5\pi/6$ in $[0, 2\pi]$.

28. The tangent is horizontal when $f'(x) = 0$:
$$f'(x) = \frac{(2-\cos x)(\cos x) - (\sin x)(\sin x)}{(2-\cos x)^2} = \frac{2\cos x - \cos^2 x - \sin^2 x}{(2-\cos x)^2} = \frac{2\cos x - 1}{(2-\cos x)^2} = 0$$
Therefore $2\cos x = 1$; $\cos x = \dfrac{1}{2}$, and $x = \pi/3, 5\pi/3$ in $[0, 2\pi]$.

29. The tangent is horizontal when $f'(x) = 0$:
$$f'(x) = \frac{(x+\cos x)(0) - (1)(1-\sin x)}{(x+\cos x)^2} = \frac{\sin x - 1}{(x+\cos x)^2} = 0$$
Therefore $\sin x = 1$, and $x = \pi/2$ in $[0, 2\pi]$.

30. The tangent is horizontal when $f'(x) = 0$. $f'(x) = \cos x - \sin x = 0$; $\sin x = \cos x$; $\tan x = 1$; therefore $x = \pi/4, 5\pi/4$ in $[0, 2\pi]$.

31. $f'(x) = \cos x$. When $x = 4\pi/3$, the slope of the tangent line is $f'(4\pi/3) = \cos 4\pi/3 = -1/2$. Thus, the slope of the normal line at $(4\pi/3, f(4\pi/3))$ or $(4\pi/3, -\sqrt{3}/2)$ is 2. The equation of the normal line is $y + \dfrac{\sqrt{3}}{2} = 2\left(x - \dfrac{4\pi}{3}\right)$ or $y = 2x - \dfrac{16\pi + 3\sqrt{3}}{6}$.

32. $f'(x) = 2\tan x \sec^2 x$. When $x = \pi/4$, the slope of the tangent line is $f'(\pi/4) = 2\tan(\pi/4)$ $\sec^2(\pi/4) = 4$. Thus, the slope of the normal line at $(\pi/4, f(\pi/4))$ or $(\pi/4, 1)$ is $-1/4$. The equation of the normal line is $y - 1 = -\dfrac{1}{4}\left(x - \dfrac{\pi}{4}\right)$ or $y = -\dfrac{1}{4}x + \dfrac{\pi + 16}{16}$.

33. $f'(x) = \cos x - x\sin x$. When $x = \pi$, the slope of the tangent line is $f'(\pi) = \cos\pi - \pi\sin\pi = -1$. Thus, the slope of the normal line at $(\pi, f(\pi))$ or $(\pi, -\pi)$ is 1. The equation of the normal line is $y + \pi = 1(x - \pi)$ or $y = x - 2\pi$.

34. $f'(x) = \dfrac{1 + \sin x - x\cos x}{(1 + \sin x)^2}$. When $x = \pi/2$, the slope of the tangent line is $f'(\pi/2) = \dfrac{1 + \sin(\pi/2) - (\pi/2)\cos(\pi/2)}{[1 + \sin(\pi/2)]^2} = \dfrac{1}{2}$. Thus, the slope of the normal line at $(\pi/2, f(\pi/2))$ or $(\pi/2, \pi/4)$ is -2. The equation of the normal line is $y - \pi/4 = -2(x - \pi/2)$ or $y = -2x + 5\pi/4$.

35. $f(x) = \sin 2x = 2\sin x\cos x$

$$f'(x) = 2[(\sin x)(-\sin x) + (\cos x)(\cos x)] = 2(\cos^2 x - \sin^2 x) = 2\cos 2x$$

36. $f(x) = \cos^2 \dfrac{x}{2} = \dfrac{1 + \cos x}{2} = \dfrac{1}{2} + \dfrac{1}{2}\cos x$; $f'(x) = -\dfrac{1}{2}\sin x$

37. $f'(x) = x\cos x + \sin x$; $f''(x) = [x(-\sin x) + \cos x] + \cos x = -x\sin x + 2\cos x$

38. $f'(x) = 3 - [(x^2)(-\sin x) + (\cos x)(2x)] = 3 + x^2\sin x - 2x\cos x$

$$f''(x) = [(x^2)(\cos x) + (\sin x)(2x)] - [(2x)(-\sin x) + (\cos x)(2)] = x^2\cos x + 4x\sin x - 2\cos x$$

39. $f'(x) = \dfrac{(x)(\cos x) - (\sin x)(1)}{x^2} = \dfrac{\cos x}{x} - \dfrac{\sin x}{x^2}$

$$f''(x) = \frac{(x)(-\sin x) - (\cos x)(1)}{x^2} - \frac{(x^2)(\cos x) - (\sin x)(2x)}{x^4} = -\frac{\sin x}{x} - \frac{2\cos x}{x^2} + \frac{2\sin x}{x^3}$$

40. $f'(x) = \dfrac{(1 + \cos x)(0) - (1)(-\sin x)}{(1 + \cos x)^2} = \dfrac{\sin x}{(1 + \cos x)^2}$

$$f''(x) = \frac{(1 + \cos x)^2(\cos x) - (\sin x)[(1 + \cos x)(-\sin x) + (1 + \cos x)(-\sin x)]}{(1 + \cos x)^4}$$

$$= \frac{(\cos x)(1 + \cos x)^2 + 2(\sin^2 x)(1 + \cos x)}{(1 + \cos x)^4} = \frac{\cos x + \cos^2 x + 2\sin^2 x}{(1 + \cos x)^3}$$

41. $f'(x) = -\csc x\cot x$

$$f''(x) = (-\csc x)(-\csc^2 x) + (\cot x)(\csc x\cot x) = \csc^3 x + \csc x\cot^2 x$$

42. $f'(x) = \sec^2 x = (\sec x)(\sec x)$
$f''(x) = (\sec x)(\sec x \tan x) + (\sec x)(\sec x \tan x) = 2\sec^2 x \tan x$

43. $y' = C_1(-\sin x) + C_2 \cos x - \frac{1}{2}[(x)(-\sin x) + (\cos x)(1)]$
$= -C_1 \sin x + C_2 \cos x + \frac{1}{2}x \sin x - \frac{1}{2}\cos x$
$y'' = -C_1 \cos x + C_2(-\sin x) + \frac{1}{2}[(x)(\cos x) + (\sin x)(1)] - \frac{1}{2}(-\sin x)$
$= -C_1 \cos x - C_2 \sin x + \frac{1}{2}x \cos x + \sin x$
Substituting into the differential equation,

$$(-C_1 \cos x - C_2 \sin x + \frac{1}{2}x \cos x + \sin x) + (C_1 \cos x + C_2 \sin x - \frac{1}{2}x \cos x) = \sin x.$$

44. $y = C_1 x^{-1/2} \cos x + C_2 x^{-1/2} \sin x$

$$y' = C_1 \left[(x^{-1/2})(-\sin x) + (\cos x)\left(-\frac{x^{-3/2}}{2}\right)\right] + C_2 \left[(x^{-1/2})(\cos x) + (\sin x)\left(-\frac{x^{-3/2}}{2}\right)\right]$$
$$= -C_1 \left(x^{-1/2} \sin x + \frac{1}{2}x^{-3/2} \cos x\right) + C_2 \left(x^{-1/2} \cos x - \frac{1}{2}x^{-3/2} \sin x\right)$$
$$= x^{-1/2}(C_2 \cos x - C_1 \sin x) - \frac{1}{2}x^{-3/2}(C_1 \cos x + C_2 \sin x)$$
$$y'' = \left[(x^{-1/2})(-C_2 \sin x - C_1 \cos x) + (C_2 \cos x - C_1 \sin x)\left(-\frac{1}{2}x^{-3/2}\right)\right]$$
$$- \frac{1}{2}\left[(x^{-3/2})(-C_1 \sin x + C_2 \cos x) + (C_1 \cos x + C_2 \sin x)\left(-\frac{3}{2}x^{-5/2}\right)\right]$$
$$= -C_2 x^{-1/2} \sin x - C_1 x^{-1/2} \cos x - \frac{1}{2}C_2 x^{-3/2} \cos x + \frac{1}{2}C_1 x^{-3/2} \sin x$$
$$+ \frac{1}{2}C_1 x^{-3/2} \sin x - \frac{1}{2}C_2 x^{-3/2} \cos x + \frac{3}{4}C_1 x^{-5/2} \cos x + \frac{3}{4}C_2 x^{-5/2} \sin x$$
$$= -C_2 x^{-1/2} \sin x - C_1 x^{-1/2} \cos x - C_2 x^{-3/2} \cos x + C_1 x^{-3/2} \sin x + \frac{3}{4}C_1 x^{-5/2} \cos x$$
$$+ \frac{3}{4}C_2 x^{-5/2} \sin x$$

Substituting into the differential equation,

$$x^2 \left(-C_2 x^{-1/2} \sin x - C_1 x^{-1/2} \cos x - C_2 x^{-3/2} \cos x + C_1 x^{-3/2} \sin x + \frac{3}{4}C_1 x^{-5/2} \cos x\right.$$
$$\left.+\frac{3}{4}C_2 x^{-5/2} \sin x\right) + x\left[x^{-1/2}(C_2 \cos x - C_1 \sin x) - \frac{1}{2}x^{-3/2}(C_1 \cos x + C_2 \sin x)\right]$$
$$+ \left(x^2 - \frac{1}{4}\right)(C_1 x^{-1/2} \cos x + C_2 x^{-1/2} \sin x)$$

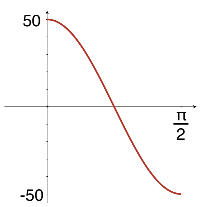
$$\begin{aligned}
&= -C_2x^{3/2}\sin x - C_1x^{3/2}\cos x - C_2x^{1/2}\cos x + C_1x^{1/2}\sin x + \frac{3}{4}C_1x^{-1/2}\cos x \\
&\quad + \frac{3}{4}C_2x^{-1/2}\sin x + C_2x^{1/2}\cos x - C_1x^{1/2}\sin x - \frac{1}{2}C_1x^{-1/2}\cos x - \frac{1}{2}C_2x^{-1/2}\sin x \\
&\quad + C_1x^{3/2}\cos x + C_2x^{3/2}\sin x - \frac{1}{4}C_1x^{-1/2}\cos x - \frac{1}{4}C_2x^{-1/2}\sin x \\
&= (-C_2x^{3/2}\sin x + C_2x^{3/2}\sin x) + (-C_1x^{3/2}\cos x + C_1x^{3/2}\cos x) \\
&\quad + (-C_2x^{1/2}\cos x + C_2x^{1/2}\cos x) + (C_1x^{1/2}\sin x - C_1x^{1/2}\sin x) \\
&\quad + \left(\frac{3}{4}C_1x^{-1/2}\cos x - \frac{1}{2}C_1x^{-1/2}\cos x - \frac{1}{4}C_1x^{-1/2}\cos x\right) \\
&\quad + \left(\frac{3}{4}C_2x^{-1/2}\sin x - \frac{1}{2}C_2x^{-1/2}\sin x - \frac{1}{4}C_2x^{-1/2}\sin x\right) = 0.
\end{aligned}$$

45. From $s = 40\cot\theta$ we obtain $\dfrac{ds}{d\theta} = -40\csc^2\theta$. When $\theta = \pi/3$ radians $\left.\dfrac{ds}{d\theta}\right|_{\theta=\pi/3} = -40\csc^2\dfrac{\pi}{3} = -40\left(\dfrac{2}{\sqrt{3}}\right)^2 = -\dfrac{160}{3}$ ft. The rate of change is negative because the length of the shadow decreases as θ increases.

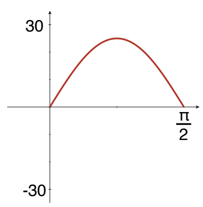
46. (a) From $\overline{PQ} = 10\sin\theta$ and $\overline{QR} = 10\cos\theta$ we find $A(\theta) = \dfrac{1}{2}(10\sin\theta)(10\cos\theta) = 50\sin\theta\cos\theta$.

(b) $A'(\theta) = (50\sin\theta)(-\sin\theta) + (\cos\theta)(50\cos\theta) = 50(\cos^2\theta - \sin^2\theta) = 50\cos 2\theta$

(c)



$\dfrac{dA}{d\theta} = 50\cos 2\theta$



$A = 50\sin\theta\cos\theta$

A is increasing while $A' > 0$ (on $[0, \pi/4]$) and decreasing while $A' < 0$ (on $[\pi/4, \pi/2]$).

(d) From the second graph in part (c) we see that the area of the graph is greatest when $\theta = \pi/4$.

47. (a) We observe by successive differentiation that since:

$$\sin x \to \cos x \to -\sin x \to -\cos x \to \sin x \to \cos x \to \ldots$$

Both $\dfrac{d^n}{dx^n}\sin x = \sin x$ and $\dfrac{d^n}{dx^n}\cos x = \cos x$ when n is a multiple of 4;
$\dfrac{d^n}{dx^n}\cos x = \sin x$ for $n = 4k+3$, k some integer ≥ 0;
$\dfrac{d^n}{dx^n}\sin x = \cos x$ for $n = 4k+1$, k some integer ≥ 0.

(b) Based on part (a), $\dfrac{d^{21}}{dx^{21}}\sin x = \dfrac{d}{dx}\left(\dfrac{d^{20}}{dx^{20}}\sin x\right) = \dfrac{d}{dx}\sin x = \cos x,$

$$\frac{d^{30}}{dx^{30}}\sin x = \frac{d^2}{dx^2}\left(\frac{d^{28}}{dx^{28}}\sin x\right) = \frac{d^2}{dx^2}\sin x = -\sin x,$$

$$\frac{d^{40}}{dx^{40}}\cos x = \cos x, \text{ and}$$

$$\frac{d^{67}}{dx^{67}}\cos x = \frac{d^3}{dx^3}\left(\frac{d^{64}}{dx^{64}}\cos x\right) = \frac{d^3}{dx^3}\cos x = \sin x.$$

48. Since $y' = -\sin x$, the slope of the tangent line at $P_1 = (x_1, \cos x_1)$ is $m_1 = -\sin x_1$ while the slope of the tangent line at $P_2 = (x_2, \cos x_2)$ is $m_2 = -\sin x_2$. Since we are looking for points such that the tangents at those points are perpendicular, we are looking for x_1, x_2 such that $\sin x_1 \sin x_2 = -1$. This is true for any $x_1 = \pi/2 + 2\pi n_1$ and $x_2 = -\pi/2 + 2\pi n_2$, n_1 and n_2 being integers. For example, $\sin(\pi/2)\sin(-\pi/2) = -1$, so the tangents at $(\pi/2, 0)$ and $(-\pi/2, 0)$ are perpendicular.

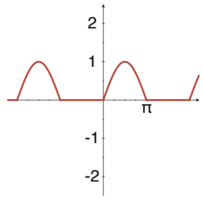
49. Since $y' = \cos x$, the slope of the tangent line at $P_1 = (x_1, \sin x_1)$ is $m_1 = \cos x_1$ while the slope of the tangent line at $P_2 = (x_2, \sin x_2)$ is $m_2 = \cos x_2$. Since we are looking for points such that the tangents at those points are parallel, we are looking for x_1, x_2 such that $\cos x_1 = \cos x_2$. This is true for any $x_1 = x_2 + 2\pi n$, n an integer, as well as any $x_1 = -x_2$. For example, the tangent lines at $(0, 0)$ and $(2\pi, 0)$ are parallel, as are $(\pi/2, 1)$ and $(-\pi/2, -1)$.

50. $$\begin{aligned}\frac{d}{dx}\cos x &= \lim_{\Delta x \to 0}\frac{\cos(x+\Delta x) - \cos x}{\Delta x} = \lim_{\Delta x \to 0}\frac{\cos x \cos \Delta x - \sin x \sin \Delta x - \cos x}{\Delta x}\\ &= \lim_{\Delta x \to 0}\frac{\cos x(\cos \Delta x - 1) - \sin x \sin \Delta x}{\Delta x}\\ &= (\cos x)\left(\lim_{\Delta x \to 0}\frac{\cos \Delta x - 1}{\Delta x}\right) - (\sin x)\left(\lim_{\Delta x \to 0}\frac{\sin \Delta x}{\Delta x}\right)\\ &= (\cos x)(0) - (\sin x)(1) = -\sin x\end{aligned}$$

51. $$\begin{aligned}\frac{d}{dx}\cot x &= \frac{d}{dx}\frac{\cos x}{\sin x} = \frac{(\sin x)\left(\frac{d}{dx}\cos x\right) - (\cos x)\left(\frac{d}{dx}\sin x\right)}{(\sin x)^2}\\ &= \frac{(\sin x)(-\sin x) - (\cos x)(\cos x)}{\sin^2 x} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}\\ &= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x\end{aligned}$$

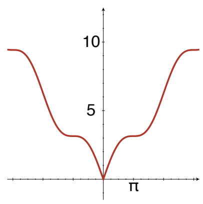
52. $$\begin{aligned}\frac{d}{dx}\csc x &= \frac{d}{dx}\frac{1}{\sin x} = \frac{(\sin x)(0) - (1)\left(\frac{d}{dx}\sin x\right)}{(\sin x)^2} = \frac{-(1)(\cos x)}{\sin^2 x} = -\frac{\cos x}{\sin^2 x}\\ &= -\frac{1}{\sin x}\cdot\frac{\cos x}{\sin x} = -\csc x \cot x\end{aligned}$$

53.



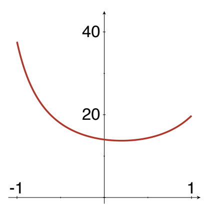
Not differentiable at $x = k\pi$ for k an integer.

54.



Not differentiable at $x = 0$.

55. (a)



(b) $\dfrac{dF}{d\theta} = \dfrac{-70(0.2)(0.2\cos\theta - \sin\theta)}{(0.2\sin\theta + \cos\theta)^2} = \dfrac{-2.8\cos\theta + 14\sin\theta}{(0.2\sin\theta + \cos\theta)^2}$

(c) Solving $-2.8\cos\theta + 14\sin\theta = 0$ we obtain $\tan\theta = 0.2$ and $\theta \approx 0.1974$ radians.

(d) $F(0.1974) \approx 13.7281$

(e) The minimum force required to pull the sled is about 13.73 pounds when θ is about 0.1974 radians or $11.3°$.

3.5 Chain Rule

1. $\dfrac{dy}{dx} = 30(-5x)^{29}(-5) = 150(5x)^{29}$

2. $y = (3x^{-1})^{14}$; $\quad \dfrac{dy}{dx} = 14(3x^{-1})^{13}(-3x^{-2}) = -\dfrac{42}{x^2}\left(\dfrac{3}{x}\right)^{13}$

3. $\dfrac{dy}{dx} = 200(2x^2 + x)^{199}(4x + 1)$

4. $y = (x - x^{-2})^5$; $\quad \dfrac{dy}{dx} = 5(x - x^{-2})^4(1 + 2x^{-3}) = \left(5 + \dfrac{10}{x^3}\right)\left(x - \dfrac{1}{x^2}\right)^4$

5. $y = (x^3 - 2x^2 + 7)^{-4}; \quad \dfrac{dy}{dx} = -4(x^3 - 2x^2 + 7)^{-5}(3x^2 - 4x)$

6. $y = 10(x^2 - 4x + 1)^{-1/2}; \quad \dfrac{dy}{dx} = -5(x^2 - 4x + 1)^{-3/2}(2x - 4)$

7. $$\begin{aligned}\frac{dy}{dx} &= (3x-1)^4[5(-2x+9)^4(-2)] + (-2x+9)^5[4(3x-1)^3(3)]\\ &= -10(3x-1)^4(-2x+9)^4 + 12(-2x+9)^5(3x-1)^3\end{aligned}$$

8. $\dfrac{dy}{dx} = x^4[6(x^2+1)^5(2x)] + (x^2+1)^6(4x^3) = 12x^5(x^2+1)^5 + 4x^3(x^2+1)^6$

9. $y = \sin(\sqrt{2}x^{1/2}); \quad \dfrac{dy}{dx} = (\cos\sqrt{2x})[(\sqrt{2}/2)x^{-1/2}] = \dfrac{\sqrt{2}}{2\sqrt{x}}\cos\sqrt{2x} = \dfrac{\cos\sqrt{2x}}{\sqrt{2x}}$

10. $\dfrac{dy}{dx} = (\sec x^2 \tan x^2)(2x) = 2x\sec x^2 \tan x^2$

11. $$y = \left(\frac{x^2-1}{x^2+1}\right)^{1/2}$$
$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{x^2-1}{x^2+1}\right)^{-1/2}\left[\frac{(x^2+1)(2x)-(x^2-1)(2x)}{(x^2+1)^2}\right] = \frac{2x}{(x^2+1)^2}\sqrt{\frac{x^2+1}{x^2-1}}$$

12. $\dfrac{dy}{dx} = \dfrac{(5x+2)^3(3) - (3x-4)[3(5x+2)^2(5)]}{(5x+2)^6} = \dfrac{3(5x+2) - 15(3x-4)}{(5x+2)^4} = \dfrac{-30x+66}{(5x+2)^4}$

13. $\dfrac{dy}{dx} = 10[x + (x^2-4)^3]^9[1 + 3(x^2-4)^2(2x)] = 10[x + (x^2-4)^3]^9[1 + 6x(x^2-4)^2]$

14. $y = (x^3 - x + 1)^{-8}; \quad \dfrac{dy}{dx} = -8(x^3 - x + 1)^{-9}(3x^2 - 1) = \dfrac{-24x^2 + 8}{(x^3 - x + 1)^9}$

15. $$\begin{aligned}\frac{dy}{dx} &= x[-4(x^{-1}+x^{-2}+x^{-3})^{-5}(-x^{-2}-2x^{-3}-4x^{-4})] + (x^{-1}+x^{-2}+x^{-3})^{-4}(1)\\ &= (x^{-1}+x^{-2}+x^{-3})^{-4} + 4x(x^{-1}+x^{-2}+x^{-3})^{-5}(x^{-2}+2x^{-3}+4x^{-4})\\ &= \frac{(x^{-1}+x^{-2}+x^{-3}) + 4x^{-1}+8x^{-2}+16x^{-3}}{(x^{-1}+x^{-2}+x^{-3})^5} = \frac{5x^{-1}+9x^{-2}+17x^{-3}}{(x^{-1}+x^{-2}+x^{-3})^5}\end{aligned}$$

16. $$\begin{aligned}y &= (2x+1)^3(3x^2-2x)^{1/2}\\ \frac{dy}{dx} &= (2x+1)^3\left[\frac{1}{2}(3x^2-2x)^{-1/2}(6x-2)\right] + (3x^2-2x)^{1/2}[3(2x+1)^2(2)]\\ &= \frac{(2x+1)^3(3x-1)}{\sqrt{3x^2-2x}} + 6(2x+1)^2\sqrt{3x^2-2x}\end{aligned}$$

17. $\dfrac{dy}{dx} = [\cos(\pi x + 1)](\pi) = \pi\cos(\pi x + 1)$

18. $\dfrac{dy}{dx} = -2\left([-\sin(-3x+7)][-3]\right) = -6\sin(-3x+7)$

19. $\dfrac{dy}{dx} = (3\sin^2 5x)(\cos 5x)(5) = 15\sin^2 5x \cos 5x$

20. $y = 4\cos^2 x^{1/2}; \quad \dfrac{dy}{dx} = (8\cos\sqrt{x})(-\sin\sqrt{x})\left(\dfrac{1}{2}x^{-1/2}\right) = -\dfrac{4\cos\sqrt{x}\sin\sqrt{x}}{\sqrt{x}}$

21. $f'(x) = (x^3)(-3x^2\sin x^3) + (\cos x^3)(3x^2) = -3x^5\sin x^3 + 3x^2\cos x^3$

22. $f'(x) = \dfrac{(\cos 6x)(\cos 5x)(5) - (\sin 5x)(-\sin 6x)(6)}{\cos^2 6x} = \dfrac{5\cos 5x\cos 6x + 6\sin 5x\sin 6x}{\cos^2 6x}$

23. $f'(x) = 10(2 + x\sin 3x)^9[(x)(\cos 3x)(3) + (\sin 3x)(1)] = 10(2 + x\sin 3x)^9(3x\cos 3x + \sin 3x)$

24. $$\begin{aligned} f'(x) &= \frac{(1+\sin 5x)^3[2(1-\cos 4x)(\sin 4x)(4)] - (1-\cos 4x)^2[3(1+\sin 5x)^2(\cos 5x)(5)]}{(1+\sin 5x)^6} \\ &= \frac{8(1+\sin 5x)^3(1-\cos 4x)(\sin 4x) - 15(1-\cos 4x)^2(1+\sin 5x)^2(\cos 5x)}{(1+\sin 5x)^6} \\ &= \frac{8(1-\cos 4x)(\sin 4x)}{(1+\sin 5x)^3} - \frac{15(1-\cos 4x)^2(\cos 5x)}{(1+\sin 5x)^4} \end{aligned}$$

25. $f(x) = \tan x^{-1}; \quad f'(x) = (\sec^2 x^{-1})(-x^{-2}) = -\dfrac{1}{x^2}\sec^2\dfrac{1}{x}$

26. $f(x) = x\cot 5x^{-2}; \quad f'(x) = (x)(-\csc^2 5x^{-2})(-10x^{-3}) + (\cot 5x^{-2})(1) = \dfrac{10}{x^2}\csc^2\dfrac{5}{x^2} + \cot\dfrac{5}{x^2}$

27. $f'(x) = (\sin 2x)(-\sin 3x)(3) + (\cos 3x)(\cos 2x)(2) = 2\cos 2x\cos 3x - 3\sin 2x\sin 3x$

28. $$\begin{aligned} f'(x) &= (\sin^2 2x)(3\cos^2 3x)(-\sin 3x)(3) + (\cos^3 3x)(2\sin 2x)(\cos 2x)(2) \\ &= -9\sin^2 2x\cos^2 3x\sin 3x + 4\cos^3 3x\sin 2x\cos 2x \end{aligned}$$

29. $f'(x) = 5(\sec 4x + \tan 2x)^4(4\sec 4x\tan 4x + 2\sec^2 2x)$

30. $$\begin{aligned} f'(x) &= 2(\csc 2x)(-\csc 2x\cot 2x)(2) - (-\csc 2x^2\cot 2x^2)(4x) \\ &= -4\csc^2 2x\cot 2x + 4x\csc 2x^2\cot 2x^2 \end{aligned}$$

31. $f'(x) = [\cos(\sin 2x)](\cos 2x)(2) = 2\cos 2x\cos(\sin 2x)$

32. $f'(x) = \left[\sec^2\left(\cos\dfrac{x}{2}\right)\right]\left(-\sin\dfrac{x}{2}\right)\left(\dfrac{1}{2}\right) = -\dfrac{1}{2}\sin\dfrac{x}{2}\sec^2\left(\cos\dfrac{x}{2}\right)$

33. $$\begin{aligned} f(x) = \cos[\sin(2x+5)^{1/2}]; \quad f'(x) &= [-\sin(\sin\sqrt{2x+5})](\cos\sqrt{2x+5})\left[\frac{1}{2}(2x+5)^{-1/2}\right](2) \\ &= -\frac{\sin(\sin\sqrt{2x+5})\cos\sqrt{2x+5}}{\sqrt{2x+5}} \end{aligned}$$

34. $f'(x) = [\sec^2(\tan x)](\sec^2 x)$

35. $f'(x) = [3\sin^2(4x^2-1)][\cos(4x^2-1)](8x) = 24x\sin^2(4x^2-1)\cos(4x^2-1)$

36. $f'(x) = [\sec(\tan^2 x^4)\tan(\tan^2 x^4)][2\tan x^4](\sec^2 x^4)(4x^3)$
$= 8x^3[\sec(\tan^2 x^4)][\tan(\tan^2 x^4)](\tan x^4)(\sec^2 x^4)$

37. $f'(x) = 6\left(1 + \{1 + [1 + (1 + x^3)^4]^5\right)^5 \{5[1 + (1 + x^3)^4]^4\}[4(1 + x^3)^3](3x^2)$
$= 360\left\{1 + [1 + (1 + x^3)^4]^5\right\}^5 [1 + (1 + x^3)^4]^4(1 + x^3)^3x^2$

38. $f(x) = [x^2 - (1 + x^{-1})^{-4}]^2;\quad f'(x) = 2[x^2 - (1 + x^{-1})^{-4}][2x + 4(1 + x^{-1})^{-5}(-x^{-2})]$
$= 2\left[x^2 - \left(1 + \frac{1}{x}\right)^{-4}\right]\left[2x - \frac{4}{x^2}\left(1 + \frac{1}{x}\right)^{-5}\right]$

39. $y' = [3(x^2 + 2)^2](2x) = 6x(x^2 + 2)^2$. When $x = -1$, the slope of the tangent line is $y'(-1) = -6(1 + 2)^2 = -54$.

40. $y = (3x+1)^{-2}$; $y' = [-2(3x+1)^{-3}](3) = -6(3x+1)^{-3}$. When $x = 0$, the slope of the tangent line is $y'(0) = -6(0 + 1)^{-3} = -6$.

41. $y' = 3\cos 3x - 20x\sin 5x + 4\cos 5x$. When $x = \pi$, the slope of the tangent line is $y'(\pi) = 3\cos 3\pi - 20\pi\sin 5\pi + 4\cos 5\pi = -7$.

42. $y' = 50 - 3(\tan^2 2x)(\sec^2 2x)(2) = 50 - 6\tan^2 2x\sec^2 2x$. When $x = \pi/6$, the slope of the tangent line is $y'(\pi/6) = 50 - 6(\sqrt{3})^2(2^2) = -22$.

43. $y' = 2\left(\frac{x}{x+1}\right)\left[\frac{(x+1)(1) - (x)(1)}{(x+1)^2}\right] = \frac{2x}{(x+1)^3}$. When $x = -1/2$, $y = \left(\frac{-1/2}{-1/2+1}\right)^2 = 1$ and $y' = \frac{2(-1/2)}{(-1/2+1)^3} = -8$. Thus, an equation of the tangent line is $y - 1 = -8(x + 1/2)$ or $y = -8x - 3$.

44. $y' = (x^2)[3(x-1)^2](1) + (x-1)^3(2x) = 3x^2(x-1)^2 + 2x(x-1)^3$. When $x = 2$, $y = 2^2(2-1)^3 = 4$ and $y' = 3(2^2)(2 - 1)^2 + 2(2)(2 - 1)^3 = 16$. Thus, an equation of the tangent line is $y - 4 = 16(x - 2)$ or $y = 16x - 28$.

45. $y' = 3\sec^2 3x$. When $x = \pi/4$, $y = \tan 3\pi/4 = -1$ and $y' = 3\sec^2 3\pi/4 = 6$. Thus, an equation of the tangent line is $y + 1 = 6\left(x - \frac{\pi}{4}\right)$ or $y = 6x - \frac{3\pi + 2}{2}$.

46. $y' = [3(-1 + \cos 4x)^2](-\sin 4x)(4) = -12(-1 + \cos 4x)^2\sin 4x$. When $x = \pi/8$, $y = (-1 + \cos\pi/2)^3 = -1$ and $y' = -12(-1 + \cos\pi/2)^2 = -12$. Thus, an equation of the tangent line is $y + 1 = -12\left(x - \frac{\pi}{8}\right)$ or $y = -12x + \frac{3\pi - 2}{2}$.

47. $y' = \sin\frac{\pi}{6x}(-2\pi x\sin\pi x^2) + \cos\pi x^2\left(-\frac{\pi}{6x^2}\cos\frac{\pi}{6x}\right) = -2\pi x\sin\frac{\pi}{6x}\sin\pi x^2 - \frac{\pi}{6x^2}\cos\pi x^2\cos\frac{\pi}{6x}$.
When $x = \frac{1}{2}$, $y = \frac{\sqrt{6}}{4}$ and $y' = -\frac{3\pi\sqrt{6} + 2\pi\sqrt{2}}{12}$. Thus, an equation of the normal line is $y - \frac{\sqrt{6}}{4} = \frac{12}{3\pi\sqrt{6} + 2\pi\sqrt{2}}\left(x - \frac{1}{2}\right)$ or $y = \frac{12}{3\pi\sqrt{6} + 2\pi\sqrt{2}}x - \frac{6}{3\pi\sqrt{6} + 2\pi\sqrt{2}} + \frac{\sqrt{6}}{4}$.

48. $y' = 3\sin^2\frac{x}{3}\left(\cos\frac{x}{3}\right)\left(\frac{1}{3}\right) = \sin^2\frac{x}{3}\cos\frac{x}{3}$. When $x = \pi$, $y = \frac{3\sqrt{3}}{8}$ and $y' = \frac{3}{8}$. Thus, an equation of the normal line is $y - \frac{3\sqrt{3}}{8} = -\frac{8}{3}(x-\pi)$ or $y = -\frac{8}{3}x - \frac{8\pi}{3} + \frac{3\sqrt{3}}{8}$.

49. $f'(x) = \pi\cos\pi x$; $\quad f''(x) = -\pi^2\sin\pi x$; $\quad f'''(x) = -\pi^3\cos\pi x$

50. $\frac{dy}{dx} = -2\sin(2x+1)$; $\quad \frac{d^2y}{dx^2} = -4\cos(2x+1)$; $\quad \frac{d^3y}{dx^3} = 8\sin(2x+1)$; $\quad \frac{d^4y}{dx^4} = 16\cos(2x+1)$;

$$\frac{d^5y}{dx^5} = -32\sin(2x+1)$$

51. $\frac{dy}{dx} = (x)(\cos 5x)(5) + (\sin 5x)(1) = 5x\cos 5x + \sin 5x$

$\frac{d^2y}{dx^2} = (5x)(-\sin 5x)(5) + (\cos 5x)(5) + (\cos 5x)(5) = -25x\sin 5x + 10\cos 5x$

$\frac{d^3y}{dx^3} = (-25x)(\cos 5x)(5) + (\sin 5x)(-25) + 10(-\sin 5x)(5) = -125x\cos 5x - 75\sin 5x$

52. $f'(x) = -2x\sin x^2$; $\quad f''(x) = (-2x)(2x\cos x^2) + (\sin x^2)(-2) = -4x^2\cos x^2 - 2\sin x^2$

53. $f'(x) = \frac{(x^2+1)^2(1) - (x)[2(x^2+1)(2x)]}{(x^2+1)^4} = \frac{1-3x^2}{(x^2+1)^3}$. To find where the tangent line is horizontal, solve $f'(x) = 0$. This gives $x = \pm\sqrt{\frac{1}{3}} = \pm\frac{\sqrt{3}}{3}$. The tangent line is horizontal at $\left(-\frac{\sqrt{3}}{3}, -\frac{3\sqrt{3}}{16}\right)$ and $\left(\frac{\sqrt{3}}{3}, \frac{3\sqrt{3}}{16}\right)$. Since x^2+1 is never 0, the graph has no vertical tangents.

54. The instantaneous rate of change is $g'(t) = \cos t - \sin 2t$. Setting $g'(t) = 0$ we obtain $\cos t = \sin 2t = 2\sin t\cos t$. Solving for t we have

$$\begin{aligned} 2\sin t\cos t - \cos t &= 0 \\ (2\sin t - 1)\cos t &= 0 \\ \sin t = \frac{1}{2}, \quad \cos t &= 0. \end{aligned}$$

Thus, $t = \pi/6 + 2k\pi$, $5\pi/6 + 2k\pi$, $\pi/2 + k\pi$, where k is an integer.

55. $f'(x) = -\frac{1}{3}\sin\frac{x}{3}$; $f''(x) = -\frac{1}{9}\cos\frac{x}{3}$. The slope of the tangent line to the graph of f' at $x = 2\pi$ is $f''(2\pi) = -\frac{1}{9}\cos\frac{2\pi}{3} = -\frac{1}{9}\left(-\frac{1}{2}\right) = \frac{1}{18}$.

56. $f'(x) = 4(1-x)^3(-1) = -4(1-x)^3$; $f''(x) = -12(1-x)^2(-1) = 12(1-x)^2$; $f'''(x) = 24(1-x)(-1) = 24x - 24$. The slope of the tangent line to the graph of f'' at $x = 2$ is $f'''(2) = 24$.

57. $\dfrac{dR}{d\theta} = 2\left(\dfrac{v_0^2}{g}\right)(\cos 2\theta)(2) = 4\left(\dfrac{v_0^2}{g}\right)\cos 2\theta.$

Setting $\dfrac{dR}{d\theta} = 0$, and assuming that $0 \le \theta \le \pi$, we obtain $\theta = \pi/4, 3\pi/4$.

58. $\dfrac{dV}{dt} = 4\pi r^2 \dfrac{dr}{dt}$. When $\dfrac{dr}{dt} = 5$, $\dfrac{dV}{dt} = 4\pi r^2(5) = 20\pi r^2$ in^3/min.

59. The volume of a sphere is $V = (4\pi/3)r^3$, so that $dV/dt = 4\pi r^2(dr/dt)$. Using $dV/dt = 10$ and $r = 2$ we obtain $dr/dt = 10/4\pi(2)^2 = 5/8\pi$ in/min.

60. (a) $\dfrac{dx}{dt} = x_0(-\sin\omega t)(\omega) + \dfrac{v_0}{\omega}(\cos\omega t)(\omega) = -\omega x_0 \sin\omega t + v_0\cos\omega t$

$$\frac{d^2x}{dt^2} = -\omega x_0(\cos\omega t)(\omega) + v_0(-\sin\omega t)(\omega) = -\omega^2 x_0\cos\omega t - v_0\omega\sin\omega t$$

Substituting into the differential equation, we get

$$(-\omega^2 x_0\cos\omega t - v_0\omega\sin\omega t) + \omega^2\left(x_0\cos\omega t + \frac{v_0}{\omega}\sin\omega t\right) =$$
$$(-\omega^2 x_0\cos\omega t - v_0\omega\sin\omega t) + \omega^2 x_0\cos\omega t + v_0\sin\omega t = 0.$$

(b) $x(0) = x_0\cos(\omega\cdot 0) + \dfrac{v_0}{\omega}\sin(\omega\cdot 0) = x_0\cos 0 + \dfrac{v_0}{\omega}\sin 0$

$$= (x_0)(1) + \left(\frac{v_0}{\omega}\right)(0) = x_0$$

$$x'(0) = -\omega x_0\sin(\omega\cdot 0) + v_0\cos(\omega\cdot 0) = (-\omega x_0)(0) + (v_0)(1) = v_0$$

61. $\dfrac{d}{dx}F(3x) = F'(3x)\dfrac{d}{dx}(3x) = 3F'(3x)$

62. $\dfrac{d}{dx}[G(-x^2)]^2 = 2[G(-x^2)]\dfrac{d}{dx}G(-x^2) = 2[G(-x^2)]G'(-x^2)\dfrac{d}{dx}(-x^2) = -4xG(-x^2)G'(-x^2)$

63. $\dfrac{d}{dx}f(-10x+7) = [f'(-10x+7)](-10) = \dfrac{1}{-10x+7}(-10) = \dfrac{10}{10x-7}$

64. $\dfrac{d}{dx}f(x^3) = [f'(x^3)](3x^2) = \dfrac{1}{1+(x^3)^2}(3x^2) = \dfrac{3x^2}{1+x^6}$

65. The derivatives for $n = 1$ to 4 are:

$$\frac{d}{dx}(1+2x)^{-1} = -2(1+2x)^{-2}; \qquad \frac{d^2}{dx^2}(1+2x)^{-1} = 2!(2^2)(1+2x)^{-3};$$
$$\frac{d^3}{dx^3}(1+2x)^{-1} = (-3!)(2^3)(1+2x)^{-4}; \qquad \frac{d^4}{dx^4}(1+2x)^{-1} = (4!)(2^4)(1+2x)^{-5}.$$

We can verify by induction that $\frac{d^n}{dx^n}(1+2x)^{-1} = (-1)^n\ n!\ 2^n\ (1+2x)^{-n-1}$:

$$\begin{aligned}\frac{d^n}{dx^n}(1+2x)^{-1} &= \frac{d}{dx}\left[\frac{d^{n-1}}{dx^{n-1}}(1+2x)^{-1}\right] = \frac{d}{dx}(-1)^{n-1}(n-1)!\ 2^{n-1}\ (1+2x)^{-(n-1)-1}\\ &= \frac{d}{dx}(-1)^{n-1}(n-1)!\ 2^{n-1}\ (1+2x)^{-n}\\ &= (-1)^{n-1}(-1)(n)(n-1)!\ 2^{n-1}\ (1+2x)^{-n-1}(2)\\ &= (-1)^n\ n!\ 2^n\ (1+2x)^{-n-1}\end{aligned}$$

66. The derivatives for $n = 1$ to 6 are:

$$\begin{aligned}&\frac{d}{dx}(1+2x)^{1/2} = (1+2x)^{-1/2}; && \frac{d^2}{dx^2}(1+2x)^{1/2} = -(1+2x)^{-3/2};\\ &\frac{d^3}{dx^3}(1+2x)^{1/2} = (1\cdot 3)(1+2x)^{-5/2}; && \frac{d^4}{dx^4}(1+2x)^{1/2} = -(3\cdot 5)(1+2x)^{-7/2};\\ &\frac{d^5}{dx^5}(1+2x)^{1/2} = (3\cdot 5\cdot 7)(1+2x)^{-9/2}; && \frac{d^6}{dx^6}(1+2x)^{1/2} = -(3\cdot 5\cdot 7)(1+2x)^{-11/2}.\end{aligned}$$

We can verify by induction that $\frac{d^n}{dx^n}\sqrt{1+2x} = (-1)^{n+1}[1\cdot 3\cdot 5\cdots(2n-3)](1+2x)^{(1-2n)/2}$:

$$\begin{aligned}\frac{d^n}{dx^n}\sqrt{1+2x} &= \frac{d}{dx}\left[\frac{d^{n-1}}{dx^{n-1}}(1+2x)^{1/2}\right]\\ &= \frac{d}{dx}(-1)^{(n-1)+1}\{1\cdot 3\cdot 5\cdots[2(n-1)-3]\}(1+2x)^{[1-2(n-1)]/2}\\ &= \frac{d}{dx}(-1)^n[1\cdot 3\cdot 5\cdots(2n-5)](1+2x)^{(3-2n)/2}\\ &= \frac{3-2n}{2}\left\{(-1)^n[1\cdot 3\cdot 5\cdots(2n-5)](1+2x)^{(3-2n-2)/2}\right\}(2)\\ &= (3-2n)(-1)^n[1\cdot 3\cdot 5\cdots(2n-5)](1+2x)^{(1-2n)/2}\\ &= (-1)(-1)^n[1\cdot 3\cdot 5\cdots(2n-5)(2n-3)](1+2x)^{(1-2n)/2}\\ &= (-1)^{n+1}[1\cdot 3\cdot 5\cdots(2n-3)](1+2x)^{(1-2n)/2}\end{aligned}$$

67. $g'(1) = h'(f(1))f'(1) = h'(3)\cdot 6 = -2\cdot 6 = -12$

68. $$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x);\quad \frac{d^2}{dx^2}f(g(x)) = f'(g(x))g''(x) + g'(x)f''(g(x))g'(x);$$

$$\begin{aligned}\left.\frac{d^2}{dx^2}f(g(x))\right|_{x=1} &= f'(g(1))g''(1) + g'(1)f''(g(1))g'(1) = [f'(2)](-1) + 3[f''(2)](3)\\ &= 4(-1) + 3(3)(3) = 23\end{aligned}$$

69. Since f is odd, $f(-x) = -f(x)$. Then $f'(x) = -\frac{d}{dx}f(-x) = -[f'(-x)](-1) = f'(-x)$.

Since $f'(-x) = f'(x)$, f' is an even function.

70. Since f is even, $f(-x) = f(x)$. Then $f'(x) = \frac{d}{dx} f(-x) = [f'(-x)](-1) = -f'(-x)$.
Since $f'(-x) = -f'(x)$, f' is an odd function.

3.6 Implicit Differentiation

1. $\frac{d}{dx} x^2 y^4 = (x^2)(4y^3 \frac{dy}{dx}) + (y^4)(2x) = 4x^2 y^3 \frac{dy}{dx} + 2xy^4$

2. $\frac{d}{dx} \frac{x^2}{y^2} = \frac{(y^2)(2x) - (x^2)(2y\ dy/dx)}{y^4} = \frac{2x}{y^2} - \frac{2x^2}{y^3}\left(\frac{dy}{dx}\right)$

3. $\frac{d}{dx} \cos y^2 = (-\sin y^2)(2y)\frac{dy}{dx} = -2y(\sin y^2)\frac{dy}{dx}$

4. $\frac{d}{dx} y \sin 3y = (y)(\cos 3y)(3)\frac{dy}{dx} + (\sin 3y)\frac{dy}{dx} = (3y\cos 3y + \sin 3y)\frac{dy}{dx}$

5. $2y\frac{dy}{dx} - 2\frac{dy}{dx} = 1 \quad \frac{dy}{dx} = \frac{1}{2y-2}$

6. $8x + 2y\frac{dy}{dx} = 0; \quad \frac{dy}{dx} = -\frac{4x}{y}$

7. $x\left(2y\frac{dy}{dx}\right) + y^2 - 2x = 0; \quad \frac{dy}{dx} = \frac{2x - y^2}{2xy}$

8. $2(y-1)\frac{dy}{dx} = 4; \quad \frac{dy}{dx} = \frac{2}{y-1}$

9. $3\frac{dy}{dx} + (-\sin y)\frac{dy}{dx} = 2x; \quad \frac{dy}{dx} = \frac{2x}{3 - \sin y}$

10. $3y^2\frac{dy}{dx} - 2\frac{dy}{dx} + 9x^2 = 4; \quad \frac{dy}{dx} = \frac{4 - 9x^2}{3y^2 - 2}$

11. $x^3\left(2y\frac{dy}{dx}\right) + y^2(3x^2) = 4x + 2y\frac{dy}{dx}; \quad \frac{dy}{dx} = \frac{4x - 3x^2y^2}{2x^3y - 2y}$

12. $5x^4 - 6x\left(3y^2\frac{dy}{dx}\right) + y^3(-6) + 4y^3\frac{dy}{dx} = 0; \quad \frac{dy}{dx} = \frac{6y^3 - 5x^4}{4y^3 - 18xy^2}$

13. $6(x^2+y^2)^5\left(2x + 2y\frac{dy}{dx}\right) = 3x^2 - 3y^2\frac{dy}{dx}$

$12y(x^2+y^2)^5\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 3x^2 - 12x(x^2+y^2)^5; \quad \frac{dy}{dx} = \frac{x^2 - 4x(x^2+y^2)^5}{y^2 + 4y(x^2+y^2)^5}$

14. $\frac{dy}{dx} = 2(x-y)\left(1 - \frac{dy}{dx}\right) = 2x - 2y - 2(x-y)\frac{dy}{dx}$

$[1 + 2(x-y)]\frac{dy}{dx} = 2x - 2y; \quad \frac{dy}{dx} = \frac{2x - 2y}{1 + 2x - 2y}$

15. $(y^{-3})(6x^5) + (x^6)\left(-3y^{-4}\dfrac{dy}{dx}\right) + (y^6)(-3x^{-4}) + (x^{-3})\left(6y^5\dfrac{dy}{dx}\right) = 2$

$$(-3x^6y^{-4} + 6x^{-3}y^5)\frac{dy}{dx} = 2 - 6x^5y^{-3} + 3x^{-4}y^6; \quad \frac{dy}{dx} = \frac{2 - 6x^5y^{-3} + 3x^{-4}y^6}{6x^{-3}y^5 - 3x^6y^{-4}}$$

16. $4y^3\dfrac{dy}{dx} - 2y\dfrac{dy}{dx} = 10; \quad (2y^3 - y)\dfrac{dy}{dx} = 5; \quad \dfrac{dy}{dx} = \dfrac{5}{2y^3 - y}$

17. $2(x-1) + 2(y+4)\dfrac{dy}{dx} = 0; \quad \dfrac{dy}{dx} = -\dfrac{x-1}{y+4} = \dfrac{1-x}{y+4}$

18. $x + y = x^2 - xy; \quad 1 + \dfrac{dy}{dx} = 2x - x\dfrac{dy}{dx} - y; \quad \dfrac{dy}{dx} + x\dfrac{dy}{dx} = 2x - y - 1; \quad \dfrac{dy}{dx} = \dfrac{2x - y - 1}{x + 1}$

19. $2y\dfrac{dy}{dx} = \dfrac{x + 2 - (x - 1)}{(x+2)^2} = \dfrac{3}{(x+2)^2}; \quad \dfrac{dy}{dx} = \dfrac{3}{2y(x+2)^2}$

20. $x^2 + y^4 = 5xy^2; \quad 2x + 4y^3\dfrac{dy}{dx} = 5x\left(2y\dfrac{dy}{dx}\right) + 5y^2; \quad 4y^3\dfrac{dy}{dx} - 10xy\dfrac{dy}{dx} = 5y^2 - 2x$

$$\frac{dy}{dx} = \frac{5y^2 - 2x}{4y^3 - 10xy}$$

21. $x\dfrac{dy}{dx} + y = [\cos(x+y)]\left(1 + \dfrac{dy}{dx}\right) = \cos(x+y) + \dfrac{dy}{dx}\cos(x+y)$

$$x\frac{dy}{dx} - \frac{dy}{dx}\cos(x+y) = \cos(x+y) - y; \quad \frac{dy}{dx} = \frac{\cos(x+y) - y}{x - \cos(x+y)}$$

22. $1 + \dfrac{dy}{dx} = -(\sin xy)\left(x\dfrac{dy}{dx} + y\right) = -x\dfrac{dy}{dx}\sin xy - y\sin xy$

$$\frac{dy}{dx} + x\frac{dy}{dx}\sin xy = -1 - y\sin xy; \quad \frac{dy}{dx} = -\frac{1 + y\sin xy}{1 + x\sin xy}$$

23. $1 = (\tan y \sec y)\dfrac{dy}{dx}; \quad \dfrac{dy}{dx} = \dfrac{1}{\tan y \sec y} = \cos y \cot y$

24. $x(\cos y)\dfrac{dy}{dx} + \sin y - y(-\sin x) - (\cos x)\dfrac{dy}{dx} = 0; \quad (x\cos y - \cos x)\dfrac{dy}{dx} = -\sin y - y\sin x$

$$\frac{dy}{dx} = \frac{\sin y + y\sin x}{\cos x - x\cos y}$$

25. $2r\dfrac{dr}{d\theta} = 2\cos 2\theta; \quad \dfrac{dr}{d\theta} = \dfrac{\cos 2\theta}{r}$

26. $r^2h = \dfrac{100}{h}; \quad r^2\dfrac{dh}{dr} + h(2r) = 0; \quad \dfrac{dh}{dr} = -\dfrac{2h}{r}$

27. $x\left(2y\dfrac{dy}{dx}\right) + y^2 + 12y^2\dfrac{dy}{dx} + 3 = 0; \quad 2xy\dfrac{dy}{dx} + 12y^2\dfrac{dy}{dx} = -3 - y^2; \quad \dfrac{dy}{dx} = \dfrac{-3 - y^2}{2xy + 12y^2}$

$$\left.\frac{dy}{dx}\right|_{(1,-1)} = \frac{-3 - (-1)^2}{2(1)(-1) + 12(-1)^2} = -\frac{2}{5}$$

28. $\dfrac{dy}{dx} = (\cos xy)\left(x\dfrac{dy}{dx} + y\right)$; $\quad \dfrac{dy}{dx} - x\dfrac{dy}{dx}\cos xy = y\cos xy$; $\quad \dfrac{dy}{dx} = \dfrac{y\cos xy}{1 - x\cos xy}$

$$\left.\frac{dy}{dx}\right|_{(\pi/2,1)} = \frac{(1)(\cos\pi/2)}{1-(\pi/2)\cos(\pi/2)} = 0$$

29. Letting $x = 1/2$ in $2y^2 + 2xy - 1 = 0$, we obtain $2y^2 + y - 1 = 0$. Factoring, we have $(2y-1)(y+1) = 0$. Thus, $y = -1, 1/2$. Differentiating gives

$$4y\frac{dy}{dx} + 2x\frac{dy}{dx} + 2y = 0; \quad (2y+x)\frac{dy}{dx} = -y; \quad \frac{dy}{dx} = -\frac{y}{2y+x}$$

$$\left.\frac{dy}{dx}\right|_{(1/2,-1)} = -\frac{-1}{-2+1/2} = -\frac{2}{3}; \quad \left.\frac{dy}{dx}\right|_{(1/2,1/2)} = -\frac{1/2}{1+1/2} = -\frac{1}{3}.$$

30. Letting $y = 1$ in $y^3 + 2x^2 = 11y$, we obtain $1 + 2x^2 = 11$ or $x^2 = 5$. Thus $x = \pm\sqrt{5}$. Differentiating gives

$$3y\frac{dy}{dx} + 4x = 11\frac{dy}{dx}; \quad 4x = (11-3y^2)\frac{dy}{dx}; \quad \frac{dy}{dx} = \frac{4x}{11-3y^2}$$

$$\left.\frac{dy}{dx}\right|_{(-\sqrt{5},1)} = \frac{-4\sqrt{5}}{8} = -\frac{\sqrt{5}}{2}; \quad \left.\frac{dy}{dx}\right|_{(\sqrt{5},1)} = \frac{4\sqrt{5}}{8} = \frac{\sqrt{5}}{2}.$$

31. $4x^3 + 3y^2\dfrac{dy}{dx} = 0$; $\dfrac{dy}{dx} = -\dfrac{4x^3}{3y^2}$. The slope of the tangent line is $\left.\dfrac{dy}{dx}\right|_{(-2,2)} = -\dfrac{4(-2)^3}{3(2)^2} = \dfrac{8}{3}$. Hence an equation of the tangent line is $y - 2 = \dfrac{8}{3}(x+2)$ or $y = \dfrac{8}{3}x + \dfrac{22}{3}$.

32. When $x = 3$, $1/3 + 1/y = 1$ or $1/y = 2/3$. Thus, $y = 3/2$. Differentiating, we see $-\dfrac{1}{x^2} - \dfrac{1}{y^2}\dfrac{dy}{dx} = 0$ or $\dfrac{dy}{dx} = -\dfrac{y^2}{x^2}$. The slope of the tangent line is $\left.\dfrac{dy}{dx}\right|_{(3,3/2)} = -\dfrac{(3/2)^2}{3^2} = -\dfrac{1}{4}$. Hence an equation of the tangent line is $y - \dfrac{3}{2} = -\dfrac{1}{4}(x-3)$ or $y = -\dfrac{1}{4}x + \dfrac{9}{4}$.

33. $(\sec^2 y)\dfrac{dy}{dx} = 1$; $\dfrac{dy}{dx} = \dfrac{1}{\sec^2 y} = \cos^2 y$. The slope of the tangent line is $\left.\dfrac{dy}{dx}\right|_{y=\pi/4} = \cos^2\pi/4 = 1/2$. When $y = \pi/4$, $\tan\pi/4 = x$. Thus, $x = 1$ and the equation of the tangent line is $y - \dfrac{\pi}{4} = \dfrac{1}{2}(x-1)$ or $y = \dfrac{1}{2}x + \dfrac{\pi-2}{4}$.

34. $3\dfrac{dy}{dx} - (\sin y)\dfrac{dy}{dx} = 2x$; $\dfrac{dy}{dx} = \dfrac{2x}{3-\sin y}$. The slope of the tangent line is $\left.\dfrac{dy}{dx}\right|_{(1,0)} = \dfrac{2}{3}$. Hence an equation of the tangent line is $y - 0 = \dfrac{2}{3}(x-1)$ or $y = \dfrac{2}{3}x - \dfrac{2}{3}$.

35. $2x - x\dfrac{dy}{dx} - y + 2y\dfrac{dy}{dx} = 0$; $(2y-x)\dfrac{dy}{dx} = y - 2x$; $\dfrac{dy}{dx} = \dfrac{y-2x}{2y-x}$. Setting $\dfrac{dy}{dx} = 0$, we obtain $y = 2x$. Substituting $y = 2x$ into the equation of the curve, we have $x^2 - x(2x) + (2x)^2 = 3$, $3x^2 = 3$, or $x = \pm 1$. Given $y = 2x$, we see that for $x = 1$, $y = 2$, and for $x = -1$, $y = -2$. Thus, the curve has horizontal tangent lines at $(1,2)$ and $(-1,-2)$.

36. $2y\dfrac{dy}{dx} = 2x - 4$; $\dfrac{dy}{dx} = \dfrac{x-2}{y}$. Setting $\dfrac{dy}{dx} = 0$, we obtain $\dfrac{x-2}{y} = 0$ or $x = 2$. Substituting $x = 2$ into the equation of the curve, we have $y^2 = 2^2 - 4(2) + 7 = 3$ or $y = \pm\sqrt{3}$. Thus, the curve has horizontal tangent lines at $(2, -\sqrt{3})$ and $(2, \sqrt{3})$.

37. $2x + 2y\dfrac{dy}{dx} = 0$; $\dfrac{dy}{dx} = -\dfrac{x}{y}$. Setting $\dfrac{dy}{dx} = \dfrac{1}{2}$ gives us $-\dfrac{x}{y} = \dfrac{1}{2}$ or $y = -2x$. Substituting this into the equation of the curve, we have $x^2 + (-2x)^2 = 25$, $5x^2 = 25$, or $x = \pm\sqrt{5}$. Since $y = -2x$, the points on the graph are $(\sqrt{5}, -2\sqrt{5})$ and $(-\sqrt{5}, 2\sqrt{5})$.

38. $2x + 2y\dfrac{dy}{dx} = 0$; $\dfrac{dy}{dx} = -\dfrac{x}{y}$. The slope of the tangent line at $(-3, 4)$ is $\left.\dfrac{dy}{dx}\right|_{(-3,4)} = \dfrac{3}{4}$ and the slope of the tangent line at $(-3, -4)$ is $\left.\dfrac{dy}{dx}\right|_{(-3,-4)} = -\dfrac{3}{4}$. Thus, an equation of the tangent line through $(-3, 4)$ is $y - 4 = \dfrac{3}{4}(x + 3)$, and an equation of the tangent line through $(-3, -4)$ is $y + 4 = -\dfrac{3}{4}(x + 3)$ or $-16 = 3x + 9$. Solving for x we obtain $x = -25/3$. Thus, the tangent lines intersect at $(-25/3, 0)$.

39. $3y^2\dfrac{dy}{dx} = 2x$; $\dfrac{dy}{dx} = \dfrac{2x}{3y^2}$. The slope of the tangent line perpendicular to $y + 3x - 5 = 0$, or $y = -3x + 5$, is $1/3$. Setting $\dfrac{dy}{dx} = \dfrac{1}{3}$ gives us $\dfrac{2x}{3y^2} = \dfrac{1}{3}$, $y^2 = 2x$, or $y = \pm\sqrt{2x}$. Substituting this into the equation of the curve, we have $(\pm\sqrt{2x})^3 = x^2$, so we can eliminate $y = -\sqrt{2x}$. Thus, $2^{3/2}x^{3/2} = x^2$, $\sqrt{x} = \sqrt{8}$, or $x = 8$. Since $y = \sqrt{2x}$, the point on the graph is $(8, 4)$.

40. $2x - x\dfrac{dy}{dx} - y + 2y\dfrac{dy}{dx} = 0$; $\dfrac{dy}{dx} = \dfrac{y - 2x}{2y - x}$. The line $y = 5$ has a slope of 0; setting $\dfrac{dy}{dx} = 0$, we obtain $\dfrac{y - 2x}{2y - x} = 0$ or $y = 2x$. Substituting this into the equation of the curve, we have $x^2 - x(2x) + (2x)^2 = 27$, $x^2 - 2x^2 + 4x^2 = 27$, $x^2 = 9$, or $x = \pm 3$. Since $y = 2x$, the points on the graph are $(-3, 6)$ and $(3, 6)$.

41. $12y^2\dfrac{dy}{dx} = 12x; \quad \dfrac{dy}{dx} = \dfrac{x}{y^2}$

$$\frac{d^2y}{dx^2} = \frac{y^2(1) - x(2y\ dy/dx)}{y^4} = \frac{y - 2x\ dy/dx}{y^3} = \frac{y - 2x(x/y^2)}{y^3} = \frac{y^3 - 2x^2}{y^5}$$

42. $x\left(4y^3\dfrac{dy}{dx}\right) + y^4 = 0; \quad \dfrac{dy}{dx} = -\dfrac{y^4}{4xy^3} = -\dfrac{y}{4x}$

$$\frac{d^2y}{dx^2} = -\frac{4x\ dy/dx - y(4)}{16x^2} = -\frac{x(-y/4x) - y}{4x^2} = \frac{5y}{16x^2}$$

43. $2x - 2y\dfrac{dy}{dx} = 0; \quad \dfrac{dy}{dx} = \dfrac{x}{y}; \quad \dfrac{d^2y}{dx^2} = \dfrac{y - x\ dy/dx}{y^2} = \dfrac{y - x(x/y)}{y^2} = \dfrac{y^2 - x^2}{y^3} = -\dfrac{25}{y^3}$

44. $2x + 8y\dfrac{dy}{dx} = 0; \quad \dfrac{dy}{dx} = -\dfrac{x}{4y}$

$$\frac{d^2y}{dx^2} = -\frac{4y - (x)(4\ dy/dx)}{16y^2} = -\frac{y - x(-x/4y)}{4y^2} = -\frac{4y^2 + x^2}{16y^3} = -\frac{16}{16y^3} = -\frac{1}{y^3}$$

45. $1 + \dfrac{dy}{dx} = (\cos y)\dfrac{dy}{dx}; \quad \dfrac{dy}{dx} = \dfrac{1}{\cos y - 1}$

$$\frac{d^2y}{dx^2} = \frac{-(-\sin y)\ dy/dx}{(\cos y - 1)^2} = \frac{(\sin y)[1/(\cos y - 1)]}{(\cos y - 1)^2} = \frac{\sin y}{(\cos y - 1)^3}$$

46. $2y\dfrac{dy}{dx} - 2x = 2\sec^2 2x; \quad \dfrac{dy}{dx} = \dfrac{\sec^2 2x + x}{y}$

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{y[2(\sec 2x)(2\sec 2x\tan 2x) + 1] - (\sec^2 2x + x)\ dy/dx}{y^2} \\ &= \frac{4y\sec^2 2x\tan 2x + y - (\sec^2 2x + x)[(\sec^2 2x + x)/y]}{y^2} \\ &= \frac{4y^2\sec^2 2x\tan 2x - (\sec^2 2x + x)^2 + y}{y^3}\end{aligned}$$

47. $2x + 2x\dfrac{dy}{dx} + 2y - 2y\dfrac{dy}{dx} = 0; \quad \dfrac{dy}{dx} = -\dfrac{x+y}{x-y} = \dfrac{y+x}{y-x}$

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{(y-x)(dy/dx + 1) - (y+x)(dy/dx - 1)}{(y-x)^2} \\ &= \frac{(y-x)\left(\dfrac{y+x}{y-x} + 1\right) - (y+x)\left(\dfrac{y+x}{y-x} - 1\right)}{(y-x)^2} \\ &= \frac{(y-x)(y+x+y-x) - (y+x)[y+x-(y-x)]}{(y-x)^3} = \frac{2y^2 - 4xy - 2x^2}{(y-x)^3} \\ &= \frac{-2(x^2 + 2xy - y^2)}{(y-x)^3} = \frac{-2}{(y-x)^3}\end{aligned}$$

48. $3x^2 + 3y^2\dfrac{dy}{dx} = 0; \quad \dfrac{dy}{dx} = -\dfrac{x^2}{y^2}$

$$\frac{d^2y}{dx^2} = -\frac{y^2(2x) - x^2(2y\ dy/dx)}{y^4} = -\frac{2xy - 2x^2(-x^2/y^2)}{y^3} = -\frac{2xy^3 + 2x^4}{y^5}$$

49. Using implicit differentiation we see

$$2x - 2y\frac{dy}{dx} = 1 \qquad \text{and} \qquad \frac{dy}{dx} = \frac{2x-1}{2y}.$$

Solving $x^2 - y^2 = x$ for y and differentiating we have

$$y = \sqrt{x^2 - x} = (x^2 - x)^{1/2} \qquad \text{and} \qquad \frac{dy}{dx} = \frac{1}{2}(x^2 - x)^{-1/2}(2x - 1).$$

Substituting the expression for y into $\dfrac{2x-1}{2y}$ we obtain

$$\frac{dy}{dx} = \frac{2x-1}{2(x^2-x)^{1/2}} = \frac{1}{2}(x^2-x)^{-1/2}(2x-1).$$

50. Using implicit differentiation we see

$$8x + 2y\frac{dy}{dx} = 0 \qquad \text{and} \qquad \frac{dy}{dx} = -\frac{4x}{y}$$

Solving $4x^2 + y^2 = 1$ for y and differentiating we have

$$y = \sqrt{1-4x^2} = (1-4x^2)^{1/2} \qquad \text{and} \qquad \frac{dy}{dx} = \frac{1}{2}(1-4x^2)^{-1/2}(-8x) = -4x(1-4x^2)^{-1/2}.$$

Substituting the expression for y into $-\dfrac{4x}{y}$ we obtain

$$\frac{dy}{dx} = -\frac{4x}{(1-4x^2)^{1/2}} = -4x(1-4x^2)^{-1/2}.$$

51. Using implicit differentiation we see

$$x^3\frac{dy}{dx} + y(3x^2) = 1 \qquad \text{and} \qquad \frac{dy}{dx} = \frac{1-3x^2y}{x^3}.$$

Solving $x^3y = x+1$ for y and differentiating we have

$$y = \frac{x+1}{x^3} \qquad \text{and} \qquad \frac{dy}{dx} = \frac{x^3-(x+1)(3x^2)}{x^6} = \frac{x-3(x+1)}{x^4} = \frac{-2x-3}{x^4}.$$

Substituting the expression for y into $\dfrac{1-3x^2y}{x^3}$ we obtain

$$\frac{dy}{dx} = \frac{1-3x^2\left(\dfrac{x+1}{x^3}\right)}{x^3} = \frac{1-\dfrac{3(x+1)}{x}}{x^3} = \frac{-2x-3}{x^4}.$$

52. Using implicit differentiation we see

$$\begin{aligned} y\cos x + (\sin x)\frac{dy}{dx} &= 1 - 2\frac{dy}{dx} \\ (2+\sin x)\frac{dy}{dx} &= 1 - y\cos x \\ \frac{dy}{dx} &= \frac{1-y\cos x}{2+\sin x}. \end{aligned}$$

Solving $y\sin x = x - 2y$ for y and differentiating we have

$$y = \frac{x}{2+\sin x} \qquad \text{and} \qquad \frac{dy}{dx} = \frac{2+\sin x - x\cos x}{(2+\sin x)^2}.$$

Substituting the expression for y into $\dfrac{1-y\cos x}{2+\sin x}$ we obtain

$$\frac{dy}{dx}=\frac{1-\dfrac{x}{2+\sin x}\cos x}{2+\sin x}=\frac{2+\sin x-x\cos x}{(2+\sin x)^2}.$$

In Problems 53–56, we solve the given equations for the appropriate values of y.

53. Since the graph lies below the line $y=1$, we have

$$y-1=-\sqrt{x-2}\qquad\text{or}\qquad y=1-\sqrt{x-2}.$$

54. Writing the equation as $y^2+xy+x^2-4=0$ and using the quadratic formula, we have

$$y=\frac{1}{2}\left(-x+\sqrt{x^2-4(x^2-4)}\right)=\frac{1}{2}\left(-x+\sqrt{16-3x^2}\right).$$

Letting $y=0$ in the original equation, we get $x=\pm 2$. The graph is in the first quadrant, so

$$y=\frac{1}{2}\left(-x+\sqrt{16-3x^2}\right),\quad 0\le x\le 2.$$

55. $y^2=4-x^2$; $y=\pm\sqrt{4-x^2}$. Since the graph is in the second and fourth quadrants,

$$y=\begin{cases}\sqrt{4-x^2}, & -2\le x<0\\ -\sqrt{4-x^2}, & 0\le x\le 2\end{cases}.$$

56. Since the graph lies above the x-axis, $y=|x|\sqrt{2-x}$.

57. $2x\dfrac{dx}{dt}+2y\dfrac{dy}{dt}=0$; $\quad\dfrac{dy}{dt}=-\left(\dfrac{x}{y}\right)\dfrac{dx}{dt}$

58. $2x\dfrac{dx}{dt}+x\dfrac{dy}{dt}+y\dfrac{dx}{dt}+2y\dfrac{dy}{dt}-\dfrac{dy}{dt}=0$; $\quad(x+2y-1)\dfrac{dy}{dt}=-(2x+y)\dfrac{dx}{dt}$
$\dfrac{dy}{dt}=-\left(\dfrac{2x+y}{x+2y-1}\right)\dfrac{dx}{dt}$

59. (a) $3x^2+3y^2\dfrac{dy}{dx}=3x\dfrac{dy}{dx}+3y$; $\dfrac{dy}{dx}=\dfrac{x^2-y}{x-y^2}$. Setting $y=x$ in $x^3+y^3=3xy$ we obtain

$$x^3+x^3=3x^2;\qquad 2x^3=3x^2;\qquad x^2(2x-3)=0;\qquad \text{so}\quad x=0,\frac{3}{2}.$$

Since $\left.\dfrac{dy}{dx}\right|_{(3/2,3/2)}=\dfrac{9/4-3/2}{3/2-9/4}=-1$, the slope of the tangent line at $(3/2,3/2)$ is -1 and the equation of the tangent line is

$$y-\frac{3}{2}=-\left(x-\frac{3}{2}\right)\qquad\text{or}\qquad y=-x+3.$$

(b) Setting $\dfrac{dy}{dx} = \dfrac{x^2 - y}{x - y^2} = 0$ we obtain $x^2 - y = 0$ or $y = x^2$. Substituting $y = x^2$ in $x^3 + y^3 = 3xy$ we obtain

$$x^3 + x^6 = 3x^3; \qquad x^3(x^3 - 2) = 0; \qquad \text{so} \quad x = 0, \sqrt[3]{2}.$$

In the first quadrant the tangent line is horizontal at $(\sqrt[3]{2}, \sqrt[3]{4})$.

60. (a) Setting $x = 1$ we have $(1 + y^2)^2 = 4(1 - y^2)$ or $y^4 + 6y^2 - 3 = 0$. From the quadratic formula, $y^2 = \dfrac{-6 \pm \sqrt{36 + 12}}{2} = -3 \pm 2\sqrt{3}$. Since y^2 cannot be negative, we obtain $y = \pm\sqrt{2\sqrt{3} - 3} \approx \pm 0.68$. Thus, the points on the graph are $(1, 0.68)$ and $(1, -0.68)$.

(b) $2(x^2 + y^2)\left(2x + 2y\dfrac{dy}{dx}\right) = 4\left(2x - 2y\dfrac{dy}{dx}\right); \quad (x^2 + y^2)\left(x + y\dfrac{dy}{dx}\right) = 2x - 2y\dfrac{dy}{dx}$

$$(x^2y + y^3 + 2y)\frac{dy}{dx} = 2x - x^3 - xy^2; \quad \frac{dy}{dx} = \frac{2x - x^3 - xy^2}{x^2y + y^3 + 2y}$$

For $y = 0.68$, the slope of the tangent line is $\left.\dfrac{dy}{dx}\right|_{(1,0.68)} \approx 0.23$ and the equation of the tangent line is $y - 0.68 = 0.23(x - 1)$ or $y = 0.23x + 0.45$. For $y = -0.68$, the slope of the tangent line is $\left.\dfrac{dy}{dx}\right|_{(1,-0.68)} \approx -0.23$, and the equation of the tangent line is $y + 0.68 = -0.23(x - 1)$ or $y = -0.23x - 0.45$.

(c) Setting $\dfrac{dy}{dx} = 0$, we find $2x - x^3 - xy^2 = 0$ or $x(2 - x^2 - y^2) = 0$. This gives $x = 0$ and $x^2 + y^2 = 2$. Substituting $x = 0$ into the original equation yields $y = 0$. However, at $(0, 0)$ $\dfrac{dy}{dx}$ is not defined. By substituting $x^2 + y^2 = 2$ into the original equation, we observe $x^2 - y^2 = 1$. Adding these last two equations, we obtain $x^2 = 3/2$ or $x = \pm\sqrt{3/2} = \pm\sqrt{6}/2$, and $y = \pm\sqrt{1/2} = \pm\sqrt{2}/2$. Thus, the points of horizontal tangency are $(\sqrt{6}/2, \sqrt{2}/2)$, $(\sqrt{6}/2, -\sqrt{2}/2)$, $(-\sqrt{6}/2, \sqrt{2}/2)$, and $(-\sqrt{6}/2, -\sqrt{2}/2)$.

61. $$2y\frac{dy}{dx} = 2x^2; \qquad \frac{dy}{dx} = \frac{3x^2}{2y}; \qquad \left.\frac{dy}{dx}\right|_{(1,1)} = \frac{3}{2}$$

$$4x + 6y\frac{dy}{dx} = 0; \qquad \frac{dy}{dx} = -\frac{2x}{3y}; \qquad \left.\frac{dy}{dx}\right|_{(1,1)} = -\frac{2}{3}$$

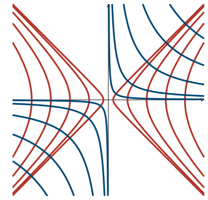
Since $\dfrac{3}{2}\left(-\dfrac{2}{3}\right) = -1$, the graphs are orthogonal at $(1, 1)$.

62. $$3y^2\frac{dy}{x} + 3x^2\frac{dy}{dx} + 6xy = 0; \qquad \frac{dy}{dx} = -\frac{2xy}{y^2 + x^2}; \qquad \left.\frac{dy}{dx}\right|_{(2,1)} = -\frac{4}{5}$$

$$4x - 4y\frac{dy}{dx} = 3; \qquad \frac{dy}{dx} = \frac{4x - 3}{4y}; \qquad \left.\frac{dy}{dx}\right|_{(2,1)} = \frac{5}{4}$$

Since $-\dfrac{4}{5}\left(\dfrac{5}{4}\right) = -1$, the graphs are orthogonal at $(2, 1)$.

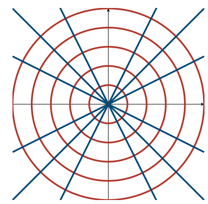
63. Solving for $\dfrac{dy}{dx}$ for each family of curves, we get



$$2x - 2y\frac{dy}{dx} = 0; \quad \frac{dy}{dx} = \frac{x}{y} \qquad \text{and} \qquad x\frac{dy}{dx} + y = 0; \quad \frac{dy}{dx} = -\frac{y}{x}$$

Since $\dfrac{x}{y}\left(-\dfrac{y}{x}\right) = -1$, the families are orthogonal trajectories of each other. The graph on the right shows $x^2 - y^2 = c_1$ in red, and $xy = c_2$ in blue, for selected values of c_1 and c_2.

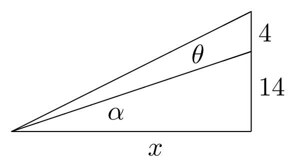
64. Solving for $\dfrac{dy}{dx}$ for each family of curves, we get



$$2x + 2y\frac{dy}{dx} = 0; \quad \frac{dy}{dx} = -\frac{x}{y} \qquad \text{and} \qquad \frac{dy}{dx} = c_2 = \frac{y}{x}$$

Since $-\dfrac{x}{y}\left(\dfrac{y}{x}\right) = -1$, the families are orthogonal trajectories of each other. The graph on the right shows $x^2 + y^2 = c_1$ in red, and $y = c_2x$ in blue, for selected values of c_1 and c_2.

65. (a) We use the formula for the tangent of the sum of two angles:



$$\tan(\alpha + \theta) = \frac{\tan\alpha + \tan\theta}{1 - \tan\alpha\tan\theta}.$$

From the figure we see that $\tan\alpha = 14/x$ and

$$\begin{aligned}
\frac{14+4}{x} = \tan(\alpha+\theta) = \frac{14/x + \tan\theta}{1 - (14/x)\tan\theta} &= \frac{14 + x\tan\theta}{x - 14\tan\theta} \\
18(x - 14\tan\theta) &= x(14 + x\tan\theta) \\
18x - 252\tan\theta &= 14x + x^2\tan\theta \\
4x &= (x^2 + 252)\tan\theta \\
\tan\theta &= \frac{4x}{x^2+252}.
\end{aligned}$$

(b) $\sec^2\theta\dfrac{d\theta}{dx} = \dfrac{(x^2+252)4 - 4x(2x)}{(x^2+252)^2} = \dfrac{1008 - 4x^2}{(x^2+252)^2}$

From $\sec^2\theta = 1 + \tan^2\theta = 1 + \dfrac{16x^2}{(x^2+252)^2} = \dfrac{(x^2+252)^2 + 16x^2}{(x^2+252)^2}$ we obtain

$$\begin{aligned}
\frac{d\theta}{dx} &= \frac{1008 - 4x^2}{(x^2+252)^2\sec^2\theta} = \frac{1008 - 4x^2}{(x^2+252)^2\{[(x^2+252)^2 + 16x^2]/(x^2+252)^2\}} \\
&= \frac{4(252 - x^2)}{(x^2+252)^2 + 16x^2}
\end{aligned}$$

(c) Setting $\dfrac{d\theta}{dx} = 0$ we obtain $x^2 = 252$ or $x = 6\sqrt{7} \approx 15.87$ ft.

66. (a) The equation of the lower half of the circle is $y = -\sqrt{1-x^2}$. Then $y' = \dfrac{x}{\sqrt{1-x^2}}$. The equation of the tangent line through the point $(x_0, -\sqrt{1-x_0^2})$ on the circle is

$$y + \sqrt{1-x_0^2} = \frac{x_0}{\sqrt{1-x_0^2}}(x - x_0), \quad y\sqrt{1-x_0^2} + 1 - x_0^2 = x_0 x - x_0^2$$
$$\text{or} \quad y\sqrt{1-x_0^2} = x_0 x - 1.$$

Since $(2, -2)$ is on the line we have $-2\sqrt{1-x_0^2} = 2x_0 - 1$. Squaring, we obtain $4(1-x_0^2) = 4x_0^2 - 4x_0 + 1$ or $8x_0^2 - 4x_0 - 3 = 0$. Solving for x_0 we obtain $x_0 = \dfrac{1}{4} \pm \dfrac{1}{4}\sqrt{7}$. Since (x_0, y_0) is in the third quadrant, we take $x_0 = \dfrac{1}{4} - \dfrac{1}{4}\sqrt{7} \approx -0.41$. Then $y_0 = -\sqrt{1-x_0^2} \approx -0.91$.

(b) The slope of the tangent line is $y'(-1/2) = -1/\sqrt{3}$ and the equation is $y + \dfrac{\sqrt{3}}{2} = -\dfrac{1}{\sqrt{3}}\left(x + \dfrac{1}{2}\right)$. Letting $y = -2$ and solving for x we obtain $x = 2(\sqrt{3} - 1)$.

67. We use implicit differentiation to find the slopes of the tangent lines at $(1, 1)$.

$$\begin{aligned} x^2 + y^2 + 4y &= 6 \\ 2x + 2y\frac{dy}{dx} + 4\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{x}{y+2} \\ \left.\frac{dy}{dx}\right|_{(1,1)} &= -\frac{1}{3} \end{aligned} \qquad \begin{aligned} x^2 + 2x + y^2 &= 4 \\ 2x + 2 + 2y\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{x+1}{y} \\ \left.\frac{dy}{dx}\right|_{(1,1)} &= -2 \end{aligned}$$

Letting $m_1 = -1/3$ and $m_2 = -2$, we have $\tan\theta = \dfrac{-1/3 + 2}{1 + 2/3} = 1$. The angle between the graphs is $\theta = \pi/4$.

68. Using implicit differentiation, we get

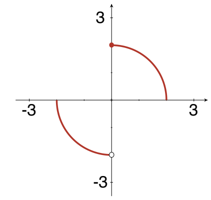
$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0; \quad \frac{dy}{dx} = -\frac{2x}{a^2}\left(\frac{b^2}{2y}\right) = -\frac{b^2 x}{a^2 y}.$$

Thus, the slope of the tangent line at (x_0, y_0) is $-\dfrac{b^2 x_0}{a^2 y_0}$ and its equation is $y - y_0 = -\dfrac{b^2 x_0}{a^2 y_0}(x - x_0)$. Dividing by b^2 and multiplying by y_0 yields

$$\frac{yy_0}{b^2} - \frac{y_0^2}{b^2} = -\frac{xx_0}{a^2} + \frac{x_0^2}{a^2}; \qquad \frac{xx_0}{a^2} + \frac{yy_0}{b^2} = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2}.$$

Since (x_0, y_0) is a point on the ellipse, then $\dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2} = 1$, and so the equation of the tangent line can be written as $\dfrac{xx_0}{a^2} + \dfrac{yy_0}{b^2} = 1$.

69. One more function defined by $x^2 + y^2 = 4$ includes the parts of the circle in the *first* and *third* quadrants (see graph at right):



$$y = \begin{cases} -\sqrt{4-x^2}, & -2 \le x < 0 \\ \sqrt{4-x^2}, & 0 \le x \le 2 \end{cases}.$$

70. In terms of y, $1 = (\cos y)\dfrac{dy}{dx}$; $\dfrac{dy}{dx} = \sec y$.

In terms of x, $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}}$ since $\sec y = \dfrac{1}{\sqrt{1-\sin^2 y}}$ and $x = \sin y$.

3.7 Derivatives of Inverse Functions

1. Since $f'(x) = 30x^2 + 8 > 0$ for all x, f is increasing on $(-\infty, \infty)$. It follows from Theorem 3.7.2 that f^{-1} exists.

2. Since $f'(x) = -35x^4 - 18x^2 - 2 < 0$ for all x, f is decreasing on $(-\infty, \infty)$. It follows that f^{-1} exists.

3. Writing $f(x) = x(x^2 + x - 2) = x(x+2)(x-1)$, we see that f has three distinct zeros. It follows that f is not one-to-one and that f^{-1} does not exist.

4. Writing $f(x) = x^2(x^2 - 2)$ we see that f has three distinct zeros. It follows that f is not one-to-one and that f^{-1} does not exist.

5. Since $f(x) = 2x^3 + 8$, $f(1/2) = 33/4$, $f'(x) = 6x^2$ and $f'(1/2) = 3/2$. Now $f(1/2) = 33/4$ implies $f^{-1}(33/4) = 1/2$. Thus, $(f^{-1})'(33/4) = 1/f'(f^{-1}(33/4)) = 1/f'(1/2) = 1/(3/2) = 2/3$.

6. Since $f(x) = -x^3 - 3x + 7$, $f(-1) = 11$, $f'(x) = -3x^2 - 3$, and $f'(-1) = -6$. Now $f(-1) = 11$ implies $f^{-1}(11) = -1$. Thus, $(f^{-1})'(11) = 1/f'(f^{-1}(11)) = 1/f'(-1) = 1/(-6) = -1/6$.

7. $f(x) = 2 + \dfrac{1}{x}$; $f'(x) = -\dfrac{1}{x^2}$; $f^{-1}(x) = \dfrac{1}{x-2}$. Using Theorem 3.7.4, we have

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{-1/\left(\dfrac{1}{x-2}\right)^2} = -\frac{1}{(x-2)^2}.$$

By direct differentiation, $(f^{-1})(x) = \dfrac{0-1}{(x-2)^2} = -\dfrac{1}{(x-2)^2}$.

8. $f'(x) = 15(5x+7)^2$; $f^{-1}(x) = \dfrac{x^{1/3} - 7}{5}$. Using Theorem 3.7.4, we have

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{15\left\{5[(x^{1/3} - 7)/5] + 7\right\}^2} = \frac{1}{15(x^{1/3})^2} = \frac{x^{-2/3}}{15}.$$

By direct differentiation, $(f^{-1})'(x) = \dfrac{1}{5}\left(\dfrac{1}{3}x^{-2/3}\right) = \dfrac{x^{-2/3}}{15}$.

9. At $x = 3$, $y = 5$, so $(5, 3)$ is the corresponding point on the graph of f^{-1}. Since $\dfrac{dy}{dx} = x^2 + 1$ and $\left.\dfrac{dy}{dx}\right|_{x=3} = 10$, then by (4) $\left.\dfrac{dx}{dy}\right|_{y=5} = \dfrac{1}{10}$. Thus, an equation of the tangent line is $y - 3 = \dfrac{1}{10}(x - 5)$ or $y = \dfrac{1}{10}x + \dfrac{5}{2}$.

10. At $x = 0$, $y = -1$, so $(-1, 0)$ is the corresponding point on the graph of f^{-1}. Since $\dfrac{dy}{dx} = -\dfrac{6}{(4x-1)^2}$ and $\left.\dfrac{dy}{dx}\right|_{x=0} = -6$, then by (4) $\left.\dfrac{dx}{dy}\right|_{y=-1} = -\dfrac{1}{6}$. Thus, an equation of the tangent line is $y - 0 = -\dfrac{1}{6}(x + 1)$ or $y = -\dfrac{1}{6}x - \dfrac{1}{6}$.

11. At $x = 1$, $y = 8$, so $(8, 1)$ is the corresponding point on the graph of f^{-1}. Since $\dfrac{dy}{dx} = 15x^4(x^5 + 1)^2$ and $\left.\dfrac{dy}{dx}\right|_{x=1} = 60$, then by (4) $\left.\dfrac{dx}{dy}\right|_{y=8} = \dfrac{1}{60}$. Thus, an equation of the tangent line is $y - 1 = \dfrac{1}{60}(x - 8)$ or $y = \dfrac{1}{60}x + \dfrac{13}{15}$.

12. At $x = -3$, $y = 14$, so $(14, -3)$ is the corresponding point on the graph of f^{-1}. Since $\dfrac{dy}{dx} = -\dfrac{2}{(x+2)^{2/3}}$ and $\left.\dfrac{dy}{dx}\right|_{x=-3} = -2$, then by (4) $\left.\dfrac{dx}{dy}\right|_{y=14} = -\dfrac{1}{2}$. Thus, an equation of the tangent line is $y + 3 = -\dfrac{1}{2}(x - 14)$ or $y = -\dfrac{1}{2}x + 4$.

13. $y' = \dfrac{5}{\sqrt{1 - (5x-1)^2}} = \dfrac{5}{\sqrt{10x - 25x^2}}$

14. $y' = \dfrac{-1/3}{\sqrt{1 - \left(\dfrac{x+1}{3}\right)^2}} = -\dfrac{1}{\sqrt{8 - 2x - x^2}}$

15. $y' = 4\left[\dfrac{-1/2}{1 + (x/2)^2}\right] = \dfrac{-2}{1 + x^2/4} = -\dfrac{8}{4 + x^2}$

16. $y' = 2 - 10\left[\dfrac{5}{5x\sqrt{(5x)^2 - 1}}\right] = 2 - \dfrac{10}{x\sqrt{25x^2 - 1}}$

17. $y' = 2\sqrt{x}\left[\dfrac{1}{1 + (\sqrt{x})^2}\left(\dfrac{1}{2\sqrt{x}}\right)\right] + (\tan^{-1}\sqrt{x})\left(\dfrac{1}{\sqrt{x}}\right) = \dfrac{1}{1+x} + \dfrac{\tan^{-1}\sqrt{x}}{\sqrt{x}}$

18. $y' = (\tan^{-1} x)\left(\dfrac{-1}{1+x^2}\right) + (\cot^{-1} x)\left(\dfrac{1}{1+x^2}\right) = \dfrac{\cot^{-1} x - \tan^{-1} x}{1 + x^2}$

19. $y' = \dfrac{(\cos^{-1} 2x)\left[\dfrac{2}{\sqrt{1-(2x)^2}}\right] - (\sin^{-1} 2x)\left[\dfrac{-2}{\sqrt{1-(2x)^2}}\right]}{(\cos^{-1} 2x)^2} = \dfrac{2(\cos^{-1} 2x + \sin^{-1} 2x)}{\sqrt{1 - 4x^2}(\cos^{-1} 2x)^2}$

20. $y' = \dfrac{(\sin x)(1/\sqrt{1-x^2}) - (\sin^{-1} x)(\cos x)}{(\sin x)^2} = \dfrac{1}{\sqrt{1-x^2}(\sin x)} - \dfrac{(\sin^{-1} x)(\cos x)}{\sin^2 x}$

21. $y = (\tan^{-1} x^2)^{-1}; \quad y' = -(\tan^{-1} x^2)^{-2}\left[\dfrac{2x}{1+(x^2)^2}\right] = -\dfrac{2x}{(\tan^{-1} x^2)^2(1+x^4)}$

22. $y = x^{-1}\sec^{-1} x; \quad y' = x^{-1}\left(\dfrac{1}{x\sqrt{x^2-1}}\right) + (\sec^{-1} x)(-x^{-2}) = \dfrac{1}{x^2\sqrt{x^2-1}} - \dfrac{\sec^{-1} x}{x^2}$

23. $y' = \dfrac{2}{\sqrt{1-x^2}} + x\left(\dfrac{-1}{\sqrt{1-x^2}}\right) + (\cos^{-1} x)(1) = \dfrac{2-x}{\sqrt{1-x^2}} + \cos^{-1} x$

24. $y' = \dfrac{-1}{1+x^2} - \left[\dfrac{1}{1+x^2/(1-x^2)}\right]\left[\dfrac{\sqrt{1-x^2} - x\dfrac{-2x}{2\sqrt{1-x^2}}}{1-x^2}\right]$

$= \dfrac{-1}{1+x^2} - \left[\sqrt{1-x^2} + \dfrac{x^2}{\sqrt{1-x^2}}\right] = -\dfrac{1}{1+x^2} - \dfrac{1}{\sqrt{1-x^2}}$

25. $y' = 3\left(x^2 - 9\tan^{-1}\dfrac{x}{3}\right)^2\left[2x - \dfrac{9}{1+(x/3)^2}\left(\dfrac{1}{3}\right)\right] = 3\left(x^2 - 9\tan^{-1}\dfrac{x}{3}\right)^2\left(2x - \dfrac{27}{9+x^2}\right)$

26. $y' = \dfrac{1}{2}[x - \cos^{-1}(x+1)]^{-1/2}\left[1 - \dfrac{-1}{\sqrt{1-(x+1)^2}}\right] = \dfrac{\sqrt{-x^2-2x}+1}{2\sqrt{x-\cos^{-1}(x+1)}\sqrt{-x^2-2x}}$

27. $F'(t) = \dfrac{1}{1+\left(\dfrac{t-1}{t+1}\right)^2}\left[\dfrac{(t+1)-(t-1)}{(t+1)^2}\right] = \dfrac{(t+1)^2}{2(t^2+1)} \cdot \dfrac{2}{(t+1)^2} = \dfrac{1}{t^2+1}$

28. $g'(t) = \dfrac{-1}{\sqrt{1-(\sqrt{3t+1})^2}}\left[\dfrac{1}{2}(3t+1)^{-1/2}(3)\right] = \dfrac{-3}{2\sqrt{1-(3t+1)}\sqrt{3t+1}}$

$= \dfrac{-3}{2\sqrt{-3t}\sqrt{3t+1}} = -\dfrac{3}{2\sqrt{9t^2-3t}}$

29. $f'(x) = \dfrac{1}{\sqrt{1-(\cos 4x)^2}}(-4\sin 4x) = \dfrac{-4\sin 4x}{\sqrt{1-\cos^2 4x}} = -\dfrac{4\sin 4x}{|\sin 4x|}$

30. $f'(x) = \dfrac{1}{1+(\sin^2 x)/4}\left(\dfrac{1}{2}\cos x\right) = \dfrac{2\cos x}{4+\sin^2 x}$

31. $f'(x) = \sec^2(\sin^{-1} x^2)\left(\dfrac{2x}{\sqrt{1-(x^2)^2}}\right) = \dfrac{2x\sec^2(\sin^{-1} x^2)}{\sqrt{1-x^4}}$

32. $f'(x) = -[\sin(x\sin^{-1} x)]\left[\dfrac{x}{\sqrt{1-x^2}} + \sin^{-1} x\right]$

33. $\dfrac{1}{1+y^2}\dfrac{dy}{dx} = 2x + 2y\dfrac{dy}{dx}; \quad \dfrac{dy}{dx} = \dfrac{2x}{1/(1+y^2) - 2y} = \dfrac{2x(1+y^2)}{1-2y-2y^3}$

34. $\dfrac{1}{\sqrt{1-y^2}}\dfrac{dy}{dx}-\dfrac{-1}{\sqrt{1-x^2}}=0;\quad \dfrac{dy}{dx}=-\dfrac{\sqrt{1-y^2}}{\sqrt{1-x^2}}=-\sqrt{\dfrac{1-y^2}{1-x^2}}$

35. $f'(x)=\dfrac{1}{\sqrt{1-x^2}}-\dfrac{1}{\sqrt{1-x^2}}=0$. Since $f'(x)=0$, $f(x)$ is constant.

36. $f'(x)=\dfrac{1}{1+x^2}-\dfrac{1/x^2}{1+(1/x)^2}=\dfrac{1}{1+x^2}-\dfrac{1}{x^2+1}=0$. Since $f'(x)=0$, $f(x)$ is constant. From $f(1)=\tan^{-1}1+\tan^{-1}1=\pi/2$, we conclude that $\tan^{-1}x+\tan^{-1}(1/x)=\pi/2$.

37. $y'=\dfrac{1/2}{\sqrt{1-x^2/4}}=\dfrac{1}{\sqrt{4-x^2}};\quad m=y'(1)=\dfrac{1}{\sqrt{3}}$

38. $y'=2(\cos^{-1}x)\left(\dfrac{-1}{\sqrt{1-x^2}}\right)=-\dfrac{2\cos^{-1}x}{\sqrt{1-x^2}}$

$m=y'(1/\sqrt{2})=-\dfrac{2\cos^{-1}(1/\sqrt{2})}{\sqrt{1-1/2}}=-\dfrac{2(\pi/4)}{\sqrt{1/2}}=-\dfrac{\sqrt{2}\pi}{2}$

39. $f'(x)=\dfrac{x}{1+x^2}+\tan^{-1}x;\quad f'(1)=\dfrac{1}{2}+\dfrac{\pi}{4}=\dfrac{2+\pi}{4};\quad f(1)=\dfrac{\pi}{4}$.

The point on the graph is $(1,\pi/4)$ and the slope is $\dfrac{2+\pi}{4}$. Thus, an equation of the tangent line is $y-\dfrac{\pi}{4}=\dfrac{2+\pi}{4}(x-1)$ or $y=\dfrac{2+\pi}{4}x-\dfrac{1}{2}$.

40. $f'(x)=\dfrac{1}{\sqrt{1-(x-1)^2}};\quad f'(1/2)=2\sqrt{3};\quad f(1/2)=\sin\left(-\dfrac{1}{2}\right)=-\dfrac{\pi}{6}$.

The point on the graph is $(1/2,-\pi/6)$ and the slope is $\dfrac{2}{\sqrt{3}}$. Thus, an equation of the tangent line is $y+\dfrac{\pi}{6}=\dfrac{2}{\sqrt{3}}\left(x-\dfrac{1}{2}\right)$ or $y=\dfrac{2\sqrt{3}}{3}x-\dfrac{\sqrt{3}}{3}-\dfrac{\pi}{6}$.

41.

42. Solving $f'(x)=\dfrac{1}{1+x^2}=\dfrac{1}{4}$, we get $1+x^2=4$ and $x=\pm\sqrt{3}$. The points whose tangents have slope $\dfrac{1}{4}$ are therefore $(\sqrt{3},\pi/3)$ and $(-\sqrt{3},-\pi/3)$. Thus, the tangent lines have equations $y\mp\dfrac{\pi}{3}=\dfrac{1}{4}(x\mp\sqrt{3})$ or $y=\dfrac{1}{4}x\mp\dfrac{\sqrt{3}}{4}\pm\dfrac{\pi}{3}$.

43. Multiple applications of (3) yields $(f^{-1})''(x)=\dfrac{d}{dx}(f^{-1})'(x)=\dfrac{d}{dx}\dfrac{1}{f'(f^{-1}(x))}$, resulting in

$$\frac{d}{dx}\frac{1}{f'(f^{-1}(x))}=\frac{-f''(f^{-1}(x))(f^{-1})'(x)}{[f'(f^{-1}(x))]^2}=\frac{-f''(f^{-1}(x))\cdot\dfrac{1}{f'(f^{-1}(x))}}{[f'(f^{-1}(x))]^2}=-\frac{f''(f^{-1}(x))}{[f'(f^{-1}(x))]^3}.$$

3.8 Exponential Functions

1. $y' = -e^{-x}$

2. $y' = 2e^{2x+3}$

3. $y' = \dfrac{e^{\sqrt{x}}}{2\sqrt{x}}$

4. $y' = 10e^{\sin 10x}\cos 10x$

5. $y' = 5^{2x}(\ln 5)(2) = 2(\ln 5)5^{2x}$

6. $y' = 10^{-3x^2}(\ln 10)(-6x) = -6x(\ln 10)10^{-3x^2}$

7. $y' = (x^3)(e^{4x})(4) + (e^{4x})(3x^2) = 4x^3e^{4x} + 3x^2e^{4x}$

8. $y' = (e^{-x})(\cos \pi x)(\pi) + (\sin \pi x)(e^{-x})(-1) = \pi e^{-x}\cos \pi x - e^{-x}\sin \pi x$

9. $f'(x) = \dfrac{(x)(-2e^{-2x}) - (e^{-2x})(1)}{x^2} = -\dfrac{(2x+1)e^{-2x}}{x^2}$

10. $f'(x) = \dfrac{(x+e^x)[(x)(e^x)+(e^x)(1)] - (xe^x)(1+e^x)}{(x+e^x)^2} = \dfrac{x^2e^x + e^{2x}}{(x+e^x)^2}$

11. $y' = \dfrac{1}{2}(1+e^{-5x})^{-1/2}(-5e^{-5x}) = -\dfrac{5e^{-5x}}{2\sqrt{1+e^{-5x}}}$

12. $y' = 10(e^{2x} - e^{-2x})^9(2e^{2x} + 2e^{-2x}) = 20(e^{2x} - e^{-2x})^9(e^{2x} + e^{-2x})$

13. $y = 2(e^{x/2} + e^{-x/2})^{-1}$; $y' = -2(e^{x/2} + e^{-x/2})^{-2}\left(\dfrac{1}{2}e^{x/2} - \dfrac{1}{2}e^{-x/2}\right) = -\dfrac{e^{x/2} - e^{-x/2}}{(e^{x/2} + e^{-x/2})^2}$

14. $$y' = \frac{(e^x - e^{-x})(e^x - e^{-x}) - (e^x + e^{-x})(e^x + e^{-x})}{(e^x - e^{-x})^2} = \frac{e^{2x} - 2 + e^{-2x} - (e^{2x} + 2 + e^{-2x})}{(e^x - e^{-x})^2}$$
$$= -\frac{4}{(e^x - e^{-x})^2}$$

15. $y = e^{8x}$; $y' = 8e^{8x}$

16. $y = e^{9x}$; $y' = 9e^{9x}$

17. $y = e^{3x-3}$; $y' = 3e^{3x-3}$

18. $y = e^{-100x}$; $y' = -100e^{-100x}$

19. $f(x) = e^{x^{1/3}} + e^{x/3}$; $f'(x) = \dfrac{1}{3}x^{-2/3}e^{x^{1/3}} + \dfrac{1}{3}e^{x/3}$

20. $$f'(x) = (2x+1)^3\left[4(1-x)^3e^{-(1-x)^4}\right] + e^{-(1-x)^4}\left[6(2x+1)^2\right]$$
$$= 4(2x+1)^3(1-x)^3e^{-(1-x)^4} + 6(2x+1)^2e^{-(1-x)^4}$$

21. $f'(x) = e^{-x}(e^x \sec^2 e^x) - e^{-x} \tan e^x = \sec^2 e^x - e^{-x} \tan e^x$

22. $f'(x) = (\sec e^{2x} \tan e^{2x})(e^{2x})(2) = 2e^{2x} \sec e^{2x} \tan e^{2x}$

23. $f'(x) = e^{x\sqrt{x^2+1}} \left[\dfrac{x}{2} 2x(x^2+1)^{-1/2} + \sqrt{x^2+1}\right] = \dfrac{2x^2+1}{\sqrt{x^2+1}} e^{x\sqrt{x^2+1}}$

24. $y' = e^{\frac{x+2}{x-2}} \dfrac{x-2-(x+2)}{(x-2)^2} = -\dfrac{4}{(x-2)^2} e^{\frac{x+2}{x-2}}$

25. $y' = 2xe^{x^2} e^{e^{x^2}}$

26. $y' = e^x + (1 - e^{-x})e^{x+e^{-x}}$

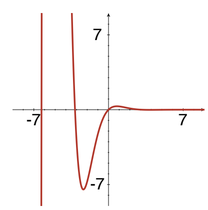
27. $y' = 2(e^x + 1)(e^x) = 2e^x(e^x + 1)$. The point on the graph is $(0, 4)$ and the slope is $y'(0) = 4$. An equation of the tangent line is $y - 4 = 4(x - 0)$ or $y = 4x + 4$.

28. $y' = -(x - 1)e^{-x} + e^{-x}$. The point on the graph is $(0, -1)$ and the slope of the normal line is $-1/y'(0) = -1/2$.

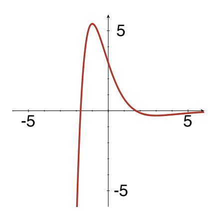
29. $y' = e^x$. The slope of the line $3x - y = 7$ is 3. Solving $e^x = 3$, we obtain $x = \ln 3$. The point on the graph of $y = e^x$ at which the tangent line is parallel to $3x - y = 7$ is $(\ln 3, e^{\ln 3})$ or $(\ln 3, 3)$.

30. $y' = 5 + 2e^{2x}$. The slope of the line $y = 6x$ is 6. Solving $5 + 2e^{2x} = 6$, we obtain $2x = \ln \dfrac{1}{2}$ or $x = -\dfrac{\ln 2}{2}$. The point on the graph of $y = 5x + e^{2x}$ at which the tangent line is parallel to $y = 6x$ is $(-\dfrac{\ln 2}{2}, -\dfrac{5 \ln 2}{2} + e^{-\ln 2})$ or $(-\dfrac{\ln 2}{2}, \dfrac{1 - 5\ln 2}{2})$.

31. $f'(x) = e^{-x}(\cos x) + -e^{-x} \sin x = e^{-x}(\cos x - \sin x)$. The tangent line is horizontal when $f'(x) = e^{-x}(\cos x - \sin x) = 0$, or when $\cos x = \sin x$; $\dfrac{\sin x}{\cos x} = 1$; $\tan x = 1$. Thus, $x = \pi/4 + k\pi$, where k is an integer.



32. $f'(x) = (3 - x^2)(-e^{-x}) - 2x(e^{-x}) = e^{-x}(x^2 - 2x - 3) = e^{-x}(x - 3)(x + 1)$. The tangent line is horizontal when $f'(x) = e^{-x}(x-3)(x+1) = 0$, or $x = 3$ or $x = -1$.



33. $\dfrac{dy}{dx} = 2xe^{x^2}$; $\dfrac{d^2y}{dx^2} = 2x(2xe^{x^2}) + 2e^{x^2} = e^{x^2}(4x^2 + 2)$

$\dfrac{d^3y}{dx^3} = e^{x^2}(8x) + 2xe^{x^2}(4x^2 + 2) = 8x^3e^{x^2} + 12xe^{x^2}$

34. $y = \dfrac{1}{1 + e^{-x}} = \dfrac{1}{1 + \dfrac{1}{e^x}} = \dfrac{e^x}{e^x + 1}$; $\dfrac{dy}{dx} = \dfrac{(e^x + 1)e^x - e^x(e^x)}{(e^x + 1)^2} = \dfrac{e^x}{(e^x + 1)^2}$

$$\frac{d^2y}{dx^2} = \frac{(e^x+1)^2e^x - e^x\left[2(e^x+1)e^x\right]}{(e^x+1)^4} = \frac{e^x(1-e^x)}{(e^x+1)^3}$$

Or, without rewriting y, $\dfrac{dy}{dx} = \dfrac{0(1+e^{-x}) - (-e^{-x})(1)}{(1+e^{-x})^2} = \dfrac{e^{-x}}{(1+e^{-x})^2}$

$$\frac{d^2y}{dx^2} = \frac{-e^{-x}(1+e^{-x})^2 - 2(1+e^{-x})(-e^{-x})(e^{-x})}{(1+e^{-x})^4} = \frac{-e^{-x}(-e^{-2x}+1)}{(1+e^{-x})^4} = \frac{-e^{-x}(1-e^{-x})}{(1+e^{-x})^3}$$

35. $\dfrac{dy}{dx} = 2e^{2x}\cos e^{2x}$; $\dfrac{d^2y}{dx^2} = 2e^{2x}(-2e^{2x}\sin e^{2x}) + 4e^{2x}\cos e^{2x} = -4e^{4x}\sin e^{2x} + 4e^{2x}\cos e^{2x}$

36. $\dfrac{dy}{dx} = x^2e^x + 2xe^x$; $\dfrac{d^2y}{dx^2} = x^2e^x + 2xe^x + 2xe^x + 2e^x = (x^2+4x+2)e^x$

$$\frac{d^3y}{dx^3} = (x^2+4x+2)e^x + (2x+4)e^x = (x^2+6x+6)e^x$$

$$\frac{d^4y}{dx^4} = (x^2+6x+6)e^x + (2x+6)e^x = (x^2+8x+12)e^x$$

37. $y' = -3C_1e^{-3x} + 2C_2e^{2x}$; $y'' = 9C_1e^{-3x} + 4C_2e^{2x}$

$$y'' + y' - 6y = 9C_1e^{-3x} + 4C_2e^{2x} - 3C_1e^{-3x} + 2C_2e^{2x} - 6C_1e^{-3x} - 6C_2e^{2x} = 0$$

38.
$$\begin{aligned}
y' &= C_1e^{-x}(-2\sin 2x) - C_1e^{-x}\cos 2x + C_2e^{-x}(2\cos 2x) - C_2e^{-x}\sin 2x\\
&= (2C_2 - C_1)e^{-x}\cos 2x - (2C_1 + C_2)e^{-x}\sin 2x\\
y'' &= (2C_2 - C_1)e^{-x}(-2\sin 2x) - (2C_2 - C_1)e^{-x}\cos 2x\\
&\qquad - (2C_1 + C_2)e^{-x}(2\cos 2x) + (2C_1 + C_2)e^{-x}\sin 2x\\
&= (4C_1 - 3C_2)e^{-x}\sin 2x - (3C_1 + 4C_2)e^{-x}\cos 2x\\
y'' + 2y' + 5y &= (4C_1 - 3C_2)e^{-x}\sin 2x - (3C_1 + 4C_2)e^{-x}\cos 2x + 2[(2C_2 - C_1)e^{-x}\cos 2x\\
&\qquad - (2C_1 + C_2)e^{-x}\sin 2x] + 5(C_1e^{-x}\cos 2x + C_2e^{-x}\sin 2x)\\
&= (4C_1 - 3C_2 - 4C_1 - 2C_2 + 5C_2)e^{-x}\sin 2x\\
&\qquad + (-3C_1 - 4C_2 + 4C_2 - 2C_1 + 5C_1)e^{-x}\cos 2x = 0
\end{aligned}$$

39. For $y = Ce^{kx}$, $y' = kCe^{kx} = k\left(Ce^{kx}\right) = ky$.

40. (a) Given that $y' = -0.01y$, we set $k = -0.01$. Since $y(0) = 100 = Ce^{-0.01(0)}$, $C = 100$, and thus the desired function is $y(x) = 100e^{-0.01x}$.

(b) Given that $\dfrac{dP}{dt} - 0.15P = 0$ and thus $\dfrac{dP}{dt} = 0.15P$, we set $k = 0.15$. Since $P(0) = P_0 = Ce^{0.15(0)}$, $C = P_0$, and thus the desired function is $P(t) = P_0e^{0.15t}$.

41. $\dfrac{dy}{dx} = \left(1 + \dfrac{dy}{dx}\right)e^{x+y}$; $\dfrac{dy}{dx} - \dfrac{dy}{dx}e^{x+y} = e^{x+y}$; $\dfrac{dy}{dx} = \dfrac{e^{x+y}}{1 - e^{x+y}}$

42. $x\dfrac{dy}{dx} + y = e^y\dfrac{dy}{dx}$; $(e^y - x)\dfrac{dy}{dx} = y$; $\dfrac{dy}{dx} = \dfrac{y}{e^y - x}$

43. $\dfrac{dy}{dx} = -\left(x\dfrac{dy}{dx} + y\right)e^{xy}\sin e^{xy}$; $\dfrac{dy}{dx} + x\dfrac{dy}{dx}e^{xy}\sin e^{xy} = -ye^{xy}\sin e^{xy}$; $\dfrac{dy}{dx} = -\dfrac{ye^{xy}\sin e^{xy}}{1 + xe^{xy}\sin e^{xy}}$

44. $\dfrac{dy}{dx} = 2(x+y)\left(1+\dfrac{dy}{dx}\right)e^{(x+y)^2}$; $\dfrac{dy}{dx} - 2(x+y)\dfrac{dy}{dx}e^{(x+y)^2} = 2(x+y)e^{(x+y)^2}$;

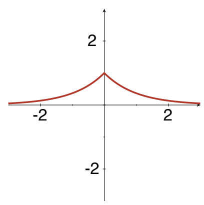
$$\frac{dy}{dx} = \frac{2x(x+y)e^{(x+y)^2}}{1-2(x+y)e^{(x+y)^2}}$$

45. $1 + 2y\dfrac{dy}{dx} = e^{x/y}\left[\dfrac{y - x(dy/dx)}{y^2}\right]$; $2y\dfrac{dy}{dx} + \dfrac{x}{y^2}\left(\dfrac{dy}{dx}\right)e^{x/y} = \dfrac{1}{y}e^{x/y} - 1$;

$2y^3\dfrac{dy}{dx} + x\dfrac{dy}{dx}e^{x/y} = ye^{x/y} - y^2$; $\dfrac{dy}{dx} = \dfrac{ye^{x/y} - y^2}{2y^3 + xe^{x/y}}$

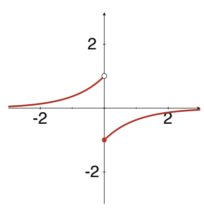
46. $e^x + e^y\dfrac{dy}{dx} = \dfrac{dy}{dx}$; $e^x = \dfrac{dy}{dx} - e^y\dfrac{dy}{dx}$; $\dfrac{dy}{dx} = \dfrac{e^x}{1-e^y}$

47. (a)



(b) Since $f(x) = e^{-|x|} = \begin{cases} e^x, & x < 0 \\ e^{-x}, & x \geq 0 \end{cases}$, we get $f'(x) = \begin{cases} e^x, & x < 0 \\ -e^{-x}, & x \geq 0 \end{cases}$.

(c)

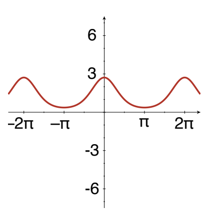


(d) Because $\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^-} e^x = 1$ and $\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+}(-e^{-x}) = -1$, the function is not differentiable at $x = 0$.

48. (a) $\cos x$ is periodic with period 2π, so $e^{\cos x}$ will also be periodic with period 2π.

(b) The tangent to f is horizontal when $f'(x) = -e^{\cos x}\sin x = 0$, and thus when $\sin x = 0$. $\sin x = 0$ for $x = k\pi$, where k is an integer, and so the tangent is horizontal at points $(k\pi, -e^{\cos k\pi}\sin k\pi)$, where k is an integer.

(c)



49. (a) $\dfrac{dP}{dt} = \dfrac{0[bP_0 + (a - bP_0)e^{-at}] - [-a(a - bP_0)e^{-at}]aP_0}{[bP_0 + (a - bP_0)e^{-at}]^2} = \dfrac{a^2 P_0(a - bP_0)e^{-at}}{[bP_0 + (a - bP_0)e^{-at}]^2}$

$$= \left(\frac{aP_0}{bP_0 + (a - bP_0)e^{-at}}\right)\left(\frac{a^2e^{-at} - abP_0e^{-at}}{bP_0 + (a - bP_0)e^{-at}}\right)$$

$$= \left(\frac{aP_0}{bP_0 + (a - bP_0)e^{-at}}\right)\left[\frac{(abP_0 - abP_0) + a^2e^{-at} - abP_0e^{-at}}{bP_0 + (a - bP_0)e^{-at}}\right]$$

$$= \left(\frac{aP_0}{bP_0 + (a - bP_0)e^{-at}}\right)\left(\frac{abP_0 + a^2e^{-at} - abP_0e^{-at} - abP_0}{bP_0 + (a - bP_0)e^{-at}}\right)$$

$$= \left(\frac{aP_0}{bP_0 + (a - bP_0)e^{-at}}\right)\left[\frac{a(bP_0 + ae^{-at} - bP_0e^{-at})}{bP_0 + (a - bP_0)e^{-at}} - \frac{b(aP_0)}{bP_0 + (a - bP_0)e^{-at}}\right]$$

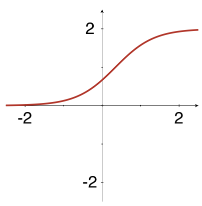
$$= \left(\frac{aP_0}{bP_0 + (a - bP_0)e^{-at}}\right)\left\{a\left[\frac{bP_0 + (a - bP_0)e^{-at}}{bP_0 + (a - bP_0)e^{-at}}\right] - b\left[\frac{aP_0}{bP_0 + (a - bP_0)e^{-at}}\right]\right\}$$

$$= P(a - bP)$$

(b) $\displaystyle\lim_{t\to-\infty} P(t) = 0$

$\displaystyle\lim_{t\to\infty} P(t) = a/b = 2$

(c)



(d) For $a = 2$, $b = 1$, and $P_0 = 1$, $P(t) = \dfrac{2}{1 + e^{-2t}}$, $P'(t) = \dfrac{4e^{-2t}}{(1 + e^{-2t})^2}$, and

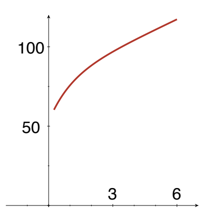
$$P''(t) = \frac{-8e^{-2t}(1 + e^{-2t})^2 - 2(-2e^{-2t})(1 + e^{-2t})(4e^{-2t})}{(1 + e^{-2t})^4}$$
$$= \frac{(1 + e^{-2t})(-8e^{-2t} - 8e^{-4t} + 16e^{-4t})}{(1 + e^{-2t})^4} = \frac{8e^{-4t} - 8e^{-2t}}{(1 + e^{-2t})^3}.$$

Thus, $P''(t) = 0$ only when $e^{-4t} = e^{-2t}$ or $t = 0$.

50. (a) $h(2) = 79.04 + 6.39(2) - e^{3.26 - 0.99(2)} \approx 88.22$ cm

(b) $h'(t) = 6.39 + 0.99e^{3.26 - 0.99t}$; $\quad h'(2) = 6.39 + 0.99e^{3.26 - 0.99(2)} \approx 9.95$ cm/yr

(c)



(d) When $h = 100$ we observe that $t \approx 3.3$ years.

51. The slope of the tangent line at $x = x_0$ is $-e^{-x_0}$. The equation of the line is $y - e^{-x_0} = -e^{-x_0}(x - x_0)$. To find the x-intercept we set $y = 0$ and solve for x: $0 - e^{-x_0} = -e^{-x_0}(x - x_0)$, so $1 = x - x_0$, and $x = x_0 + 1$. Thus, the x-intercept is $x_0 + 1$.

52. The slope of the tangent to $y = e^x$ at $x = 0$ is $e^x = e^0 = 1$. The slope of the tangent to $y = e^{-x}$ at $x = 0$ is $-e^{-x} = -e^{-0} = -1$. Thus, the tangents are perpendicular to each other.

53. Rewriting $2x + y = 1$ as $y = 1 - 2x$, we see that the slope of this line is -2. Since the derivative of $y = e^x$ is e^x, which is never less than zero, then there is no point on the graph of $y = e^x$ at which the tangent line is parallel to $2x + y = 1$.

54. $f'(x) = e^x$. At any given $x = x_0$, $y = y' = e^{x_0}$, and thus an equation of the tangent line at (x_0, e^{x_0}) is $y - e^{x_0} = e^{x_0}(x - x_0)$ or $y = e^{x_0}x + (1 - x_0)e^{x_0}$. Substituting the origin $(0, 0)$ for (x, y) and solving for x_0, we get $0 = e^{x_0}(0) + (1 - x_0)e^{x_0}$, $0 = (1 - x_0)e^{x_0}$, $x_0 = 1$. Thus, the only tangent line to the graph of $f(x) = e^x$ that passes through the origin has the equation $y = ex$.

55. The derivatives for $n = 1$ to 6 are:

$$\frac{d}{dx}\sqrt{e^x} = \frac{1}{2}(e^x)^{-1/2}e^x = \frac{1}{2}\sqrt{e^x}; \qquad \frac{d^2}{dx^2}\sqrt{e^x} = \frac{1}{4}(e^x)^{-1/2}e^x = \frac{1}{4}\sqrt{e^x};$$
$$\frac{d^3}{dx^3}\sqrt{e^x} = \frac{1}{8}(e^x)^{-1/2}e^x = \frac{1}{8}\sqrt{e^x}; \qquad \frac{d^4}{dx^4}\sqrt{e^x} = \frac{1}{16}(e^x)^{-1/2}e^x = \frac{1}{16}\sqrt{e^x};$$
$$\frac{d^5}{dx^5}\sqrt{e^x} = \frac{1}{32}(e^x)^{-1/2}e^x = \frac{1}{32}\sqrt{e^x}; \qquad \frac{d^6}{dx^6}\sqrt{e^x} = \frac{1}{64}(e^x)^{-1/2}e^x = \frac{1}{64}\sqrt{e^x}.$$

We can verify by induction that $\dfrac{d^n}{dx^n}\sqrt{e^x} = \dfrac{1}{2^n}\sqrt{e^x}$:

$$\frac{d^n}{dx^n}\sqrt{e^x} = \frac{d}{dx}\left(\frac{d^{n-1}}{dx^{n-1}}\sqrt{e^x}\right) = \frac{d}{dx}\frac{1}{2^{n-1}}\sqrt{e^x} = \frac{1}{2}\cdot\frac{1}{2^{n-1}}(e^x)^{-1/2}\cdot e^x = \frac{1}{2^n}\sqrt{e^x}$$

56. The derivatives for $n = 1$ to 6 are:

$$\frac{d}{dx}xe^{-x} = -xe^{-x} + e^{-x} = (1 - x)e^{-x}; \qquad \frac{d^2}{dx^2}xe^{-x} = (x - 2)e^{-x};$$
$$\frac{d^3}{dx^3}xe^{-x} = (3 - x)e^{-x}; \qquad \frac{d^4}{dx^4}xe^{-x} = (x - 4)e^{-x};$$
$$\frac{d^5}{dx^5}xe^{-x} = (5 - x)e^{-x}; \qquad \frac{d^6}{dx^6}xe^{-x} = (x - 6)e^{-x}.$$

We can verify by induction that $\dfrac{d^n}{dx^n}xe^{-x} = (-1)^n(x - n)e^{-x}$:

$$\begin{aligned}\frac{d^n}{dx^n}xe^{-x} &= \frac{d}{dx}\left(\frac{d^{n-1}}{dx^{n-1}}xe^{-x}\right) = \frac{d}{dx}(-1)^{n-1}[x - (n - 1)]e^{-x} = \frac{d}{dx}(-1)^{n-1}(x - n + 1)e^{-x}\\ &= -(-1)^{n-1}(x - n + 1)e^{-x} + (-1)^{n-1}e^{-x} = (-1)^{n-1}(-x + n - 1 + 1)e^{-x}\\ &= -(-1)^{n-1}(x - n)e^{-x} = (-1)^n(x - n)e^{-x}\end{aligned}$$

57.

$h \to 0$	0.1	0.01	0.001	0.0001	0.00001	0.000001
$\dfrac{(1.5)^h - 1}{h}$	0.413797	0.406288	0.405547	0.405473	0.405466	0.405465

$$m(b) = \lim_{h\to 0} \frac{b^h - 1}{h} \approx 0.405465$$

58.

$h \to 0$	0.1	0.01	0.001	0.0001	0.00001	0.000001
$\dfrac{2^h - 1}{h}$	0.717735	0.695555	0.693387	0.693171	0.693150	0.693147

$$m(b) = \lim_{h\to 0} \frac{b^h - 1}{h} \approx 0.693147$$

59.

$h \to 0$	0.1	0.01	0.001	0.0001	0.00001	0.000001
$\dfrac{3^h - 1}{h}$	1.161232	1.104669	1.099216	1.098673	1.098618	1.098613

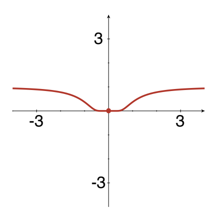
$$m(b) = \lim_{h\to 0} \frac{b^h - 1}{h} \approx 1.098613$$

60.

$h \to 0$	0.1	0.01	0.001	0.0001	0.00001	0.000001
$\dfrac{5^h - 1}{h}$	1.746189	1.622459	1.610734	1.609567	1.609451	1.609439

$$m(b) = \lim_{h\to 0} \frac{b^h - 1}{h} \approx 1.609439$$

61. $f'(x) = 2x^{-3}e^{-1/x^2}$ exists for all $x \neq 0$, and so is differentiable on $(-\infty, 0)$ and $(0, \infty)$. For $x = 0$, we use the definition of the derivative:



$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{e^{-1/(0+h)^2} - 0}{h}$$
$$= \lim_{h\to 0} \frac{e^{-1/h^2}}{h} = 0$$

The limit exists, and thus f is differentiable for all x.

3.9 Logarithmic Functions

1. $y' = \dfrac{10}{x}$

2. $y' = \dfrac{10}{10x} = \dfrac{1}{x}$

3. $y = \dfrac{1}{2}\ln x; \quad y' = \dfrac{1}{2x}$

4. $y' = \dfrac{1}{2}(\ln x)^{-1/2}\dfrac{1}{x} = \dfrac{1}{2x\sqrt{\ln x}}$

5. $y' = \dfrac{4x^3 + 6x}{x^4 + 3x^2 + 1}$

6. $y = 20\ln(x^2+1); \quad y' = \dfrac{40x}{x^2+1}$

7. $y = 3x^2 \ln x; \quad y' = 3x^2\left(\dfrac{1}{x}\right) + 6x\ln x = 3x + 6x\ln x$

8. $y' = 1 - \dfrac{5}{5x+1}$

9. $y' = \dfrac{x\left(\dfrac{1}{x}\right) - \ln x}{x^2} = \dfrac{1-\ln x}{x^2}$

10. $y' = x\left[2(\ln x)\left(\dfrac{1}{x}\right)\right] + (\ln x)^2 = 2\ln x + (\ln x)^2$

11. $y = \ln x - \ln(x+1); \quad y' = \dfrac{1}{x} - \dfrac{1}{x+1} = \dfrac{1}{x(x+1)}$

12. $y' = \dfrac{(\ln 2x)\left(\dfrac{4}{4x}\right) - (\ln 4x)\left(\dfrac{2}{2x}\right)}{(\ln 2x)^2} = \dfrac{\ln 2x - \ln 4x}{x(\ln 2x)^2}$

13. $y' = -\dfrac{-\sin x}{\cos x} = \tan x$

14. $y' = \dfrac{1}{3}\left(\dfrac{3\cos 3x}{\sin 3x}\right) = \cot 3x$

15. $y' = \dfrac{-1/x}{(\ln x)^2} = -\dfrac{1}{x(\ln x)^2}$

16. $y = \ln 1 - \ln x; \quad y' = -\dfrac{1}{x}$

17. $f'(x) = \dfrac{x\left(\dfrac{1}{x}\right) + \ln x}{x \ln x} = \dfrac{1 + \ln x}{x \ln x}$

18. $f'(x) = \dfrac{1}{\ln(\ln x)} \cdot \dfrac{1}{\ln x} \cdot \dfrac{1}{x} = \dfrac{1}{x \ln x \ln(\ln x)}$

19. $g(x) = \left(\dfrac{1}{2} \ln x\right)^{1/2} = \dfrac{1}{\sqrt{2}}(\ln x)^{1/2}$; $\quad g'(x) = \dfrac{1}{\sqrt{2}} \cdot \dfrac{1}{2}(\ln x)^{-1/2} \cdot \dfrac{1}{x} = \dfrac{1}{2x\sqrt{2 \ln x}}$

20. $w'(\theta) = \theta \cos(\ln 5\theta)\left(\dfrac{5}{5\theta}\right) + \sin(\ln 5\theta) = \cos(\ln 5\theta) + \sin(\ln 5\theta)$

21. $H(t) = 2 \ln t + \ln(3t^2 + 6)$; $\quad H'(t) = \dfrac{2}{t} + \dfrac{6t}{3t^2 + 6} = \dfrac{2}{t} + \dfrac{2t}{t^2 + 2}$

22. $G(t) = \dfrac{1}{2} \ln(5t + 1) + 6 \ln(t^3 + 4)$; $\quad G'(t) = \dfrac{5}{10t + 2} + \dfrac{18t^2}{t^3 + 4}$

23. $f(x) = \ln(x + 1) + \ln(x + 2) - \ln(x + 3)$; $\quad f'(x) = \dfrac{1}{x + 1} + \dfrac{1}{x + 2} - \dfrac{1}{x + 3}$

24. $f(x) = \dfrac{1}{2}[5 \ln(3x + 2) - \ln(x^4 + 7)]$; $\quad f'(x) = \dfrac{15}{6x + 4} - \dfrac{2x^3}{x^4 + 7}$

25. $y' = 1/x$. The point on the graph is $(1, 0)$ and the slope of the line is $y'(1) = 1$. Thus, an equation of the tangent line is $y - 0 = 1(x - 1)$ or $y = x - 1$.

26. $y' = 2x/(x^2 - 3)$. The point on the graph is $(2, 0)$ and the slope of the line is $y'(2) = 4$. Thus, an equation of the tangent line is $y - 0 = 4(x - 2)$ or $y = 4x - 8$.

27. $y' = \dfrac{3e^{3x} + 1}{e^{3x} + x}$. The slope of the tangent to the graph of y at $x = 0$ is $\dfrac{3e^{3\cdot 0} + 1}{e^{3\cdot 0} + 0} = 4$.

28. $y = \ln x + \ln e^{-x^3} = \ln x - x^3$. Thus, $y' = \dfrac{1}{x} - 3x^2$. The slope of the tangent to the graph of y at $x = 1$ is $\dfrac{1}{1} - 3(1^2) = -2$.

29. $f'(x) = 2/x$. The slope of the tangent to the graph of f is 4 when $x = 1/2$. Since $f''(x) = -2/x^2$, the slope of the tangent to the graph of f' at $x = 1/2$ is $f''(1/2) = -8$.

30. The slope of $x + 4y = 1$ is $-1/4$. The slope of the desired tangent line is then 4. Solving $y' = 1/x = 4$, we obtain $x = 1/4$. The point on the graph is $(1/4, \ln 1/2)$.

31. The tangent line to f is horizontal when $f'(x) = \dfrac{\left(\dfrac{1}{x}\right)x - \ln x}{x^2} = \dfrac{1 - \ln x}{x^2}$ is 0. Solving for x, we get $\dfrac{1 - \ln x}{x^2} = 0$; $\ln x = 1$; $x = e$. Thus, the tangent is horizontal at $(e, \dfrac{\ln e}{e}) = (e, 1/e)$.

32. The tangent line to f is horizontal when $f'(x) = x^2(1/x) + 2x \ln x = x(1 + 2 \ln x)$ is 0. Solving for x, we get $x(1 + 2 \ln x) = 0$; because x must be positive, it is understood that $x > 0$, so $x = e^{-1/2}$. Thus, the tangent is horizontal at $(e^{-1/2}, (e^{-1/2})^2 \ln e^{-1/2}) = (e^{-1/2}, -1/2e)$.

33. $$\frac{d}{dx}\ln\left(x+\sqrt{x^2-1}\right)=\frac{1+2x(x^2-1)^{-1/2}\left(\frac{1}{2}\right)}{x+\sqrt{x^2-1}}=\frac{1+\frac{x}{\sqrt{x^2-1}}}{x+\sqrt{x^2-1}}=\frac{\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}}}{x+\sqrt{x^2-1}}$$
$$=\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}}\cdot\frac{1}{x+\sqrt{x^2-1}}=\frac{1}{\sqrt{x^2-1}}$$

34. $$\frac{d}{dx}\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)=\frac{d}{dx}[\ln(1+\sqrt{1-x^2})-\ln x]=\frac{(1/2)(1-x^2)^{-1/2}\cdot -2x}{1+\sqrt{1-x^2}}-\frac{1}{x}$$
$$=\frac{-x}{\sqrt{1-x^2}+1-x^2}-\frac{1}{x}=\frac{-x^2-\sqrt{1-x^2}-1+x^2}{x(\sqrt{1-x^2}+1-x^2)}$$
$$=\frac{-\sqrt{1-x^2}-1}{x(\sqrt{1-x^2}+1-x^2)}$$

35. $$\frac{d}{dx}\ln(\sec x+\tan x)=\frac{\sec x\tan x+\sec^2 x}{\sec x+\tan x}=\frac{(\sec x)(\tan x+\sec x)}{\sec x+\tan x}=\sec x$$

36. $$\frac{d}{dx}\ln(\csc x-\cot x)=\frac{-\csc x\cot x-(-\csc^2 x)}{\csc x-\cot x}=\frac{\csc x(-\cot x+\csc x)}{\csc x-\cot x}=\csc x$$

37. $$\frac{dy}{dx}=\frac{1}{x};\quad \frac{d^2y}{dx^2}=-\frac{1}{x^2};\quad \frac{d^3y}{dx^3}=\frac{2}{x^3}$$

38. $$\frac{dy}{dx}=x/x+\ln x=1+\ln x;\quad \frac{d^2y}{dx^2}=\frac{1}{x}$$

39. $$\frac{dy}{dx}=\frac{2\ln|x|}{x};\quad \frac{d^2y}{dx^2}=\frac{\left(\frac{2}{x}\right)x-2\ln|x|}{x^2}=\frac{2(1-\ln|x|)}{x^2}$$

40. $$\frac{dy}{dx}=\frac{5}{5x-3}=5(5x-3)^{-1};\qquad \frac{d^2y}{dx^2}=-25(5x-3)^{-2};$$
$$\frac{d^3y}{dx^3}=250(5x-3)^{-3};\qquad \frac{d^4y}{dx^4}=-3750(5x-3)^{-4}$$

41. $$y'=-\frac{1}{2}C_1x^{-3/2}+C_2\left(x^{-3/2}-\frac{1}{2}x^{-3/2}\ln x\right)$$
$$y''=\frac{3}{4}C_1x^{-5/2}+C_2\left(-\frac{3}{2}x^{-5/2}-\frac{1}{2}x^{-5/2}+\frac{3}{4}x^{-5/2}\ln x\right)$$
$$=\frac{3}{4}C_1x^{-5/2}+C_2\left(-2x^{-5/2}+\frac{3}{4}x^{-5/2}\ln x\right)$$

$$4x^2y'' + 8xy' + y = 4x^2\left[\frac{3}{4}C_1x^{-5/2} + C_2\left(-2x^{-5/2} + \frac{3}{4}x^{-5/2}\ln x\right)\right]$$
$$+ 8x\left[-\frac{1}{2}C_1x^{-3/2} + C_2\left(x^{-3/2} - \frac{1}{2}x^{-3/2}\ln x\right)\right] + (C_1x^{-1/2} + C_2x^{-1/2}\ln x)$$
$$= 3C_1x^{-1/2} - 8C_2x^{-1/2} + 3C_2x^{-1/2}\ln x - 4C_1x^{-1/2} + 8C_2x^{-1/2} - 4C_2x^{-1/2}\ln x$$
$$+ C_1x^{-1/2} + C_2x^{-1/2}\ln x$$
$$= (3C_1x^{-1/2} - 4C_1x^{-1/2} + C_1x^{-1/2} - 8C_2x^{-1/2} + 8C_2x^{-1/2})$$
$$+ (3C_2x^{-1/2}\ln x - 4C_2x^{-1/2}\ln x + C_2x^{-1/2}\ln x) = 0$$

42. $$y' = C_1x^{-1}[-\sin(\sqrt{2}\ln x)]\frac{\sqrt{2}}{x} - C_1x^{-2}\cos(\sqrt{2}\ln x)$$
$$+ C_2x^{-1}[\cos(\sqrt{2}\ln x)]\frac{\sqrt{2}}{x} - C_2x^{-2}\sin(\sqrt{2}\ln x)$$
$$= -\sqrt{2}C_1x^{-2}\sin(\sqrt{2}\ln x) - C_1x^{-2}\cos(\sqrt{2}\ln x)$$
$$+ \sqrt{2}C_2x^{-2}\cos(\sqrt{2}\ln x) - C_2x^{-2}\sin(\sqrt{2}\ln x)$$
$$y'' = -\sqrt{2}C_1x^{-2}[\cos(\sqrt{2}\ln x)]\frac{\sqrt{2}}{x} + 2\sqrt{2}C_1x^{-3}\sin(\sqrt{2}\ln x) + C_1x^{-2}[\sin(\sqrt{2}\ln x)]\frac{\sqrt{2}}{x}$$
$$+ 2C_1x^{-3}\cos(\sqrt{2}\ln x) - \sqrt{2}C_2x^{-2}[\sin(\sqrt{2}\ln x)]\frac{\sqrt{2}}{x} - 2\sqrt{2}C_2x^{-3}\cos(\sqrt{2}\ln x)$$
$$- C_2x^{-2}[\cos(\sqrt{2}\ln x)]\frac{\sqrt{2}}{x} + 2C_2x^{-3}\sin(\sqrt{2}\ln x)$$
$$= 3\sqrt{2}C_1x^{-3}\sin(\sqrt{2}\ln x) - 3\sqrt{2}C_2x^{-3}\cos(\sqrt{2}\ln x)$$
$$x^2y'' + 3xy' + 3y = 3\sqrt{2}C_1x^{-1}\sin(\sqrt{2}\ln x) - 3\sqrt{2}C_2x^{-1}\cos(\sqrt{2}\ln x)$$
$$- 3\sqrt{2}C_1x^{-1}\sin(\sqrt{2}\ln x) - 3C_1x^{-1}\cos(\sqrt{2}\ln x) + 3\sqrt{2}C_2x^{-1}\cos(\sqrt{2}\ln x)$$
$$- 3C_2x^{-1}\sin(\sqrt{2}\ln x) + 3C_1x^{-1}\cos(\sqrt{2}\ln x) + 3C_2x^{-1}\sin(\sqrt{2}\ln x) = 0$$

43. $y^2 = \ln x + \ln y$; $\quad 2yy' = \dfrac{1}{x} + \dfrac{1}{y}\cdot y'$; $\quad 2xy^2y' = y + xy'$; $\quad (2xy^2 - x)y' = y$; $\quad y' = \dfrac{y}{2xy^2 - x}$

44. $y' = \dfrac{1 + y'}{x + y}$; $\quad (x + y)y' = 1 + y'$; $\quad y' = \dfrac{1}{x + y - 1}$

45. $x + y^2 = \ln x - \ln y$; $\quad 1 + 2yy' = \dfrac{1}{x} - \dfrac{1}{y}\cdot y'$; $\quad xy + 2xy^2y' = y - xy'$; $\quad y' = \dfrac{y - xy}{2xy^2 + x}$

46. $y = \ln x + 2\ln y$; $\quad y' = \dfrac{1}{x} + \dfrac{2}{y}\cdot y'$; $\quad xyy' = y + 2xy'$; $\quad y' = \dfrac{y}{xy - 2x}$

47. $xy' + y = \dfrac{2x + 2yy'}{x^2 + y^2}$; $\quad (x^3 + xy^2)y' + x^2y + y^3 = 2x + 2yy'$; $\quad y' = \dfrac{2x - x^2y - y^3}{x^3 + xy^2 - 2y}$

48. $x^2 + y^2 = 2\ln(x + y)$; $2x + 2yy' = \dfrac{2(1 + y')}{x + y}$; $x^2 + xy + (xy + y^2)y' = 1 + y'$; $y' = \dfrac{1 - x^2 - xy}{xy + y^2 - 1}$

49. $\ln y = (\sin x)\ln x;\quad \frac{1}{y}\left(\frac{dy}{dx}\right) = (\sin x)\left(\frac{1}{x}\right) + (\cos x)(\ln x);\quad \frac{dy}{dx} = x^{sinx}\left[\frac{\sin x}{x} + (\cos x)\ln x\right]$

50. $\ln y = x\ln(\ln|x|);\quad \frac{1}{y}\left(\frac{dy}{dx}\right) = x\left(\frac{1}{\ln|x|}\cdot\frac{1}{x}\right) + \ln(\ln|x|)$

$$\frac{dy}{dx} = (\ln|x|)^x\left[\frac{1}{\ln|x|} + \ln(\ln|x|)\right],\quad |x| > 1$$

51. $\ln y = \ln x + x\ln(x-1);\quad \frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{x} + x\left(\frac{1}{x-1}\right) + \ln(x-1)$

$$\frac{dy}{dx} = x(x-1)^x\left[\frac{1}{x} + \frac{x}{x-1} + \ln(x-1)\right]$$

52. $\ln y = x\ln(x^2+1) - 2\ln x;\quad \frac{1}{y}\left(\frac{dy}{dx}\right) = x\left(\frac{2x}{x^2+1}\right) + \ln(x^2+1) - \frac{2}{x}$

$$\frac{dy}{dx} = \frac{(x^2+1)^x}{x^2}\left[\frac{2x^2}{x^2+1} + \ln(x^2+1) - \frac{2}{x}\right]$$

53. $\ln|y| = \frac{1}{2}\ln|2x+1| + \frac{1}{2}\ln|3x+2| - \ln|4x+3|$

$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{2}{2(2x+1)} + \frac{3}{2(3x+2)} - \frac{4}{4x+3}$$

$$\frac{dy}{dx} = \frac{\sqrt{(2x+1)(3x+2)}}{4x+3}\left(\frac{1}{2x+1} + \frac{3}{6x+4} - \frac{4}{4x+3}\right)$$

54. $\ln|y| = 10\ln|x| + \frac{1}{2}\ln(x^2+5) - \frac{1}{3}\ln(8x^2+2)$

$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{10}{x} + \frac{2x}{2(x^2+5)} - \frac{16x}{3(8x^2+2)}$$

$$\frac{dy}{dx} = \frac{x^{10}\sqrt{x^2+5}}{\sqrt[3]{8x^2+2}}\left(\frac{10}{x} + \frac{x}{x^2+5} - \frac{16x}{24x^2+6}\right)$$

55. $\ln|y| = 5\ln|x^3-1| + 4\ln|x^4+3x^3| - 9\ln|7x+5|$

$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{5(3x^2)}{x^3-1} + \frac{4(4x^3+9x^2)}{x^4+3x^3} - \frac{9(7)}{7x+5}$$

$$\frac{dy}{dx} = \frac{(x^3-1)^5(x^4+3x^3)^4}{(7x+5)^9}\left(\frac{15x^2}{x^3-1} + \frac{16x+36}{x^2+3x} - \frac{63}{7x+5}\right)$$

56. $\ln|y| = \ln|x| + \frac{1}{2}\ln|x+1| + \frac{1}{3}\ln(x^2+2)$

$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{x} + \frac{1}{2(x+1)} + \frac{2x}{3(x^2+2)}$$

$$\frac{dy}{dx} = x\sqrt{x+1}\,\sqrt[3]{x^2+2}\left(\frac{1}{x} + \frac{1}{2x+2} + \frac{2x}{3x^2+6}\right)$$

57. $\ln y = (x+2)\ln x$; $\quad \dfrac{1}{y}\left(\dfrac{dy}{dx}\right) = (x+2)\dfrac{1}{x} + \ln x$; $\quad \dfrac{dy}{dx} = x^{x+2}\left(\dfrac{x+2}{x} + \ln x\right)$

The point on the graph is $(1,1)$ and the slope of the tangent line is $y'(1) = 3$. An equation of the tangent line is $y - 1 = 3(x-1)$ or $y = 3x - 2$.

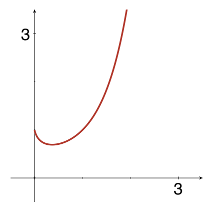
58. $\ln y = \ln x + x\ln(\ln x)$; $\quad \dfrac{1}{y}\left(\dfrac{dy}{dx}\right) = \dfrac{1}{x} + x\left(\dfrac{1}{\ln x}\cdot\dfrac{1}{x}\right) + \ln(\ln x)$

$$\frac{dy}{dx} = x(\ln x)^x\left[\frac{1}{x} + \frac{1}{\ln x} + \ln(\ln x)\right]$$

The point on the graph is (e,e) and the slope of the tangent line is $y'(e) = 1+e$. An equation of the tangent line is $y - e = (1+e)(x-e)$ or $y = (1+e)x - e^2$.

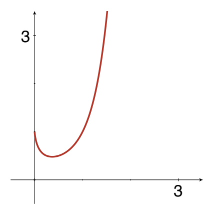
59. $\ln y = x\ln x$; $\quad \dfrac{1}{y}\left(\dfrac{dy}{dx}\right) = x\left(\dfrac{1}{x}\right) + \ln x$; $\quad \dfrac{dy}{dx} = x^x(1+\ln x)$

The tangent line is horizontal when $dy/dx = x^x(1+\ln x) = 0$. x^x is never 0, so we need only solve $1 + \ln x = 0$, which yields $\ln x = -1$ and $x = 1/e$. Thus, the tangent line is horizontal at $(1/e, 1/e^{1/e})$.



60. $\ln y = 2x\ln x$; $\quad \dfrac{1}{y}\left(\dfrac{dy}{dx}\right) = 2x\left(\dfrac{1}{x}\right) + 2\ln x$; $\quad \dfrac{dy}{dx} = 2x^{2x}(1+\ln x)$

The tangent line is horizontal when $dy/dx = 2x^{2x}(1+\ln x) = 0$. x^{2x} is never 0, so we need only solve $1+\ln x = 0$, which yields $\ln x = -1$ and $x = 1/e$. Thus, the tangent line is horizontal at $(1/e, 1/e^{2/e})$.



61. From Problem 59, $\dfrac{d}{dx}x^x = x^x(1+\ln x)$:

(a) $\dfrac{dy}{dx} = (\sec^2 x^x)\dfrac{d}{dx}x^x = x^x(1+\ln x)\sec^2 x^x$

(b) $\dfrac{dy}{dx} = x^x e^{x^x}[x^x(1+\ln x)] + e^{x^x}[x^x(1+\ln x)] = (1+x^x)e^{x^x}x^x(1+\ln x)$

(c) $\ln y = x^x\ln x$; $\dfrac{1}{y}\left(\dfrac{dy}{dx}\right) = x^x\left(\dfrac{1}{x}\right) + x^x(1+\ln x)\ln x$; $\dfrac{dy}{dx} = x^{x^x}x^x\left(\dfrac{1}{x} + \ln x + \ln^2 x\right)$

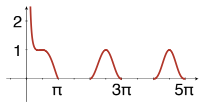
62. $y = (x^{1/2})^x = x^{x/2}$; $\quad \ln y = \dfrac{x}{2}\ln x$; $\quad \dfrac{1}{y}\left(\dfrac{dy}{dx}\right) = \dfrac{1}{2}\left[x\left(\dfrac{1}{x}\right) + \ln x\right]$; $\quad \dfrac{dy}{dx} = \dfrac{\sqrt{x}^x}{2}(1+\ln x)$;

$$\frac{d^2y}{dx^2} = \frac{\sqrt{x}^x}{2}\left(\frac{1}{x}\right) + \frac{\sqrt{x}^x}{4}(1+\ln x)(1+\ln x) = \frac{\sqrt{x}^x}{2}\left[\frac{1}{x} + \frac{(1+\ln x)^2}{2}\right]$$

63. $g(x) = |\ln x|$ is also not differentiable at $x = 1$. When $x > 1$, $\ln x > 0$, so $g(x) = \ln x$, $g'(x) = 1/x$, and $g'_+(1) = 1$. When $x < 1$, $\ln x < 0$, so $g(x) = -\ln x$, $g'(x) = -1/x$, and $g'_-(1) = -1$. Since $g'_+(1) \neq g'_-(1)$, g is not differentiable at 1.

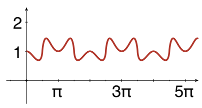
64. By the change of base formula for logarithms, $\log_x e = \ln e/\ln x = 1/\ln x$. Thus, $\dfrac{d}{dx}\log_x e = \dfrac{d}{dx}\left(\dfrac{1}{\ln x}\right) = \dfrac{d}{dx}(\ln x)^{-1} = -(\ln x)^{-2}\left(\dfrac{1}{x}\right) = -\dfrac{1}{x\ln^2 x}$.

65. (a)



(b) The function is not defined at intervals $[k\pi, (k+1)\pi]$, where k is an odd positive integer, because $\sin x < 0$ at those intervals.

66. (a)

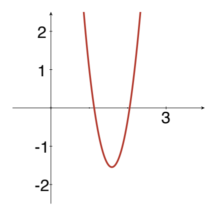


(b) $\ln y = \cos x \ln|\cos x|$; $\quad \dfrac{1}{y}\left(\dfrac{dy}{dx}\right) = (\cos x)\left(\dfrac{-\sin x}{\cos x}\right) + (-\sin x)(\ln|\cos x|)$

$$\frac{dy}{dx} = -|\cos x|^{\cos x}(\sin x)(1 + \ln|\cos x|)$$

The tangent to the graph is horizontal when $y' = -|\cos x|^{\cos x}(\sin x)(1 + \ln|\cos x|) = 0$, which is true in $[0, 5\pi]$ when $x \approx 1.2, 1.95, 4.33, 5.09, 7.47, 8.23, 10.62, 11.37, 13.76,$ and 14.51, and when $x = k\pi$, k an integer.

67. $f(x)$ is smallest at the sole point where its tangent is horizontal; $f'(x) = 3x^2 - 12/x$, so solving $3x^2 - 12/x = 0$ we get $x^3 = 4$ or $x = \sqrt[3]{4}$.



3.10 Hyperbolic Functions

1. $\cosh x = \sqrt{1 + \sinh^2 x} = \sqrt{1 + (-1/2)^2} = \dfrac{\sqrt{5}}{2}$; $\quad \tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{-1/2}{\sqrt{5}/2} = -\dfrac{\sqrt{5}}{5}$

$\coth x = \dfrac{\cosh x}{\sinh x} = -\sqrt{5}$; $\quad \operatorname{sech} x = \dfrac{1}{\cosh x} = \dfrac{1}{\sqrt{5}/2} = \dfrac{2\sqrt{5}}{5}$

$\operatorname{csch} x = \dfrac{1}{\sinh x} = \dfrac{1}{-1/2} = -2$

2. $\sinh x = \pm\sqrt{\cosh^2 x - 1} = \pm\sqrt{3 - 1} = \pm\sqrt{2}$; $\quad \tanh x = \dfrac{\sinh x}{\cosh x} = \pm\dfrac{\sqrt{2}}{3}$

$\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{3}{\pm\sqrt{2}} = \pm\dfrac{3\sqrt{2}}{2}$; $\operatorname{sech} x = \dfrac{1}{\cosh x} = \dfrac{1}{3}$; $\operatorname{csch} x = \dfrac{1}{\sinh x} = \dfrac{1}{\pm\sqrt{2}} = \pm\dfrac{\sqrt{2}}{2}$

3. $y' = 10 \sinh 10x$

4. $y' = -8 \operatorname{sech} 8x \tanh 8x$

5. $y' = \dfrac{1}{2\sqrt{x}} \operatorname{sech}^2 \sqrt{x}$

6. $y' = \dfrac{1}{x^2}\operatorname{csch}\dfrac{1}{x}\coth\dfrac{1}{x}$

7. $y' = -6(3x-1)\operatorname{sech}(3x-1)^2\tanh(3x-1)^2$

8. $y' = 2xe^{x^2}\cosh e^{x^2}$

9. $y' = 3\sinh 3x[-\operatorname{csch}^2(\cosh 3x)] = -3\sinh 3x\operatorname{csch}^2(\cosh 3x)$

10. $y' = 3x^2\cosh x^3\operatorname{sech}^2(\sinh x^3)$

11. $y' = (\sinh 2x)(3\sinh 3x) + (\cosh 3x)(2\cosh 2x) = 3\sinh 2x\sinh 3x + 2\cosh 2x\cosh 3x$

12. $$\begin{aligned} y' &= (\operatorname{sech} x)(-4\operatorname{csch}^2 4x) + (\coth 4x)(-\operatorname{sech} x\tanh x) \\ &= -4\operatorname{sech} x\operatorname{csch}^2 4x - \tanh x\operatorname{sech} x\coth 4x \end{aligned}$$

13. $y' = x(2x\sinh x^2) + \cosh x^2 = 2x^2\sinh x^2 + \cosh x^2$

14. $y' = \dfrac{x\cosh x - \sinh x}{x^2}$

15. $y' = 3\sinh^2 x\cosh x$

16. $y' = 4(\cosh^3\sqrt{x})\dfrac{\sinh\sqrt{x}}{2\sqrt{x}} = \dfrac{2(\cosh^3\sqrt{x})(\sinh\sqrt{x})}{\sqrt{x}}$

17. $f'(x) = \dfrac{2}{3}(x-\cosh x)^{-1/3}(1-\sinh x)$

18. $f'(x) = \dfrac{1}{2}(4+\tanh 6x)^{-1/2}(6\operatorname{sech}^2 6x) = \dfrac{3\operatorname{sech}^2 6x}{\sqrt{4+\tanh 6x}}$

19. $f'(x) = \dfrac{4\sinh 4x}{\cosh 4x} = 4\tanh 4x$

20. $f'(x) = 2[\ln(\operatorname{sech} x)]\left(\dfrac{-\operatorname{sech} x\tanh x}{\operatorname{sech} x}\right) = -2\tanh x\ln(\operatorname{sech} x)$

21. $f'(x) = \dfrac{(1+\cosh x)e^x - e^x\sinh x}{(1+\cosh x)^2} = \dfrac{e^x(1+\cosh x-\sinh x)}{(1+\cosh x)^2} = \dfrac{1+e^x}{(1+\cosh x)^2}$

22. $f'(x) = \dfrac{(x^2+\sinh x)(1/x) - (\ln x)(2x+\cosh x)}{(x^2+\sinh x)^2} = \dfrac{x^2+\sinh x - 2x^2\ln x - x\ln x\cosh x}{x(x^2+\sinh x)^2}$

23. $F'(t) = e^{\sinh t}\cosh t$

24. $H(t) = e^{t+\operatorname{csch} t^2}$; $\quad H'(t) = e^{t+\operatorname{csch} t^2}(1-2t\operatorname{csch} t^2\coth t^2)$

25. $g'(t) = \dfrac{(1+\sinh 2t)\cos t - 2\sin t\cosh 2t}{(1+\sinh 2t)^2}$

26. $$w'(t) = \frac{(1+\cosh t)^2 \operatorname{sech}^2 t - (\tanh t)[2(1+\cosh t)\sinh t]}{(1+\cosh t)^4} = \frac{(1+\cosh t)\operatorname{sech}^2 t - 2\sinh t \tanh t}{(1+\cosh t)^3}$$

27. $y' = 3\cosh 3x$. The point on the graph is $(0,0)$ and the slope of the tangent line is $y'(0) = 3$. An equation of the tangent line is $y = 3x$.

28. $y' = \sinh x$. The point on the graph is $\left(1, \dfrac{e^2+1}{2e}\right)$. The slope of the tangent line is $y'(1) = \sinh 1 = \dfrac{e^2-1}{2e}$. An equation of the tangent line is $y - \dfrac{e^2+1}{2e} = \dfrac{e^2-1}{2e}(x-1)$ or $y = \dfrac{e^2-1}{2e}x + \dfrac{1}{e}$.

29. The tangent is horizontal when $f'(x) = (x^2-2)\sinh x + 2x\cosh x - 2x\cosh x - 2\sinh x = (x^2-4)\sinh x = 0$, or when $x = 0$ or ± 2, yielding the points $(0,-2)$, $(2, -e^2+3e^{-2})$, and $(-2, -e^2+3e^{-2})$.

30. The tangent is horizontal when $f'(x) = \cos x\sinh x - \sin x\cosh x - \sin x\cosh x - \cos x\sinh x = -2\sin x\cosh x$, or when $x = k\pi$, k an integer, yielding the points $(0,1)$, $(\pi, -[e^\pi + e^{-\pi}]/2)$, $(-\pi, -[e^\pi + e^{-\pi}]/2)$ and so on.

31. $\dfrac{dy}{dx} = \operatorname{sech}^2 x;\quad \dfrac{d^2y}{dx^2} = 2(\operatorname{sech} x)(-\operatorname{sech} x\tanh x) = -2\operatorname{sech}^2 x\tanh x$

32. $\dfrac{dy}{dx} = -\operatorname{sech} x\tanh x;\ \dfrac{d^2y}{dx^2} = -\operatorname{sech} x\operatorname{sech}^2 x + \tanh x\operatorname{sech} x\tanh x = \tanh^2 x\operatorname{sech} x - \operatorname{sech}^3 x$

33. $y' = kC_1\sinh kx + kC_2\cosh kx;\quad y'' = k^2C_1\cosh kx + k^2C_2\sinh kx$
$y'' - k^2y = k^2C_1\cosh kx + k^2C_2\sinh kx - k^2(C_1\cosh kx + C_2\sinh kx) = 0$

34. $$\begin{aligned}
y' &= -kC_1\sin kx + kC_2\cos kx + kC_3\sinh kx + kC_4\cosh kx\\
y'' &= -k^2C_1\cos kx - k^2C_2\sin kx + k^2C_3\cosh kx + k^2C_4\sinh kx\\
y''' &= k^3C_1\sin kx - k^3C_2\cos kx + k^3C_3\sinh kx + k^3C_4\cosh kx\\
y^{(4)} &= k^4C_1\cos kx + k^4C_2\sin kx + k^4C_3\cosh kx + k^4C_4\sinh kx\\
y^{(4)} - k^4y &= (k^4C_1\cos kx + k^4C_2\sin kx + k^4C_3\cosh kx + k^4C_4\sinh kx)\\
&\qquad - k^4(C_1\cos kx + C_2\sin kx + C_3\cosh kx + C_4\sinh kx) = 0
\end{aligned}$$

35. $y' = \dfrac{3}{\sqrt{9x^2+1}}$

36. $y' = \dfrac{1/2}{\sqrt{x^2/4-1}} = \dfrac{1}{\sqrt{x^2-4}}$

37. $y' = \dfrac{-2x}{1-(1-x^2)^2}$

38. $y' = \dfrac{-1/x^2}{1-(1/x)^2} = \dfrac{1}{1-x^2}$

39. $y' = \dfrac{-\csc x \cot x}{1-\csc^2 x} = \dfrac{-\csc x \cot x}{-\cot^2 x} = \sec x$

40. $y' = \dfrac{\cos x}{\sqrt{\sin^2 x + 1}}$

41. $y' = x\dfrac{3x^2}{\sqrt{x^6+1}} + \sinh^{-1} x^3 = \dfrac{3x^3}{\sqrt{x^6+1}} + \sinh^{-1} x^3$

42. $y' = x^2\dfrac{-1}{|x|\sqrt{1+x^2}} + 2x\operatorname{csch}^{-1} x = \dfrac{-|x|}{\sqrt{1+x^2}} + 2x\operatorname{csch}^{-1} x$

43. $y' = \dfrac{x\left(\dfrac{-1}{x\sqrt{1-x^2}}\right) - \operatorname{sech}^{-1} x}{x^2} = \dfrac{1+\sqrt{1-x^2}\operatorname{sech}^{-1} x}{x^2\sqrt{1-x^2}}$

44. $y' = \dfrac{e^{2x}\left[\dfrac{2e^{2x}}{1-(e^{2x})^2}\right] - (\coth^{-1} e^{2x})2e^{2x}}{(e^{2x})^2} = \dfrac{2e^{2x} - 2(1-e^{4x})\coth^{-1} e^{2x}}{e^{2x}(1-e^{4x})}$

45. $y' = \dfrac{1}{\operatorname{sech}^{-1} x} \cdot \dfrac{-1}{x\sqrt{1-x^2}} = \dfrac{-1}{x\sqrt{1-x^2}\operatorname{sech}^{-1} x}$

46. $y' = x\left(\dfrac{1}{1-x^2}\right) + \tanh^{-1} x + \dfrac{-2x(1/2)(1-x^2)^{-1/2}}{\sqrt{1-x^2}} = \dfrac{x}{1-x^2} + \tanh^{-1} x - \dfrac{x}{1-x^2} = \tanh^{-1} x$

47. $y' = \dfrac{1}{2}(\cosh^{-1} 6x)^{-1/2}\dfrac{6}{\sqrt{36x^2-1}} = \dfrac{3}{\sqrt{\cosh^{-1} 6x}\sqrt{36x^2-1}}$

48. $y' = -3(\tanh^{-1} 2x)^{-4}\dfrac{2}{1-4x^2} = \dfrac{-6}{(\tanh^{-1} 2x)^4(1-4x^2)}$

49. (a) $\dfrac{dv}{dt} = \sqrt{\dfrac{mg}{k}}\sqrt{\dfrac{kg}{m}}\operatorname{sech}^2\left(\sqrt{\dfrac{kg}{m}}t\right) = g\operatorname{sech}^2\left(\sqrt{\dfrac{kg}{m}}t\right)$

$$\begin{aligned} m\frac{dv}{dt} - mg + kv^2 &= mg\operatorname{sech}^2\left(\sqrt{\frac{kg}{m}}t\right) - mg + k\left[\frac{mg}{k}\tanh^2\left(\sqrt{\frac{kg}{m}}t\right)\right] \\ &= mg\left[\operatorname{sech}^2\left(\sqrt{\frac{kg}{m}}t\right) - 1 + \tanh^2\left(\sqrt{\frac{kg}{m}}t\right)\right] \\ &= mg\left[\operatorname{sech}^2\left(\sqrt{\frac{kg}{m}}t\right) - \operatorname{sech}^2\left(\sqrt{\frac{kg}{m}}t\right)\right] = 0 \end{aligned}$$

(b) From Figure 3.10.2a in the text we see that $\lim_{t\to\infty} \tanh t = 1$, so $\lim_{t\to\infty} v(t) = \sqrt{mg/k}$.

(c) Using $v_{\text{ter}} = \sqrt{mg/k}$, $m = 80$, and $k = 0.25$, we find $v_{\text{ter}} = \sqrt{80(9.8)/0.25} = 56$ m/s.

50. (a) $x = a\,\mathrm{sech}^{-1}\dfrac{y}{a} - \sqrt{a^2 - y^2}$

(b) Differentiating with respect to x gives:

$$\begin{aligned}
1 &= a\left(\frac{1}{\frac{y}{a}\sqrt{1-\frac{y^2}{a^2}}}\right)\left(-\frac{1}{a}\frac{dy}{dx}\right) - \frac{1}{2}\left(\frac{1}{\sqrt{a^2-y^2}}\right)\left(-2y\frac{dy}{dx}\right) \\
&= \frac{dy}{dx}\left(\frac{-a^2}{y\sqrt{a^2-y^2}} + \frac{y}{\sqrt{a^2-y^2}}\right) \\
&= \frac{dy}{dx}\left[\frac{-(a^2-y^2)}{y\sqrt{a^2-y^2}}\right] \\
&= \frac{dy}{dx}\left(\frac{-\sqrt{a^2-y^2}}{y}\right) \\
\frac{dy}{dx} &= \frac{-y}{\sqrt{a^2-y^2}}
\end{aligned}$$

(c) From the figure, we see that $\dfrac{dy}{dx} = \dfrac{-y}{\sqrt{a^2-y^2}}$ is the slope of the taut line at (x, y). In other words, the taut line is tangent to the curve at (x, y).

51. $\cosh(\ln 4) = \dfrac{e^{\ln 4} + e^{-\ln 4}}{2} = \dfrac{4 + 1/4}{2} = 2.125$

52. $\sinh(\ln 0.5) = \dfrac{e^{\ln 0.5} - e^{-\ln 0.5}}{2} = \dfrac{0.5 - 1/0.5}{2} = -0.75$

53. $\sinh(\ln x) = \dfrac{e^{\ln x} - e^{-\ln x}}{2} = \dfrac{x - 1/x}{2} = \dfrac{x^2-1}{2x}, \quad x > 0$

54. $\tanh(3\ln x) = \dfrac{e^{3\ln x} - e^{-3\ln x}}{e^{3\ln x} + e^{-3\ln x}} = \dfrac{x^3 - x^{-3}}{x^3 + x^{-3}} = \left(\dfrac{x^6-1}{x^3}\right)\left(\dfrac{x^3}{x^6+1}\right) = \dfrac{x^6-1}{x^6+1}, \quad x > 0$

55. First, we evaluate $\cosh x + \sinh x = \dfrac{e^x + e^{-x}}{2} + \dfrac{e^x - e^{-x}}{2} = e^x$. Applying this result twice, we have $(\cosh x + \sinh x)^n = (e^x)^n = e^{nx} = \cosh nx + \sinh nx$.

Chapter 3 in Review

A. True/False

1. False. Consider $y = |x|$ at $x = 0$.

2. True

3. False. $y = x^{1/3}$ has a vertical tangent at $x = 0$, but is not differentiable there.
4. True
5. True
6. False. Consider the product rule for differentiation.
7. True
8. False. The derivative gives the values of the slopes of tangent lines at points on the graph; it is not the tangent line.
9. True
10. False. For $f(x) = x^2$ and $g(x) = x^2 + 3$, $f'(x) = g'(x) = 2x$, but $f(x) \neq g(x)$.
11. True. $f'(x) = \cos x$, whose range is $[-1, 1]$.
12. True. $\dfrac{d}{dx}\tan^{-1} x = \dfrac{1}{1+x^2}$, which is greater than 0 for all x.
13. False. $\dfrac{d}{dx}\cos^{-1} x = \dfrac{-1}{\sqrt{1-x^2}}$.
14. True, since $f'(x) > 0$ for all x.
15. True
16. False. $f(x) = x$ is increasing, but $f'(x) = 1$ is not.
17. False. Trivially, $f(x) = 0$ is the same as its derivative, but so is $f(x) = ce^x$ for any non-zero constant c.
18. False. $\dfrac{d}{dx}\ln|x| = \dfrac{1}{x}$, $x \neq 0$.
19. True. $\dfrac{d}{dx}\cosh^2 x = 2\cosh x \sinh x = \dfrac{d}{dx}\sinh^2 x$.
20. True

B. Fill in the Blanks

1. 0
2. $\dfrac{-1}{-1/2} = 2$
3. $-1/4$
4. x^n
5. $y = -\dfrac{5}{4}x - \dfrac{3}{2}$

6. 18

7. -3

8. 15

9. 5

10. $(x^3)^2 \cdot 3x^2 = 3x^8$

11. $16(\cos^2 4x)F''(\sin 4x) - 16(\sin 4x)F'(\sin 4x)$

12. $\cot 0$ is undefined

13. $a = 6,\ b = -9$

14. $\dfrac{\tan 2x}{2}$

15. $(1, 5)$

16. $2^x \ln 2$

17. $\dfrac{1}{x \ln 10}$

18. all $x \neq 2$

19. catenary

20. 0

C. Exercises

1. $f(x) = \dfrac{4}{5}x^{0.1}; \quad f'(x) = \dfrac{0.4}{5}x^{-0.9} = 0.08x^{-0.9}$

2. $y' = -\dfrac{3x^2 + 8x - 6}{(x^3 + 4x^2 - 6x + 11)^2}$

3. $F'(t) = 10[t + (t^2+1)^{1/2}]^9[1 + (1/2)(t^2+1)^{-1/2}(2t)] = 10[t + (t^2+1)^{1/2}]^9[1 + t(t^2+1)^{-1/2}]$

4. $h'(\theta) = \theta^{1.5}[0.5(\theta^2+1)^{-0.5}(2\theta)] + (\theta^2+1)^{0.5}(1.5\theta^{0.5}) = \theta^{2.5}(\theta^2+1)^{-0.5} + 1.5\theta^{0.5}(\theta^2+1)^{0.5}$

5. $y = (x^4+16)^{1/4}(x^3+8)^{1/3}$

$$\begin{aligned} y' &= (x^4+16)^{1/4}\left[\frac{1}{3}(x^3+8)^{-2/3}(3x^2)\right] + (x^3+8)^{1/3}\left[\frac{1}{4}(x^4+16)^{-3/4}(4x^3)\right] \\ &= x^2(x^4+16)^{1/4}(x^3+8)^{-2/3} + x^3(x^4+16)^{-3/4}(x^3+8)^{1/3} \end{aligned}$$

6. $g(u) = \left(\dfrac{6u-1}{u+7}\right)^{1/2}$

$$g'(u) = \frac{1}{2}\left(\frac{6u-1}{u+7}\right)^{-1/2}\left[\frac{(u+7)\cdot 6-(6u-1)\cdot 1}{(u+7)^2}\right] = \frac{1}{2}\left(\frac{6u-1}{u+7}\right)^{-1/2}\left[\frac{43}{(u+7)^2}\right]$$
$$= \frac{43(6u-1)^{-1/2}}{2(u+7)^{3/2}}$$

7. $y' = \dfrac{(4x+1)(-4\sin 4x)-(\cos 4x)(4)}{(4x+1)^2} = -\dfrac{16x\sin 4x + 4\sin 4x + 4\cos 4x}{(4x+1)^2}$

8. $y' = 10(-\csc^2 8x)(8) = -80\csc^2 8x$

9. $f'(x) = x^3(2\sin 5x)(\cos 5x)(5) + (\sin^2 5x)(3x^2) = 10x^3\sin 5x\cos 5x + 3x^2\sin^2 5x$

10. $y' = 2[\tan(\cos 2x)][\sec^2(\cos 2x)](-\sin 2x)(2) = -4\sin 2x\tan(\cos 2x)\sec^2(\cos 2x)$

11. $y' = \dfrac{-3/x^2}{\sqrt{1-(3/x)^2}} = -\dfrac{3}{\dfrac{x^2}{|x|}\sqrt{x^2-9}} = -\dfrac{3}{|x|\sqrt{x^2-9}}$

12. $y' = \cos x\left(-\dfrac{1}{\sqrt{1-x^2}}\right) + \cos^{-1}x(-\sin x) = -\dfrac{\cos x}{\sqrt{1-x^2}} - \sin x\cos^{-1}x$

13. $y' = -(\cot^{-1}x)^{-2}\left(-\dfrac{1}{1+x^2}\right) = \dfrac{1}{(1+x^2)(\cot^{-1}x)^2}$

14. $y' = \dfrac{2}{|2x-1|\sqrt{(2x-1)^2-1}}$

15. $y' = -\dfrac{2}{\sqrt{1-x^2}} + 2x\dfrac{-2x}{2\sqrt{1-x^2}} + 2\sqrt{1-x^2} = \dfrac{-2(1+x^2)}{\sqrt{1-x^2}} + 2\sqrt{1-x^2} = \dfrac{-4x^2}{\sqrt{1-x^2}}$

16. $y' = x^2\dfrac{1}{1+x^2-1}\cdot\dfrac{2x}{2\sqrt{x^2-1}} + 2x\tan^{-1}\sqrt{x^2-1} = \dfrac{x}{\sqrt{x^2-1}} + 2x\tan^{-1}\sqrt{x^2-1}$

17. $y = \dfrac{x+1}{e^x}$; $\quad y' = \dfrac{e^x-(x+1)e^x}{e^{2x}} = \dfrac{-xe^x}{e^{2x}} = -xe^{-x}$

18. $y' = (e+e^2)^x\ln(e+e^2)$;

19. $y' = 7x^6 + 7^x\ln 7 + 7e^{7x}$

20. $y' = -e(e^x+1)^{-e-1}e^x = -e^{x+1}(e^x+1)^{-e-1}$

21. $y = \ln x + \dfrac{1}{2}\ln(4x-1)$; $\quad y' = \dfrac{1}{x} + \dfrac{4}{2(4x-1)} = \dfrac{1}{x} + \dfrac{2}{4x-1}$

22. $y = (2\ln\cos x)^2 = 4(\ln\cos x)^2$; $\quad y' = 8(\ln\cos x)\dfrac{-\sin x}{\cos x} = -8(\tan x)\ln\cos x$

23. $y' = \dfrac{1/\sqrt{1-x^2}}{\sqrt{(\sin^{-1}x)^2+1}} = \dfrac{1}{\sqrt{1-x^2}\sqrt{(\sin^{-1}x)^2+1}}$

24. $y' = (\tan^{-1}x)\dfrac{1}{1-x^2} + (\tanh^{-1}x)\dfrac{1}{1+x^2} = \dfrac{\tan^{-1}x}{1-x^2} + \dfrac{\tanh^{-1}x}{1+x^2}$

25. $$y' = xe^{x\cosh^{-1}x}\left(x\frac{1}{\sqrt{x^2-1}} + \cosh^{-1}x\right) + e^{x\cosh^{-1}x}$$
$$= \left(\frac{x^2}{\sqrt{x^2-1}} + x\cosh^{-1}x + 1\right)e^{x\cosh^{-1}x}$$

26. $y' = \dfrac{1}{\sqrt{x^2-1+1}} \cdot \dfrac{2x}{2\sqrt{x^2-1}} = \dfrac{x}{|x|\sqrt{x^2-1}}$

27. $y' = 3x^2e^{x^3}\sinh e^{x^3}$

28. $y' = -(\tanh 5x)^{-2}(\operatorname{sech} 5x)(5) = -\dfrac{5\operatorname{sech}^2 5x}{\tanh^2 5x} = -5\operatorname{csch}^2 5x$

29. $\dfrac{dy}{dx} = \dfrac{5}{2}(3x+1)^{3/2}(3) = \dfrac{15}{2}(3x+1)^{3/2}$; $\dfrac{d^2y}{dx^2} = \dfrac{45}{4}(3x+1)^{1/2}(3) = \dfrac{135}{4}(3x+1)^{1/2}$;
$\dfrac{d^3y}{dx^3} = \dfrac{135}{8}(3x+1)^{-1/2}(3) = \dfrac{405}{8}(3x+1)^{-1/2}$

30. $$y' = [\cos(x^3-2x)](3x^2-2);$$
$$y'' = [\cos(x^3-2x)](6x) + (3x^2-2)[-\sin(x^3-2x)](3x^2-2)$$
$$= 6x\cos(x^3-2x) - (3x^2-2)^2\sin(x^3-2x)$$

31. $\dfrac{ds}{dt} = 2t - 2t^{-3}$; $\dfrac{d^2s}{dt^2} = 2 + 6t^{-4}$; $\dfrac{d^3s}{dt^3} = -24t^{-5}$; $\dfrac{d^4s}{dt^4} = 120t^{-6}$

32. $\dfrac{dW}{dv} = \dfrac{v+1-(v-1)}{(v+1)^2} = 2(v+1)^{-2}$; $\dfrac{d^2W}{dv^2} = -4(v+1)^{-3}$; $\dfrac{d^3W}{dv^3} = 12(v+1)^{-4}$

33. $$y' = e^{\sin 2x}(\cos 2x)(2) = 2e^{\sin 2x}\cos 2x$$
$$y'' = 2e^{\sin 2x}(-\sin 2x)(2) + 2e^{\sin 2x}(\cos 2x)(2)(\cos 2x) = 4e^{\sin 2x}(\cos^2 2x - \sin 2x)$$

34. $f'(x) = x^2(1/x) + 2x\ln x = x + 2x\ln x$; $f''(x) = 1 + 2x(1/x) + 2\ln x = 3 + 2\ln x$
$f'''(x) = 2/x, \quad x > 0$

35. $$y = \ln|(x+5)^4(2-x)^3| - \ln|(x+8)^{10}\sqrt[3]{6x+4}|$$
$$= \ln(x+5)^4 + \ln|(2-x)^3| - \ln(x+8)^{10} - \ln|\sqrt[3]{6x+4}|$$
$$= 4\ln|x+5| + 3\ln|2-x| - 10\ln|x+8| - \frac{1}{3}\ln|6x+4|$$
$$y' = \frac{4}{x+5} - \frac{3}{2-x} - \frac{10}{x+8} - \frac{1}{3x+2}$$

36. $\ln y = x^2 \ln 5 + (\sin 2x)\ln x;\quad \frac{1}{y}\left(\frac{dy}{dx}\right) = 2x\ln 5 + (\sin 2x)\frac{1}{x} + 2\cos 2x \ln x$

$$\frac{dy}{dx} = 5^{x^2} x^{\sin 2x}\left(2x\ln 5 + \frac{\sin 2x}{x} + 2\cos 2x\ln x\right)$$

37. $y' = 3x^2 + 1$. The slope of the tangent at $x = 1$ is 4, so the slope of the tangent to the inverse of y is $1/4$.

38. $f^{-1}(x) = \sqrt[3]{1 - \frac{8}{x}};\quad (f^{-1})'(x) = \frac{1}{3}\left(1 - \frac{8}{x}\right)^{-2/3}\left(\frac{8}{x^2}\right) = \frac{8}{3x^2}\left(\frac{x-8}{x}\right)^{-2/3}$

39. $x(2yy') + y^2 = e^x - e^y y';\quad (2xy + e^y)y' = e^x - y^2;\quad y' = \frac{e^x - y^2}{2xy + e^y}$

40. $y = \ln x + \ln y;\quad y' = \frac{1}{x} + \frac{y'}{y};\quad \left(1 - \frac{1}{y}\right)y' = \frac{1}{x};\quad y' = \frac{y}{x(y-1)}$

41. A line that is perpendicular to $y = -3x$ will have slope $\frac{1}{3}$. Thus, we need x such that $f'(x) = 3x^2 = \frac{1}{3}$, so $x = \pm\frac{1}{3}$. Equations of tangent lines that are perpendicular to $y = -3x$ are $y \pm \frac{1}{27} = \frac{1}{3}\left(x \pm \frac{1}{3}\right)$ or $y = \frac{1}{3}x \pm \frac{2}{27}$.

42. $f'(x) = x - 5;\quad f''(x) = 1$

(a) For $f''(x) = f(x)$, we have $1 = \frac{1}{2}x^2 - 5x + 1$. Solving, we get $\frac{1}{2}x^2 - 5x = 0$, $x\left(\frac{1}{2}x - 5\right) = 0$, or $x = 0, 10$. Thus, the points for which $f''(x) = f(x)$ are $(0, 1)$ and $(10, 1)$.

(b) For $f''(x) = f'(x)$, we have $1 = x - 5$ or $x = 6$. Thus, the point for which $f''(x) = f'(x)$ is $(6, 1)$.

43. $y' = 2x$. If (a, a^2) is the point of tangency, then the slope of the tangent line through $(0, -9)$ is $\frac{a^2 + 9}{a - 0}$. The slope is also $y'(a) = 2a$. Thus, $\frac{a^2 + 9}{a} = 2a$, $9 = a^2$, and $a = \pm 3$. The tangent line through $(-3, 9)$ is $y - 9 = -6(x + 3)$ or $y = -6x - 9$. The tangent line through $(3, 9)$ is $y - 9 = 6(x - 3)$ or $y = 6x - 9$.

44. (a) $y' = 2x$. At $x = 1$, $y = 1$ and $y' = 2$. The equation of the tangent line is $y - 1 = 2(x - 1)$ or $y = 2x - 1$. Setting $y = 0$ we find the x-intercept to be $x = 1/2$.

(b) The equation of the line through $(1/2, 0)$ with slope $-\frac{1}{2}$ is $y - 0 = -\frac{1}{2}\left(x - \frac{1}{2}\right)$ or $y = -\frac{1}{2}x + \frac{1}{4}$.

(c) Substituting $y = x^2$ into $y = -\frac{1}{2}x + \frac{1}{4}$, we obtain $x^2 = -\frac{1}{2}x + \frac{1}{4}$ or $4x^2 + 2x - 1 = 0$. Then $x = \frac{1}{8}(-2 \pm \sqrt{4 + 16}) = -\frac{1}{4} \pm \frac{1}{4}\sqrt{5}$ and the line intersects the graph at about $(-0.81, 0.65)$ and $(0.31, 0.01)$.

45. $f'(x) = \dfrac{1}{2\sqrt{x}}$. The slope of the line through $(1, 1)$ and $(9, 3)$ is $1/4$. Solving $\dfrac{1}{2\sqrt{x}} = \dfrac{1}{4}$ we obtain $x = 4$. The point on the graph is $(4, 2)$.

46. $f'(x) = -x^{-2}$; $f''(x) = 2x^{-3}$; $f'''(x) = -6x^{-4}$. The slope of the tangent line to f'' at $x = 2$ is $f'''(2) = -6/16 = -3/8$.

47. $f'(x) = -2\sin x - 2\sin 2x$. The slope of a horizontal line is 0, so we solve $-2\sin x - 2\sin 2x = 0$. Using $\sin 2x = 2\sin x\cos x$ we have $\sin x + 2\sin x\cos x = 0$, $(\sin x)(1 + 2\cos x) = 0$, and x must therefore satisfy $\sin x = 0$ or $\cos x = -1/2$. For $0 \le x \le 2\pi$, this gives $x = 0$, π, 2π, $2\pi/3$, and $4\pi/3$.

48. $y' = 1/x$. Let $(a, \ln 2a)$ be the point on the graph such that the tangent line passes through the origin. The slope of the line is $1/a$, and since it passes through $(0, 0)$, the equation is $y = \dfrac{1}{a}x$. The tangent line also passes through $(a, \ln 2a)$, so $\ln 2a = \dfrac{1}{a}a = 1$. Solving for a, we obtain $2a = e^1$ and $a = e/2$. The point on the graph is $(e/2, \ln e)$ or $(e/2, 1)$.

49. Evaluating $q(t)$ when $t = 0$, we get

$$\begin{aligned} q(0) &= E_0C + (q_0 - E_0C)\left(\frac{k_1}{k_1 + k_2 \cdot 0}\right)^{1/Ck_2} \\ &= E_0C + (q_0 - E_0C)(1) = q_0. \end{aligned}$$

So $q(t)$ satisfies the initial condition $q(0) = q_0$. Now, rewriting

$$q(t) = E_0C + (q_0 - E_0C)k_1^{1/Ck_2}(k_1 + k_2t)^{-1/Ck_2},$$

we get

$$\frac{dq}{dt} = -\frac{(q_0 - E_0C)k_1^{1/Ck_2}}{Ck_2}(k_1 + k_2t)^{-1/Ck_2 - 1}(k_2) = -\frac{q_0 - E_0C}{C(k_1 + k_2t)}\left(\frac{k_1}{k_1 + k_2t}\right)^{1/Ck_2}.$$

Evaluating $(k_1 + k_2t)\dfrac{dq}{dt}$ yields:

$$(k_1 + k_2t)\left[-\frac{q_0 - E_0C}{C(k_1 + k_2t)}\left(\frac{k_1}{k_1 + k_2t}\right)^{1/Ck_2}\right] = -\left(\frac{q_0 - E_0C}{C}\right)\left(\frac{k_1}{k_1 + k_2t}\right)^{1/Ck_2}$$

Then, evaluating $\dfrac{1}{C}q$ results in:

$$\frac{1}{C}\left[E_0C + (q_0 - E_0C)\left(\frac{k_1}{k_1 + k_2t}\right)^{1/Ck_2}\right] = E_0 + \left(\frac{q_0 - E_0C}{C}\right)\left(\frac{k_1}{k_1 + k_2t}\right)^{1/Ck_2}$$

Substituting into the left side of the differential equation, we find

$$-\left(\frac{q_0 - E_0C}{C}\right)\left(\frac{k_1}{k_1 + k_2t}\right)^{1/Ck_2} + E_0 + \left(\frac{q_0 - E_0C}{C}\right)\left(\frac{k_1}{k_1 + k_2t}\right)^{1/Ck_2} = E_0.$$

50.
$$
\begin{aligned}
y &= C_1x + C_2\left[\frac{x}{2}\ln(x-1) - \frac{x}{2}\ln(x+1) + 1\right]\\
y' &= C_1 + \frac{C_2x}{2(x-1)} + \frac{C_2\ln(x-1)}{2} - \frac{C_2x}{2(x+1)} - \frac{C_2\ln(x+1)}{2}\\
&= C_1 + \frac{C_2x[(x+1)-(x-1)]}{2(x^2-1)} + \frac{C_2\ln(x-1)}{2} - \frac{C_2\ln(x+1)}{2}\\
&= C_1 + \frac{C_2x}{x^2-1} + \frac{C_2\ln(x-1)}{2} - \frac{C_2\ln(x+1)}{2} = C_1 + \frac{C_2x}{x^2-1} + \frac{C_2}{2}\ln\left(\frac{x-1}{x+1}\right)\\
y'' &= \frac{C_2(x^2-1) - 2x(C_2x)}{(x^2-1)^2} + \frac{C_2}{2(x-1)} - \frac{C_2}{2(x+1)}\\
&= \frac{-C_2(x^2+1)}{(x^2-1)^2} + \frac{C_2}{2(x-1)} - \frac{C_2}{2(x+1)} = \frac{-C_2(x^2+1)}{(x^2-1)^2} + \frac{C_2(x+1-x+1)}{2(x^2-1)}\\
&= \frac{-C_2(x^2+1)}{(x^2-1)^2} + \frac{C_2}{x^2-1} = \frac{C_2(-x^2-1+x^2-1)}{(x^2-1)^2} = \frac{-2C_2}{(x^2-1)^2}
\end{aligned}
$$

$$
\begin{aligned}
(1-x^2)y'' - 2xy' + 2y &= (1-x^2)\frac{-2C_2}{(x^2-1)^2} - 2x\left[C_1 + \frac{C_2x}{x^2-1} + \frac{C_2}{2}\ln\left(\frac{x-1}{x+1}\right)\right]\\
&\qquad + 2\left\{C_1x + C_2\left[\frac{x}{2}\ln\left(\frac{x-1}{x+1}\right) + 1\right]\right\}\\
&= \frac{2C_2}{x^2-1} - 2C_1x - \frac{2C_2x^2}{x^2-1} - C_2x\ln\left(\frac{x-1}{x+1}\right) + 2C_1x + C_2x\ln\left(\frac{x-1}{x+1}\right) + 2C_2\\
&= \frac{2C_2}{x^2-1} - \frac{2C_2x^2}{x^2-1} + 2C_2 = \frac{2C_2 - 2C_2x^2 + 2C_2x^2 - 2C_2}{x^2-1} = 0
\end{aligned}
$$

51.
$$
\begin{aligned}
y' &= -C_1e^{-x} + C_2e^x + C_3[x(-e^{-x}) + e^{-x}] + C_4(xe^x + e^x)\\
&= (C_3 - C_1)e^{-x} + (C_2 + C_4)e^x - C_3xe^{-x} + C_4xe^x\\
y'' &= -(C_3 - C_1)e^{-x} + (C_2 + C_4)e^x - C_3[x(-e^{-x}) + e^{-x}] + C_4(xe^x + e^x)\\
&= (C_1 - 2C_3)e^{-x} + (C_2 + 2C_4)e^x + C_3xe^{-x} + C_4xe^x\\
y''' &= -(C_1 - 2C_3)e^{-x} + (C_2 + 2C_4)e^x + C_3[x(-e^{-x}) + e^{-x}] + C_4(xe^x + e^x)\\
&= (C_3 - C_1)e^{-x} + (C_2 + 3C_4)e^x - C_3xe^{-x} + C_4xe^x\\
y^{(4)} &= -(C_3 - C_1)e^{-x} + (C_2 + 3C_4)e^x - C_3[x(-e^{-x}) + e^{-x}] + C_4(xe^x + e^x)\\
&= (C_1 - 2C_3)e^{-x} + (C_2 + 4C_4)e^x + C_3xe^{-x} + C_4xe^x
\end{aligned}
$$

$$
\begin{aligned}
y^{(4)} - 2y'' + y &= [(C_1 - 2C_3)e^{-x} + (C_2 + 4C_4)e^x + C_3xe^{-x} + C_4xe^x] - 2[(C_1 - 2C_3)e^{-x}\\
&\qquad + (C_2 + 2C_4)e^x + C_3xe^{-x} + C_4xe^x] + C_1e^{-x} + C_2e^x + C_3xe^{-x} + C_4xe^x\\
&= (C_1 - 2C_3 - 2C_1 + 2C_3 + C_1)e^{-x} + (C_2 + 4C_4 - 2C_2 - 4C_4 + C_2)e^x\\
&\qquad + (C_3 - 2C_3 + C_3)xe^{-x} + (C_4 - 2C_4 + C_4)xe^x = 0
\end{aligned}
$$

52.
$$\begin{aligned}
y' &= -C_1\sin x + C_2\cos x + C_3[x(-\sin x)+\cos x] + C_4(x\cos x+\sin x)\\
&= (C_4-C_1)\sin x + (C_2+C_3)\cos x - C_3x\sin x + C_4x\cos x\\
y'' &= (C_4-C_1)\cos x + (C_2+C_3)(-\sin x) - C_3(x\cos x+\sin x) + C_4[x(-\sin x)+\cos x]\\
&= (2C_4-C_1)\cos x - (C_2+2C_3)\sin x - C_3x\cos x - C_4x\sin x\\
y''' &= (2C_4-C_1)(-\sin x) - (C_2+2C_3)\cos x - C_3[x(-\sin x)+\cos x] - C_4(x\cos x+\sin x)\\
&= (C_1-3C_4)\sin x - (C_2+3C_3)\cos x + C_3x\sin x - C_4x\cos x\\
y^{(4)} &= (C_1-3C_4)\cos x - (C_2+3C_3)(-\sin x) + C_3(x\cos x+\sin x) - C_4[x(-\sin x)+\cos x]\\
&= (C_1-4C_4)\cos x + (C_2+4C_3)\sin x + C_3x\cos x + C_4x\sin x
\end{aligned}$$

$$\begin{aligned}
y^{(4)}+2y''+y &= [(C_1-4C_4)\cos x + (C_2+4C_3)\sin x + C_3x\cos x + C_4x\sin x]\\
&\quad + 2[(2C_4-C_1)\cos x - (C_2+2C_3)\sin x - C_3x\cos x - C_4x\sin x]\\
&\quad + (C_1\cos x + C_2\sin x + C_3x\cos x + C_4x\sin x)\\
&= (C_1-4C_4+4C_4-2C_1+C_1)\cos x + (C_2+4C_3-2C_2-4C_3+C_2)\sin x\\
&\quad + (C_3-2C_3+C_3)x\cos x + (C_4-2C_4+C_4)x\sin x = 0
\end{aligned}$$

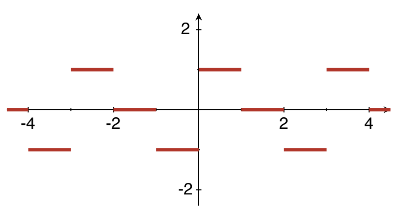
53. (a) Setting $x=2$ we have $y^3-y-2^2-4=y(y+1)(y-1)=0$. Thus, we find that $(2,0)$, $(2,1)$, and $(2,-1)$ lie on the graph.

(b) Using implicit differentiation we obtain

$$3y^2y'-y'+2x=0; \qquad y'(3y^2-1)=-2x; \qquad y'=\frac{2x}{1-3y^2}.$$

Thus $y'\,|_{x=2,y=0}=4$, $y'\,|_{x=2,y=1}=-2$, and $y'\,|_{x=2,y=-1}=-2$.

54.



55. Setting $x=\dfrac{1}{8}$ we have $\left(\dfrac{1}{8}\right)^{2/3}+y^{2/3}=1$; $y^{2/3}=\dfrac{3}{4}$. Thus, the points corresponding to $x=1/8$ are $(1/8,\pm3\sqrt{3}/8)$. Using implicit differentiation we obtain $\dfrac{2}{3}x^{-1/3}+\dfrac{2}{3}y^{-1/3}y'=0$; $y'=-x^{-1/3}y^{1/3}$. Thus $y'\Big|_{x=1/8,y=3\sqrt{3}/8}=-\sqrt{3}$ and $y'\Big|_{x=1/8,y=-3\sqrt{3}/8}=-\sqrt{3}$. Equations of the tangent lines to the graph at these points are $y\pm\dfrac{3\sqrt{3}}{8}=\pm\sqrt{3}(x-1/8)$ or $y=\pm\sqrt{3}x\mp\dfrac{\sqrt{3}}{2}$.

56. From Exercise 55, $y' = -x^{-1/3}y^{1/3}$.

$$\begin{aligned} y'' &= -x^{-1/3}\left(\frac{1}{3}y^{-2/3}y'\right) + y^{1/3}\left(\frac{1}{3}x^{-4/3}\right) = \frac{1}{3}x^{-4/3}y^{1/3} - \left(\frac{1}{3}x^{-1/3}y^{-2/3}\right)\left(-x^{-1/3}y^{1/3}\right) \\ &= \frac{1}{3}x^{-4/3}y^{1/3} + \frac{1}{3}x^{-2/3}y^{-1/3} = \frac{1}{3}x^{-4/3}y^{-1/3}(y^{2/3} + x^{2/3}) = \frac{1}{3}x^{-4/3}y^{-1/3} \end{aligned}$$

57. For $x \neq 0$, $f'(x) = \begin{cases} 2x, & x < 0 \\ \dfrac{1}{2}x^{-1/2}, & x > 0. \end{cases}$ Using (2) of Section 3.1:

$$\begin{aligned} f'_{-}(0) &= \lim_{h\to 0^-}\frac{(0+h)^2 - 0^2}{h} = \lim_{h\to 0^-}\frac{h^2}{h} = \lim_{h\to 0^-} h = 0 \\ f'_{+}(0) &= \lim_{h\to 0^+}\frac{\sqrt{0+h} - \sqrt{0}}{h} = \lim_{h\to 0^+}\frac{\sqrt{h}}{h} = \lim_{h\to 0^+}\frac{1}{\sqrt{h}} = \infty \end{aligned}$$

$f'_{+}(0)$ does not exist, so $f'(0)$ does not exist.

Chapter 4

Applications of the Derivative

4.1 Rectilinear Motion

1. $s(1/2) = -1$, $s(3) = 19$; $v(t) = 8t - 6$, $v(1/2) = -2$, $v(3) = 18$, $|v(1/2)| = 2$, $|v(3)| = 18$; $a(t) = 8$, $a(1/2) = 8$, $a(3) = 8$

2. $s(1) = 16$, $s(4) = 4$; $v(t) = 2(2t - 6)(2) = 8t - 24$, $v(1) = -16$, $v(4) = 8$, $|v(1)| = 16$, $|v(4)| = 8$; $a(t) = 8$, $a(1) = 8$, $a(4) = 8$

3. $s(-2) = 18$, $s(2) = 6$; $v(t) = -3t^2 + 6t + 1$, $v(-2) = -23$, $v(2) = 1$, $|v(-2)| = 23$, $|v(2)| = 1$; $a(t) = -6t + 6$, $a(-2) = 18$, $a(2) = -6$

4. $s(-1) = 1$, $s(3) = 57$; $v(t) = 4t^3 - 3t^2 + 1$, $v(-1) = -6$, $v(3) = 82$, $|v(-1)| = 6$, $|v(3)| = 82$; $a(t) = 12t^2 - 6t$, $a(-1) = 18$, $a(3) = 90$

5. $s(1/4) = -15/4$, $s(1) = 0$; $v(t) = 1 + 1/t^2$, $v(1/4) = 17$, $v(1) = 2$, $|v(1/4)| = 17$, $|v(1)| = 2$; $a(t) = -2/t^3$, $a(1/4) = -128$, $a(1) = -2$

6. $s(-1) = -1$, $s(0) = 0$; $v(t) = 2/(t + 2)^2$, $v(-1) = 2$, $v(0) = 1/2$, $|v(-1)| = 2$, $|v(0)| = 1/2$; $a(t) = -4/(t + 2)^3$, $a(-1) = -4$, $a(0) = -1/2$

7. $s(1) = 1$, $s(3/2) = 1/2$; $v(t) = 1 + \pi \cos \pi t$, $v(1) = 1 - \pi$, $v(3/2) = 1$, $|v(1)| = \pi - 1$, $|v(3/2)| = 1$; $a(t) = -\pi^2 \sin \pi t$, $a(1) = 0$, $a(3/2) = \pi^2$

8. $s(1/2) = 0$, $s(1) = -1$; $v(t) = -\pi t \sin \pi t + \cos \pi t$, $v(1/2) = -\pi/2$, $v(1) = -1$, $|v(1/2)| = \pi/2$, $|v(1)| = 1$; $a(t) = -\pi^2 t \cos \pi t - 2\pi \sin \pi t$, $a(1/2) = -2\pi$, $a(1) = \pi^2$

9. $v(t) = 2t - 4$
   (a) Solving $t^2 - 4t - 5 = 0$ gives $t = -1, 5$. The velocity when $s(t) = 0$ is $v(-1) = -6$, $v(5) = 6$.
   (b) Solving $t^2 - 4t - 5 = 7$ gives $t = -2, 6$. The velocity when $s(t) = 7$ is $v(-2) = -8$, $v(6) = 8$.

10. $v(t) = 2t + 6$; $a(t) = 2$

(a) Setting $t^2+6t+10=2t+6$ we obtain $t^2+4t+4=0$ or $(t+2)^2=0$. Thus $t=-2$ and $s(-2)=2$.

(b) Solving $2t+6=-2$ gives $t=-4$. The velocity when $v(t)=-a(t)$ is $v(-4)=-2$.

11. $v(t)=3t^2-4; \quad a(t)=6t$

(a) Solving $3t^2-4=2$ gives $t=\pm\sqrt{2}$. When $v(t)=2$, $a(-\sqrt{2})=-6\sqrt{2}$, $a(\sqrt{2})=6\sqrt{2}$.

(b) Solving $6t=18$ gives $t=3$. Then $s(3)=15$.

(c) Solving $t^3-4t=t(t+2)(t-2)=0$ gives $t=0,\pm 2$. Then $v(0)=-4$, $v(-2)=8$, $v(2)=8$.

12. $v(t)=3t^2-6t; \quad a(t)=6t-6$

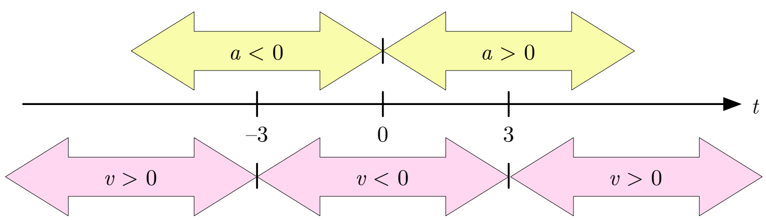
(a) Solving $3t^2-6t=0$ gives $t=0,2$. Then $s(0)=8$, $s(2)=4$.

(b) Solving $6t-6=0$ gives $t=1$. Then $s(1)=6$.

(c) The particle is slowing down when its velocity and acceleration have opposite algebraic signs. Since two numbers will have opposite signs when their product is negative, we consider $v(t)a(t)=(3t^2-6t)(6t-6)=18t(t-2)(t-1)<0$. Solving this inequality, we see that $v(t)$ and $a(t)$ will have opposite signs when $t<0$ and $1<t<2$.

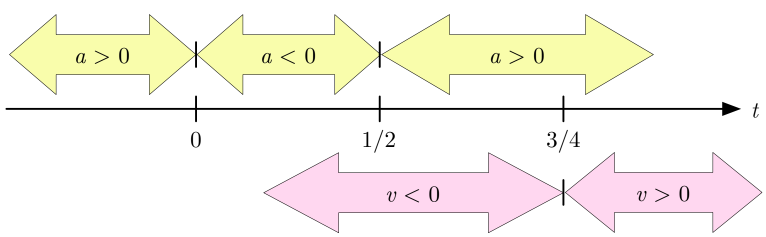
The particle is speeding up when its velocity and acceleration have the same algebraic sign. Since two numbers will have the same sign when their product is positive, we consider $v(t)a(t)=18t(t-2)(t-1)>0$. Solving this inequality, we see that $v(t)$ and $a(t)$ will have the same signs when $t>2$ or $0<t<1$.

13. $v(t)=3t^2-27=3(t-3)(t+3); \quad a(t)=6t$



The particle is slowing down on $(-\infty,-3)$ and on $(0,3)$; it is speeding up on $(-3,0)$ and on $(3,\infty)$.

14. $v(t)=4t^3-3t^2=t^2(4t-3); \quad a(t)=12t^2-6t=6t(2t-1)$

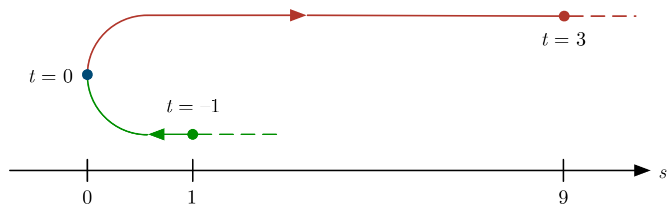


The particle is slowing down on $(-\infty,0)$ and on $(1/2,3/4)$; it is speeding up on $(0,1/2)$ and on $(3/4,\infty)$.

In order to draw the graphs in Problems 15–28 we need to determine when the particle changes direction. For a continuous position function, this will occur when the velocity is 0. This is a *necessary* condition; it is not *sufficient*. That is, the velocity may be 0 without the particle changing direction (see, for example, Problem 16). The arrows $\rightarrow$ and $\leftarrow$ in the charts indicate the direction of motion on the specified interval, as determined by the sign of the velocity on that interval.

15. $v(t) = 2t$; $\quad a(t) = 2$. Solving $v = 0$ we obtain $t = 0$.

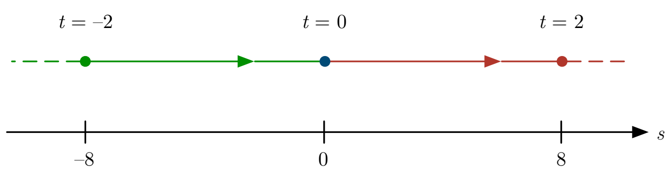
t	-1		0		3
s	1	$\leftarrow$	0	$\rightarrow$	9
v	$-$		0	$+$	
a			$+$		



The particle is slowing down on $(-\infty, 0)$ and speeding up on $(0, \infty)$.

16. $v(t) = 3t^2$; $\quad a(t) = 6t$. Solving $v = 0$ we obtain $t = 0$; solving $a = 0$ we obtain $t = 0$.

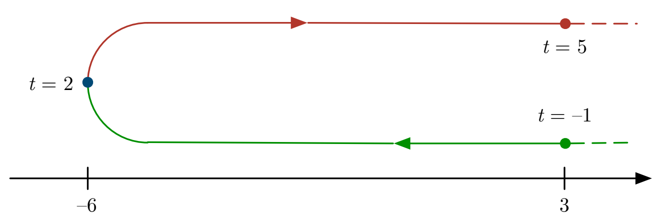
t	-2		0		2
s	-8	$\rightarrow$	0	$\rightarrow$	8
v	$+$		0	$+$	
a	$-$		0	$+$	



The particle is slowing down on $(-\infty, 0)$ and speeding up on $(0, \infty)$.

17. $v(t) = 2t - 4$; $\quad a(t) = 2$. Solving $v = 0$ we obtain $t = 2$.

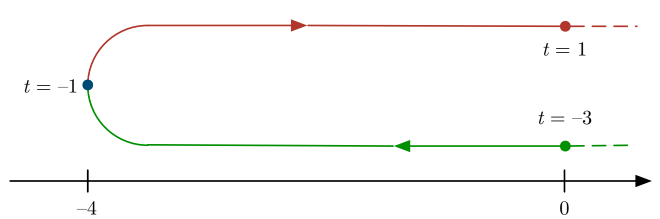
t	-1		2		5
s	3	$\leftarrow$	-6	$\rightarrow$	3
v	$-$		0	$+$	
a			$+$		



The particle is slowing down on $(-\infty, 2)$ and speeding up on $(2, \infty)$.

18. $s(t) = t^2 + 2t - 3$; $\quad v(t) = 2t + 2$; $\quad a(t) = 2$. Solving $v = 0$ we obtain $t = -1$.

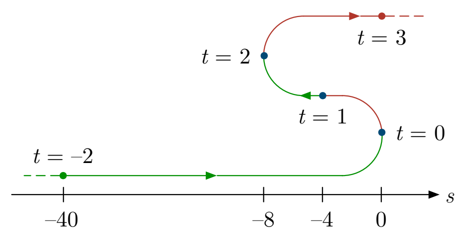
t	-3		-1		1
s	0	$\leftarrow$	-4	$\rightarrow$	0
v	$-$		0	$+$	
a			$+$		



The particle is slowing down on $(-\infty, -1)$ and speeding up on $(-1, \infty)$.

19. $v(t) = 6t^2 - 12t = 6t(t-2)$; $\quad a(t) = 12t - 12$. Solving $v = 0$ we obtain $t = 0, 2$; solving $a = 0$ we obtain $t = 1$.

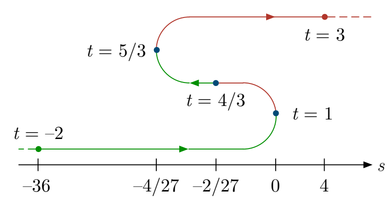
t	-2		0		1		2		3
s	-40	$\rightarrow$	0	$\leftarrow$	-4	$\leftarrow$	-8	$\rightarrow$	0
v	$+$		0	$-$			0	$+$	
a	$-$				0	$+$			



The particle is slowing down on $(-\infty, 0)$ and $(1, 2)$; it is speeding up on $(0, 1)$ and $(2, \infty)$.

20. $v(t) = (t-1)^2 + (t-2)[2(t-1)] = (t-1)(3t-5)$; $\quad a(t) = (t-1)(3) + (3t-5)(1) = 6t - 8$. Solving $v = 0$ we obtain $t = 1, 5/3$; solving $a = 0$ we obtain $t = 4/3$.

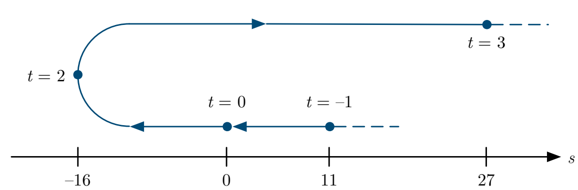
t	-2		1		$4/3$		$5/3$		3
s	-36	$\rightarrow$	0	$\leftarrow$	$-2/27$	$\leftarrow$	$-4/27$	$\rightarrow$	4
v	$+$		0	$-$			0	$+$	
a	$-$				0	$+$			



The particle is slowing down on $(-\infty, 1)$ and $(0, 5/3)$; it is speeding up on $(1, 4/3)$ and $(5/3, \infty)$.

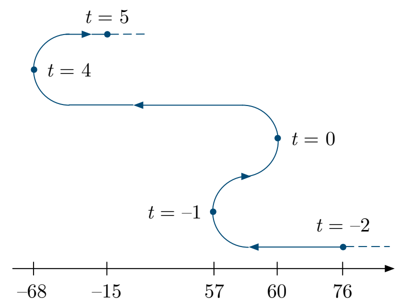
21. $v(t) = 12t^3 - 24t^2 = 12t^2(t-2)$; $\quad a(t) = 36t^2 - 48t$. Solving $v = 0$ we obtain $t = 0, 2$.

t	-1		0		2		3
s	11	$\leftarrow$	0	$\leftarrow$	-16	$\rightarrow$	27
v	$-$		0	$-$	0	$+$	



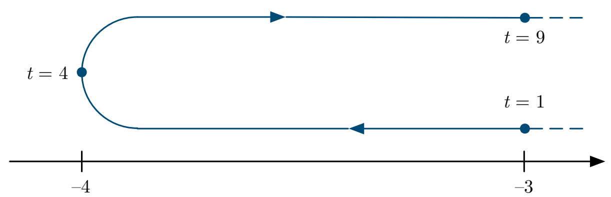
22. $v(t) = 4t^3 - 12t^2 - 16t = 4t(t+1)(t-4)$; $\quad a(t) = 12t^2 - 24t - 16$. Solving $v = 0$ we obtain $t = -1, 0, 4$.

t	-2		-1		0		4		5
s	76	$\leftarrow$	57	$\rightarrow$	60	$\leftarrow$	-68	$\rightarrow$	-15
v	$-$		0	$+$	0	$-$	0	$+$	



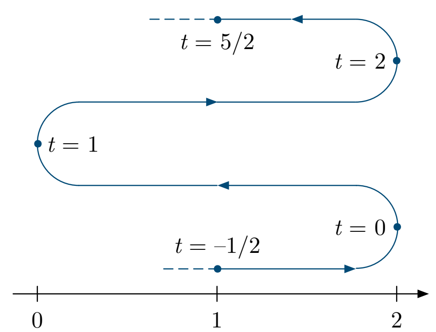
23. $v(t) = 1 - 2t^{-1/2}$; $\quad a(t) = t^{-3/2}$. Setting $v = 0$ we obtain $2/\sqrt{t} = 1$, so $\sqrt{t} = 2$ and $t = 4$.

t	1		4		9
s	-3	$\leftarrow$	-4	$\rightarrow$	-3
v	$-$		0	$+$	



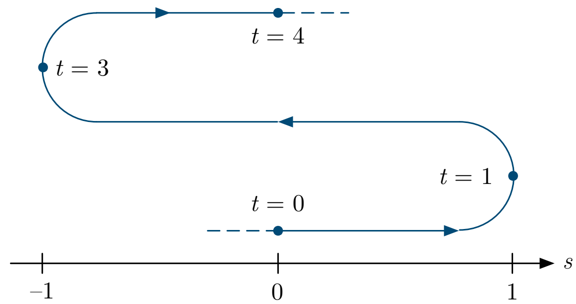
24. $v(t) = -\pi \sin \pi t$; $\quad a(t) = -\pi^2 \cos \pi t$. Setting $v = 0$ we obtain $\sin \pi t = 0$. Thus, for $-1/2 \leq t \leq 5/2$, we have $t = 0, 1, 2$.

t	$-1/2$		0		1		2		$5/2$
s	1	$\rightarrow$	2	$\leftarrow$	0	$\rightarrow$	2	$\leftarrow$	1
v	$+$		0	$-$	0	$+$	0	$-$	



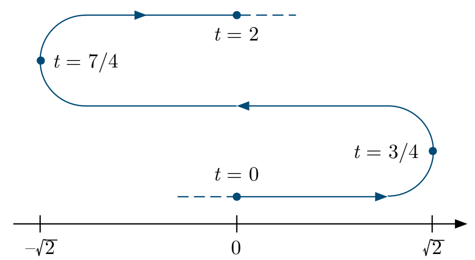
25. $v(t) = \dfrac{\pi}{2} \cos \dfrac{\pi}{2} t$; $\quad a(t) = -\dfrac{\pi^2}{4} \sin \dfrac{\pi}{2} t$. Setting $v = 0$ we obtain $\cos \dfrac{\pi}{2} t = 0$. Thus, for $0 \leq t \leq 4$, we have $t = 1, 3$.

t	0		1		3		4
s	0	$\rightarrow$	1	$\leftarrow$	-1	$\rightarrow$	0
v		+	0	$-$	0	+	



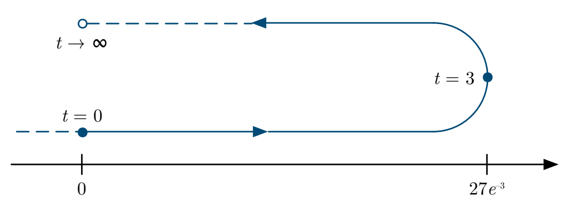
26. $v(t) = \pi\cos\pi t + \pi\sin\pi t$; $\quad a(t) = \pi^2\sin\pi t + \pi^2\cos\pi t$. Setting $v = 0$ we obtain $\cos\pi t = -\sin\pi t$ or $\tan\pi t = -1$. Thus, for $0 \le t \le 2$, we have $t = 3/4, 7/4$.

t	0		3/4		7/4		2
s	-1	$\rightarrow$	$\sqrt{2}$	$\leftarrow$	$-\sqrt{2}$	$\rightarrow$	-1
v		+	0	$-$	0	+	



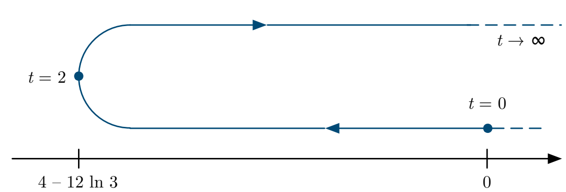
27. $v(t) = t^3(-e^{-t}) + 3t^2(e^{-t}) = e^{-t}(3t^2 - t^3) = t^2e^{-t}(3-t)$; $\quad a(t) = e^{-t}(6t - 3t^2) + (-e^{-t})(3t^2 - t^3) = te^{-t}(t^2 - 6t + 6)$. Setting $v = 0$ we obtain $t = 0, 3$. In addition, $\lim\limits_{t\to\infty} v(t) = 0$.

t	0		3		∞
s	0	$\rightarrow$	$27e^{-3}$	$\leftarrow$	0
v	0	+	0	$-$	



28. $v(t) = 2t - \dfrac{12}{t+1}$; $\quad a(t) = 2 + 12(t+1)^{-2}$. Setting $v = 0$ we obtain $t = 2$ for $0 \le t < \infty$. In addition, $\lim\limits_{t\to\infty} v(t) = \infty$.

t	0		2		∞
s	0	$\leftarrow$	$4 - 12\ln 3$	$\rightarrow$	∞
v		$-$	0	+	

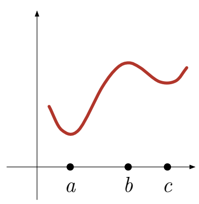


29.

Interval	$v(t)$	$a(t)$
(a, b)	$+$	$-$
(b, c)	0	0
(c, d)	$+$	$+$
(d, e)	$+$	$-$
(e, f)	$-$	$-$
(f, g)	$-$	$+$

The particle is slowing down on (a, b), (d, e), and (f, g); it is speeding up on (c, d) and (e, f).

30.



31. (a) $v(t) = -32t + 48$. Solving $v = 0$ we obtain $t = 3/2$. The velocity is positive on $(-\infty, 3/2)$ and negative on $(3/2, \infty)$.
    (b) The maximum is attained when the velocity is 0. This height is then $s(3/2) = 42$ ft.

32. $v(t) = -2t + 10$. Solving $v = 0$ we obtain $t = 5$. For $t < 5$, we have $v > 0$ and the particle is moving to the right. On $[1, 5]$ it moves $|s(5) - s(1)| = |5 - (-11)| = 16$ cm. For $t > 5$, we have $v < 0$ and the particle is moving to the left. On $[5, 6]$ it moves $|s(6) - s(5)| = |4 - 5| = 1$ cm. Thus, the total distance traveled is 17 cm.

33. $s(t) = 16t^2 \sin 30° = 8t^2$; $\quad v(t) = 16t$; $\quad a(t) = 16$. At the bottom of the hill, $s = 8t^2 = 256$ and $t = \sqrt{32} = 4\sqrt{2}$. The velocity is $v(4\sqrt{2}) = 64\sqrt{2}$ ft/s and the acceleration is $a(4\sqrt{2}) = 16$ ft/s^2.

34. $s(t) = 16t^2 \sin\theta$. Since $\sin\theta = 3/5$ in this case, $s(t) = \dfrac{48}{5}t^2$. Then $v(t) = \dfrac{96}{5}t$ and $a(t) = \dfrac{96}{5}$. At the bottom of the hill, $s = \dfrac{48}{5}t^2 = 500$ ft and $t = \dfrac{50}{4\sqrt{3}} = \dfrac{25}{2\sqrt{3}}s$. The velocity is $v(25/2\sqrt{3}) = 240/\sqrt{3}$ ft/s and the acceleration is $a(25/2\sqrt{3}) = 96/5$ ft/s^2.

35. We are given $\theta = 16t^2$. Since the circle has radius 1, $y = \sin\theta = \sin 16t^2$ and $dy/dt = 32t\cos 16t^2$. For $t = \sqrt{\pi}/4$, $dy/dt = 8\sqrt{\pi}\cos\pi = -8\sqrt{\pi}$ ft/s. Since dy/dt is negative, the y-coordinate is decreasing.

36.
$$\begin{aligned} F &= \frac{d}{dt}(mv) = \frac{d}{dt}\left(\frac{m_0 v}{\sqrt{1 - v^2/c^2}}\right) \\ &= \frac{(1 - v^2/c^2)^{1/2} m_0 \dfrac{dv}{dt} - m_0 v\left[\dfrac{1}{2}(1 - v^2/c^2)^{-1/2}(-2v/c^2)\dfrac{dv}{dt}\right]}{1 - v^2/c^2} \\ &= \frac{(1 - v^2/c^2)m_0 a - m_0 v(v/c^2)a}{(1 - v^2/c^2)^{3/2}} = \frac{m_0 a}{(1 - v^2/c^2)^{3/2}} = \frac{m_0 a}{\sqrt{(1 - v^2/c^2)^3}} \end{aligned}$$

4.2 Related Rates

1. Let V be the volume and x the length of a side. Then $V = x^3$ and $\dfrac{dV}{dt} = 3x^2\dfrac{dx}{dt}$.

2. From $V = xyz$ we find
$$\frac{dV}{dt} = x\frac{d}{dt}(yz) + yz\frac{dx}{dt} = x\left(y\frac{dz}{dt} + z\frac{dy}{dt}\right) + yz\frac{dx}{dt} = xy\frac{dz}{dt} + xz\frac{dy}{dt} + yz\frac{dx}{dt}.$$
Using the given sides and rates, we obtain $\dfrac{dV}{dt} = 1(2)(10) + 1(3)(10) + 2(3)(10) = 110\ \text{cm}^3/\text{s}$.

3. Let A be the area and let x be the length of a side. Then $A = \dfrac{\sqrt{3}x^2}{4}$ and $\dfrac{dA}{dt} = \dfrac{\sqrt{3}}{2}x\dfrac{dx}{dt}$. When $\dfrac{dx}{dt} = 2$ and $x = 8$ we have $\dfrac{dA}{dt} = \dfrac{\sqrt{3}}{2}(8)(2) = 8\sqrt{3}\ \text{cm}^2/\text{h}$.

4. From Problem 3 we have $A = \dfrac{\sqrt{3}x^2}{4}$ and $\dfrac{dA}{dt} = \dfrac{\sqrt{3}}{2}x\dfrac{dx}{dt}$. Since $A = \sqrt{75}$, we have $x^2 = 20$ and $x = 2\sqrt{5}$. When $\dfrac{dx}{dt} = 2$ cm/h and $x = 2\sqrt{5}$ cm, $\dfrac{dA}{dt} = \dfrac{\sqrt{3}}{2}(2\sqrt{5})(2) = 2\sqrt{15}\ \text{cm}^2/\text{h}$.

5. Let x be the length, y the width, and s the diagonal of the rectangle. Then $s^2 = x^2 + y^2$ or $y^2 = s^2 - x^2$, and $2y\dfrac{dy}{dt} = 2s\dfrac{ds}{dt} - 2x\dfrac{dx}{dt}$ or $\dfrac{dy}{dt} = \dfrac{s}{y}\left(\dfrac{ds}{dt}\right) - \dfrac{x}{y}\left(\dfrac{dx}{dt}\right)$. When $x = 8$ in and $y = 6$ in, $s = 10$ in. Then $\dfrac{dy}{dt} = \dfrac{10}{6}(1) - \dfrac{8}{6}\left(\dfrac{1}{4}\right) = \dfrac{4}{3}$ in/h.

6. Let x be the side of a cube and s the diagonal. Then $s^2 = 3x^2$ and $2s\dfrac{ds}{dt} = 6x\dfrac{dx}{dt}$ or $\dfrac{ds}{dt} = 3\left(\dfrac{x}{z}\right)\left(\dfrac{dx}{dt}\right)$. When $\dfrac{dx}{dt} = 5$ cm/h, $\dfrac{ds}{dt} = 3\left(\dfrac{x}{\sqrt{3}x}\right)(5) = 5\sqrt{3}$ cm/h.

7. $\sin\theta = x/s$ or $x = s\sin\theta$. Differentiating with respect to t gives $\dfrac{dx}{dt} = s\dfrac{d}{dt}\sin\theta + (\sin\theta)\dfrac{ds}{dt} = s\cos\theta\dfrac{d\theta}{dt} + \sin\theta\dfrac{ds}{dt}$.

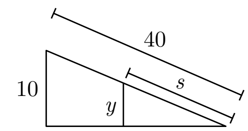
8. $\dfrac{dy}{dt} = 2x\dfrac{dx}{dt} + 4\dfrac{dx}{dt}$. When $x = 2$ cm and $\dfrac{dx}{dt} = 3$ cm/min, $\dfrac{dy}{dt} = 2(2)(3) + 4(3) = 24$ cm/min. When $y = 6$, we have $x^2 + 4x + 1 = 6$, $x^2 + 4x - 5 = 0$, and $(x+5)(x-1) = 0$. Thus $x = -5, 1$. Since $\dfrac{dy}{dt} = 2x\dfrac{dx}{dt} + 4\dfrac{dx}{dt}$, then for $\dfrac{dx}{dt} = 3$,
$$\left.\frac{dy}{dt}\right|_{x=1} = 2(1)(3) + 4(3) = 18\ \text{cm/min} \quad \text{and} \quad \left.\frac{dy}{dt}\right|_{x=-5} = 2(-5)(3) + 4(3) = -18\ \text{cm/min}.$$

9. $2y\dfrac{dy}{dt} = \dfrac{dx}{dt}$; $\dfrac{dy}{dt} = \dfrac{dx/dt}{2y}$. From $y^2 = x+1$ we see that for $x = 8$, $y = \pm 3$. Since $\dfrac{dx}{dt} = 4x+4$, we have $\dfrac{dy}{dt} = \dfrac{4x+4}{2y}$. Thus $\left.\dfrac{dy}{dt}\right|_{y=3} = \dfrac{4(8)+4}{2(3)} = 6$ and $\left.\dfrac{dy}{dt}\right|_{y=-3} = \dfrac{4(8)+4}{2(-3)} = -6$.

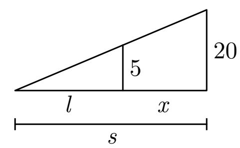
10. $4\dfrac{dy}{dt} = 2x\dfrac{dx}{dt} + \dfrac{dx}{dt}$. Since $\dfrac{dx}{dt} = \dfrac{dy}{dt}$, we cancel the derivatives and obtain $4 = 2x + 1$ or $x = \dfrac{3}{2}$. From $4y = x^2 + x$ we see that for $x = \dfrac{3}{2}$, $y = \dfrac{15}{16}$. Hence the point on the graph is $(3/2, 15/16)$.

11. If T is the area of the triangle then $T = \dfrac{1}{2}xy = \dfrac{1}{2}x^{4/3}$ and $\dfrac{dT}{dt} = \dfrac{2}{3}x^{1/3}\dfrac{dx}{dt}$. When $x = 8$, then $\dfrac{dT}{dt} = \dfrac{2}{3}(8)^{1/3}\left(\dfrac{1}{3}\right) = \dfrac{4}{9}$ cm^2/h.

12. Using similar triangles, $y/s = 10/40$ or $y = s/4$. Then $\dfrac{dy}{dt} = \dfrac{1}{4} \cdot \dfrac{ds}{dt}$ and since $\dfrac{ds}{dt} = 2$ ft/s, $\dfrac{dy}{dt} = \dfrac{1}{4}(2) = \dfrac{1}{2}$ ft/s.



13. (a) Since the lengths of corresponding sides in similar triangles are proportional, $\dfrac{5}{20} = \dfrac{l}{l+x}$ or $l = x/3$. When $\dfrac{dx}{dt} = 3$ ft/s, differentiating gives $\dfrac{dl}{dt} = \dfrac{1}{3} \cdot \dfrac{dx}{dt} = \dfrac{1}{3}(3) = 1$ ft/s.



(b) Differentiating $s = l + x$ gives $\dfrac{ds}{dt} = \dfrac{dl}{dt} + \dfrac{dx}{dt}$. Since $\dfrac{dx}{dt} = 3$ ft/s and from (a) $\dfrac{dl}{dt} = 1$ ft/s, the tip of the shadow is moving at a rate of $\dfrac{ds}{dt} = 1 + 3 = 4$ ft/s.

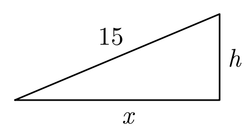
14. (a) Since $D = 2r$, we have $\dfrac{dD}{dt} = 2\dfrac{dr}{dt} = 2 \cdot 2 = 4$ ft/s.

(b) Since $C = 2\pi r$, we have $\dfrac{dC}{dt} = 2\pi\dfrac{dr}{dt} = 2\pi \cdot 2 = 4\pi$ ft/s.

(c) Since $A = \pi r^2$ we have $\dfrac{dA}{dt} = 2\pi r\dfrac{dr}{dt}$. When $\dfrac{dr}{dt} = 2$ ft/s and $r = 3$ ft, $\dfrac{dA}{dt} = 2\pi \cdot 3 \cdot 2 = 12\pi$ ft^2/s.

(d) Since $A = \pi r^2 = 8\pi$ we have $r = 2\sqrt{2}$. When $\dfrac{dr}{dt} = 2$ ft/s and $r = 2\sqrt{2}$ ft, $\dfrac{dA}{dt} = 2\pi(2\sqrt{2})(2) = 8\pi\sqrt{2}$ ft^2/s.

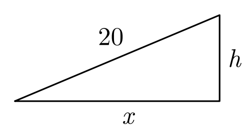
15. From the Pythagorean Theorem, $15^2 = h^2 + x^2$. Differentiating gives



$$0 = 2h\frac{dh}{dt} + 2x\frac{dx}{dt} \qquad \text{or} \qquad \frac{dh}{dt} = -\frac{x}{h} \cdot \frac{dx}{dt}.$$

When $x = 5$ ft, $h = 10\sqrt{2}$ ft, and $\dfrac{dx}{dt} = 2$ ft/min, we have $\dfrac{dh}{dt} = -\dfrac{5}{10\sqrt{2}}(2) = -\dfrac{1}{\sqrt{2}}$ ft/min.

16. From the Pythagorean Theorem, $20^2 = h^2 + x^2$. Differentiating gives

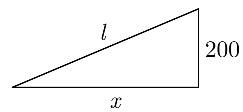


$$0 = 2h\frac{dh}{dt} + 2x\frac{dx}{dt} \qquad \text{or} \qquad \frac{dx}{dt} = -\frac{h}{x} \cdot \frac{dh}{dt}.$$

When $h = 18$ ft, $x = 2\sqrt{19}$ ft, and $\dfrac{dh}{dt} = -\dfrac{1}{2}$ ft/min, we have $\dfrac{dx}{dt} = -\dfrac{18}{2\sqrt{19}}\left(-\dfrac{1}{2}\right) = \dfrac{9}{2\sqrt{19}}$ ft/min.

17. Since $\theta_1 = \dfrac{\pi}{2} - \theta_2$, $\dfrac{d\theta_1}{dt} = -\dfrac{d\theta_2}{dt}$ and θ_1 is increasing at the same rate θ_2 is decreasing.

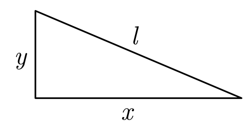
18. From the Pythagorean Theorem, $l^2 = x^2 + 200^2$. Differentiating gives



$$2l\frac{dl}{dt} = 2x\frac{dx}{dt} \qquad \text{or} \qquad \frac{dx}{dt} = \frac{l}{x}\cdot\frac{dl}{dt}.$$

We are given $\dfrac{dl}{dt} = 3$ ft/s, and when $l = 400$ ft we have $x^2 = 400^2 - 200^2$ or $x = 200\sqrt{3}$ ft. Then the kite moves at a rate of $\dfrac{dx}{dt} = \dfrac{400}{200\sqrt{3}}(3) = 2\sqrt{3}$ ft/s.

19. From the Pythagorean Theorem, $x^2 + y^2 = l^2$. Differentiating gives

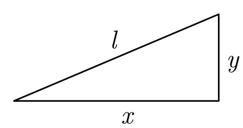


$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2l\frac{dl}{dt}.$$

We are given $\dfrac{dx}{dt} = 10$ knots and $\dfrac{dy}{dt} = 15$ knots, so $\dfrac{dl}{dt} = \dfrac{10x + 15y}{l}$. At 2:00 PM, $x = 20$ nautical miles, $y = 15$ nautical miles, and $l = \sqrt{20^2 + 15^2} = 25$ nautical miles. Thus

$$\frac{dl}{dt} = \frac{10(20) + 15(15)}{25} = 17 \text{ knots.}$$

20. From the Pythagorean Theorem, $x^2 + y^2 = l^2$. Differentiating gives

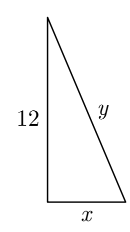


$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2l\frac{dl}{dt}.$$

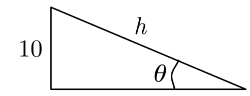
We are given $\dfrac{dy}{dt} = -9$ km/h and $\dfrac{dx}{dt} = 12$ km/h. Then $\dfrac{dl}{dt} = \dfrac{12x - 9y}{l}$. At 9:20 AM, $y = 20 - \dfrac{4}{3}(9) = 8$ km, $x = \dfrac{4}{3}(12) = 16$ km, and $l = \sqrt{16^2 + 8^2} = 8\sqrt{5}$ km. Therefore

$$\frac{dl}{dt} = \frac{12(16) + 9(8)}{8\sqrt{5}} = 3\sqrt{5} \text{ km/h.}$$

21. Let x be the distance from the boat to the base of the dock and y be the distance from the boat to the pulley. From the Pythagorean Theorem, $x^2 + 144 = y^2$. Thus, $2x\dfrac{dx}{dt} = 2y\dfrac{dy}{dt}$ and $\dfrac{dx}{dt} = \left(\dfrac{y}{x}\right)\dfrac{dy}{dt}$. We are given $\dfrac{dy}{dt} = -1$ and $x = 16$. At this time, $y = 20$ and $\dfrac{dx}{dt} = \dfrac{20}{16}(-1) = -\dfrac{5}{4}$ ft/s. Hence, the boat is approaching the dock at a rate of $\dfrac{5}{4}$ ft/s.

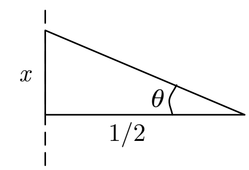


22. We are given $\frac{dh}{dt} = -1$ and we want to find $\frac{d\theta}{dt}$ when $h = 30$. Then $\theta = \csc^{-1}\frac{h}{10}$, $\frac{d\theta}{dt} = \frac{-1/10}{(h/10)\sqrt{(h/10)^2-1}} \cdot \frac{dh}{dt}$, and

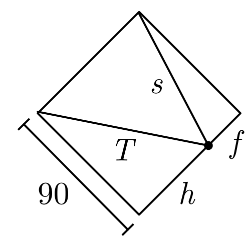


$$\left.\frac{d\theta}{dt}\right|_{h=30} = \frac{-1}{30\sqrt{9-1}}(-1) = \frac{\sqrt{2}}{120} \text{ rad/s.}$$

23. We are given $\frac{dx}{dt} = 15$ and we want to find $\frac{d\theta}{dt}$ when $x = 1/2$. Then $\theta = \tan^{-1}\frac{x}{1/2} = \tan^{-1} 2x$, $\frac{d\theta}{dt} = \frac{2}{1+4x^2} \cdot \frac{dx}{dt}$, and $\left.\frac{d\theta}{dt}\right|_{x=1/2} = 1(15) = 15$ rad/h.



24. From the Pythagorean Theorem, $s^2 = f^2 + 90^2$. Differentiating gives



$$2s\frac{ds}{dt} = 2f\frac{df}{dt} \qquad \text{or} \qquad \frac{ds}{dt} = \frac{f}{s} \cdot \frac{df}{dt}.$$

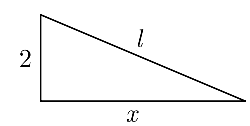
We are given $\frac{df}{dt} = -20$ ft/s, and when the runner is 60 ft from home base, $f = 90 - 60 = 30$ and $s = \sqrt{30^2 + 90^2} = \sqrt{9000} = 30\sqrt{10}$. Then

$$\frac{ds}{dt} = \frac{30}{30\sqrt{10}}(-20) = -\frac{20}{\sqrt{10}} \approx -6.325 \text{ ft/s.}$$

The distance from the runner to second base is decreasing at a rate of approximately 6.325 ft/s. Similarly, $T^2 = h^2 + 90^2$ and $\frac{dT}{dt} = \frac{h}{T} \cdot \frac{dh}{dt}$. We are given $\frac{dh}{dt} = 20$ ft/s, and when $h = 60$, $T = \sqrt{60^2 + 90^2} = \sqrt{11{,}700} = 30\sqrt{13}$. Then

$$\frac{dT}{dt} = \frac{60}{30\sqrt{13}}(20) = \frac{40}{\sqrt{13}} \approx 11.094 \text{ ft/s.}$$

25. From the Pythagorean Theorem, $l^2 = x^2 + 4$. Differentiating gives



$$2l\frac{dl}{dt} = 2x\frac{dx}{dt} \qquad \text{or} \qquad \frac{dl}{dt} = \frac{x}{l} \cdot \frac{dx}{dt}.$$

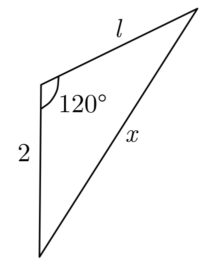
We are given $\frac{dx}{dt} = -600$ mi/h, and when $x = 1.5$ mi, $l = 2.5$ mi. Then

$$\frac{dl}{dt} = \frac{1.5}{2.5}(-600) = -360 \text{ mi/h.}$$

Thus the distance is decreasing at a rate of 360 mi/h.

26. From the law of cosines, $x^2 = 2^2 + l^2 - 2(2)l\cos 120° = 4 + l^2 + 2l$. Differentiating gives

$$2x\frac{dx}{dt} = (2l+2)\frac{dl}{dt} \qquad \text{or} \qquad \frac{dx}{dt} = \frac{l+1}{x}\cdot\frac{dl}{dt}.$$

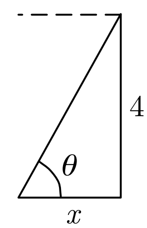


We are given $\frac{dl}{dt} = 600$ mi/h. After one minute, the plane has travelled 10 miles. When $l = 10$ mi, $x^2 = 4 + 10^2 + 2(10)$ and $x = 2\sqrt{31}$ mi. Thus,

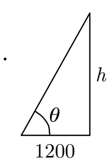
$$\frac{dx}{dt} = \frac{10+1}{2\sqrt{31}}(600) = \frac{3300}{\sqrt{31}} \approx 592.70 \text{ mi/h}.$$

27. Differentiating $x = 4\cot\theta$, we obtain $\frac{dx}{dt} = -4\csc^2\theta\frac{d\theta}{dt}$. Converting 30° to $\pi/6$ radians, we are given $\frac{d\theta}{dt} = -\frac{\pi}{6}$. Thus, when $\theta = 60°$,

$$\frac{dx}{dt} = -4(\csc^2 60°)\left(-\frac{\pi}{6}\right) = \frac{8\pi}{9} \approx 2.79 \text{ km/min}.$$



28. Differentiating $x = 1200\tan\theta$ gives $\frac{dx}{dt} = 1200\sec^2\theta\frac{d\theta}{dt}$. We are given $\frac{d\theta}{dt} = 0.1$. When $\theta = \frac{\pi}{6}$, $\frac{dx}{dt} = 1200\left(\sec^2\frac{\pi}{6}\right)(0.1) = 1200(4/3)(0.1) = 160$ km/s.



29. Let y be the altitude of the rocket, x the distance along the ground from the point of launch, and s the distance the rocket has travelled.

   (a) $y = s\sin 60° = \frac{\sqrt{3}}{2}s$; $\frac{dy}{dt} = \frac{\sqrt{3}}{2}\cdot\frac{ds}{dt}$. When $\frac{ds}{dt} = 1000$, $\frac{dy}{dt} = \frac{\sqrt{3}}{2}(1000) = 500\sqrt{3}$ mi/h.

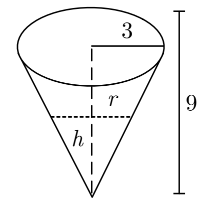
   (b) $x = s\cos 60° = \frac{1}{2}s$; $\frac{dx}{dt} = \frac{1}{2}\cdot\frac{ds}{dt}$. When $\frac{ds}{dt} = 1000$, $\frac{dx}{dt} = \frac{1}{2}(1000) = 500$ mi/h.

30. $V = \pi r^2 h$. Since r is a constant, differentiating with respect to t gives $\frac{dV}{dt} = \pi r^2\frac{dh}{dt}$. When $r = 40/2 = 20$ ft and $\frac{dh}{dt} = -3/2$ ft/min, $\frac{dV}{dt} = \pi(20)^2(-3/2) = -600\pi$ ft^3/min. Thus the volume is decreasing at a rate of 600π ft^3/min.

31. $V = \pi r^2 h$. Since r is a constant, differentiating with respect to t gives $\frac{dV}{dt} = \pi r^2 h\frac{dh}{dt}$ or $\frac{dh}{dt} = \frac{1}{\pi r^2}\cdot\frac{dV}{dt}$. When $r = 8$ m and $\frac{dV}{dt} = 10$ m^3/min, the oil level rises at a rate of $\frac{dh}{dt} = \frac{10}{\pi(8)^2} = \frac{5}{32\pi}$ m/min.

32. The volume of water is $V = 5xh$, so $\dfrac{dV}{dt} = 5x\dfrac{dh}{dt} + 5h\dfrac{dx}{dt}$. We are given $\dfrac{dV}{dt} = 1$, $\dfrac{dx}{dt} = \dfrac{1}{12}$, and $x = 4$. From $V = 40$ and $x = 4$ we see that $h = 2$. Then $1 = 5(4)\dfrac{dh}{dt} + 5(2)\dfrac{1}{12}$ and $\dfrac{dh}{dt} = \dfrac{1}{120}$ ft/min $= \dfrac{1}{10}$ in/min. The water is rising at this instant.

33. (a) Since the lengths of corresponding sides in similar triangles are proportional, $\dfrac{h}{9} = \dfrac{r}{3}$ or $h = 3r$. The volume of water is $V = \dfrac{1}{3}\pi r^2 h = \dfrac{1}{3}\pi\left(\dfrac{h}{3}\right)^2 h = \dfrac{1}{27}\pi h^3$. Differentiating gives



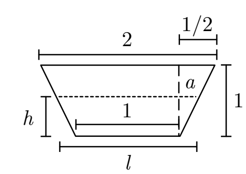
$$\frac{dV}{dt} = \frac{1}{9}\pi h^2\frac{dh}{dt} \qquad \text{or} \qquad \frac{dh}{dt} = \frac{9}{\pi h^2}\cdot\frac{dV}{dt}.$$

We are given $\dfrac{dV}{dt} = -1$. When $h = 6$, the water level is changing at a rate of $\dfrac{dh}{dt} = \dfrac{9}{\pi(36)}(-1) = -\dfrac{1}{4\pi}$ ft/min.

(b) From $h = 3r$ we have $\dfrac{dh}{dt} = 3\dfrac{dr}{dt}$, so when $h = 6$, the radius of water is changing at a rate of $\dfrac{dr}{dt} = \dfrac{1}{3}\cdot\dfrac{dh}{dt} = \dfrac{1}{3}\left(-\dfrac{1}{4\pi}\right) = -\dfrac{1}{12\pi}$ ft/min.

(c) The initial volume of water is $V_0 = \dfrac{1}{3}\pi(3)^2(9) = 27\pi$ ft^3. At time t the volume of water is $V(t) = V_0 - t = 27\pi - t$. We also have from part (a) that $V = \dfrac{1}{3}\pi r^2 h = \dfrac{1}{3}\pi r^2(3r) = \pi r^3$. Thus, $\pi r^3 = 27\pi - t$ or $r = \left(27 - \dfrac{t}{\pi}\right)^{1/3}$. Then $\dfrac{dr}{dt} = -\dfrac{1}{3\pi}\left(27 - \dfrac{t}{\pi}\right)^{-2/3}$ and $\left.\dfrac{dr}{dt}\right|_{t=6} = -\dfrac{1}{3\pi(27 - 6/\pi)^{2/3}} \approx -0.0124$ ft/min.

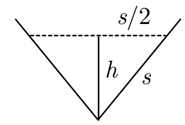
34. Since the lengths of corresponding sides in similar triangles are proportional, $a/h = 1/2$ and $l = 1 + 2a = 1 + h$. The volume of water is $V = \dfrac{h(1+l)}{2}(4) = 2h(2+h) = 4h + 2h^2$. Differentiating gives



$$\frac{dV}{dt} = (4 + 4h)\frac{dh}{dt} \qquad \text{or} \qquad \frac{dh}{dt} = \frac{1}{4+4h}\cdot\frac{dV}{dt}.$$

When $h = \dfrac{1}{4}$ m and $\dfrac{dV}{dt} = \dfrac{1}{2}$ m^3/s, the rate at which the water level rises is $\dfrac{dh}{dt} = \dfrac{1}{4+4(1/4)}\cdot\dfrac{1}{2} = \dfrac{1}{10}$ m/s.

35. (a) From the Pythagorean Theorem, $s^2 = h^2 + (s/2)^2$ or $s = \dfrac{2h}{\sqrt{3}}$. The volume of the water is $V = \dfrac{1}{2}sh(20) = 10\dfrac{2h}{\sqrt{3}}h = \dfrac{20h^2}{\sqrt{3}}$. Differentiating with respect to t gives $\dfrac{dV}{dt} = \dfrac{40h}{\sqrt{3}} \cdot \dfrac{dh}{dt}$ so $\dfrac{dh}{dt} = \dfrac{\sqrt{3}}{40h} \cdot \dfrac{dV}{dt}$. When $h = 1$ ft and $\dfrac{dV}{dt} = 4$ ft^3/min, the rate at which the water level rises is $\dfrac{dh}{dt} = \dfrac{\sqrt{3}}{10}$ ft/min.



(b) From part (a) we see that the initial volume of water is $V_0 = \dfrac{20h_0^2}{\sqrt{3}}$. At time t, the volume of water is $V = 4t + V_0 = 4t + \dfrac{20h_0^2}{\sqrt{3}}$. In terms of h, we saw in part (a) that $V = \dfrac{20h^2}{\sqrt{3}}$. Thus, $\dfrac{20h^2}{\sqrt{3}} = 4t + \dfrac{20h_0^2}{\sqrt{3}}$. Solving for h and differentiating, we find

$$h = \sqrt{\frac{\sqrt{3}}{5}t + h_0^2} \quad \text{and} \quad \frac{dh}{dt} = \frac{1}{2}\left(\frac{\sqrt{3}}{5}t + h_0^2\right)^{-1/2}\frac{\sqrt{3}}{5} = \frac{\sqrt{3}}{10}\left(\frac{\sqrt{3}}{5}t + h_0^2\right)^{-1/2}.$$

(c) Setting $h = 5$ and $h_0 = \dfrac{1}{2}$ in part (b), we have $5 = \sqrt{\dfrac{\sqrt{3}}{5}t + \dfrac{1}{4}}$ or $t = \dfrac{165\sqrt{3}}{4} \approx 71.45$ min. The rate at which the water is rising when $t = \dfrac{165\sqrt{3}}{4}$ is

$$\left.\frac{dh}{dt}\right|_{t=165\sqrt{3}/4} = \frac{\sqrt{3}}{10}\left(\frac{\sqrt{3}}{5} \cdot \frac{165\sqrt{3}}{4} + \frac{1}{4}\right)^{-1/2} = \frac{\sqrt{3}}{50} \approx 0.035 \text{ ft/min.}$$

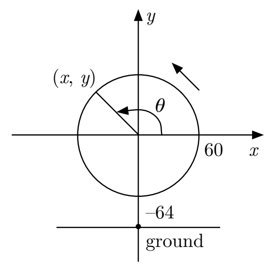
36. The volume between the spheres is $V = \dfrac{4}{3}\pi r_0^3 - \dfrac{4}{3}\pi r_i^3$. Differentiating gives $\dfrac{dV}{dt} = 4\pi r_0^2\dfrac{dr_0}{dt} - 4\pi r_i^2\dfrac{dr_i}{dt}$. For $\dfrac{dr_0}{dt} = 2$ m/hr, $\dfrac{dr_i}{dt} = -\dfrac{1}{2}$ m/hr, $r_0 = 3$ m, and $r_i = 1$ m, we have $\dfrac{dV}{dt} = 4\pi(9)(2) - 4\pi(1)(-1/2) = 74\pi$ m^3/h.

37. The volume of a sphere is $V = \dfrac{4}{3}\pi r^3$. Differentiating gives $\dfrac{dV}{dt} = 4\pi r^2\dfrac{dr}{dt}$. The surface area of a sphere is $S = 4\pi r^2$, so $\dfrac{dV}{dt} = S\dfrac{dr}{dt}$. Since we are given that $\dfrac{dV}{dt} = kS$, we have $\dfrac{dr}{dt} = k$. Thus, the radius changes at a constant rate.

38. The volume is $V = \dfrac{4}{3}\pi r^3$, so $\dfrac{dV}{dt} = 4\pi r^2\dfrac{dr}{dt} = k$, where k is a constant. Now, the surface area is $S = 4\pi r^2$ and $\dfrac{dS}{dt} = 8\pi r\dfrac{dr}{dt}$. From the formula for $\dfrac{dV}{dt}$, we have $\dfrac{dr}{dt} = \dfrac{k}{4\pi r^2}$. Thus, $\dfrac{dS}{dt} = 8\pi r\left(\dfrac{k}{4\pi r^2}\right) = \dfrac{2k}{r}$, and the rate of change of the surface area is inversely proportional to the radius.

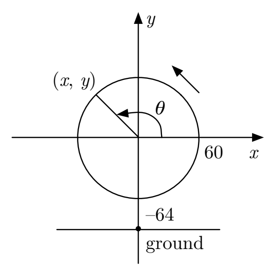
39. $V = x^3$, $\dfrac{dV}{dt} = 3x^2\dfrac{dx}{dt}$. The surface area of the cube is $S = 6x^2$, so when $S = 54$, $x = 3$. Now when $\dfrac{dV}{dt} = -\dfrac{1}{4}$ we have $-\dfrac{1}{4} = 3(3)^2\dfrac{dx}{dt}$ and $\dfrac{dx}{dt} = -\dfrac{1}{108}$. From $\dfrac{dS}{dt} = 12x\dfrac{dx}{dt}$ we use $x = 3$ and $\dfrac{dx}{dt} = -\dfrac{1}{108}$ to compute $\dfrac{dS}{dt} = 12(3)\left(-\dfrac{1}{108}\right) = -\dfrac{1}{3}$ in^2/min.

40. Place the origin at the center of the ferris wheel with the x-axis parallel to the ground. To find the vertical and horizontal rates, we use $y = 60\sin\theta$ and $x = 60\cos\theta$. Differentiating, we have $\dfrac{dy}{dt} = 60\cos\theta\dfrac{d\theta}{dt}$ and $\dfrac{dx}{dt} = -60\sin\theta\dfrac{d\theta}{dt}$. Assuming the wheel revolves counter-clockwise once every two minutes, $\dfrac{d\theta}{dt} = \pi$ radians per minute. Thus, $\dfrac{dy}{dt} = 60\pi\cos\theta$ and $\dfrac{dx}{dt} = -60\pi\sin\theta$.



When the passenger is 64 feet above the ground, $\theta = 0$, $\sin\theta = 0$, and $\cos\theta = 1$. Thus, the passenger is rising $\dfrac{dy}{dt} = 60\pi$ ft/min, and is moving horizontally $\dfrac{dx}{dt} = 0$ ft/min.

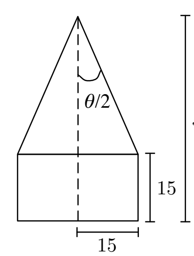
41. Place the origin at the center of the ferris wheel with the x-axis parallel to the ground. The coordinates of the point P are $x = 60\cos\theta$ and $y = 60\sin\theta$. If the coordinates of the point Q are $(q, -64)$ then the slope of the line through PQ is $\dfrac{60\sin\theta + 64}{60\cos\theta - q}$. Since this line is perpendicular to the line through the center of the wheel and P, its slope is also $-\dfrac{1}{\tan\theta}$.



Solving $\dfrac{60\sin\theta + 64}{60\cos\theta - q} = -\dfrac{1}{\tan\theta}$ for q we obtain $q = 60\cos\theta + 64\tan\theta + 60\sin\theta\tan\theta$ and $\dfrac{dq}{dt} = \left(-60\sin\theta + 64\sec^2\theta + 60\sin\theta\sec^2\theta + 60\sin\theta\right)\dfrac{d\theta}{dt}$. When $\theta = \pi/4$ and $\dfrac{d\theta}{dt} = \pi$ we have $\left.\dfrac{dq}{dt}\right|_{\theta=\pi/4} = (60\sqrt{2} + 128)\pi \approx 668.7$ radians/min.

42. (a) From the figure, we can see that $\tan\theta/2 = \dfrac{15}{s - 15}$ and so

$$\theta/2 = \tan^{-1}\frac{15}{s-15} \qquad \text{and} \qquad \theta = 2\tan^{-1}\frac{15}{s-15}.$$



(b) Differentiating the expression in (a) yields

$$\frac{d\theta}{dt} = \frac{2}{1 + \left(\dfrac{15}{s-15}\right)^2} \cdot \frac{-15}{(s-15)^2} \cdot \frac{ds}{dt} = -\frac{30}{(s-15)^2 + 225} \cdot \frac{ds}{dt}.$$

From $s = -16t^2 - t + 200$ we get $\dfrac{ds}{dt} = -32t - 1$, so when $t = 3$, $s = 53$ and $\dfrac{ds}{dt} = -97$. Thus, $\left.\dfrac{d\theta}{dt}\right|_{t=3} = -\dfrac{30}{(53-15)^2 + 225}(-97) \approx 1.74$ rad/s.

(c) As $s \to 15$, $\theta \to 2 \lim_{s\to 15} \tan^{-1} \dfrac{15}{s-15} = 2 \lim_{x\to\infty} \tan^{-1} x = 2(\pi/2) = \pi$.

(d) To find when the diver hits the water, we solve $-16t^2 - t + 200 = 15$, obtaining $t \approx 3.37$ s. Then $\left.\dfrac{d\theta}{dt}\right|_{t=3.37;\ s=15} = -\dfrac{30}{225}[-32(3.37) - 1] \approx 14.51$ rad/s.

43. $R^{-1} = R_1^{-1} + R_2^{-1}$. Differentiating with respect to t gives

$$-R^{-2}\frac{dR}{dt} = -R_1^{-2}\frac{dR_1}{dt} - R_2^{-2}\frac{dR_2}{dt} \quad \text{and} \quad \frac{1}{R^2}\cdot\frac{dR}{dt} = \frac{1}{R_1^2}\cdot\frac{dR_1}{dt} + \frac{1}{R_2^2}\cdot\frac{dR_2}{dt}$$

so $\dfrac{dR}{dt} = \dfrac{R^2}{R_1^2}\cdot\dfrac{dR_1}{dt} + \dfrac{R^2}{R_2^2}\cdot\dfrac{dR_2}{dt}$.

44. From $PV^{1.4} = k$ we obtain $P\dfrac{d}{dt}V^{1.4} + V^{1.4}\dfrac{dP}{dt} = 0$ and $P(1.4V^{0.4})\dfrac{dV}{dt} + V^{1.4}\dfrac{dP}{dt} = 0$ so $\dfrac{dP}{dt} = -\dfrac{1.4PV^{0.4}}{V^{1.4}}\cdot\dfrac{dV}{dt}$. When $P = 100$ lb/in^2, $V = 32$ in^3 and $\dfrac{dV}{dt} = -2$ in^3/s, so $\dfrac{dP}{dt} = -\dfrac{1.4(100)(32)^{0.4}}{(32)^{1.4}}(-2) = 8.75$ lb/in^2/s.

45. (a) From $R = \dfrac{C}{T} = \dfrac{0.493T - 0.913}{T} = 0.493 - \dfrac{0.913}{T}$ we find $\dfrac{dR}{dt} = \dfrac{0.913}{T^2}\cdot\dfrac{dT}{dt} > 0$. Thus, the ratio increases.

(b) To find the value of T when $C = \dfrac{T}{3}$, we solve $\dfrac{T}{3} = 0.493T - 0.913$, obtaining $T \approx 5.718$. Then $\left.\dfrac{dR}{dt}\right|_{T=5.718} \approx \dfrac{0.913}{(5.718)^2}(1) \approx 0.028 = 2.8\%$/day.

46. The rate of change of length is $\dfrac{dL}{dt} = \dfrac{18-10}{20} = 0.4$ cm per million years. From $E = 0.007P^{2/3} = 0.007(0.12L^{2.53})^{2/3} \approx 0.0017L^{1.68667}$ we obtain $\dfrac{dE}{dt} \approx 0.0029L^{0.68667}\dfrac{dL}{dt} = 0.0029L^{0.68667}(0.4) = 0.00115L^{0.68667}$. To determine the value of L when the fish was half its final body weight, we note that the final body weight is $P = 0.12(18)^{2.53}$ and solve $\dfrac{1}{2}(0.12)(18)^{2.53} = 0.12L^{2.53}$. This gives $L \approx 13.69$ mm. Thus, the rate at which the species's brain was growing is

$$\left.\frac{dE}{dt}\right|_{L=13.69} \approx 0.00115(13.69)^{0.68667} \approx 0.0069 \text{ g/million years.}$$

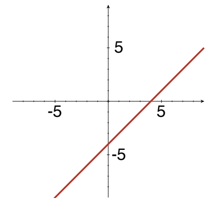
47. (a) From $\dfrac{dP}{dt} = 800\dfrac{dm}{dt}$ and $\dfrac{dm}{dt} = 30$ kg/h, we see that the momentum is changing at a rate of $800(400) = 24,000$ kg km/h.

(b) In this case, both m and v are variables so $\dfrac{dP}{dt} = m\dfrac{dv}{dt} + v\dfrac{dm}{dt}$. At $t = 1$ hour the mass of the airplane is $10^5 + 30 =$ 100,030 kg and the velocity is 750 km/h. Thus,

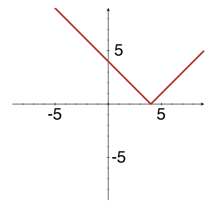
$$\left.\frac{dP}{dt}\right|_{t=1} = 100{,}030(20) + 750(30) = 2{,}023{,}100 \text{ kg km/h}^2.$$

4.3 Extrema of Functions

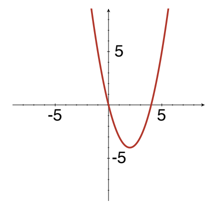
1. (a) Absolute maximum: $f(2) = -2$; absolute minimum: $f(-1) = -5$
   (b) Absolute maximum: $f(7) = 3$; absolute minimum: $f(3) = -1$
   (c) No extrema
   (d) Absolute maximum: $f(4) = 0$; absolute minimum: $f(1) = -3$



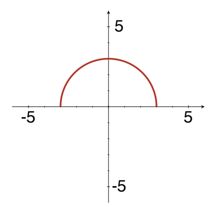
2. (a) Absolute maximum: $f(-1) = 5$; absolute minimum: $f(2) = 2$
   (b) Absolute maximum: $f(7) = 3$; absolute minimum: $f(4) = 0$
   (c) Absolute minimum: $f(4) = 0$
   (d) Absolute maximum: $f(1) = 3$; absolute minimum: $f(4) = 0$



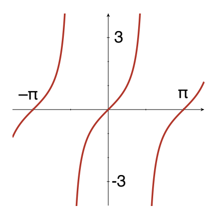
3. (a) Absolute maximum: $f(4) = 0$; absolute minimum: $f(2) = -4$
   (b) Absolute maximum: $f(1) = f(3) = -3$;
   absolute minimum: $f(2) = -4$
   (c) Absolute minimum: $f(2) = -4$
   (d) Absolute maximum: $f(5) = 5$



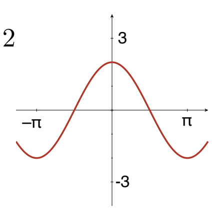
4. (a) Absolute maximum: $f(0) = 3$; absolute minimum: $f(-3) = f(3) = 0$
   (b) Absolute maximum: $f(0) = 3$
   (c) Absolute minimum: $f(0) = 3$
   (d) Absolute maximum: $f(0) = 3$;
   absolute minimum: $f(-1) = f(1) = 2\sqrt{2}$



5. (a) No extrema
   (b) Absolute maximum: $f(\pi/4) = 1$; absolute minimum: $f(-\pi/4) = -1$
   (c) Absolute maximum: $f(\pi/3) = \sqrt{3}$; absolute minimum: $f(0) = 0$
   (d) No extrema



6. (a) Absolute maximum: $f(0) = 2$; absolute minimum: $f(-\pi) = f(\pi) = -2$
   (b) Absolute maximum: $f(0) = 2$;
   absolute minimum: $f(-\pi/2) = f(\pi/2) = 0$
   (c) Absolute maximum: $f(\pi/3) = 1$; absolute minimum: $f(2\pi/3) = -1$
   (d) Absolute maximum: $f(0) = 2$; absolute minimum: $f(\pi) = -2$



7. Solving $f'(x) = 4x - 6 = 0$ we obtain critical number $3/2$.

8. Since $f'(x) = 3x^2 + 1 > 0$ for all x, the function has no critical numbers.

9. Solving $f'(x) = 6x^2 - 30x - 36 = 6(x-6)(x+1) = 0$ we obtain the critical numbers 6 and -1.

10. Solving $f'(x) = 4x^3 - 12x^2 = 4x^2(x-3) = 0$ we obtain the critical numbers 0 and 3.

11. Solving $f'(x) = (x-2)^2(1) + (x-1)[2(x-2)] = (x-2)[(x-2)+2(x-1)] = (x-2)(3x-4) = 0$ we obtain the critical numbers 2 and 4/3.

12. Solving $f'(x) = x^2[3(x+1)^2] + (x+1)^3(2x) = x(x+1)^2[3x+2(x+1)] = x(x+1)^2(5x+2) = 0$ we obtain the critical numbers 0, -1, and $-2/5$.

13. Solving $f'(x) = \dfrac{x^{1/2} - (1+x)\left(\frac{1}{2}x^{-1/2}\right)}{x} = \dfrac{x-1}{2x^{3/2}}$ we obtain the critical number 1. $f'(x)$ does not exist when $x = 0$, but 0 is not in the domain of $f(x)$, so the only critical number is 1.

14. Solving $f'(x) = \dfrac{2-x^2}{(x^2+2)^2} = 0$ we obtain the critical numbers $\sqrt{2}$ and $-\sqrt{2}$.

15. We note that $f'(x) = \dfrac{4}{3(4x-3)^{2/3}} \neq 0$ for all x and $f'(3/4)$ does not exist. Since 3/4 is in the domain of $f(x)$, it is a critical number.

16. Solving $f'(x) = \dfrac{2}{3}x^{-1/3} + 1 = 0$ we obtain the critical number $-8/27$. We note that $f'(x)$ does not exist when $x = 0$. Since 0 is in the domain of $f(x)$, it is a critical number.

17. Solving $f'(x) = \dfrac{1}{3}(x-1)^2(x+2)^{-2/3} + 2(x+2)^{1/3}(x-1) = 0$ we observe $(x-1)^2(x+1)^{-2/3} + 6(x+2)^{1/3}(x-1) = 0$ or $(x+2)^{1/3}(x-1)[(x-1)(x+2)^{-1} + 6] = 0$. Thus, -2 and 1 are critical numbers. Since we also have $\dfrac{x-1}{x+2} + 6 = 0$ or $x - 1 = -6(x+2)$, then $x = -11/7$ and $-11/7$ is also a critical number.

18. Solving

$$f'(x) = \frac{(x+1)^{1/3} - (x+4)\left(\frac{1}{3}\right)(x+1)^{-2/3}}{(x+1)^{2/3}} = \frac{3(x+1)-(x+4)}{3(x+1)^{4/3}} = \frac{2x-1}{3(x+1)^{4/3}} = 0$$

we obtain the critical number $x = 1/2$. The value $x = -1$ is not in the domain of $f(x)$, so the only critical number is 1/2.

19. Solving $f'(x) = -1 + \cos x = 0$ we obtain the critical numbers $2n\pi$ where $n = 0, \pm1, \pm2, \ldots$.

20. Solving $f'(x) = -4\sin 4x = 0$ we obtain the critical numbers $n\pi/4$, where $n = 0, \pm1, \pm2, \ldots$.

21. Solving $f'(x) = 2x - 8/x = 0$ we obtain the critical number 2. $f'(x) = 0$ when $x = -2$, but -2 is not in the domain of $f(x)$, so it is not a critical number. $f'(x)$ does not exist when $x = 0$, but 0 is not in the domain of $f(x)$, so the only critical number is 2.

22. Solving $f'(x) = e^{-x} + 2 = 0$ we obtain the critical number $\ln 1/2 \approx -0.693$.

23. Solving $f'(x) = -2x + 6 = 0$ we obtain the critical number 3. The absolute maximum is $f(3) = 9$ and the absolute minimum is $f(1) = 5$.

x	1	3	4
$f(x)$	5	9	8

24. Solving $f'(x) = 2(x-1) = 0$ we obtain the critical number 1 which is not in $[2, 5]$. The absolute maximum is $f(5) = 16$ and the absolute minimum is $f(2) = 1$.

x	2	5
$f(x)$	1	16

25. We note that $f'(x) = \dfrac{2}{3}x^{-1/3}$ does not exist at $x = 0$. Since 0 is in the domain, it is a critical number. The absolute maximum is $f(8) = 4$ and the absolute minimum is $f(0) = 0$.

x	-1	0	8
$f(x)$	1	0	4

26. We note that $f'(x) = \dfrac{8}{3}x^{5/3} - \dfrac{2}{3}x^{-1/3} = (8x^2 - 2)/3x^{1/3}$ does not exist when $x = 0$, but 0 is in the domain of $f(x)$ so it is a critical number. Solving $f'(x) = 0$, we obtain $x = -1/2, 1/2$. The absolute maximum is 0 and occurs at -1, 0, and 1. The absolute minimum is $-3/2^{8/3}$ and occurs at $-1/2$ and $1/2$.

x	-1	$-1/2$	0	$1/2$	1
$f(x)$	0	$-3/2^{8/3}$	0	$-3/2^{8/3}$	0

27. Solving $f'(x) = 3x^2 - 12x = 0$ we obtain the critical numbers 0 and 4. However, only 0 is in $[-3, 2]$. The absolute maximum is $f(0) = 2$ and the absolute minimum is $f(-3) = -79$.

x	-3	0	2
$f(x)$	-79	2	-14

28. Solving $f'(x) = -3x^2 - 2x + 5 = 0$ we obtain the critical numbers $-5/3$ and 1. The absolute maximum is $f(1) = 3$ and the absolute minimum is $f(-5/3) = -175/27$.

x	-2	$-5/3$	1	2
$f(x)$	-6	$-175/27$	3	-2

29. Solving $f'(x) = 3x^2 - 6x + 3 = 0$ we obtain the critical number 1. The absolute maximum is $f(3) = 8$ and the absolute minimum is $f(-4) = -125$.

x	-4	1	3
$f(x)$	-125	0	8

30. Solving $f'(x) = 4x^3 + 12x^2 = 0$ we obtain the critical numbers 0 and -3. However, only 0 is in $[0, 4]$. The absolute maximum is $f(4) = 502$ and the absolute minimum is $f(0) = -10$.

x	0	4
$f(x)$	-10	502

31. Write $f(x) = x^6 - 2x^5 + x^4$. Then solving $f'(x) = 6x^5 - 10x^4 + 4x^3 = 2x^3(3x-2)(x-1) = 0$ we obtain 0, 2/3, and 1. The absolute maximum is $f(2) = 16$ and the absolute minimum is 0 and occurs at $x = 0$ and $x = 1$.

x	-1	0	$2/3$	1	2
$f(x)$	4	0	$16/729$	0	16

32. Solving $f'(x) = \dfrac{1-3x^2}{2\sqrt{x}(x^2+1)^2} = 0$ we obtain the critical numbers $\pm\sqrt{1/3}$. Neither is in $[1/4, 1/2]$. The absolute maximum is $f(1/2) = 2\sqrt{2}/5$ and the absolute minimum is $f(1/4) = 8/17$.

x	$1/4$	$1/2$
$f(x)$	$8/17$	$2\sqrt{2}/5$

33. Solving $f'(x) = -4\sin 2x + 4\sin 4x = -4\sin 2x + 8\sin 2x\cos 2x = 4\sin 2x(2\cos 2x - 1) = 0$ on $[0, 2\pi]$ we obtain the critical numbers 0, $\pi/2$, π, $3\pi/2$, 2π, $\pi/6$, $5\pi/6$, $7\pi/6$, and $11\pi/6$. The absolute maximum is 3/2 and occurs at $x = \pi/6$, $5\pi/6$, $7\pi/6$, and $11\pi/6$. The absolute minimum is -3 and occurs at $x = \pi/2$ and $3\pi/2$.

x	0	$\pi/6$	$\pi/2$	$5\pi/6$	π	$7\pi/6$	$3\pi/2$	$11\pi/6$	2π
$f(x)$	1	$3/2$	-3	$3/2$	1	$3/2$	-3	$3/2$	1

34. Solving $f'(x) = 15\cos 3x = 0$ on $[0, \pi/2]$ we obtain the critical numbers $\pi/6$ and $\pi/2$. The absolute maximum is $f(\pi/6) = 6$ and the absolute minimum is $f(\pi/2) = -4$.

x	0	$\pi/6$	$\pi/2$
$f(x)$	1	6	-4

35. Solving $f'(x) = 96\sin 24x\cos 24x = 48\sin 48x = 0$ on $[0, \pi]$ we obtain the critical numbers $k\pi/48$, where k is an integer from 0 to 48. The absolute maximum is 5 and occurs when k is odd. The absolute minimum is 3 and occurs when k is even.

x	0	$\pi/48$	$\pi/24$	$\ldots$	$23\pi/24$	$47\pi/48$	π
$f(x)$	3	5	3	$\ldots$	3	5	3

36. Solving $f'(x) = 2 - \sec^2 x = 0$ we obtain the critical numbers $-\pi/4$ and $\pi/4$. The absolute maximum is $f(\pi/4) = \pi/2 - 1 \approx 0.57$ and the absolute minimum is $f(1.5) \approx -11.10$.

x	-1	$-\pi/4$	$\pi/4$	1.5
$f(x)$	-0.44	-0.57	0.57	-11.10

37. Solving $f'(x) = \begin{cases} 2x+2, & x < 0 \\ 2x-2, & x > 0 \end{cases} = 0$ we obtain the critical numbers 1 and -1. We note that $f'(x)$ does not exist when $x = 0$. Since 0 is in the domain of $f(x)$, it is also a critical number. The absolute minimum is $f(-1) = f(1) = -1$, the endpoint absolute maximum is $f(3) = 3$, and the relative maximum is $f(0) = 0$.

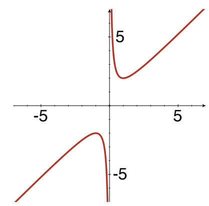
x	-2	-1	0	1	3
$f(x)$	0	-1	0	-1	3

38. Solving $f'(x) = \begin{cases} 4, & -5 \le x < -2 \\ 2x, & -2 < x \le 1 \end{cases} = 0$ we obtain the critical number 0. We note that $f'(x)$ does not exist when $x = -2$. Since -2 is in the domain of $f(x)$, it is also a critical number. The absolute minimum is $f(-2) = 4$, the endpoint absolute maximum is $f(-5) = -8$, and the relative maximum is $f(0) = 0$.

x	-5	-2	0	1
$f(x)$	-8	4	0	1

39. (a) c_1, c_3, c_4, c_{10}
(b) c_2, c_5, c_6, c_7, c_8, c_9
(c) Endpoint absolute maximum: $f(b)$; absolute minimum: $f(c_7)$
(d) Relative maxima: $f(c_3)$, $f(c_5)$, $f(c_9)$; relative minima: $f(c_2)$, $f(c_4)$, $f(c_7)$, $f(c_{10})$

40. Solving $f'(x) = 1 - \dfrac{1}{x^2} = 0$ we obtain the critical numbers 1 and -1. From the graph of $f(x)$ we see that $f(-1) = -2$ is a relative minimum and $f(1) = 2$ is a relative maximum. Thus, the relative minimum is greater than the relative maximum.



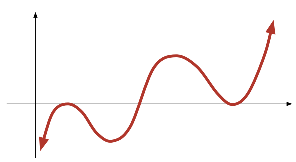
41. (a) $-16t^2 + 320t$ is negative outside the interval $[0, 20]$.
(b) Solving $s'(t) = -32t + 320 = 0$ we obtain the critical number $t = 10$. From the data in the accompanying table, we see that the maximum height attained by the projectile on $[0, 20]$ is 1600 ft.

t	0	10	20
$s(t)$	0	1600	0

42. (a) From the diagram, we see that v is defined for r in $[0, R]$.
(b) Holding R constant we have $v'(r) = -Pr/2vl$. Solving $v'(r) = 0$ we obtain the critical number $r = 0$. From the data in the accompanying table, we see that the maximum velocity is $PR^2/4vl$ cm/s and the minimum velocity is 0 cm/s.

r	0	R
$v(r)$	$PR^2/4vl$	0

43.



44. $f(x) = c$, where c is a constant.

45. For every x that is not an integer, $f'(x) = 0$, and for every integer value of x, $f'(x)$ does not exist. Therefore, since $f(x)$ is defined for all real x, every value of x is a critical number.

46. Setting $f'(x) = \dfrac{(cx+d)\cdot a - (ax+b)\cdot c}{(cx+d)^2} = \dfrac{ad-bc}{(cx+d)^2} = 0$, we obtain $ad - bc = 0$. Hence, if $ad - bc \ne 0$, there are no critical points. When $ad - bc = 0$, $b/a = d/c$ and $f(x) = \dfrac{a(x+b/a)}{x(x+d/c)} = \dfrac{a}{c}$. Thus $f(x)$ is a constant function of $(-\infty, -d/c) \cup (-d/c, \infty)$.

47. Solving $f'(x) = nx^{n-1} = 0$, we see that 0 is a critical number and $f(0) = 0$ is the only possible relative extremum. When n is even, $f(x)$ is positive for all non-zero x, and $f(0) = 0$ is a relative minimum. When n is odd, $f(x) < 0$ for $x < 0$ and $f(x) > 0$ for $x > 0$, so $f(0) = 0$ is not a relative extremum in this case.

48. The derivative of an n-th degree polynomial is a polynomial of degree $n-1$, and hence has at most $n-1$ zeros. The n-th degree polynomial therefore has at most $n-1$ critical numbers.

49. Since $f(a)$ is a relative minimum, there is an interval (c_1, c_2) around a in which $f(x) \geq f(a)$. Consider the interval $(-c_2, -c_1)$ around $-a$. Since $f(x)$ is even, $f(-x) = f(x)$, and for $-x$ in $(-c_2, -c_1)$, $f(-x) = f(x) \geq f(a) = f(-a)$. Therefore $f(-a)$ is a relative minimum.

50. Since $f(a)$ is a relative maximum, there is an interval (c_1, c_2) around a in which $f(x) \leq f(a)$. Consider the interval $(-c_2, -c_1)$ around $-a$. Since $f(x)$ is odd, $f(-x) = -f(x)$, and for $-x$ in $(-c_2, -c_1)$, $f(-x) = -f(x) \geq -f(a) = f(-a)$. Therefore $f(-a)$ is a relative minimum.

51. Since $f(x)$ is even and everywhere differentiable, we have $f(-x) = f(x)$ and $-f'(-x) = f'(x)$ through implicit differentiation. When $x = 0$, we have $-f'(0) = f'(0)$ and so $f'(0) = 0$. Thus, 0 is a critical number of f.

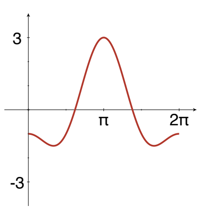
52. (a) $D_x(k + f(x)) = f'(x)$; c

    (b) $D_x(kf(x)) = kf'(x)$; c

    (c) $D_x(f(x + k)) = f'(x + k)$; $c - k$

    (d) $D_x(f(kx)) = kf'(kx)$; c/k

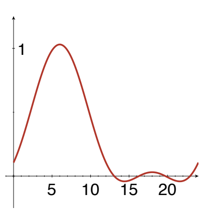
53. (a)



    (b) Solving $f'(x) = 2\sin x - 2\sin 2x = 2\sin x - 4\sin x\cos x = 2\sin x(1 - 2\cos x) = 0$ on $[0, 2\pi]$, we obtain the critical numbers 0, $\pi/3$, π, $5\pi/3$, and 2π.

    (c) Computing $f(0) = f(2\pi) = -1$, $f(\pi/3) = f(-5\pi/3) = -3/2$, and $f(\pi) = 3$, we see that the absolute maximum is $f(\pi) = 3$ and the absolute minimum is $f(\pi/3) = f(-5\pi/3) = -3/2$.

54. (a)



(b) Setting $I'(t) = \frac{b\omega}{2}\cos\omega t + \frac{4b\omega}{3\pi}\sin 2\omega t = \frac{b\omega}{2}\cos\omega t + \frac{8b\omega}{3\pi}\sin\omega t\cos\omega t = 0$ we obtain $\cos\omega t = 0$ and $\sin\omega t = -\frac{3\pi}{16}$. In the former case, we have $\omega t = \frac{\pi}{2}$ and $\frac{3\pi}{2}$. Setting $\omega = \frac{\pi}{12}$ we find $t = 6$ and $t = 18$. From $\sin\omega = -\frac{3\pi}{16}$ we find $\omega t \approx -0.6299$. We want $\omega t > 0$ so we use the fact that $\sin(\pi - \omega t) = \sin(2\pi + \omega t) = \sin\omega t$. Then $\omega t \approx \pi + 0.6299 \approx 3.7715$ and $t \approx 14.406$, and $\omega t \approx 2\pi - 0.6299 \approx 5.6533$ and $t \approx 21.594$. The critical numbers are 6, 14.406, 18, and 21.594.

4.4 Mean Value Theorem

1. $f(x)$ is continuous and differentiable on $[-2, 2]$ and $f(-2) = f(2) = 0$, so Rolle's Theorem applies. Solving $f'(c) = 2c = 0$ we obtain $c = 0$.

2. $f(x)$ is continuous and differentiable on $[1, 5]$ and $f(1) = f(5) = 0$, so Rolle's Theorem applies. Solving $f'(c) = 2c - 6 = 0$ we obtain $c = 3$.

3. Since $f(-2) = 19 \neq 0$, Rolle's Theorem does not apply.

4. $f(x)$ is continuous and differentiable on $[0, 4]$ and $f(0) = f(4) = 0$, so Rolle's Theorem applies. Solving $f'(c) = 3c^2 - 10c + 4 = 0$ we obtain $c = (5 \pm \sqrt{13})/3$. Both of these values are in $(0, 4)$.

5. $f(x)$ is continuous and differentiable on $[-1, 0]$ and $f(-1) = f(0) = 0$, so Rolle's Theorem applies. Solving $f'(c) = 3c^2 + 2c = 0$ we obtain $c = 0, -2/3$. Only $c = -2/3$ is in the interval $(-1, 0)$.

6. $f(x)$ is continuous and differentiable on $[0, 1]$ and $f(0) = f(1) = 0$, so Rolle's Theorem applies. Writing $f(x) = x^3 - 2x^2 + x$ we obtain $f'(c) = 3c^2 - 4c + 1 = (3c - 1)(c - 1)$. Thus $f'(c) = 0$ on $(0, 1)$ for $c = 1/3$.

7. $f(x)$ is continuous and differentiable on $[-\pi, 2\pi]$ and $f(-\pi) = f(2\pi) = 0$, so Rolle's Theorem applies. Solving $f'(c) = \cos c = 0$ on $(-\pi, 2\pi)$, we obtain $c = -\pi/2$, $\pi/2$, $3\pi/2$.

8. $f(x)$ is not continuous at $\pi/2$ so Rolle's Theorem does not apply.

9. Since $f'(x) = x^{-1/3}$, $f(x)$ is not differentiable on $(-1, 1)$ and Rolle's Theorem does not apply.

10. $f(x)$ is continuous and differentiable on $[1, 8]$ and $f(1) = f(8) = 0$, so Rolle's Theorem applies. Solving $f'(c) = \frac{2}{3}c^{-1/3} - c^{-2/3} = 0$ we obtain $\frac{2}{3}c^{1/3} - 1 = 0$ or $c = \frac{27}{8}$.

11. $f(a) \neq 0$

12. f is not differentiable at every point in (a, b).

13. $f(x)$ is continuous and differentiable on $[-1, 7]$, so the Mean Value Theorem applies. Setting $f'(c) = 2c = \frac{f(7) - f(-1)}{7 + 1} = 6$, we obtain $c = 3$.

14. $f(x)$ is continuous and differentiable on $[2,3]$, so the Mean Value Theorem applies. Setting $f'(c) = -2c+8 = \dfrac{f(3)-f(2)}{3-2} = 3$, we obtain $c = 5/2$.

15. $f(x)$ is continuous and differentiable on $[2,5]$, so the Mean Value Theorem applies. Setting $f'(c) = 3c^2+1 = \dfrac{f(5)-f(2)}{5-2} = 40$, we obtain $3c^2 = 39$. Then on $(2,5)$, $c = \sqrt{13}$.

16. $f(x)$ is continuous and differentiable on $[-3,3]$, so the Mean Value Theorem applies. Setting $f'(c) = 4c^3-4c = \dfrac{f(3)-f(-3)}{3+3} = 0$, we obtain $4c(c^2-1) = 0$. Then $c = -1, 0, 1$.

17. $f(x)$ is not continuous at 0, so the Mean Value Theorem does not apply.

18. $f(x)$ is continuous and differentiable on $[1,5]$, so the Mean Value Theorem applies. Setting $f'(c) = 1-1/c^2 = \dfrac{f(5)-f(1)}{5-1} = 4/5$, we obtain $1/5 = 1/c^2$ or $c^2 = 5$. Then on $(1,5)$, $c = \sqrt{5}$.

19. $f(x)$ is continuous and differentiable on $[0,9]$, so the Mean Value Theorem applies. Setting $f'(c) = 1/2\sqrt{c} = \dfrac{f(9)-f(0)}{9-0} = 1/3$, we obtain $c = 9/4$.

20. $f(x)$ is continuous and differentiable on $[2,6]$, so the Mean Value Theorem applies. Setting $f'(c) = 2/\sqrt{4c+1} = \dfrac{f(6)-f(2)}{6-2} = 1/2$, we obtain $4 = \sqrt{4c+1}$ or $16 = 4c+1$. Then $c = 15/4$.

21. $f(x)$ is continuous and differentiable on $[-2,-1]$, so the Mean Value Theorem applies. Setting $f'(c) = -2/(c-1)^2 = \dfrac{f(-1)-f(-2)}{-1+2} = -1/3$, we obtain $(c-1)^2 = 6$. Then on $[-2,-1]$, $c = 1-\sqrt{6}$.

22. Since $f'(x) = \dfrac{1}{3}x^{-2/3} - 1$, $f(x)$ is not differentiable at 0 and the Mean Value Theorem does not apply.

23. f is not continuous at b.

24. f is not differentiable at every point in (a,b).

25. $f'(x) = 2x$. Solving $f'(x) = 0$, we obtain the critical number 0. The function is decreasing on $(-\infty, 0]$ and increasing on $[0,\infty)$.

x		0	
f	$\searrow$		$\nearrow$
f'	$-$	0	$+$

26. $f'(x) = 3x^2$. Solving $f'(x) = 0$, we obtain the critical number 0. The function is decreasing on $(-\infty, 0]$ and increasing on $[0,\infty)$.

x		0	
f	$\nearrow$		$\nearrow$
f'	$+$	0	$+$

27. $f'(x) = 2x + 6$. Solving $f'(x) = 0$, we obtain the critical number -3. The function is decreasing on $(-\infty, -3]$ and increasing on $[-3, \infty)$.

x		-3	
f	$\searrow$		$\nearrow$
f'	$-$	0	$+$

28. $f'(x) = -2x + 10$. Solving $f'(x) = 0$, we obtain the critical number 5. The function is decreasing on $[5, \infty)$ and increasing on $(-\infty, 5]$.

x		5	
f	$\nearrow$		$\searrow$
f'	$+$	0	$-$

29. $f'(x) = 3x^2 - 6x = 3x(x - 2)$. Solving $f'(x) = 0$, we obtain the critical numbers 0 and 2. The function is decreasing on $[0, 2]$ and increasing on $(-\infty, 0]$ and $[2, \infty)$.

x		0		2	
f	$\nearrow$		$\searrow$		$\nearrow$
f'	$+$	0	$-$	0	$+$

30. $f'(x) = x^2 - 2x - 8 = (x - 4)(x + 2)$. Solving $f'(x) = 0$, we obtain the critical numbers -2 and 4. The function is decreasing on $[-2, 4]$ and increasing on $(-\infty, -2]$ and $[4, \infty)$.

x		-2		4	
f	$\nearrow$		$\searrow$		$\nearrow$
f'	$+$	0	$-$	0	$+$

31. $f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$. Solving $f'(x) = 0$, we obtain the critical numbers 0 and 3. The function is decreasing on $(-\infty, 0]$ and $[0, 3]$, and increasing on $[3, \infty)$.

x		0		3	
f	$\searrow$		$\searrow$		$\nearrow$
f'	$-$	0	$-$	0	$+$

32. $f'(x) = 20x^4 - 40x^3 = 20x^3(x - 2)$. Solving $f'(x) = 0$, we obtain the critical numbers 0 and 2. The function is decreasing on $[0, 2]$ and increasing on $(-\infty, 0]$ and $[2, \infty)$.

x		0		2	
f	$\nearrow$		$\searrow$		$\nearrow$
f'	$+$	0	$-$	0	$+$

33. $f'(x) = -\dfrac{1}{3}x^{-2/3}$. The only critical number is at 0 where $f'(x)$ does not exist. The function is decreasing on $(-\infty, 0]$ and $[0, \infty)$.

x		0	
f	$\searrow$		$\searrow$
f'	$-$	undefined	$-$

34. $f'(x) = \dfrac{2}{3}x^{-1/3} - \dfrac{2}{3}x^{-2/3} = \dfrac{2}{3}(x^{1/3} - 1)/x^{2/3}$. Solving $f'(x) = 0$, we obtain the critical number 1. Since $f'(x)$ does not exist at 0, the critical numbers are 0 and 1. The function is decreasing on $(-\infty, 0]$ and $[0, 1)$, and increasing on $[1, \infty)$.

x		0		1	
f	$\searrow$		$\searrow$		$\nearrow$
f'	$-$	undefined	$-$	0	$+$

35. $f'(x) = 1 - 1/x^2 = (x^2 - 1)/x^2$. Solving $f'(x) = 0$, we obtain the critical numbers -1 and 1. At $x = 0$, $f(x)$ is undefined. The function is decreasing on $[-1, 0)$ and $(0, 1]$, and increasing on $(-\infty, 1]$ and $[1, \infty)$.

x		-1		0		1	
f	$\nearrow$		$\searrow$	undefined	$\searrow$		$\nearrow$
f'	$+$	0	$-$	undefined	$-$	0	$+$

36. $f'(x) = -1/x^2 - 2/x^3 = -(x+2)/x^3$. Solving $f'(x) = 0$, we obtain the critical number -2. At $x = 0$, $f(x)$ is undefined. The function is decreasing on $(-\infty, -2]$ and $(0, \infty)$, and increasing on $[-2, 0)$.

x		-2		0	
f	$\searrow$		$\nearrow$	undefined	$\searrow$
f'	$-$	0	$+$	undefined	$-$

37. $f'(x) = x\left(\dfrac{-2x}{2\sqrt{8-x^2}}\right) + \sqrt{8-x^2} = \dfrac{-x^2+8-x^2}{\sqrt{8-x^2}} = \dfrac{8-2x^2}{\sqrt{8-x^2}}$. Solving $f'(x) = 0$, we obtain the critical numbers -2 and 2. Also, $f(x)$ is only defined for $-\sqrt{8} \le x \le \sqrt{8}$ and $f'(x)$ is only defined for $-\sqrt{8} < x < \sqrt{8}$. The function is decreasing on $[-\sqrt{8}, -2]$ and $[2, \sqrt{8}]$, and increasing on $[-2, 2]$.

x	$-\sqrt{8}$		-2		2		$\sqrt{8}$
f		$\searrow$		$\nearrow$		$\searrow$	
f'	undefined	$-$	0	$+$	0	$-$	undefined

38. $f'(x) = \dfrac{\sqrt{x^2+1} - (x+1)(2x/2\sqrt{x^2+1})}{x^2+1} = \dfrac{x^2+1-(x+1)x}{(x^2+1)\sqrt{x^2+1}} = \dfrac{1-x}{(x^2+1)\sqrt{x^2+1}}$. Solving $f'(x) = 0$, we obtain the critical number 1. The function is decreasing on $[1, \infty)$ and increasing on $(-\infty, 1]$.

x		1	
f	$\nearrow$		$\searrow$
f'	$+$	0	$-$

39. $f'(x) = -10x/(x^2+1)^2$. Solving $f'(x) = 0$, we obtain the critical number 0. The function is decreasing on $[0, \infty)$ and increasing on $(-\infty, 0]$.

x		0	
f	$\nearrow$		$\searrow$
f'	$+$	0	$-$

40. $f'(x) = \dfrac{(x+1)(2x) - x^2}{(x+1)^2} = \dfrac{x^2+2x}{(x+1)^2} = \dfrac{x(x+2)}{(x+1)^2}$. Solving $f'(x) = 0$, we obtain the critical numbers 0 and -2. At $x = -1$, $f(x)$ is undefined. The function is decreasing on $[-2, -1)$ and $(-1, 0]$, and increasing on $(-\infty, -2]$ and $[0, \infty)$.

x		-2		-1		0	
f	$\nearrow$		$\searrow$	undefined	$\searrow$		$\nearrow$
f'	$+$	0	$-$	undefined	$-$	0	$+$

41. $f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3)$. Solving $f'(x) = 0$, we obtain the critical numbers 1 and 3. The function is decreasing on $[1, 3]$ and increasing on $(-\infty, 1]$ and $[3, \infty)$.

x		1		3	
f	$\nearrow$		$\searrow$		$\nearrow$
f'	$+$	0	$-$	0	$+$

42. $f'(x) = 3(x^2-1)^2(2x)$. Solving $f'(x) = 0$, we obtain the critical numbers 0 and 1. The function is decreasing on $(-\infty, 0]$ and increasing on $[0, 1]$ and $[1, \infty)$.

x		0		1	
f	$\searrow$		$\nearrow$		$\nearrow$
f'	$-$	0	$+$	0	$+$

43. $f'(x) = \cos x$. Solving $f'(x) = 0$, we obtain the critical numbers $\pi/2+k\pi$ for $k = 0, \pm1, \pm2, \ldots$. The sign of $f'(x) = \cos x$ is positive on $(-\pi/2 + 2k\pi, \pi/2 + 2k\pi)$ for $k = 0, \pm1, \pm2, \ldots$, and negative on the other intervals. Thus, $f(x) = \sin x$ is increasing on $[-\pi/2 + 2k\pi, \pi/2 + 2k\pi]$ and decreasing on $[\pi/2 + 2k\pi, 3\pi/2 + 2k\pi]$ for $k = 0, \pm1, \pm2, \ldots$.

44. $f'(x) = -1 + \sec^2 x$. When $x = \pi/2 + k\pi$, $k = 0, \pm1, \pm2, \ldots$, $f(x)$ is undefined. Solving $f'(x) = 0$, we obtain the critical numbers $k\pi$ for $k = 0, \pm1, \pm2, \ldots$. Since $\sec^2 x \geq 1$ for all x in the domain of $f(x)$, $f'(x)$ is always nonnegative. Thus, $f(x)$ is increasing on $(-\pi/2 + k\pi, k\pi]$ and $[k\pi, \pi/2 + k\pi)$ for $k = 0, \pm1, \pm2, \ldots$.

45. $f'(x) = 1 - e^{-x}$. Solving $f'(x) = 0$, we obtain the critical number 0. The function is decreasing on $(-\infty, 0]$ and increasing on $[0, \infty)$.

x		0	
f	$\searrow$		$\nearrow$
f'	$-$	0	$+$

46. $f'(x) = e^{-x}(-x^2 + 2x) = e^{-x}x(2 - x)$. Solving $f'(x) = 0$, we obtain the critical numbers 0 and 2. The function is decreasing on $(-\infty, 0]$ and $[2, \infty)$, and increasing on $[0, 2]$.

x		0		2	
f	$\searrow$		$\nearrow$		$\searrow$
f'	$-$	0	$+$	0	$-$

47. Since $f'(x) = 12x^2 + 1 > 0$ for all x, the function is increasing and has relative extrema.

48. Since $f(x) = -1 - 1/2\sqrt{2-x} < 0$ for all x, the function is decreasing and has no relative extrema.

49. Let $s(t)$ denote the distance travelled since 1:15 P.M. At 2:15 P.M., we have $t = 1$. By the Mean Value Theorem applied to the interval $[0, 1]$, we have $v(c) = s'(c) = \dfrac{s(1) - s(0)}{1 - 0} = 70$, for some c between 0 and 1. That is, at some time between 1:15 and 2:15 P.M. the motorist was speeding at 70 mi/h.

50. $V'(r) = -kr^4 + 4kr^3(r_0 - r) = -kr^3(5r - 4r_0)$. Solving $V'(r) = 0$ we obtain the critical number $r = 4r_0/5$ on $[r_0/2, r_0]$. V is increasing on $[r_0/2, 4r_0/5]$ and decreasing on $[4r_0/5, r_0]$. The volume of air flow will be maximum for $r = 4r_0/5$.

r	$r_0/2$		$4r_0/5$		r_0
V		$\nearrow$		$\searrow$	
V'		$+$	0	$-$	

51. $f'(x) = 4x^3 + 3x^2 - 1$. Since f and f' are continuous and differentiable on $[-1, 1]$ and $f(-1) = f(1) = 0$, Rolle's Theorem applies. Thus, there exists c in $(-1, 1)$ such that $f'(c) = 4c^3 + 3c^2 - 1 = 0$.

52. Suppose $x_1 < x_2$ on $[a, b]$. Since $f'(x) > 0$ and $g'(x) > 0$ for all x in (a, b), then both f and g are increasing on $[a, b]$, and so $f(x_1) < f(x_2)$ and $g(x_1) < g(x_2)$. Then $(f + g)(x_1) = f(x_1) + g(x_1) < f(x_2) + g(x_2) = (f + g)(x_2)$ and $f + g$ is increasing on $[a, b]$.

53. We want $(fg)'(x) = f(x)g'(x)+f'(x)g(x) > 0$ for all x in (a, b). Since $f'(x) > 0$ and $g'(x) > 0$, the condition will hold if $f(x) > 0$ and $g(x) > 0$ for all x in (a, b).

54. We have $f'(x) = 3ax^2 + b > 0$ since $a > 0$ and $b > 0$. If there exist r_1 and r_2 such that $f(r_1) = f(r_2) = 0$, then the hypotheses of Rolle's Theorem are satisfied. Thus, there exists $\bar{c}$ between r_1 and r_2 such that $f'(\bar{c}) = 0$. Since $f'(x) > 0$ for all x, the function $f(x) = ax^3+bx+c$ cannot have two distinct real roots.

55. We have $f'(x) = 2ax + b$. If there exist numbers $r_1 < r_2 < r_3$ such that $f(r_1) = f(r_2) = f(r_3) = 0$, then by Rolle's Theorem there exist c_1 in (r_1, r_2) and c_2 in (r_2, r_3) such that $f'(c_1) = f'(c_2) = 0$. But this is impossible since $f'(x)$ has only the single real root $-b/2a$. Therefore $f(x)$ can have at most two real roots.

56. We have $f'(x) = 2ax + b$. Since $f(x)$ is continuous and differentiable on $[x_1, x_2]$, the Mean Value Theorem applies. Now,

$$f'(x_3) = f'\left(\frac{x_1 + x_2}{2}\right) = a(x_1 + x_2) + b$$

and

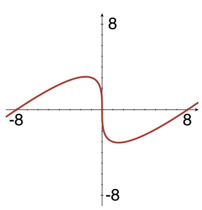
$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{ax_2^2 + bx_2 + c - (ax_1^2 + bx_1 + c)}{x_2 - x_1} = \frac{a(x_2^2 - x_1^2) + b(x_2 - x_1)}{x_2 - x_1}$$
$$= a(x_2 + x_1) + b, \quad \text{so} \quad f'(x_3) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

57. The polynomial function f is continuous and differentiable everywhere; if it has four distinct x-intercepts, then there exist values $x_1 < x_2 < x_3 < x_4$ such that $f(x_1) = f(x_2) = f(x_3) = f(x_4) = 0$. Since the values are distinct, there are three distinct intervals (x_1, x_2), (x_2, x_3), and (x_3, x_4) for which f satisfies the hypotheses of Rolle's Theorem, and so there exist c_1 in (x_1, x_2), c_2 in (x_2, x_3), and c_3 in (x_3, x_4) such that $f'(c_1) = f'(c_2) = f'(c_3) = 0$. Because $x_1 < x_2 < x_3 < x_4$, then $c_1 < c_2 < c_3$, and as such there are at least three points at which a tangent line to the graph of f is horizontal.

58. Any quadratic function with an interval centered around its vertex satisfies the given conditions. As a specific example, consider $f(x) = x^2 + 6x + 10$ on $[-6, 0]$. $f(x)$ is continuous on $[-6, 0]$ and differentiable on $(-6, 0)$, and $f(-6) = f(0) = 10$. Solving $f'(x) = 2x + 6 = 0$, we get $c = -3$.

59. $f'(x) = x\cos x + \sin x$. The hypotheses of Rolle's Theorem apply on $[0, \pi]$, so for some x in $(0, \pi)$, $x\cos x + \sin x = 0$ or $\cot x = -1/x$.

60. (a)



(b) f is continuous on $[-8, 8]$, and $f(-8) = f(8) = 0$. However, f is not differentiable on $(-8, 8)$, since there is a vertical tangent at the origin.

(c) $f'(x) = 1 - \frac{4}{3}x^{-2/3}$. Solving $f'(c) = 1 - \frac{4}{3}c^{-2/3} = 0$ we obtain $\frac{1}{\sqrt[3]{c^2}} = \frac{3}{4}$, $c^2 = \frac{64}{27}$, and $c = \pm 8/3\sqrt{3} \approx \pm 1.5396$.

61. We have $f'(x) = -2\sin 2x$. Setting $f'(c) = -2\sin 2c = \frac{f(\pi/4) - f(0)}{\pi/4 - 0} = -4/\pi$, we obtain $\sin 2c = 2/\pi$. Then $2c \approx 0.6901$ and $c \approx 0.3451$.

62. We have $f'(x) = \cos x$. Setting $f'(c) = \cos c = \dfrac{f(\pi/2) - f(\pi/4)}{\pi/2 - \pi/4} \approx \dfrac{2 - 1.7071}{\pi/4} \approx 0.3729$, we obtain $c \approx 1.1886$.

4.5 Limits Revisited — L'Hôpital's Rule

In this exercise set, the symbol "$\overset{h}{=}$" is used to denote the fact that L'Hôpital's Rule was applied to obtain the equality.

1. $\displaystyle\lim_{x\to 0} \frac{\cos x - 1}{x} \overset{h}{=} \frac{-\sin x}{1} = 0$

2. $\displaystyle\lim_{t\to 3} \frac{t^3 - 27}{t - 3} \overset{h}{=} \lim_{h\to 3} \frac{3t^2}{1} = 27$

3. $\displaystyle\lim_{x\to 1} \frac{2x - 2}{\ln x} \overset{h}{=} \lim_{x\to 1} \frac{2}{1/x} = 2$

4. $\displaystyle\lim_{x\to 0^+} \frac{\ln 2x}{\ln 3x} \overset{h}{=} \lim_{x\to 0^+} \frac{1/x}{1/x} = 1$

5. $\displaystyle\lim_{x\to 0} \frac{e^{2x} - 1}{3x + x^2} \overset{h}{=} \lim_{x\to 0} \frac{2e^{2x}}{3 + 2x} = \frac{2}{3}$

6. $\displaystyle\lim_{x\to 0} \frac{\tan x}{2x} \overset{h}{=} \lim_{x\to 0} \frac{\sec^2 x}{2} = \frac{1}{2}$

7. $\displaystyle\lim_{t\to \pi} \frac{5\sin^2 t}{1 + \cos t} \overset{h}{=} \lim_{t\to \pi} \frac{10\sin t\cos t}{-\sin t} = \lim_{t\to \pi} \frac{10\cos t}{-1} = 10$

8. $\displaystyle\lim_{\theta\to 1} \frac{\theta^2 - 1}{e^{\theta^2} - e} \overset{h}{=} \lim_{\theta\to 1} \frac{2\theta}{2\theta e^{\theta^2}} = \frac{1}{e}$

9. $\displaystyle\lim_{x\to 0} \frac{6 + 6x + 6x^2 - 6e^x}{x - \sin x} \overset{h}{=} \lim_{x\to 0} \frac{6 + 6x - 6e^x}{1 - \cos x} \overset{h}{=} \lim_{x\to 0} \frac{6 - 6e^x}{\sin x} \overset{h}{=} \lim_{x\to 0} \frac{-6e^x}{\cos x} = -6$

10. It is not necessary to use L'Hôpital's Rule here. Instead, multiply the numerator and denominator by $1/x^3$.

$$\lim_{x\to\infty} \frac{3x^2 - 4x^3}{5x + 7x^3} = \lim_{x\to\infty} \frac{3/x - 4}{5/x^2 + 7} = -\frac{4}{7}$$

11. It is not necessary to use L'Hôpital's Rule here.

$$\lim_{x\to 0^+} \frac{\cot 2x}{\cot x} = \lim_{x\to 0^+} \frac{\cos 2x \sin x}{\sin 2x \cos x} = \lim_{x\to 0^+} \frac{(\cos^2 x - \sin^2 x)\sin x}{2\sin x\cos x\cos x} = \lim_{x\to 0^+} \frac{\cos^2 x - \sin^2 x}{2\cos^2 x} = \frac{1}{2}$$

12. $\displaystyle\lim_{x\to 0} \frac{\arcsin(x/6)}{\arctan(x/2)} \overset{h}{=} \lim_{x\to 0} \frac{1/\sqrt{36 - x^2}}{2/(4 + x^2)} = \frac{1/6}{2/4} = \frac{1}{3}$

13. $\lim_{t\to 2} \frac{t^2+3t-10}{t^3-2t^2+t-2} \stackrel{h}{=} \lim_{t\to 2} \frac{2t+3}{3t^2-4t+1} = \frac{7}{5}$

14. $\lim_{r\to -1} \frac{r^3-r^2-5r-3}{(r+1)^2} \stackrel{h}{=} \lim_{r\to -1} \frac{3r^2-2r-5}{2(r+1)} \stackrel{h}{=} \lim_{r\to -1} \frac{6r-2}{2} = -4$

15. $\lim_{x\to 0} \frac{x-\sin x}{x^3} \stackrel{h}{=} \lim_{x\to 0} \frac{1-\cos x}{3x^2} \stackrel{h}{=} \lim_{x\to 0} \frac{\sin x}{6x} \stackrel{h}{=} \lim_{x\to 0} \frac{\cos x}{6} = \frac{1}{6}$

16. L'Hôpital's Rule does not apply. $\lim_{x\to 1} \frac{x^2+4}{x^2+1} = \frac{1+4}{1+1} = \frac{5}{2}$

17. $\lim_{x\to 0} \frac{\cos 2x}{x^2}$ has the form $1/0$ and does not exist.

18. $\lim_{x\to\infty} \frac{2e^{4x}+x}{e^{4x}+3x} \stackrel{h}{=} \lim_{x\to\infty} \frac{8e^{4x}+1}{4e^{4x}+3} \stackrel{h}{=} \lim_{x\to\infty} \frac{32e^{4x}}{16e^{4x}} = 2$

19. $\lim_{x\to 1^+} \frac{\ln\sqrt{x}}{x-1} = \lim_{x\to 1^+} \frac{\frac{1}{2}\ln x}{x-1} \stackrel{h}{=} \lim_{x\to 1^+} \frac{1/2x}{1} = \frac{1}{2}$

20. $\lim_{x\to\infty} \frac{\ln(3x^2+5)}{\ln(5x^2+1)} \stackrel{h}{=} \lim_{x\to\infty} \frac{6x/(3x^2+5)}{10x/(5x^2+1)} = \lim_{x\to\infty} \frac{3(5x^2+1)}{5(3x^2+5)} \stackrel{h}{=} \lim_{x\to\infty} \frac{30x}{30x} = \lim_{x\to\infty} 1 = 1$

21. $\lim_{x\to 2} \frac{e^{x^2}-e^{2x}}{x-2} \stackrel{h}{=} \lim_{x\to 2} \frac{2xe^{x^2}-2e^{2x}}{1} = 2e^4$

22. $\lim_{x\to 0} \frac{4^x-3^x}{x} \stackrel{h}{=} \lim_{x\to 0} \frac{4^x\ln 4-3^x\ln 3}{1} = \ln\frac{4}{3}$

23. $\lim_{x\to\infty} \frac{x\ln x}{x^2+1} \stackrel{h}{=} \frac{1+\ln x}{2x} \stackrel{h}{=} \lim_{x\to\infty} \frac{1/x}{2} = 0$

24. $\lim_{t\to 0} \frac{1-\cosh t}{t^2} \stackrel{h}{=} \lim_{t\to 0} \frac{-\sinh t}{2t} \stackrel{h}{=} \lim_{t\to 0} \frac{-\cosh t}{2} = -\frac{1}{2}$

25. $\lim_{x\to 0} \frac{1-\tan^{-1} x}{x^3} \stackrel{h}{=} \lim_{x\to 0} \frac{1-\dfrac{1}{1+x^2}}{3x^2} = \lim_{x\to 0} \frac{x^2}{3(x^2+x^4)} = \lim_{x\to 0} \frac{1}{3(1+x^2)} = \frac{1}{3}$

26. $\lim_{x\to 0} \frac{(\sin 2x)^2}{x^2} \stackrel{h}{=} \lim_{x\to 0} \frac{4\sin 2x\cos 2x}{2x} = \lim_{x\to 0} \frac{\sin 4x}{x} \stackrel{h}{=} \lim_{x\to 0} \frac{4\cos 4x}{1} = 4$

27. $\lim_{x\to\infty} \frac{e^x}{x^4} \stackrel{h}{=} \lim_{x\to\infty} \frac{e^x}{4x^3} \stackrel{h}{=} \lim_{x\to\infty} \frac{e^x}{12x^2} \stackrel{h}{=} \lim_{x\to\infty} \frac{e^x}{24x} \stackrel{h}{=} \lim_{x\to\infty} \frac{e^x}{24}$
The limit has the form $\infty/24$ and does not exist.

28. $\lim_{x\to\infty} \frac{e^{1/x}}{\sin(1/x)}$ has the form $1/0$ and does not exist.

29. $\displaystyle \lim_{x\to 0} \frac{x-\tan^{-1}x}{x-\sin^{-1}x} \stackrel{h}{=} \lim_{x\to 0} \frac{1-\dfrac{1}{1+x^2}}{1-\dfrac{1}{\sqrt{1-x^2}}} = \lim_{x\to 0} \frac{1-(1+x^2)^{-1}}{1-(1-x^2)^{-1/2}}$

$$\stackrel{h}{=} \lim_{x\to 0} \frac{2x(1+x^2)^{-2}}{-x(1-x^2)^{-3/2}} = \lim_{x\to 0} \frac{2(1-x^2)^{3/2}}{-(1+x^2)^2} = -2$$

30. $\displaystyle \lim_{t\to 1} \frac{t^{1/3}-t^{1/2}}{t-1} \stackrel{h}{=} \lim_{t\to 1} \frac{\frac{1}{3}t^{-2/3}-\frac{1}{2}t^{-1/2}}{1} = \lim_{t\to 1} \frac{2t^{1/2}-3t^{2/3}}{6t^{7/6}} = -\frac{1}{6}$

31. $\displaystyle \lim_{u\to\pi/2} \frac{\ln(\sin u)}{(2u-\pi)^2} \stackrel{h}{=} \lim_{u\to\pi/2} \frac{\cot u}{4(2u-\pi)} \stackrel{h}{=} \lim_{u\to\pi/2} \frac{-\csc^2 u}{8} = -\frac{1}{8}$

32. $\displaystyle \lim_{\theta\to\pi/2} \frac{\tan\theta}{\ln(\cos\theta)} \stackrel{h}{=} \lim_{\theta\to\pi/2} \frac{\sec^2\theta}{-\tan\theta} = \lim_{\theta\to\pi/2} \frac{-1}{\cos\theta\sin\theta}$

The limit has the form $-1/0$ and does not exist.

33. $\displaystyle \lim_{x\to-\infty} \frac{1+e^{-2x}}{1-e^{-2x}} \stackrel{h}{=} \lim_{x\to-\infty} \frac{-2e^{-2x}}{2e^{-2x}} = -1$

34. $\displaystyle \lim_{x\to 0} \frac{e^x-x-1}{2x^2} \stackrel{h}{=} \lim_{x\to 0} \frac{e^x-1}{4x} \stackrel{h}{=} \lim_{x\to 0} \frac{e^x}{4} = \frac{1}{4}$

35. $\displaystyle \lim_{r\to 0} \frac{r-\cos r}{r-\sin r}$ has the form $-1/0$ and does not exist.

36. $\displaystyle \lim_{t\to\pi} \frac{\csc 7t}{\csc 2t} = \lim_{t\to\pi} \frac{\sin 2t}{\sin 7t} \stackrel{h}{=} \lim_{t\to\pi} \frac{2\cos 2t}{7\cos 7t} = -\frac{2}{7}$

37. $\displaystyle \lim_{x\to 0^+} \frac{x^2}{\ln^2(1+3x)} \stackrel{h}{=} \lim_{x\to 0^+} \frac{2x}{\dfrac{6\ln(1+3x)}{1+3x}} = \lim_{x\to 0^+} \frac{x(1+3x)}{3\ln(1+3x)} \stackrel{h}{=} \lim_{x\to 0^+} \frac{1+6x}{\dfrac{9}{1+3x}}$

$$= \lim_{x\to 0^+} \frac{(1+6x)(1+3x)}{9} = \frac{1}{9}$$

38. $\displaystyle \lim_{x\to 3} \left(\frac{\ln x-\ln 3}{x-3}\right)^2 = \lim_{x\to 3} \frac{(\ln x)^2-2\ln 3\ln x-(\ln 3)^2}{x^2-6x+9} \stackrel{h}{=} \lim_{x\to 3} \frac{(2\ln x)/x-(2\ln 3)/x}{2x-6}$

$$= \lim_{x\to 3} \frac{\ln x-\ln 3}{x(x-3)} \stackrel{h}{=} \lim_{x\to 3} \frac{1/x}{2x-3} = \frac{1}{9}$$

Alternatively, note that

$$\lim_{x\to 3} \left(\frac{\ln x-\ln 3}{x-3}\right)^2 = \left[\lim_{x\to 3} \frac{\ln x-\ln 3}{x-3}\right]^2 = \left(\frac{d}{dx}\ln x\bigg|_{x=3}\right)^2 = \left(\frac{1}{x}\bigg|_{x=3}\right)^2 = \frac{1}{9}.$$

39. $\displaystyle \lim_{x\to 0} \frac{3x^2+e^x-e^{-x}-2\sin x}{x\sin x} \stackrel{h}{=} \lim_{x\to 0} \frac{6x+e^x+e^{-x}-2\cos x}{x\cos x+\sin x} \stackrel{h}{=} \lim_{x\to 0} \frac{6+e^x-e^{-x}+2\sin x}{-x\sin x+2\cos x} = 3$

40. $\displaystyle \lim_{x\to 8} \frac{\sqrt{x+1}-3}{x^2-64} \stackrel{h}{=} \lim_{x\to 8} \frac{1/2\sqrt{x+1}}{2x} = \frac{1/6}{16} = \frac{1}{96}$

41. Indeterminate form: $\infty - \infty$

$$\lim_{x\to 0}\left(\frac{1}{e^x-1}-\frac{1}{x}\right)=\lim_{x\to 0}\frac{x-e^x+1}{xe^x-x}\overset{h}{=}\lim_{x\to 0}\frac{1-e^x}{xe^x+e^x-1}\overset{h}{=}\lim_{x\to 0}\frac{-e^x}{xe^x+2e^x}=-\frac{1}{2}$$

42. Indeterminate form: $\infty - \infty$

$$\lim_{x\to 0^+}(\cot x-\csc x)=\lim_{x\to 0^+}\left(\frac{\cos x}{\sin x}-\frac{1}{\sin x}\right)=\lim_{x\to 0^+}\frac{\cos x-1}{\sin x}\overset{h}{=}\lim_{x\to 0^+}\frac{-\sin x}{\cos x}=0$$

43. Indeterminate form: $\infty - \infty$

$$\lim_{x\to\infty}x(e^{1/x}-1)=\lim_{x\to\infty}\frac{e^{1/x}-1}{1/x}\overset{h}{=}\lim_{x\to\infty}\frac{-e^{1/x}/x^2}{-1/x^2}=\lim_{x\to\infty}e^{1/x}=1$$

44. Indeterminate form: $0\cdot\infty$

$$\lim_{x\to 0^+}x\ln x=\lim_{x\to 0^+}\frac{\ln x}{1/x}\overset{h}{=}\lim_{x\to 0^+}\frac{1/x}{-1/x^2}=\lim_{x\to 0^+}(-x)=0$$

45. Indeterminate form: 0^0. Set $y=x^x$. Then $\ln y=x\ln x$ and $\lim\limits_{x\to 0^+}x\ln x=\lim\limits_{x\to 0^+}\dfrac{\ln x}{1/x}\overset{h}{=}$ $\lim\limits_{x\to 0^+}\dfrac{1/x}{-1/x^2}=\lim\limits_{x\to 0^+}(-x)=0$. Thus, $\lim\limits_{x\to 0^+}x^x=e^0=1$.

46. Indeterminate form: 1^∞. Set $y=x^{1/(1-x)}$. Then $\ln y=\dfrac{\ln x}{1-x}$ and $\lim\limits_{x\to 1^-}\dfrac{\ln x}{1-x}\overset{h}{=}\lim\limits_{x\to 1^-}\dfrac{1/x}{-1}=$ -1. Thus, $\lim\limits_{x\to 1^-}x^{1/(1-x)}=e^{-1}$.

47. Indeterminate form: $\infty - \infty$

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right)=\lim_{x\to 0}\frac{\sin x-x}{x\sin x}\overset{h}{=}\lim_{x\to 0}\frac{\cos x-1}{x\cos x+\sin x}\overset{h}{=}\lim_{x\to 0}\frac{-\sin x}{-x\sin x+2\cos x}=0$$

48. Indeterminate form: $\infty - \infty$

$$\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{\cos 3x}{x^2}\right)=\lim_{x\to 0}\frac{1-\cos 3x}{x^2}\overset{h}{=}\lim_{x\to 0}\frac{3\sin 3x}{2x}\overset{h}{=}\lim_{x\to 0}\frac{9\cos 3x}{2}=\frac{9}{2}$$

49. Indeterminate form: $\infty - \infty$

$$\lim_{t\to 3}\left(\frac{\sqrt{t+1}}{t^2-9}-\frac{2}{t^2-9}\right)=\lim_{t\to 3}\frac{\sqrt{t+1}-2}{t^2-9}\overset{h}{=}\lim_{t\to 3}\frac{1/2\sqrt{t+1}}{2t}=\lim_{t\to 3}\frac{1}{4t\sqrt{t+1}}=\frac{1}{24}$$

50. Indeterminate form: $\infty - \infty$

$$\begin{aligned}\lim_{x\to 0^+}\left[\frac{1}{x}-\frac{1}{\ln(x+1)}\right]&=\lim_{x\to 0^+}\frac{\ln(x+1)-x}{x\ln(x+1)}\overset{h}{=}\lim_{x\to 0^+}\frac{1/(x+1)-1}{x/(x+1)+\ln(x+1)}\\&=\lim_{x\to 0^+}\frac{1-(x+1)}{x+(x+1)\ln(x+1)}\overset{h}{=}\lim_{x\to 0^+}\frac{-1}{1+1+\ln(x+1)}=-\frac{1}{2}\end{aligned}$$

51. Indeterminate form: $0 \cdot \infty$

$$\lim_{\theta \to 0} \theta \csc 4\theta = \lim_{\theta \to 0} \frac{\theta}{\sin 4\theta} \stackrel{h}{=} \lim_{\theta \to 0} \frac{1}{4\cos 4\theta} = \frac{1}{4}$$

52. Indeterminate form: 1^∞. Set $y = (\sin^2 x)^{\tan x} = (\sin x)^{2\tan x}$. Then $\ln y = 2\tan x \ln \sin x$ and

$$\lim_{x \to \pi/2^-} 2\tan x \ln \sin x = \lim_{x \to \pi/2^-} \frac{2\ln \sin x}{\cot x} \stackrel{h}{=} \lim_{x \to \pi/2^-} \frac{2\cos x \sin x}{-\csc^2 x} = \lim_{x \to \pi/2^-} (-2\cos x \sin x) = 0.$$

Thus, $\lim_{x \to \pi/2^-} (\sin^2 x)^{\tan x} = e^0 = 1$.

53. Indeterminate form: ∞^0. Set $y = (2 + e^x)^{e^{-x}}$. Then $\ln y = e^{-x} \ln(2 + e^x)$ and

$$\lim_{x \to \infty} e^{-x} \ln(2 + e^x) = \lim_{x \to \infty} \frac{\ln(2 + e^x)}{e^x} \stackrel{h}{=} \lim_{x \to \infty} \frac{e^x/(2 + e^x)}{e^x} = \lim_{x \to \infty} \frac{1}{2 + e^x} = 0.$$

Thus, $\lim_{x \to \infty} (2 + e^x)^{e^{-x}} = e^0 = 1$.

54. Indeterminate form: 0^0. Set $y = (1 - e^x)^{x^2}$. Then $\ln y = x^2 \ln(1 - e^x)$ and

$$\begin{aligned}\lim_{x \to 0^-} x^2 \ln(1 - e^x) &= \lim_{x \to 0^-} \frac{\ln(1 - e^x)}{x^{-2}} \stackrel{h}{=} \lim_{x \to 0^-} \frac{-e^x/(1 - e^x)}{-2/x^3} = \lim_{x \to 0^-} \frac{x^3 e^x}{2(1 - e^x)} \\ &\stackrel{h}{=} \lim_{x \to 0^-} \frac{x^3 e^x + 3x^2 e^x}{-2e^x} = 0.\end{aligned}$$

Thus, $\lim_{x \to 0^-} (1 - e^x)^{x^2} = e^0 = 1$.

55. Indeterminate form: 1^∞. Set $y = (1 + 3/t)^t$. Then $\ln y = t \ln(1 + 3/t)$ and

$$\lim_{t \to \infty} t \ln(1 + 3/t) = \lim_{t \to \infty} \frac{\ln(1 + 3/t)}{1/t} \stackrel{h}{=} \lim_{t \to \infty} \frac{\frac{-3/t^2}{1 + 3/t}}{-1/t^2} = \lim_{t \to \infty} \frac{3}{1 + 3/t} = 3.$$

Thus, $\lim_{t \to \infty} (1 + 3/t)^t = e^3$.

56. Indeterminate form: 1^∞. Set $y = (1+2h)^{4/h}$. Then $\ln y = \frac{4}{h} \ln(1+2h)$ and $\lim_{h \to 0} \frac{4\ln(1 + 2h)}{h} \stackrel{h}{=} \lim_{h \to 0} \frac{8/(1 + 2h)}{1} = 8$. Thus, $\lim_{h \to 0} (1 + 2h)^{4/h} = e^8$.

57. Indeterminate form: 0^0. Set $y = x^{(1 - \cos x)}$. Then $\ln y = (1 - \cos x)\ln x$ and

$$\begin{aligned}\lim_{x \to 0} (1 - \cos x)\ln x &= \lim_{x \to 0} \frac{\ln x}{1/(1 - \cos x)} \stackrel{h}{=} \lim_{x \to 0} \frac{1/x}{-\sin x/(1 - \cos x)^2} = \lim_{x \to 0} \frac{-(1 - \cos x)^2}{x \sin x} \\ &\stackrel{h}{=} \lim_{x \to 0} \frac{-2(1 - \cos x)\sin x}{x\cos x + \sin x} = \lim_{x \to 0} \frac{-2\sin x + \sin 2x}{x \cos x + \sin x} \\ &\stackrel{h}{=} \lim_{x \to 0} \frac{-2\cos x + 2\cos 2x}{-x\sin x + 2\cos x} = 0.\end{aligned}$$

Thus, $\lim_{x \to 0} x^{(1 - \cos x)} = e^0 = 1$.

58. Indeterminate form: 1^∞. Set $y = (\cos 2\theta)^{1/\theta^2}$. Then $\ln y = \dfrac{1}{\theta^2}\ln(\cos 2\theta)$ and

$$\lim_{\theta\to 0}\frac{\ln(\cos 2\theta)}{\theta^2} \overset{h}{=} \lim_{\theta\to 0}\frac{-2\sin 2\theta/\cos 2\theta}{2\theta} = \lim_{\theta\to 0}\frac{-\tan 2\theta}{\theta} \overset{h}{=} \lim_{\theta\to 0}\frac{-2\sec^2 2\theta}{1} = -2.$$

Thus, $\lim\limits_{\theta\to 0}(\cos 2\theta)^{1/\theta^2} = e^{-2}$.

59. Indeterminate form: $0\cdot\infty$

$$\begin{aligned}\lim_{x\to\infty}\frac{1}{x^2\sin^2(2/x)} &= \lim_{x\to\infty}\frac{1/x^2}{\sin^2(2/x)} \overset{h}{=} \lim_{x\to\infty}\frac{-2/x^3}{[2\sin(2/x)\cos(2/x)](-2/x^2)}\\ &= \lim_{x\to\infty}\frac{1}{2x\sin(2/x)\cos(2/x)} = \lim_{x\to\infty}\frac{1/x}{\sin(4/x)} \overset{h}{=} \lim_{x\to\infty}\frac{-1/x^2}{\cos(4/x)(-4/x^2)}\\ &= \lim_{x\to\infty}\frac{1}{4\cos(4/x)} = \frac{1}{4}\end{aligned}$$

60. $\lim\limits_{x\to 1}(x^2-1)^{x^2}$ does not exist since for $x<1$, x^2-1 is negative and even roots of negative numbers do not exist.

61. Indeterminate form: $\infty-\infty$

$$\begin{aligned}\lim_{x\to 1}\left(\frac{1}{x-1}-\frac{5}{x^2+3x-4}\right) &= \lim_{x\to 1}\left[\frac{1}{x-1}-\frac{5}{(x-1)(x+4)}\right] = \lim_{x\to 1}\frac{x+4-5}{(x-1)(x+4)}\\ &= \lim_{x\to 1}\frac{x-1}{(x-1)(x+4)} = \lim_{x\to 1}\frac{1}{x+4} = \frac{1}{5}\end{aligned}$$

62. Indeterminate form: $\infty-\infty$. $\lim\limits_{x\to 0}\left(\dfrac{1}{x^2}-\dfrac{1}{x}\right) = \lim\limits_{x\to 0}\dfrac{1-x}{x^2}$. The limit has the form $1/0$ and does not exist.

63. Indeterminate form: $0\cdot\infty$

$$\lim_{x\to\infty}x^5e^{-x} = \lim_{x\to\infty}\frac{x^5}{e^x} \overset{h}{=} \lim_{x\to\infty}\frac{5x^4}{e^x} \overset{h}{=} \cdots \overset{h}{=} \lim_{x\to\infty}\frac{5!}{e^x} = 0$$

64. Indeterminate form: ∞^0. Set $y=(x+e^x)^{2/x}$. Then $\ln y = \dfrac{2}{x}\ln(x+e^x)$ and

$$\lim_{x\to\infty}\frac{2\ln(x+e^x)}{x} \overset{h}{=} \lim_{x\to\infty}\frac{2(1+e^x)/(x+e^x)}{1} = \lim_{x\to\infty}\frac{2(1+e^x)}{x+e^x} \overset{h}{=} \lim_{x\to\infty}\frac{2e^x}{1+e^x} \overset{h}{=} \lim_{x\to\infty}\frac{2e^x}{e^x} = 2.$$

Thus, $\lim\limits_{x\to\infty}(x+e^x)^{2/x} = e^2$.

65. Indeterminate form: $0\cdot\infty$

$$\lim_{x\to\infty}x\left(\frac{\pi}{2}-\arctan x\right) = \lim_{x\to\infty}\frac{\pi/2-\tan^{-1}x}{1/x} \overset{h}{=} \lim_{x\to\infty}\frac{-1/(1+x^2)}{-1/x^2} = \lim_{x\to\infty}\frac{x^2}{1+x^2} \overset{h}{=} \lim_{x\to\infty}\frac{2x}{2x} = 1$$

66. Indeterminate form: $0 \cdot \infty$

$$\lim_{t\to\pi/4}(t-\pi/4)\tan 2t = \lim_{t\to\pi/4}\frac{t-\pi/4}{\cot 2t} \stackrel{h}{=} \lim_{t\to\pi/4}\frac{1}{-2\csc^2 2t} = -\frac{1}{2}$$

67. Indeterminate form: $0 \cdot \infty$

$$\lim_{x\to\infty} x\tan\left(\frac{5}{x}\right) = \lim_{x\to\infty}\frac{\tan(5/x)}{1/x} \stackrel{h}{=} \lim_{x\to\infty}\frac{\sec^2(5/x)(-5/x^2)}{-1/x^2} = 5\lim_{x\to\infty}\sec^2(5/x) = 5$$

68. Indeterminate form: $0 \cdot \infty$

$$\lim_{x\to 0^+} x\ln(\sin x) = \lim_{x\to 0^+}\frac{\ln(\sin x)}{1/x} \stackrel{h}{=} \lim_{x\to 0^+}\frac{\cos x/\sin x}{-1/x^2} = \lim_{x\to 0^+}\left(\frac{-x^2}{\tan x}\right) \stackrel{h}{=} \lim_{x\to 0^+}\left(\frac{-2x}{\sec^2 x}\right) = 0$$

69. Indeterminate form: $\infty - \infty$. $\displaystyle\lim_{x\to-\infty}\left(\frac{1}{e^x} - x^2\right) = \lim_{x\to-\infty}\frac{1-x^2e^x}{e^x}$. Now

$$\lim_{x\to-\infty} x^2e^x = \lim_{x\to-\infty}\frac{x^2}{e^{-x}} \stackrel{h}{=} \lim_{x\to-\infty}\frac{2x}{-e^{-x}} \stackrel{h}{=} \lim_{x\to-\infty}\frac{2}{e^{-x}} = 0,$$

so $\displaystyle\lim_{x\to-\infty}\frac{1-x^2e^x}{e^x}$ has the form $1/0$ and $\displaystyle\lim_{x\to-\infty}\left(\frac{1}{e^x} - x^2\right)$ does not exist.

70. Indeterminate form: 1^∞. Set $y = (1+5\sin x)^{\cot x}$. Then $\ln y = \cot x\ln(1+5\sin x)$ and

$$\lim_{x\to 0}\cot x\ln(1+5\sin x) = \lim_{x\to 0}\frac{\ln(1+5\sin x)}{\tan x} \stackrel{h}{=} \lim_{x\to 0}\frac{5\cos x/(1+5\sin x)}{\sec^2 x} = \lim_{x\to 0}\frac{5\cos^3 x}{1+5\sin x} = 5.$$

Thus, $\displaystyle\lim_{x\to 0}(1+5\sin x)^{\cot x} = e^5$.

71. Indeterminate form: 1^∞. Set $y = \left(\dfrac{3x}{3x+1}\right)^x$. Then $\ln y = x\ln\left(\dfrac{3x}{3x+1}\right)$ and

$$\begin{aligned}\lim_{x\to\infty} x\ln\left(\frac{3x}{3x+1}\right) &= \lim_{x\to\infty}\frac{\ln\left(\frac{3x}{3x+1}\right)}{1/x} \stackrel{h}{=} \lim_{x\to\infty}\frac{\left(\frac{3x+1}{3x}\right)\left[\frac{3(3x+1)-3(3x)}{(3x+1)^2}\right]}{-1/x^2}\\ &= \lim_{x\to\infty}\left(\frac{-x}{3x+1}\right) \stackrel{h}{=} \lim_{x\to\infty}\frac{-1}{3} = -\frac{1}{3}.\end{aligned}$$

Thus, $\displaystyle\lim_{x\to\infty}\left(\frac{3x}{3x+1}\right)^x = e^{-1/3}$.

72. Indeterminate form: $\infty - \infty$

$$\begin{aligned}\lim_{\theta\to\pi/2^-}(\sec^3\theta - \tan^3\theta) &= \lim_{\theta\to\pi/2^-}\left(\frac{1}{\cos^3\theta} - \frac{\sin^3\theta}{\cos^3\theta}\right) = \lim_{\theta\to\pi/2^-}\frac{1-\sin^3\theta}{\cos^3\theta}\\ &\stackrel{h}{=} \lim_{\theta\to\pi/2^-}\frac{-3\sin^2\theta\cos\theta}{-3\cos^2\theta\sin\theta} = \lim_{\theta\to\pi/2^-}\frac{\sin\theta}{\cos\theta}\end{aligned}$$

This limit has the form $1/0$ and does not exist.

73. Indeterminate form: 0^0. Set $y = (\sinh x)^{\tan x}$. Then $\ln y = \tan x \ln(\sinh x)$ and

$$\lim_{x\to 0} \tan x \ln(\sinh x) = \lim_{x\to 0} \frac{\ln(\sinh x)}{\cot x} \stackrel{h}{=} \lim_{x\to 0} \frac{\cosh x/\sinh x}{-\csc^2 x} = \lim_{x\to 0} \frac{-\sin^2 x}{\tanh x}$$
$$\stackrel{h}{=} \lim_{x\to 0} \frac{-2\sin x\cos x}{\operatorname{sech}^2 x} = 0.$$

Thus, $\lim\limits_{x\to 0} (\sinh x)^{\tan x} = e^0 = 1$.

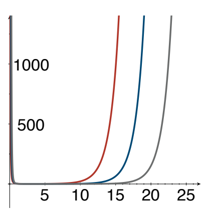
74. Set $y = x^{\ln^2 x}$. Then $\ln y = \ln^2 x \ln x = \ln^3 x$ and $\lim\limits_{x\to 0+} \ln^3 x = -\infty$. Thus, $\lim\limits_{x\to 0+} x^{\ln^2 x} = \lim\limits_{t\to -\infty} e^t = 0$.

75. Since $\lim\limits_{x\to 0^+} \dfrac{e^x-1}{x} \stackrel{h}{=} \lim\limits_{x\to 0^+} \dfrac{e^x}{1} = 1$, $\lim\limits_{x\to 0^+} \ln \dfrac{e^x-1}{x} = 0$, and $\dfrac{1}{x}\ln\left(\dfrac{e^x-1}{x}\right)$ has the form $0/0$ as $x \to 0$. Now

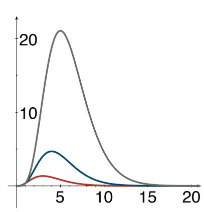
$$\lim_{x\to 0^+} \frac{\ln[(e^x-1)/x]}{x} = \lim_{x\to 0^+} \frac{\ln(e^x-1)-\ln x}{x} \stackrel{h}{=} \lim_{x\to 0^+} \frac{\dfrac{e^x}{e^x-1} - \dfrac{1}{x}}{1} = \lim_{x\to 0^+} \frac{xe^x - e^x + 1}{xe^x - x}$$
$$\stackrel{h}{=} \lim_{x\to 0^+} \frac{xe^x}{xe^x + e^x - 1} \stackrel{h}{=} \lim_{x\to 0^+} \frac{xe^x + e^x}{xe^x + 2e^x} = \frac{1}{2}.$$

76. Since $\lim\limits_{x\to\infty} \dfrac{e^x-1}{x} \stackrel{h}{=} \lim\limits_{x\to\infty} \dfrac{e^x}{1} = \infty$, $\lim\limits_{x\to 0^+} \ln \dfrac{e^x-1}{x} = \infty$, and $\dfrac{1}{x}\ln\left(\dfrac{e^x-1}{x}\right)$ has the form ∞/∞ as $x \to \infty$. Now

$$\lim_{x\to\infty} \frac{\ln\left(\dfrac{e^x-1}{x}\right)}{x} \stackrel{h}{=} \lim_{x\to\infty} \frac{(x-1)e^x+1}{x(e^x-1)} \stackrel{h}{=} \lim_{x\to\infty} \frac{xe^x}{xe^x+e^x-1}$$
$$\stackrel{h}{=} \lim_{x\to\infty} \frac{xe^x+e^x}{xe^x+2e^x} = \lim_{x\to\infty} \frac{x+1}{x+2} = 1.$$

77. 

In all three cases, it appears that $\lim\limits_{x\to\infty} f(x) = \infty$.

78. 

In all three cases, it appears that $\lim\limits_{x\to\infty} f(x) = 0$.

79. Since n is a positive integer, then by repeated applications of L'Hôpital's Rule,

$$\lim_{x\to\infty}\frac{x^n}{e^x}\overset{h}{=}\lim_{x\to\infty}\frac{nx^{n-1}}{e^x}\overset{h}{=}\lim_{x\to\infty}\frac{n(n-1)x^{n-2}}{e^x}\overset{h}{=}\cdots\overset{h}{=}\lim_{x\to\infty}\frac{n!}{e^x}=0.$$

80. Since n is a positive integer, then by repeated applications of L'Hôpital's Rule,

$$\lim_{x\to\infty}\frac{e^x}{x^n}\overset{h}{=}\lim_{x\to\infty}\frac{e^x}{nx^{n-1}}\overset{h}{=}\lim_{x\to\infty}\frac{e^x}{n(n-1)x^{n-2}}\overset{h}{=}\cdots\overset{h}{=}\lim_{x\to\infty}\frac{e^x}{n!}=\infty.$$

81. (a) Letting r be the radius of the circle, we see that the area of the circular sector is $A_C=\frac{1}{2}r^2(2\theta)=r^2\theta$ and the area of the isosceles triangle is $A_T=2\left(\frac{1}{2}r\sin\theta\cdot r\cos\theta\right)=\frac{1}{2}r^2\sin 2\theta$. Then the area of the shaded region is $A=A_C-A_T=r^2\left(\theta-\frac{1}{2}\sin 2\theta\right)$. Now the length of the arc is 5, so $r\theta=5$ and $r=5/\theta$. Thus, $A=25\left(\theta-\frac{1}{2}\sin 2\theta\right)/\theta^2$.

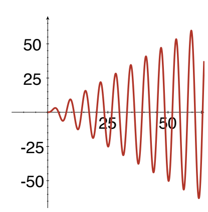
(b) $\lim_{\theta\to 0}A\overset{h}{=}\lim_{\theta\to 0}\frac{25(1-\cos 2\theta)}{2\theta}\overset{h}{=}\lim_{\theta\to 0}\frac{50\sin 2\theta}{2}=0$

(c) $$\frac{dA}{d\theta}=25\left[\frac{\theta^2(1-\cos 2\theta)-2\theta\left(\theta-\frac{1}{2}\sin 2\theta\right)}{\theta^4}\right]=25\left(\frac{-\theta-\theta\cos 2\theta+\sin 2\theta}{\theta^3}\right)$$

$$\begin{aligned}\lim_{\theta\to 0}\frac{dA}{dh}&\overset{h}{=}25\lim_{\theta\to 0}\left[\frac{-1-(-2\theta\sin 2\theta+\cos 2\theta)+2\cos 2\theta}{3\theta^2}\right]\\&=25\lim_{\theta\to 0}\left(\frac{-1+2\theta\sin 2\theta+\cos 2\theta}{3\theta^2}\right)\overset{h}{=}25\lim_{\theta\to 0}\left(\frac{4\theta\cos 2\theta+2\sin 2\theta-2\sin 2\theta}{6\theta}\right)\\&=25\lim_{\theta\to 0}\left(\frac{2\cos 2\theta}{3}\right)=\frac{50}{3}\end{aligned}$$

82. (a) $$\begin{aligned}x(t)&=\lim_{\gamma\to\omega}\frac{F_0}{\omega(\omega^2-\gamma^2)}(-\gamma\sin\omega t+\omega\sin\gamma t)=\frac{F_0}{\omega}\lim_{\gamma\to\omega}\frac{-\gamma\sin\omega t+\omega\sin\gamma t}{\omega^2-\gamma^2}\\&\overset{h}{=}\frac{F_0}{\omega}\lim_{\gamma\to\omega}\frac{-\sin\omega t+\omega t\cos\gamma t}{-2\gamma}=\frac{F_0(-\sin\omega t+\omega t\cos\omega t)}{-2\omega^2}\end{aligned}$$

(b) Plotting $x(t)=\frac{2(-\sin t+t\cos t)}{-2}=\sin t-t\cos t$, we get the graph on the right. As $t\to\infty$, the spring/mass oscillates to greater and greater displacements.



83. (a) From $p_1v_1^\gamma = k = p_2v_2^\gamma$ we have $p_2 = p_1(v_1/v_2)^\gamma$. Then

$$W = \frac{p_2v_2 - p_1v_1}{1-\gamma} = \frac{p_1(v_1/v_2)^\gamma v_2 - p_1v_1}{1-\gamma} = p_1v_1\left[\frac{(v_1/v_2)^{\gamma-1}-1}{1-\gamma}\right]$$
$$= p_1v_1\left[\frac{(v_2/v_1)^{1-\gamma}-1}{1-\gamma}\right].$$

(b) $\displaystyle\lim_{\gamma\to1}\left\{p_1v_1\left[\frac{(v_2/v_1)^{1-\gamma}-1}{1-\gamma}\right]\right\} \overset{h}{=} p_1v_1\lim_{\gamma\to1}\frac{(v_2/v_1)^{1-\gamma}\ln(v_2/v_1)(-1)}{-1} = p_1v_1\ln\left(\frac{v_2}{v_1}\right).$

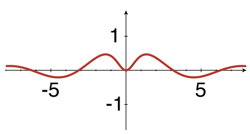
84. (a) Letting $p = 2$ we obtain $\sigma = \dfrac{1-10^{-0.05(4)}}{0.115(4)} \times 100 \approx 80.23\%$

(b) $$\lim_{p\to0}\frac{1-10^{-0.05p^2}}{0.115p^2}\times100 \overset{h}{=} \lim_{p\to0}\frac{0.1p10^{-0.05p^2}\ln10}{0.23p}\times100$$
$$= \lim_{p\to0}\frac{0.1\cdot10^{-0.05p^2}\ln10}{0.23}\times100 = \frac{0.1\ln10}{0.23}\times100 \approx 100.11\%$$

A possible explanation for why this percentage is more than 100% is that the formula may be only an approximation with diminished accuracy near $p = 0$.

85. $$\lim_{h\to0}\frac{f(x+h)-2f(x)+f(x-h)}{h^2} \overset{h}{=} \lim_{h\to0}\frac{f'(x+h)-f'(x-h)}{2h}$$
$$\overset{h}{=} \lim_{h\to0}\frac{f''(x+h)+f''(x-h)}{2} = f''(x)$$

86. (a)



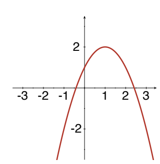
(b) $\displaystyle\lim_{x\to\infty} f(x) = 0$

(c) $\displaystyle\lim_{x\to\infty} x\sin x$ does not exist.

4.6 Graphing and the First Derivative

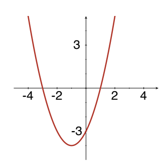
1. The x-intercepts are $1 \pm \sqrt{2}$. The y-intercept is 1. Solving $f'(x) = -2x + 2 = 0$ we obtain the critical number 1. The relative maximum is $f(1) = 2$.

x		1	
f	$\nearrow$	2	$\searrow$
f'	$+$	0	$-$



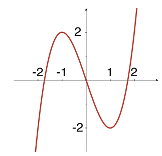
2. The x-intercepts are -3 and 1. The y-intercept is -3. Solving $f'(x) = 2x+2 = 0$ we obtain the critical number -1. The relative maximum is $f(-1) = -4$.

x		-1	
f	$\searrow$	-4	$\nearrow$
f'	$-$	0	$+$



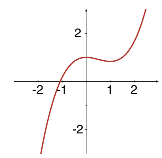
3. The x-intercepts are 0 and $\pm\sqrt{3}$. The y-intercept is 0. Solving $f'(x) = 3x^2 - 3 = 0$ we obtain the critical numbers -1 and 1. The relative maximum is $f(-1) = 2$ and the relative minimum is $f(1) = -2$.

x		-1		1	
f	$\nearrow$	2	$\searrow$	-2	$\nearrow$
f'	$+$	0	$-$	0	$+$



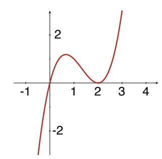
4. The y-intercept is 1. (The x-intercepts are not easily determined.) Solving $f'(x) = x^2 - x = x(x-1) = 0$ we obtain the critical numbers 0 and 1. The relative maximum is $f(0) = 1$ and the relative minimum is $f(1) = 5/6$.

x		0		1	
f	$\nearrow$	1	$\searrow$	$5/6$	$\nearrow$
f'	$+$	0	$-$	0	$+$



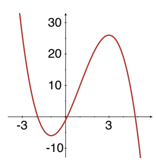
5. The x-intercepts are 0 and 2. The y-intercept is 0. Writing $f(x) = x^3 - 4x^2 + 4x$ and solving $f'(x) = 3x^2 - 8x + 4 = (3x - 2)(x - 2) = 0$ we obtain the critical numbers $2/3$ and 2. The relative maximum is $f(2/3) = 32/27$ and the relative minimum is $f(2) = 0$.

x		$2/3$		2	
f	$\nearrow$	$32/27$	$\searrow$	0	$\nearrow$
f'	$+$	0	$-$	0	$+$

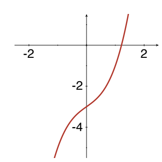


6. The y-intercept is -1. Solving $f'(x) = -3x^2 + 6x + 9 = -3(x - 3)(x + 1) = 0$ we obtain the critical numbers -1 and 3. The relative maximum is $f(3) = 26$ and the relative minimum is $f(-1) = -6$.

x		-1		3	
f	$\searrow$	-6	$\nearrow$	26	$\searrow$
f'	$-$	0	$+$	0	$-$

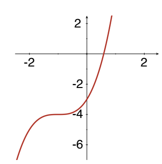


7. There is no easily determined x-intercept. The y-intercept is -3. Since $f'(x) = 3x^2 + 1 > 0$ for all x, there are no critical numbers and no relative extrema.



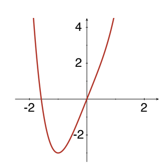
8. Solving $f(x) = (x + 1)^3 - 4 = 0$ we see that the x-intercept is $\sqrt[3]{4} - 1$. The y-intercept is -3. Solving $f'(x) = 3x^2 + 6x + 3 = 3(x + 1)^2 = 0$ we obtain the critical number -1. There are no relative extrema.

x		-1	
f	$\nearrow$	-4	$\nearrow$
f'	$+$	0	$+$



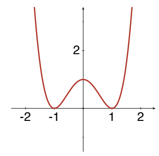
9. The x-intercepts are 0 and $-4^{1/3}$. The y-intercept is 0. Solving $f'(x) = 4x^3 + 4 = 0$ we obtain the critical number -1. The relative minimum is $f(-1) = -3$.

x		-1	
f	$\searrow$	-3	$\nearrow$
f'	$-$	0	$+$



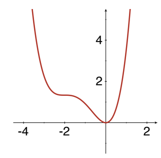
10. The x-intercepts are ± 1. The y-intercept is 1. Solving $f'(x) = 4x(x^2 - 1) = 0$ we obtain the critical numbers -1, 0, and 1. The relative maximum is $f(0) = 1$ and the relative minima are $f(-1) = f(1) = 0$.

x		-1		0		1	
f	$\searrow$	0	$\nearrow$	1	$\searrow$	0	$\nearrow$
f'	$-$	0	$+$	0	$-$	0	$+$



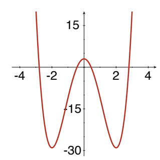
11. The x- and y-intercepts are 0. Solving $f'(x) = x^3 + 4x^2 + 4x = x(x + 2)^2 = 0$ we obtain the critical numbers -2 and 0. The relative minimum is $f(0) = 0$.

x		-2		0	
f	$\searrow$	$4/3$	$\searrow$	0	$\nearrow$
f'	$-$	0	$-$	0	$+$



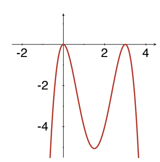
12. The x-intercepts are $\pm\sqrt{4 \pm \sqrt{58}/2}$. The y-intercept is 3. Solving $f'(x) = 8x^3 - 32x = 8x(x^2 - 4) = 0$ we obtain the critical numbers -2, 0, and 2. The relative maximum is $f(0) = 3$ and the relative minima are $f(-2) = f(2) = -29$.

x		-2		0		2	
f	$\searrow$	-29	$\nearrow$	3	$\searrow$	-29	$\nearrow$
f'	$-$	0	$+$	0	$-$	0	$+$



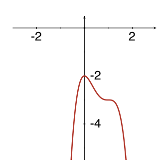
13. The x-intercepts are 0 and 3. The y-intercept is 0. Solving $f'(x) = -2x^2(x - 3) + (x - 3)(-2x) = -2x(x - 3)(2x - 3) = 0$ we obtain the critical numbers 0, 3/2, and 3. The relative maxima are $f(0) = f(3) = 0$ and the relative minimum is $f(3/2) = -81/16$.

x		0		$3/2$		3	
f	$\nearrow$	0	$\searrow$	$-81/16$	$\nearrow$	0	$\searrow$
f'	$+$	0	$-$	0	$+$	0	$-$



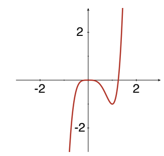
14. There are no easily determined x-intercepts. The y-intercept is -2. Solving $f'(x) = -12x^3 + 24x^2 - 12x = -12x(x - 1)^2 = 0$ we obtain the critical numbers 0 and 1. The relative maximum is $f(0) = -2$.

x		0		1	
f	$\nearrow$	-2	$\searrow$	-3	$\searrow$
f'	$+$	0	$-$	0	$-$



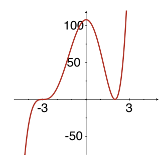
15. The x-intercepts are 0 and $5/4$. The y-intercept is 0. Solving $f'(x) = 20x^4 - 20x^3 = 20x^3(x-1) = 0$ we obtain the critical numbers 0 and 1. The relative maximum is $f(0) = 0$ and the relative minimum is $f(1) = -1$.

x		0		1	
f	$\nearrow$	0	$\searrow$	-1	$\nearrow$
f'	$+$	0	$-$	0	$+$



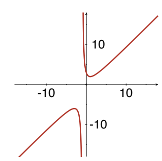
16. The x-intercepts are -3 and 2. The y-intercept is 108. Solving $f'(x) = 3(x-2)^2(x+3)^2 + 2(x-2)(x+3)^3 = 5x(x-2)(x+3)^2 = 0$ we obtain the critical numbers -3, 0, and 2. The relative maximum is $f(0) = 108$ and the relative minimum is $f(2) = 0$.

x		-3		0		2	
f	$\nearrow$	0	$\nearrow$	108	$\searrow$	0	$\nearrow$
f'	$+$	0	$+$	0	$-$	0	$+$



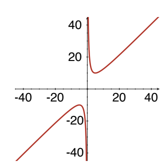
17. The y-intercept is 3. The function is undefined for $x = -1$. Solving $f'(x) = (x+2x-3)/(x+1)^2 = (x+3)(x-1)/(x+1)^2 = 0$ we obtain the critical numbers -3 and 1. The relative maximum is $f(-3) = -6$ and the relative minimum is $f(1) = 2$.

x		-3		-1		1	
f	$\nearrow$	-6	$\searrow$	undefined	$\searrow$	2	$\nearrow$
f'	$+$	0	$-$	undefined	$-$	0	$+$



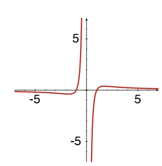
18. There are no intercepts. The function is undefined for $x = 0$. Solving $f'(x) = 1 - 25/x^2 = (x-25)/x^2 = 0$ we obtain the critical numbers -5 and 5. The relative maximum is $f(-5) = -10$ and the relative minimum is $f(5) = 10$.

x		-5		0		5	
f	$\nearrow$	-10	$\searrow$	undefined	$\searrow$	10	$\nearrow$
f'	$+$	0	$-$	undefined	$-$	0	$+$



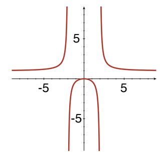
19. We write $f(x) = (x^2-1)/x^3$. The x-intercepts are ± 1. The function is undefined for $x = 0$. Solving $f'(x) = -1/x^2 + 3/x^4 = (3-x^2)/x^4 = 0$ we obtain the critical numbers $-\sqrt{3}$ and $\sqrt{3}$. The relative maximum is $f(\sqrt{3}) = 2/3\sqrt{3}$ and the relative minimum is $f(-\sqrt{3}) = -2/3\sqrt{3}$.

x		$-\sqrt{3}$		0		$\sqrt{3}$	
f	$\searrow$	$-2/3\sqrt{3}$	$\nearrow$	undefined	$\nearrow$	$2/3\sqrt{3}$	$\searrow$
f'	$-$	0	$+$	undefined	$+$	0	$-$



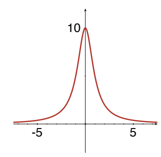
20. The x- and y-intercepts are 0. The function is undefined for $x = \pm 2$. Solving $f'(x) = -8/(x^2-4)^2 = 0$ we obtain the critical number 0. The relative maximum is $f(0) = 0$.

x		-2		0		2	
f	$\nearrow$	undefined	$\nearrow$	0	$\searrow$	undefined	$\searrow$
f'	$+$	undefined	$+$	0	$-$	undefined	$-$



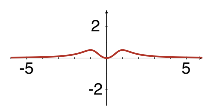
21. The y-intercept is 10. Solving $f'(x) = -20x/(x^2+1) = 0$ we obtain the critical number 0. The relative maximum is $f(0) = 10$.

x		0	
f	$\nearrow$	10	$\searrow$
f'	$+$	0	$-$



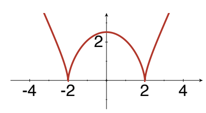
22. The x- and y-intercepts are 0. Solving $f'(x) = 2x(1-x^4)/(x^4+1)^2 = 0$ we obtain the critical numbers -1, 0, and 1. The relative maxima are $f(-1) = f(1) = 1/2$ and the relative minimum is $f(0) = 0$.

x		-1		0		1	
f	$\nearrow$	$1/2$	$\searrow$	0	$\nearrow$	$1/2$	$\searrow$
f'	$+$	0	$-$	0	$+$	0	$-$



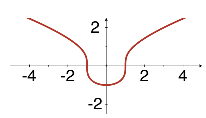
23. The x-intercepts are 2 and -2. The y-intercept is $2\sqrt[3]{2}$. Solving $f'(x) = (4/3)x(x^2-4)^{-1/3} = 0$ we obtain the critical number 0. Also, $f'(x)$ is undefined for $x = \pm 2$, which are critical numbers. The relative maximum is $f(0) \approx 2.5$ and the relative minima are $f(\pm 2) = 0$.

x		-2		0		2	
f	$\searrow$	0	$\nearrow$	2.5	$\searrow$	0	$\nearrow$
f'	$-$	undefined	$+$	0	$-$	undefined	$+$



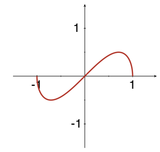
24. The x-intercepts are ± 1. The y-intercept is -1. Solving $f'(x) = 2x/3(x^2-1)^{2/3} = 0$ we obtain the critical number 0. Also, $f'(x)$ is undefined for $x = \pm 1$, which are critical numbers. The relative minimum is $f(0) = -1$.

x		-1		0		1	
f	$\searrow$	0	$\searrow$	-1	$\nearrow$	0	$\nearrow$
f'	$-$	undefined	$-$	0	$+$	undefined	$+$



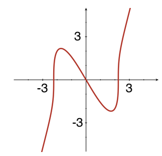
25. The x-intercepts are -1, 0, and 1. The y-intercept is 0. The function is undefined for $x < -1$ and $x > 1$. Solving $f'(x) = (1-2x^2)/\sqrt{1-x^2} = 0$ we obtain the critical numbers $-1/\sqrt{2}$ and $1/\sqrt{2}$. The relative maximum is $f(1/\sqrt{2}) = 1/2$ and the relative minimum is $f(-1/\sqrt{2}) = -1/2$.

x	-1		$-1/\sqrt{2}$		$1/\sqrt{2}$		1
f	0	$\searrow$	$-1/2$	$\nearrow$	$1/2$	$\searrow$	0
f'	undefined	$-$	0	$+$	0	$-$	undefined



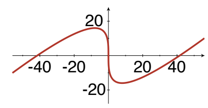
26. The x-intercepts are $-\sqrt{5}$, 0, and $\sqrt{5}$. The y-intercept is 0. Solving $f'(x) = (5x^2-15)/(x^2-5)^{2/3} = 0$ we obtain the critical numbers $-\sqrt{3}$ and $\sqrt{3}$. Also, $f'(x)$ is undefined for $x = \pm\sqrt{5}$, which are critical numbers. The relative maximum is $f(-\sqrt{3}) = 108^{1/6}$ and the relative minimum is $f(\sqrt{3}) = -108^{1/6}$.

x		$-\sqrt{5}$		$-\sqrt{3}$		$\sqrt{3}$		$\sqrt{5}$	
f	$\nearrow$	0	$\nearrow$	$108^{1/6}$	$\searrow$	$-108^{1/6}$	$\nearrow$	0	$\nearrow$
f'	$+$	undefined	$+$	0	$-$	0	$+$	undefined	$+$



27. The x-intercepts are $-24\sqrt{3}$, 0, and $24\sqrt{3}$. The y-intercept is 0. Solving $f'(x) = 1 - 4x^{-2/3} = (x^{2/3}-4)/x^{2/3} = 0$ we obtain the critical numbers -8 and 8. Also, $f'(x)$ is undefined for $x = 0$, which is a critical number. The relative maximum is $f(-8) = 16$ and the relative minimum is $f(8) = -16$.

x		-8		0		8	
f	$\nearrow$	16	$\searrow$	0	$\searrow$	-16	$\nearrow$
f'	$+$	0	$-$	undefined	$-$	0	$+$

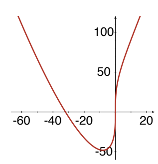


28. We have $f(x) = x^{1/3}(x+32)$. The x-intercepts are 0 and -32. The y-intercept is 0. Solving

$$f'(x) = \frac{4}{3}x^{1/3} + \frac{32}{3}x^{-2/3} = \frac{4x+32}{3x^{2/3}} = 0$$

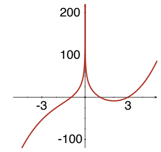
we obtain the critical number -8. Also, $f'(x)$ is undefined for $x = 0$, which is a critical number. The relative minimum is $f(-8) = -48$.

x		-8		0	
f	$\searrow$	-48	$\nearrow$	0	$\nearrow$
f'	$-$	0	$+$	undefined	$+$



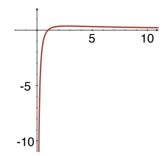
29. We have $f(x) = \begin{cases} x^3 - 24\ln x, & x > 0 \\ x^3 - 24\ln(-x), & x < 0 \end{cases}$. There are no easily-determined x-intercepts. The function is undefined for $x = 0$. Solving $f'(x) = 3x^2 - 24/x = 0$ we obtain the critical number 2. Also, $f'(x)$ is undefined for $x = 0$, which is a critical number. The relative minimum is $f(2) = 8 - 24\ln 2 \approx -8.6355$.

x		0		2	
f	$\nearrow$	undefined	$\searrow$	-8.6355	$\nearrow$
f'	$+$	undefined	$-$	0	$+$



30. The x-intercept is 1. The function is undefined for $x \le 0$. Solving $f'(x) = (1 - \ln x)/x^2 = 0$ we obtain the critical number e. The relative maximum is $f(e) = 1/e$.

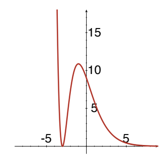
x	0		e	
f	undefined	$\nearrow$	$1/e$	$\searrow$
f'	undefined	$+$	0	$-$



31. The x-intercept is 3. The y-intercept is 9. Solving

$$\begin{aligned} f'(x) &= -(x+3)^2e^{-x} + 2e^{-x}(x+3) \\ &= e^{-x}(-x^2 - 4x - 3) \\ &= -e^{-x}(x+1)(x+3) = 0 \end{aligned}$$

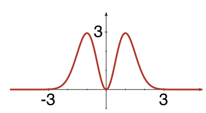
x		-3		-1	
f	$\searrow$	0	$\nearrow$	$4e$	$\searrow$
f'	$-$	0	$+$	0	$-$

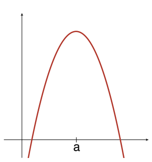


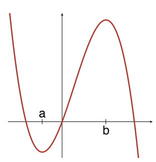
we obtain the critical numbers -3 and -1. The relative maximum is $f(-1) = 4e$ and the relative minimum is $f(-3) = 0$.

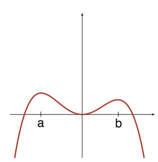
32. The x-intercept is 0. The y-intercept is 0. Solving $f'(x) = 8x^2[(-2x)e^{-x^2}] + 16x(e^{-x^2}) = 16xe^{-x^2}(1 - x^2) = 16xe^{-x^2}(1+x)(1-x) = 0$ we obtain the critical numbers -1, 0, and 1. The relative maxima are $f(-1) = f(1) = 8/e$ and the relative minimum is $f(0) = 0$.

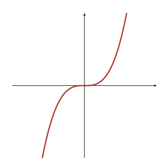
x		-1		0		1	
f	$\nearrow$	$8/e$	$\searrow$	0	$\nearrow$	$8/e$	$\searrow$
f'	$+$	0	$-$	0	$+$	0	$-$

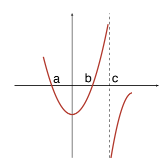


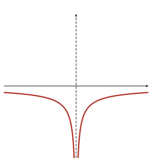
33. 

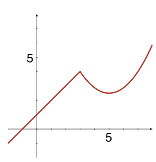
34. 

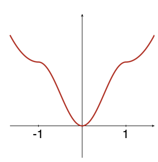
35. 

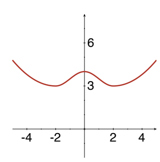
36. 

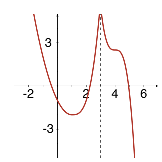
37. 

38. 

39. 

40. 

41. 

42. 

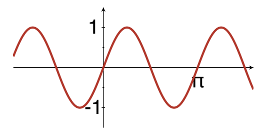
43. The slopes of the tangent lines are given by $f'(x) = 3x^2 + 12x - 1$. Solving $f''(x) = 6x + 12 = 0$ we obtain the point -2. The relative minimum is $f'(-2) = -13$.

x		-2	
f'	$\searrow$	-13	$\nearrow$
f''	$-$	0	$+$

44. The slopes of the tangent lines are given by $f'(x) = 4x^3 - 12x$. Solving $f''(x) = 12x^2 - 12 = 0$ we obtain the points -1 and 1. The relative maximum is $f'(-1) = 8$ and the relative minimum is $f'(1) = -8$.

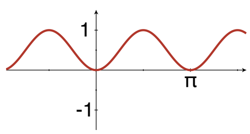
x		-1		1	
f'	$\nearrow$	8	$\searrow$	-8	$\nearrow$
f''	$+$	0	$-$	0	$+$

45. (a) $g(x) > 0$ for x in $(k\pi, k\pi + \pi/2)$, where k is an integer. $g(x) < 0$ for x in $(k\pi - \pi/2, k\pi)$, where k is an integer.



(b) Solving $f'(x) = 2\sin x \cos x = \sin 2x = g(x) = 0$ we obtain the critical numbers $k\pi/2$, where k is an integer. The relative maxima of $f(x)$ occur at $k\pi + \pi/2$, where k is an integer. The relative minima of $f(x)$ occur at $k\pi$, where k is an integer.

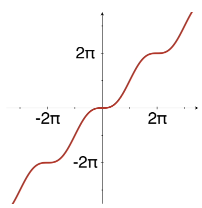
(c)



46. (a) Solving $f'(x) = 1 - \cos x = 0$ we obtain the critical numbers $2\pi k$, where k is an integer.

(b) Since $f'(x) \geq 0$ for all x, the sign of $f'(x)$ cannot change around a critical number. Thus $f(x)$ has no relative extrema.

(c)



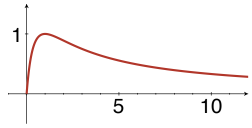
47. (a) Setting $f'(x) = 2(x-a_1)+2(x-a_2)+\cdots+2(x-a_n) = 0$ we obtain $nx = a_1+a_2+\cdots+a_n$ or $x = (a_1 + a_2 + \cdots + a_n)/n = \bar{x}$. Thus, $\bar{x}$ is a critical number of $f(x)$.

(b) To see that $f(\bar{x})$ is a relative minimum write $f(x) = nx^2 - 2(a_1 + a_2 + \cdots + a_n)x + (a_1^2 + a_2^2 + \cdots + a_n^2)$. Then $f'(x) = 2nx - 2(a_1 + a_2 + \cdots + a_n) = 2n(x - \bar{x})$. When $x < \bar{x}$, $f'(x) < 0$ and when $x > \bar{x}$, $f'(x) > 0$. Thus, $f(\bar{x})$ is a relative minimum.

48. (a) $T(1/r) = \dfrac{4/r}{(1/r+1)^2} = \dfrac{4}{r[(1+r)/r]^2} = \dfrac{4}{(1+r)^2/r} = \dfrac{4r}{(r+1)^2} = T(r)$. Since r is a ratio, $T(r) = T(1/r)$ means that the fraction of energy transmitted is the same in both directions.

(b) Solving $T'(r) = \dfrac{4(r+1)^2 - 8r(r+1)}{(r+1)^4} = -\dfrac{4(r-1)}{(r+1)^3} = 0$ we obtain the critical number 1. Since $T'(0) > 0$ and $T'(2) < 0$ we see that $T(1) = 1$ is a relative maximum.

(c)



49. Solving $f'(x) = 2ax + b = 0$ we obtain the critical number $-b/2a$. We want $-b/2a = 2$ or $b = -4a$. We are given $f(2) = 4a+2b+c = 6$ and $f(0) = c = 4$. Solving these three equations, we obtain $a = -1/2$, $b = 2$, and $c = 4$. Thus $f(x) = -\dfrac{1}{2}x^2 + 2x + 4$.

50. $f'(x) = 3ax^2 + 2bx + c$. Since 0 is a critical number, $f'(0) = c = 0$ and $f'(x) = 3ax^2 + 2bx$. Since 1 is a critical number, $f'(1) = 3a + 2b = 0$. We also have $f(0) = d = -3$ and $f(1) = a + b + c + d = a + b - 3 = 4$. Solving the system

$$\begin{aligned} 3a + 2b &= 0 \\ a + b &= 7, \end{aligned}$$

we obtain $a = -14$ and $b = 21$. Thus $f(x) = -14x^3 + 21x^2 - 3$.

51. If $f'(0) > 0$, then $f'(x) > 0$ on some interval $(-a, a)$. But then $f(x)$ is increasing on $(-a, a)$ and cannot be symmetric about the y-axis. Similarly, if $f'(0) < 0$, then $f(x)$ is decreasing on some open interval containing 0 and cannot be symmetric about the y-axis. Since $f'(0)$ exists, we must have $f'(0) = 0$. Since $f(x)$ is neither increasing nor decreasing in an open interval around 0, it must have a relative extremum at 0.

52. Solving $f'(x) = x^m[n(x-1)^{n-1}] + (x-1)^n(mx^{m-1}) = x^{m-1}(x-1)^{n-1}[(m+n)x - m] = 0$ we see that $f(x)$ has critical numbers at 0, $\dfrac{m}{m+n}$, and 1. If n is odd, then $(x-1)^{n-1} > 0$ for all $x \neq 1$ and the relative minimum is $f\left(\dfrac{m}{m+n}\right)$.

x		$\frac{m}{m+n}$		1	
f	$\searrow$		$\nearrow$		
f'	$-$	0	$+$	0	

If n is even, then $(x-1)^{n-1} < 0$ for $x < 1$ and the relative minimum is $f(1)$.

x		$\frac{m}{m+n}$		1	
f			$\searrow$		$\nearrow$
f'		0	$-$	0	$+$

53. (a) Since f and g are differentiable and have a critical number at c, $f'(c) = g'(c) = 0$. Then $(f+g)'(c) = f'(c) + g'(c) = 0$, $(f-g)'(c) = f'(c) - g'(c) = 0$, and $(fg)'(c) = f(c)g'(c) + f'(c)g(c) = 0$. Thus, $f+g$, $f-g$, and fg have critical numbers c.

    (b) Suppose $f'(x) > 0$ and $g'(x) > 0$ for $a < x < c$, and $f'(x) < 0$ and $g'(x) < 0$ for $c < x < b$. Then $(f+g)'(x) = f'(x) + g'(x) > 0$ for $a < x < c$ and $(f+g)'(x) < 0$ for $c < x < b$. Thus, $f+g$ has a relative maximum at c.

    To see that neither $f-g$ nor fg necessarily have relative maxima at c let $f(x) = -x^4$ and $g(x) = -x^2$. Both have relative maxima at $c = 0$. However $(f-g)(x) = -x^4 + x^2$ and $(fg)(x) = x^6$ both have relative minima at $c = 0$.

4.7 Graphing and the Second Derivative

1. $f'(x) = -2x + 7$; $f''(x) = -2$. Since $f''(x) < 0$ for all x, the graph is concave downward on $(-\infty, \infty)$.

2. $f'(x) = -2(x+2)$; $f''(x) = -2$. Since $f''(x) < 0$ for all x, the graph is concave downward on $(-\infty, \infty)$.

3. $f'(x) = -3x^2 + 12x + 1$; $f''(x) = -6x + 12$. Solving $f''(x) = 0$ we obtain $x = 2$. The graph is concave upward on $(-\infty, 2)$ and concave downward on $(2, \infty)$.

x		2	
f''	+	0	−

4. $f'(x) = 3(x+5)^2$; $f''(x) = 6(x+5)$. Solving $f''(x) = 0$ we obtain $x = -5$. The graph is concave upward on $(-5, \infty)$ and concave downward on $(-\infty, -5)$.

x		-5	
f''	−	0	+

5. $f'(x) = (x-4)^2(4x-4)$; $f''(x) = 12(x-4)(x-2)$. Solving $f''(x) = 0$ we obtain $x = 2$ and 4. The graph is concave upward on $(-\infty, 2)$, $(4, \infty)$, and concave downward on $(2, 4)$.

x		2		4	
f''	+	0	−	0	+

6. $f'(x) = 24x^3 + 6x^2 - 24x$; $f''(x) = 72x^2 + 12x - 24 = 12(3x+2)(2x-1)$. Solving $f''(x) = 0$ we obtain $x = -2/3$ and $1/2$. The graph is concave upward on $(-\infty, -2/3)$, $(1/2, \infty)$, and concave downward on $(-2/3, 1/2)$.

x		$-2/3$		$1/2$	
f''	+	0	−	0	+

7. $f'(x) = \dfrac{1}{3}x^{-2/3} + 2$; $f''(x) = -\dfrac{2}{9}x^{-5/3}$. $f''(x)$ is undefined for $x = 0$. The graph is concave upward on $(-\infty, 0)$ and concave downward on $(0, \infty)$.

x		0	
f''	+	undefined	−

8. $f'(x) = \dfrac{8}{3}x^{5/3} - \dfrac{40}{3}x^{-1/3}$; $f''(x) = \dfrac{40}{9}x^{2/3} + \dfrac{40}{9}x^{-4/3} = \dfrac{40}{9}(x^2 + 1)/x^{4/3}$. $f''(x)$ is undefined for $x = 0$. The graph is concave upward on $(-\infty, 0)$ and $(0, \infty)$.

x		0	
f''	+	undefined	+

9. $f'(x) = 1 - 9/x^2$; $f''(x) = 18/x^3$. $f''(x)$ is undefined for $x = 0$. The graph is concave upward on $(0, \infty)$ and concave downward on $(-\infty, 0)$.

x		0	
f''	−	undefined	+

10. $f'(x) = x/\sqrt{x^2+10}$; $f''(x) = 10/(x^2+10)^{3/2}$. Since $f''(x) > 0$ for all x, the graph is concave upward on $(-\infty, \infty)$.

11. $f'(x) = \dfrac{-2x}{(x^2+3)^2}$; $f''(x) = \dfrac{(x^2+3)^2(-2) + 2x[4x(x^2+3)]}{(x^2+3)^4} = \dfrac{6(x^2-1)}{(x^2+3)^3}$.

Solving $f''(x) = 0$ we obtain $x = \pm 1$. The graph is concave upward on $(-\infty, -1)$ and $(1, \infty)$, and concave downward on $(-1, 1)$.

x		-1		1	
f''	+	0	−	0	+

12. $f'(x) = 3/(x+2)^2$; $f''(x) = -6/(x+2)^3$. $f''(x)$ is undefined for $x = -2$. The graph is concave upward on $(-\infty, -2)$ and concave downward on $(-2, \infty)$.

x		-2	
f''	+	undefined	−

13. f' is increasing on $(-2, 2)$.
f' is decreasing on $(-\infty, -2)$ and $(2, \infty)$.

x		-2		2	
f''	−	0	+	0	−

14. f' is increasing on $(-\infty, 0)$.
f' is decreasing on $(0, \infty)$.

x		0	
f''	+	0	−

15. f' is increasing on $(-\infty, -1)$ and $(4, \infty)$.
f' is decreasing on $(-1, 4)$.

x		−1		4	
f''	+	0	−	0	+

16. f' is increasing on $(-2, 0)$ and $(2, \infty)$.
f' is decreasing on $(-\infty, -2)$ and $(0, 2)$.

x		−2		0		2	
f''	−	0	+	0	−	0	+

17. $f'(x) = \sec x \tan x$;

$f''(x) = (\sec x)(\sec^2 x) + (\tan x)(\sec x \tan x) = (\sec x)(\sec^2 x + \tan^2 x) = (1 + \sin^2 x)/\cos^3 x$. $f''(x)$ is positive when $\cos x > 0$ and negative when $\cos x < 0$. Thus, the graph of $f(x) = \sec x$ is concave upward when $\cos x > 0$ and concave downward when $\cos x < 0$.

18. $f'(x) = -\csc x \cot x$;

$f''(x) = -(\csc x)(-\csc^2 x)+(\cot x)(\csc x \cot x) = (\csc x)(\csc^2 x+\cot^2 x) = (1+\cos^2 x)/\sin^3 x$. $f''(x)$ is positive when $\sin x > 0$ and negative when $\sin x < 0$. Thus, the graph of $f(x) = \csc x$ is concave upward when $\sin x > 0$ and concave downward when $\sin x < 0$.

19. $f'(x) = 4x^3-24x+1$; $f''(x) = 12x^2-24$. Solving $f''(x) = 0$ we obtain $x = -\sqrt{2}$ and $\sqrt{2}$. The inflection points are $(-\sqrt{2}, -21 - \sqrt{2})$ and $(\sqrt{2}, -21 + \sqrt{2})$.

x		$-\sqrt{2}$		$\sqrt{2}$	
f''	+	0	−	0	+

20. $f'(x) = \dfrac{5}{3}x^{2/3} + 4$; $f''(x) = \dfrac{10}{9}x^{-1/3}$. $f''(x)$ is undefined for $x = 0$. The inflection point is $(0, 0)$.

x		0	
f''	−	undefined	+

21. $f'(x) = \cos x$; $f''(x) = -\sin x$. Solving $f''(x) = 0$ we obtain $x = k\pi$, where k is an integer. Since $f(x) = \sin x$ is 2π-periodic, the graph has inflection points at $(k\pi, 0)$, where k is an integer.

x	0		π		2π
f''	0	−	0	+	0

22. $f'(x) = -\sin x$; $f''(x) = -\cos x$. Solving $f''(x) = 0$ we obtain $x = \pi/2 + k\pi$, where k is an integer.

x	$-\pi/2$		$\pi/2$		$3\pi/2$		$5\pi/2$
f''	0	−	0	+	0	−	0

Since $f(x) = \cos x$ is 2π-periodic, the graph has inflection points at $(\pi/2 + k\pi, (-1)^k)$, where k is an integer.

23. $f'(x) = 1 - \cos x$; $f''(x) = \sin x$. Solving $f''(x) = 0$ we obtain $x = k\pi$, where k is an integer. Since the sign of $f''(x) = \sin x$ changes around each $k\pi$, the graph of $f(x) = x - \sin x$ has inflection points at $(k\pi, k\pi)$, where k is an integer.

24. $f'(x) = \sec^2 x$; $f''(x) = 2\sec^2 x \tan x = 2\sin x/\cos^3 x$. Solving $f''(x) = 0$ on $(-\pi/2, \pi/2)$ we obtain $x = 0$. The graph of $f(x)$ has an inflection point at $(0, 0)$, and since $f(x) = \tan x$ is π-periodic, it has inflection points at $(k\pi, 0)$, where k is an integer.

x		0	
f''	−	0	+

25. $f'(x) = 1 - xe^{-x} + e^{-x} = 1 + (1-x)e^{-x}$; $f''(x) = -(1-x)e^{-x} - e^{-x} = (x-2)e^{-x}$. Solving $f''(x) = 0$ we obtain $x = 2$. The inflection point is $(2, 2 + 2/e^2)$.

x		2	
f''	$-$	0	$+$

26. $f'(x) = -2x^2e^{-x^2} + e^{-x^2} = (1-2x^2)e^{-x^2}$;

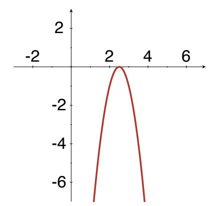
$$f''(x) = -2x(1-2x^2)e^{-x^2} - 4xe^{-x^2}$$
$$= (4x^3 - 6x)e^{-x^2} = (4x^2 - 6)xe^{-x^2}.$$

x		$-\sqrt{3/2}$		0		$\sqrt{3/2}$	
f''	$-$	0	$+$	0	$-$	0	$+$

Solving $f''(x) = 0$ we obtain $x = 0$, $-\sqrt{3/2}$, and $\sqrt{3/2}$. The inflection points are $(0,0)$, $(-\sqrt{3/2}, -\sqrt{3/2}e^{-3/2})$, and $(\sqrt{3/2}, \sqrt{3/2}e^{-3/2})$.

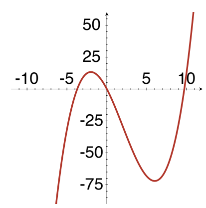
27. The x-intercept is $5/2$. The y-intercept is -25. $f'(x) = -4(2x-5)$; $f''(x) = -8$. Solving $f'(x) = 0$ we obtain the critical number $5/2$. The relative maximum is $(5/2, 0)$.

x		$5/2$	
f		0	
f'		0	
f''	$-$	$-$	$-$



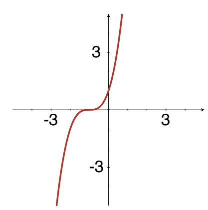
28. The x-intercepts are $3 - 3\sqrt{5}$, 0, and $3 + 3\sqrt{5}$. The y-intercept is 0. $f'(x) = x^2 - 4x - 12 = (x-6)(x+2)$; $f''(x) = 2x - 4$. Solving $f'(x) = 0$ we obtain the critical numbers -2 and 6. Solving $f''(x) = 0$ we obtain $x = 2$. The relative maximum is $(-2, 40/3)$, the relative minimum is $(6, -72)$, and the inflection point is $(2, -88/3)$.

x		-2		2		6	
f		$40/3$		$-88/3$		-72	
f'		0				0	
f''	$-$	$-$	$-$	0	$+$	$+$	$+$



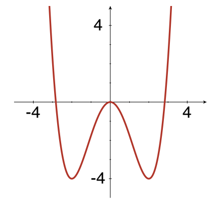
29. $f(x) = (x+1)^3$. The x-intercept is -1. The y-intercept is 1. $f'(x) = 3(x+1)^2$; $f''(x) = 6(x+1)$. Solving $f'(x) = 0$ we obtain the critical number -1. Solving $f''(x) = 0$ we obtain $x = -1$. Since $f''(-1) = 0$ the second derivative test does not apply. There are no relative extrema. The inflection point is $(-1, 0)$.

x		-1	
f	$\nearrow$	0	$\nearrow$
f'	$+$	0	$+$
f''	$-$	0	$+$



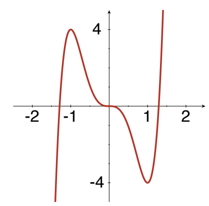
30. $f(x) = \dfrac{1}{4}x^2(x^2 - 8)$. The x-intercepts are $-\sqrt{8}$, 0, and $\sqrt{8}$. The y-intercept is 0. $f'(x) = x^3 - 4x = x(x^2 - 4)$; $f''(x) = 3x^2 - 4$. Solving $f'(x) = 0$ we obtain the critical numbers -2, 0, and 2. Solving $f''(x) = 0$ we obtain $-2/\sqrt{3}$ and $2/\sqrt{3}$. The relative minima are $(-2, -4)$ and $(2, -4)$. The relative maximum is $(0, 0)$. The inflection points are $(-2/\sqrt{3}, -20/9)$ and $(2/\sqrt{3}, -20/9)$.

x		-2		$-2/\sqrt{3}$		0		$2/\sqrt{3}$		2	
f		-4		$-20/9$		0		$-20/9$		-4	
f'		0				0				0	
f''	$+$	$+$	$+$	0	$-$	$-$	$-$	0	$+$	$+$	$+$



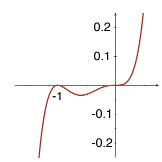
31. $f(x) = 2x^3(3x^2 - 5)$. The x-intercepts are $-\sqrt{5/3}$, 0, and $\sqrt{5/3}$. The y-intercept is 0. $f'(x) = 30x^4 - 30x^2 = 30x^2(x^2 - 1)$; $f''(x) = 120x^3 - 60x = 60x(2x^2 - 1)$. Solving $f'(x) = 0$ we obtain the critical numbers -1, 0, and 1. Solving $f''(x) = 0$ we obtain $x = -1/\sqrt{2}$, 0, and $1/\sqrt{2}$. The second derivative test does not apply at 0. The relative maximum is $(-1, 4)$ and the relative minimum is $(1, -4)$. The inflection points are $(-1/\sqrt{2}, 7/\sqrt{8})$, $(0, 0)$, and $(1/\sqrt{2}, -7/\sqrt{8})$.

x		-1		$-1/\sqrt{2}$		0		$1/\sqrt{2}$		1	
f		4		$7/\sqrt{8}$	$\searrow$	0	$\searrow$	$-7/\sqrt{8}$		-4	
f'		0			$-$	0	$-$			0	
f''	$-$	$-$	$-$	0	$+$	0	$-$	0	$+$	$+$	$+$



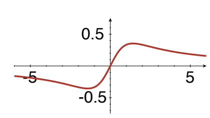
32. The x-intercepts are -1 and 0. The y-intercept is 0. $f'(x) = x^2(x+1)(5x+3)$; $f''(x) = 2x(10x^2 + 12x + 3)$. Solving $f'(x) = 0$ we obtain the critical numbers -1, $-3/5$, and 0. Solving $f''(x) = 0$ we obtain $(-6 - \sqrt{6})/10$, $(-6 + \sqrt{6})/10$, and 0. The second derivative test does not apply at 0. The relative maximum is $(-1, 0)$ and the relative minimum is $(-3/5, -0.03)$. The inflection points are $(0, 0)$, $((-6 - \sqrt{6})/10, -0.01)$, and $((-6 + \sqrt{6})/10, -0.02)$.

x		-1		$\dfrac{-6-\sqrt{6}}{10}$		$-3/5$		$\dfrac{-6+\sqrt{6}}{10}$		0	
f		0							$\nearrow$	0	$\nearrow$
f'		0				0			$+$	0	$+$
f''	$-$	$-$	$-$	0	$+$	$+$	$+$	0	$-$	0	$+$



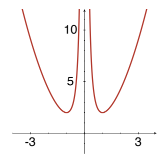
33. The x- and y-intercepts are 0. $f'(x) = \dfrac{2 - 2x^2}{(x^2+2)^2}$; $f''(x) = \dfrac{2x(x^2 - 6)}{(x^2+2)^3}$. Solving $f'(x) = 0$ we obtain the critical numbers $-\sqrt{2}$ and $\sqrt{2}$. Solving $f''(x) = 0$ we obtain $-\sqrt{6}$, 0, and $\sqrt{6}$. The relative maximum is $(\sqrt{2}, \sqrt{2}/4)$ and the relative minimum is $(-\sqrt{2}, -\sqrt{2}/4)$. The inflection points are $(-\sqrt{6}, -\sqrt{6}/8)$, $(0, 0)$, and $(\sqrt{6}, \sqrt{6}/8)$.

x		$-\sqrt{6}$		$-\sqrt{2}$		0		$\sqrt{2}$		$\sqrt{6}$	
f		$-\sqrt{6}/8$		$-\sqrt{2}/4$		0		$\sqrt{2}/4$		$\sqrt{6}/8$	
f'				0				0			
f''	$-$	0	$+$	$+$	$+$	0	$-$	$-$	$-$	0	$+$



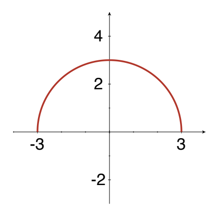
34. There are no x- or y-intercepts. $f'(x) = 2x - 2/x^3 = 2(x^4 - 1)/x^3$; $f''(x) = 2 + 6/x^4$. Solving $f'(x) = 0$ we obtain the critical numbers -1 and 1. The function is undefined at $x = 0$. $f''(x) = 0$ has no real solution. The relative minima are $(-1, 2)$ and $(1, 2)$.

x		-1		0		1	
f		2		undefined		2	
f'		0		undefined		0	
f''	$+$	$+$	$+$	undefined	$+$	$+$	$+$



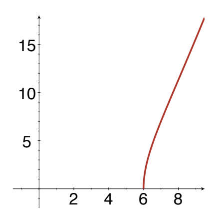
35. The domain of the function is $[-3, 3]$. The x-intercepts are -3 and 3. The y-intercept is 3. $f'(x) = -x/\sqrt{9 - x^2}$; $f''(x) = -9/(9 - x^2)^{3/2}$. Solving $f'(x) = 0$ we obtain the critical number 0. $f''(x) = 0$ has no real solution. The relative maximum is $(0, 3)$.

x	-3		0		3
f	0		3		0
f'	undefined		0		undefined
f''	undefined	$-$	$-$	$-$	undefined



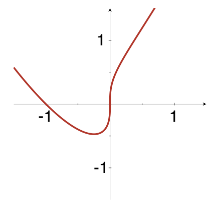
36. The domain of the function is $[6, \infty)$. The x-intercept is 0. $f'(x) = \dfrac{3x - 12}{2\sqrt{x - 6}}$; $f''(x) = \dfrac{3x - 24}{4(x - 6)^{3/2}}$. $f'(x) = 0$ has no solution in $[6, \infty)$. Solving $f''(x) = 0$ we obtain $x = 8$. The inflection point is $(8, 8\sqrt{2})$.

x	6		8	
f	0		$8\sqrt{2}$	
f'	undefined	$+$	$+$	$+$
f''	undefined	$-$	0	$+$



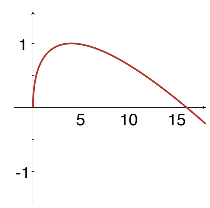
37. The x-intercepts are -1 and 0. The y-intercept is 0. $f'(x) = \dfrac{4x + 1}{3x^{2/3}}$; $f''(x) = \dfrac{4x - 2}{9x^{5/3}}$. Solving $f'(x) = 0$ we obtain the critical point $-1/4$. Also, $f'(x)$ is undefined for $x = 0$. Solving $f''(x) = 0$ we obtain $x = 1/2$. The relative minimum is $(-1/4, -3/4^{4/3})$. The inflection points are $(1/2, 3/2^{4/3})$ and $(0, 0)$.

x		$-1/4$		0		$1/2$	
f		$-3/4^{4/3}$		0		$3/2^{4/3}$	
f'		0		undefined			
f''	$+$	$+$	$+$	undefined	$-$	0	$+$

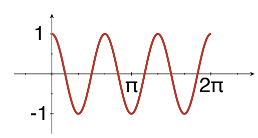


38. The domain of the function is $[0, \infty)$. The x-intercepts are 0 and 16. The y-intercept is 0. $f'(x) = 1/2\sqrt{x} - 1/4 = \dfrac{2-\sqrt{x}}{4\sqrt{x}}$; $f''(x) = -1/4x^{3/2}$. Solving $f'(x) = 0$ we obtain the critical number 4. $f''(x) = 0$ has no solution. The relative maximum is $(4, 1)$.

x	0		4	
f			1	
f'	undefined		0	
f''	undefined	$-$	$-$	$-$

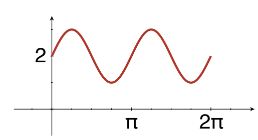


39. The x-intercepts are $(k\pi/6, 0)$ for $k = 1, 3, 5, 7, 9,$ and 11. The y-intercept is 1. $f'(x) = -3\sin 3x$; $f''(x) = -9\cos 3x$. Solving $f'(x) = 0$ we obtain the critical numbers $k\pi/3$ for $k = 0, 1, 2, 3, 4, 5,$ and 6. Computing $f''(x)$ at these values we see that $f(x)$ has relative maxima at $(2\pi/3, 1)$ and $(4\pi/3, 1)$, and relative minima at $(\pi/3, -1)$, $(\pi, -1)$, and $(5\pi/3, -1)$.



Solving $f''(x) = 0$ we obtain $(k\pi/6, 0)$ for $k = 1, 3, 5, 7, 9,$ and 11. These are all points of inflection since the sign of $f''(x)$ changes around each one.

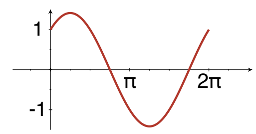
40. There are no x-intercepts. The y-intercept is 2. $f'(x) = 2\cos 2x$; $f''(x) = -4\sin 2x$. Solving $f'(x) = 0$ we obtain the critical numbers $\pi/4$, $3\pi/4$, $5\pi/4$, and $7\pi/4$. Computing $f''(x)$ at these values we see that $f(x)$ has relative maxima at $(\pi/4, 3)$ and $(5\pi/4, 3)$, and relative minima at $(3\pi/4, 1)$ and $(7\pi/4, 1)$.



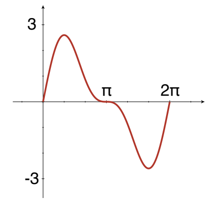
Solving $f''(x) = 0$ we obtain $(k\pi/2, 2)$ for $k = 0, 1, 2, 3,$ and 4. These are all points of inflection since the sign of $f''(x)$ changes around each one.

41. Solving $\sin x = -\cos x$ or $\tan x = -1$ we obtain the x-intercepts $3\pi/4$ and $7\pi/4$. The y-intercept is 1. $f'(x) = -\sin x + \cos x$; $f''(x) = -\cos x - \sin x$. Solving $f'(x) = 0$ we obtain the critical numbers $\pi/4$ and $5\pi/4$. Solving $f''(x) = 0$ we obtain $x = 3\pi/4$ and $7\pi/4$. The relative maximum is $(\pi/4, \sqrt{2})$ and the relative minimum is $(5\pi/4, -\sqrt{2})$. The inflection points are $(3\pi/4, 0)$ and $(7\pi/4, 0)$.

x	0		$\pi/4$		$3\pi/4$		$5\pi/4$		$7\pi/4$		2π
f	1		$\sqrt{2}$		0		$-\sqrt{2}$		0		1
f'			0				0				
f''		$-$	$-$	$-$	0	$+$	$+$	$+$	0	$-$	



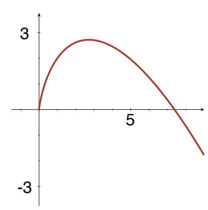
42. Solving $2\sin x + \sin 2x = 2\sin x + 2\sin x\cos x = (2\sin x)(1+\cos x) = 0$ we obtain the x-intercepts 0, π, and 2π. The y-intercept is 0. $f'(x) = 2\cos x + 2\cos 2x = 2\cos x + 2(2\cos^2 x - 1) = 2(2\cos^2 x + \cos x - 1) = 2(2\cos x - 1)(\cos x + 1)$. $f''(x) = -2\sin x - 4\sin 2x = -2(\sin x + 4\sin x\cos x) = -2(\sin x)(1+4\cos x)$. Solving $f'(x)=0$ we obtain the critical numbers $\pi/3$, π, and $5\pi/3$. Solving $f''(x)=0$ we obtain $x=0$, π, 2π, and c, where $\cos c = -1/4$. Using a calculator, we find $c \approx 1.82$ and 4.46. The relative maximum is $(\pi/3, 3\sqrt{3}/2)$ and the relative minimum is $(5\pi/3, -3\sqrt{3}/2)$. The inflection points are $(0,0)$, $(1.82, 1.45)$, and $(4.46, -1.45)$.



x	0		$\pi/3$		1.82		π		4.46		$5\pi/3$		2π
f	0		$3\sqrt{3}/2$		1.45		0		-1.45		$-3\sqrt{3}/2$		0
f'			0	$-$	$-$	$-$	0	$-$	$-$	$-$	0		
f''	0	$-$	$-$	$-$	0	$+$	0	$-$	0	$+$	$+$	$+$	0

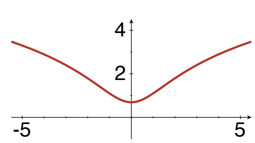
43. The domain of the function is $(0,\infty)$. Solving $2x = x\ln x$ or $\ln x = 2$ we obtain the x-intercept e^2. There is no y-intercept. $f'(x) = 1 - \ln x$; $f''(x) = -1/x$. Solving $f'(x)=0$ we obtain the critical number e. $f''(x)=0$ has no solution. The relative maximum is (e,e).

x	0		e		e^2
f	undefined		e		0
f'	undefined		0		
f''	undefined	$-$	$-$	$-$	$-$



44. There are no x-intercepts. The y-intercept is $\ln 2$. $f'(x) = \dfrac{2x}{x^2+2}$; $f''(x) = \dfrac{4-2x^2}{(x^2+2)^2} = \dfrac{2(2-x^2)}{(x^2+2)^2}$. Solving $f'(x)=0$ we obtain the critical number 0. Solving $f''(x)=0$ we obtain $x = -\sqrt{2}$ and $\sqrt{2}$. The relative minimum is $(0, \ln 2)$. The inflection points are $(-\sqrt{2}, \ln 4)$ and $(\sqrt{2}, \ln 4)$.

x		$-\sqrt{2}$		0		$\sqrt{2}$	
f		$\ln 4$		$\ln 2$		$\ln 4$	
f'				0			
f''	$-$	0	$+$	$+$	$+$	0	$-$



45. $f(x) = \dfrac{1}{2}\sin 2x$; $f'(x) = \cos 2x$; $f''(x) = -2\sin 2x$. Since $f''(\pi/4) = -2 < 0$, the function has a relative maximum at $(\pi/4, 1/2)$.

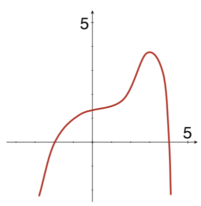
46. $f'(x) = x\cos x + \sin x$; $f''(x) = -x\sin x + 2\cos x$. Since $f''(0) = 2 > 0$, the function has a relative minimum at $(0, 0)$.

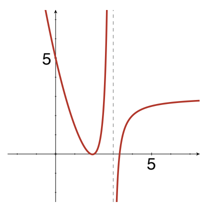
47. $f'(x) = 2\tan x \sec^2 x$; $f''(x) = 2\sec^4 x + 4\tan^2 x\sec^2 x$. Since $f''(\pi) = 2 > 0$, the function has a relative minimum at $(\pi, 0)$.

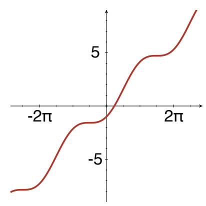
48. $f'(x) = 12(\cos 4x)(1 + \sin 4x)^2$;

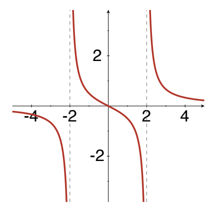
$$f''(x) = 12(\cos 4x)[2(1 + \sin 4x)(4\cos 4x)] + (1 + \sin 4x)^2(-48\sin 4x).$$

Since $f''(\pi/8) = -192 < 0$, the function has a relative maximum at $(\pi/8, 8)$.

49.


50.


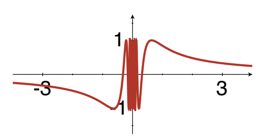
51.


52.


53. $f'(x) = 3ax^2 + 2bx + c$; $f''(x) = 6ax + 2b$. Since the graph has an inflection point at $(1, 1)$, $f''(1) = 6a + 2b = 0$. Using the fact that $(-1, 0)$ and $(1, 1)$ lie on the graph, we have $-a + b - c = 0$ and $a + b + c = 1$. Solving these three equations, we obtain $a = -1/6$, $b = 1/2$, and $c = 2/3$.

54. $f'(x) = 3ax^2 + 2bx + c$; $f''(x) = 6ax + 2b$. Since the graph has a horizontal tangent at $(1, 1)$, $f'(1) = 3a + 2b + c = 0$. Since the graph has an inflection point at $(1, 1)$, $f''(1) = 6a + 2b = 0$. Using the fact that $(1, 1)$ is on the graph, we have $f(1) = a + b + c = 1$. Solving these three equations, we obtain $a = 1$, $b = -3$, and $c = 3$.

55. Since $f(x)$ is an odd function, the graph is symmetric with respect to the origin. $f'(x) = -\cos(1/x)/x^2$; $f''(x) = [2x\cos(1/x) - \sin(1/x)]/x^4$. Solving $f'(x) = 0$ we obtain the positive critical numbers $2/\pi$, $2/3\pi$, $2/5\pi$, $2/7\pi$, $\ldots$. Since the sign of $f''(x)$ alternates at these points, they are alternately relative maxima and relative minima. For $x > 3\pi$, $f''(x) > 0$ and the graph is concave upwards.



56. $f'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \ldots + a_2 x + a_1$;

$$f''(x) = n(n-1)a_n x^{n-2} + (n-1)(n-2)a_{n-1}x^{n-3} + \ldots + a_2.$$

Since a polynomial of degree $n - 2$ can have at most $n - 2$ zeroes, $f(x)$ can have at most $n - 2$ points of inflection.

57. $f'(x) = n(x - x_0)^{n-1}$; $f''(x) = n(n-1)(x - x_0)^{n-2}$.

(a) If n is odd, then the sign of $f''(x)$ changes around x_0 and $(x_0, 0)$ is a point of inflection.

(b) If n is even, then $f''(x) > 0$ for $x \neq x_0$ and $(x_0, 0)$ is not a point of inflection. Using the first derivative test, we see that $f'(x) < 0$ for $x < x_0$ and $f'(x) > 0$ for $x > x_0$. Thus $(x_0, 0)$ is a relative minimum.

58. $f'(x) = 2ax + b$; $f''(x) = 2a$. If $a > 0$ then $f''(x) > 0$ and the graph of $f(x)$ is concave upward on $(-\infty, \infty)$. If $a < 0$ then $f''(x) < 0$ and the graph of $f(x)$ is concave downward on $(-\infty, \infty)$.

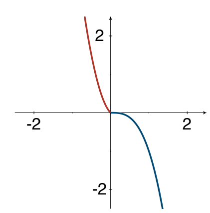
59. Since $f'''(c) \neq 0$, the graph of $f''(x)$ is either increasing (when $f'''(c) > 0$) or decreasing (when $f'''(c) < 0$) through $(c, 0)$. Thus, the sign of $f''(x)$ changes around $x = c$, and $(c, f(c))$ is a point of inflection.

60. $f(x) = x$

61. The statement is false: Consider $f(x) = x^{1/3}$. Since $f'(x) = 1/3x^{2/3}$ is not defined at $x = 0$, f' cannot have a critical number at 0. (Recall that a critical number must be in the domain of the function.) From $f''(x) = -2/9x^{5/3}$ we see that $f''(x) > 0$ for $x < 0$, and $f''(x) < 0$ for $x > 0$. Thus $(0, 0)$ is a point of inflection.

62. $f'(x) = 20x - 1 + e^x$; $f''(x) = 20 + e^x$. $f''(x) = 0$ has no solution, and therefore $f(x)$ cannot have a point of inflection.

63. The graph of $f(x) = \begin{cases} 4x^2 - x, & x \leq 0 \\ -x^3, & x > 0 \end{cases}$ is shown on the right. The function has no tangent line at $(0, 0)$ since $f'_+(0) = 0$ and $f'_-(0) = -1$. Although concavity changes at $(0, 0)$, according to Definition 4.7.2 there is no point of inflection because there is no tangent line at the point.



64. (a) Since f is a polynomial function of degree 3, then f' is a polynomial function of degree 2, and its graph is a parabola. We are given that c_1 and c_2 are *distinct* critical numbers; thus, c_1 and c_2 are *not* on the f' parabola's vertex. Since $f'' = 0$ only at the f' parabola's vertex, then $f''(c_1)$ and $f''(c_2)$ cannot be zero, and therefore $f(c_1)$ and $f(c_2)$ must be relative extrema.

(b) Let c be the x coordinate of the f' parabola's vertex; thus, $f''(c) = 0$, and $(c, f(c))$ is the point of inflection for the graph of f. c must therefore be $\dfrac{c_1 + c_2}{2}$.

65. As stated in *Notes from the Classroom*, textbooks disagree on the precise definition of a point of inflection. With sufficient research, the student should find definitions for which some function f will have different points of inflection.

4.8 Optimization

1. Let x and $60 - x$ be the two numbers. We want to maximize $P(x) = x(60 - x) = 60x - x^2$ on $[0, 60]$. Solving $P'(x) = 60 - 2x = 0$ we obtain the critical number 30. Since $P''(x) = -2 < 0$, the product is maximized by the numbers 30, 30.

2. Let x and $50/x$ be the two numbers. We want to minimize $S(x) = x+50/x$ on $(0,\infty)$. Solving $S'(x) = 1 - 50/x^2 = 0$ we obtain the critical number $\sqrt{50}$. Since $S''(\sqrt{50}) > 0$, the sum is minimized by the numbers $\sqrt{50}$, $\sqrt{50}$.

3. Let x be the number. We want to maximize $f(x) = x - x^2$. Solving $f'(x) = 1 - 2x = 0$ we obtain the critical number $1/2$. Since $f''(x) = -2 < 0$, the product is maximized by the number $1/2$.

4. Let x and $S-x$ be the numbers. We want to maximize $P(x) = x^m(S-x)^n$ on $[0,S]$. Solving
$$P'(x) = x^m[n(S-x)^{n-1}(-1)] + (S-x)^n(mx^{m-1}) = x^{m-1}(S-x)^{n-1}[mS-(m+n)x] = 0$$
we obtain the critical numbers 0, S, and $\dfrac{mS}{m+n}$. Since $P(0) = P(S) = 0$, the product is maximized by the numbers $x = \dfrac{mS}{m+n}$ and $S - x = \dfrac{nS}{m+n}$.

5. Let x and $1-x$ be the two numbers. We want to maximize $S(x) = x^2+2(1-x)^2 = 2-4x+3x^2$. Solving $S'(x) = -4+6x = 0$ we obtain the critical number $2/3$. Since $S''(x) = 6 > 0$, the sum is minimized by $x = 2/3$ and $1 - x = 1/3$.

6. Let x and $1/x$ be the numbers. We want to maximize $S(x) = x + 1/x$ on $(0,\infty)$. Solving $S'(x) = 1 - 1/x^2 = 0$ we obtain the critical number 1. Since $S''(1) = 2 > 0$, the minimum sum is $S(1) = 2$.

7. Let $(x, \sqrt{6x})$ be on the graph. We will minimize the square of the distance.

   For $(5,0)$, $D(x) = (x-5)^2 + (\sqrt{6x} - 0)^2 = x^2 - 4x + 25$. Solving $D'(x) = 2x - 4 = 0$ we obtain the critical number 2. Since $D''(x) = 2 > 0$, the distance is minimized by the points $(2, \pm 2\sqrt{3})$ on the graph.

   For $(3,0)$, $D(x) = (x-3)^2 + (\sqrt{6x} - 0)^2 = x^2 + 9$. Solving $D'(x) = 2x = 0$ we obtain the critical number 0. Since $D''(x) = 2 > 0$, the distance is minimized by the point $(0,0)$ on the graph.

8. Let $(x, 1-x)$ be a point on the graph. We will minimize the square of the distance to $(2,3)$: $D(x) = (x-2)^2 + (1-x-3)^2 = 2x^2 + 8$. Solving $D'(x) = 4x = 0$ we obtain the critical number 0. Since $D''(x) = 4 > 0$, the distance is minimized by the point $(0,1)$ on the graph.

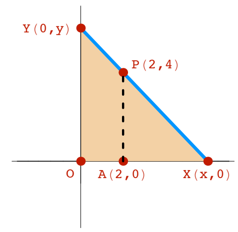
9. The slope of the tangent line at x is $s(x) = 3x^2 - 8x$. To minimize $s(x)$, we solve $s'(x) = 6x - 8 = 0$. This gives $x = 4/3$. Since $s''(x) = 6 > 0$, the slope is minimized at the point $(4/3, -128/27)$.

10. The slope of the tangent line at x is $s(x) = 16x - 1/x^2$. Since $\lim\limits_{x\to\infty} s(x) = +\infty$, the tangent line to the graph does not have a maximum slope.

11. Let (x,y) be the corner of the rectangle lying on the line. Then $y = 2 - \dfrac{2}{3}x$, and we want to maximize $A(x) = xy = 2x - \dfrac{2}{3}x^2$ on $[0,3]$. Solving $A'(x) = 2 - \dfrac{4}{3}x = 0$ we obtain the critical number $3/2$. Since $A''(x) = -4/3 < 0$, the area is maximized when the base of the rectangle is $3/2$ and the height is 1.

12. Let the width of the rectangle be $2x$ and the height be y. Then $y = 24 - x^2$, and we want to maximize $A(x) = 2xy = 2x(24 - x^2)$ on $[0, \sqrt{24}]$. Solving $A'(x) = 48 - 6x^2 = 0$ we obtain the critical number $2\sqrt{2}$. Since $A''(2\sqrt{2}) < 0$ the area is maximized when the rectangle is $4\sqrt{2}$ units wide and 16 units high.

13. Triangle OXY is similar to triangle AXP, so



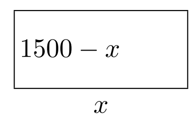
$$\frac{y}{x} = \frac{4}{x-2} \qquad \text{and} \qquad y = \frac{4x}{x-2}.$$

We want to minimize the area of the triangle $A(x) = \frac{1}{2}xy = \frac{1}{2}x\left(\frac{4x}{x-2}\right) = \frac{2x^2}{x-2}$ on $(2, \infty)$.

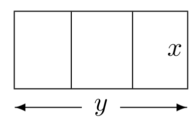
Solving $A'(x) = \dfrac{4x(x-2) - 2x^2}{(x-2)^2} = \dfrac{2x(x-4)}{(x-2)^2} = 0$ we obtain the critical number 4. Since $A''(4) > 0$, the area is minimized when the vertices are $(4, 0)$ and $(0, 8)$.

14. We want to maximize $s(x) = 1 - x - (x^2 - 1) = 2 - x - x^2$ on $[-2, 1]$. Solving $s'(x) = -1 - 2x = 0$ we obtain the critical number $-1/2$. Since $s''(x) = -2 < 0$, the distance is maximized when $x = -1/2$. The maximum distance is $s(-1/2) = 9/4$.

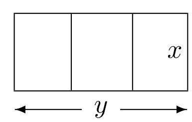
15. Let x and $1500 - x$ be the two sides of the corral. We want to maximize $A(x) = x(1500 - x)$ on $[0, 1500]$. Solving $A'(x) = 1500 - 2x = 0$ we obtain the critical number 750. Since $A'(x) = -2 < 0$, the area is maximized when the corral is 750 ft × 750 ft.



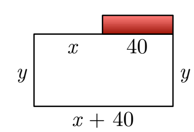
16. Let x and y be the lengths shown in the figure. Since $xy = 4000$, $y = 4000/x$, and we want to minimize $P(x) = 4x + 2y = 4x + 8000/x$. Solving $P'(x) = 4 - 8000/x^2$ we obtain the critical number $20\sqrt{5}$. Since $P''(20\sqrt{5}) > 0$, the amount of fence is minimized when $x = 20\sqrt{5}$ m and $y = 40\sqrt{5}$ m.



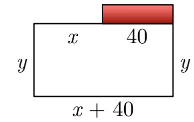
17. Let x and y be as shown in the figure. Since $4x + 2y = 8000$, $y = 4000 - 2x$, and we want to maximize $A(x) = xy = 4000x - 2x^2$. Solving $A'(x) = 4000 - 4x = 0$ we obtain the critical number 1000. Since $A''(x) = -4 < 0$, the area is maximized when $x = 1000$ m and $y = 2000$ m.



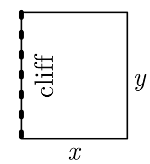
18. Let x and y be as shown in the figure. Then $2x + 40 + 2y = 160$ and $y = 60 - x$. We want to maximize $A(x) = (x + 40)y = (x + 40)(60 - x) = 2400 + 20x - x^2$ for x in $[0, 60]$. Solving $A'(x) = 20 - 2x = 0$ we obtain the critical number 10. Since $A''(x) = -2 < 0$, the maximum area is obtained when $x = 10$ ft and $y = 50$ ft. Thus, the enclosed yard will be a square 50 feet on each side.



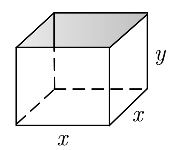
19. Let x and y be as shown in the figure. Then $2x + 40 + 2y = 80$ and $y = 20 - x$. We want to maximize $A(x) = (x + 40)y = (x + 40)(20 - x) = 800 - 20x - x^2$ for x in $[0, 20]$. Solving $A'(x) = -20 - 2x = 0$ we obtain the critical number -10. Comparing $A(0) = 800$ and $A(20) = 0$ we see that the maximum area is obtained when $x = 0$ and $y = 20$. Thus, the yard will be a rectangle 40 feet long by 20 feet wide.



20. Let x be the lengths of the sides perpendicular to the cliff and y the length of the sides parallel to the cliff. Then $xy = 128,000$ and $y = 128,000/x$. We want to minimize $C(x) = 2(2.5x) + 2.5y + 1.5y = 5x + 512,000/x$. Solving $C'(x) = 5 - 512,000/x^2 = 0$ we obtain the critical number 320. Since $C''(320) > 0$, the cost is minimized when $x = 320$ ft and $y = 400$ ft.

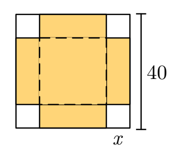


21. Let x be a side of the base and y the height of the box. Then $x^2y = 32,000$ and $y = 32,000/x^2$. We want to minimize $A(x) = x^2 + 4xy = x^2 + 128,000/x$. Solving $A'(x) = 2x - 128,000/x^2 = 0$ we obtain the critical number 40. Since $A''(40) > 0$, the amount of material is minimized when $x = 40$ cm and $y = 20$ cm.



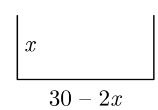
22. In this case, we want to minimize $A(x) = 2x^2 + 4xy = 2x^2 + 128,000/x$. Solving $A'(x) = 4x - 128,000/x^2$ we obtain the critical number $20\sqrt[3]{4}$. Since $A''(20\sqrt[3]{4}) > 0$, the amount of material is minimized when $x = y = 20\sqrt[3]{4}$ cm.

23. Let x be the side of the square cut-out. We want to maximize $V(x) = x(40-2x)^2$ on $[0, 20]$. Solving $V'(x) = 12x^2 - 320x + 1600 = 4(3x-20)(x-20) = 0$ we obtain the critical numbers $20/3$ and 20. Since $V(20) = 0$ and $V''(20) < 0$, the volume is maximized when the height is $20/3$ cm and the base is $80/3$ cm $\times$ $80/3$ cm. The maximum volume is $V(20/3) = 128,000/27$ cm^3.

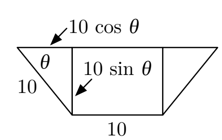


24. $V(x) = x(30-4x)(20-2x) = 600x - 140x^2 + 8x^3$. Solving $V'(x) = 600 - 280x + 24x^2 = 8(3x^2 - 35x + 75) = 0$ we obtain the critical number $(35 - 5\sqrt{13})/6 \approx 2.83$ and $(35 + 5\sqrt{13})/6 \approx 8.84$. Since $0 \le x \le 7.5$ and $V''(2.83) = -280 + 48(2.83) < 0$, we see that the volume is maximized when $x = (35 - 5\sqrt{13})/6 \approx 2.83$. The dimensions of this box are $20/3 + 10\sqrt{13}/3 \approx 18.69$ in long, $25/3 + 5\sqrt{13}/3 \approx 14.34$ in wide by $35/6 - 5\sqrt{13}/6 \approx 2.83$ in high. The maximum volume is the product of these dimensions or approximately 758.08 in^3.

25. Let x be the height of the gutter. We want to maximize $A(x) = x(30 - 2x)$ on $[0, 15]$. Solving $A'(x) = 30 - 4x = 0$ we obtain the critical number $15/2$. Since $A''(x) = -4 < 0$, the cross-sectional area and hence the volume, is maximized when the gutter is 7.5 cm high and 15 cm wide.



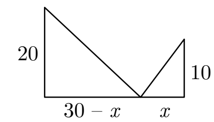
26. The total area of the two triangles is $100 \sin\theta \cos\theta$. We want to maximize $A(\theta) =$ area of triangles $+$ area of a rectangle $= 100 \sin\theta \cos\theta + 100 \sin\theta$ on $[0, \pi/2]$. Solving



$$\begin{aligned} A'(\theta) &= 100(\cos^2\theta - \sin^2\theta) + 100\cos\theta \\ &= 100(2\cos^2\theta - 1) + 100\cos\theta = 100(2\cos\theta - 1)(\cos\theta + 1) = 0 \end{aligned}$$

on $[0, \pi/2]$ we obtain the critical number $\pi/3$. Comparing $A(0) = 0$, $A(\pi/3) = 75\sqrt{3}$, and $A(\pi/2) = 100$, we see that the cross-sectional area and hence the volume, is maximized when $\theta = \pi/3$.

27. Let x be the distance from the 10-foot flagpole. We want to minimize $L(x) = \sqrt{400+(30-x)^2} + \sqrt{100+x^2}$ on $[0, 30]$. Setting

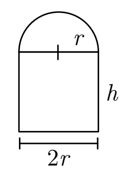


$$L'(x) = \frac{-(30-x)}{\sqrt{400+(30-x)^2}} + \frac{x}{\sqrt{100+x^2}} = 0$$

we obtain $x^2+20x-300 = (x+30)(x-10) = 0$. The positive critical number is 10. Comparing $L(0) = 10+10\sqrt{13}$, $L(10) = 30\sqrt{2}$, and $L(30) = 20+10\sqrt{10}$, we see that the length of wire is minimized when it is attached 10 feet from the 10-foot flagpole.

28. Let x be the radius of the semicircle and y the length of the rectangle. Then $2y + 2\pi x = 2$ and $y = 1 - \pi x$. We want to maximize $A(x) = 2xy = 2x(1-\pi x) = 2x - 2\pi x^2$. Solving $A'(x) = 2 - 4\pi x = 0$ we obtain the critical number $1/2\pi$. Since $A''(x) = -4\pi < 0$, the area is maximized when the semicircle has radius $1/2\pi \approx 0.1592$ km $= 159.2$ m and the length of the rectangle is $1 - \pi(1/2\pi) = 0.5$ km $= 500$ m.

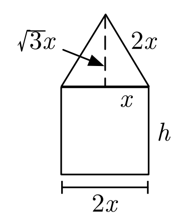
29. Let the radius of the semicircle be r and the height of the rectangle h. Then the perimeter is $2r + 2h + \pi r = 10$ and $h = \dfrac{1}{2}[10 - (2+\pi)r]$. We want to maximize



$$A(r) = 2rh + \frac{\pi r^2}{2} = 10r - \left(2 + \frac{\pi}{2}\right) r^2.$$

Solving $A'(r) = 10 - (4+\pi)r = 0$ we obtain the critical number $\dfrac{10}{4+\pi}$. Since $A''(r) = -4-\pi < 0$, the area is maximized when the base of the window is $\dfrac{20}{4+\pi}$ m, the height of the rectangular portion is $\dfrac{10}{4+\pi}$ m, and the radius of the circular portion is $\dfrac{10}{4+\pi}$ m.

30. Let the base of the window be $2x$ and the height of the rectangle h. Then the perimeter is $6x+2h = 10$ and $h = 5-3x$. We want to maximize $A(x) = 2xh + \sqrt{3}x^2 = 10x - 6x^2 + \sqrt{3}x^2$. Solving $A'(x) = 10 - (12-2\sqrt{3})x = 0$ we obtain the critical number $\dfrac{5}{6-\sqrt{3}}$. Since $A''(x) = -12+2\sqrt{3} < 0$, the area is maximized when the base of the window is $\dfrac{10}{6-\sqrt{3}} \approx 2.34$ m and the height of the rectangular portion is $5 - 3\left(\dfrac{5}{6-\sqrt{3}}\right) = \dfrac{15-5\sqrt{3}}{6-\sqrt{3}} \approx 1.49$ m.



31. By the Pythagorean Theorem, $L^2 = (x+5)^2 + y^2$. Using similar triangles, we have $\dfrac{y}{10} = \dfrac{x+5}{x}$ and $y = \dfrac{10(x+5)}{x}$. We want to minimize

$$L^2 = (x+5)^2 + \frac{100(x+5)^2}{x^2} = (x+5)^2\left(1 + \frac{100}{x^2}\right)$$

for $x > 0$. Setting $\dfrac{dL^2}{dx} = 0$ we obtain

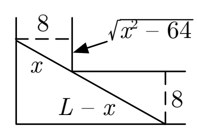
$$\begin{aligned}\frac{dL^2}{dx} &= (x+5)^2\left(-\frac{200}{x^3}\right) + \left(1+\frac{100}{x^2}\right)[2(x+5)] = 2(x+5)\left[1+\frac{100}{x^2} - (x+5)\frac{100}{x^3}\right] \\ &= 2(x+5)\left(1-\frac{500}{x^3}\right) = 2(x+5)\left(\frac{x^3-500}{x^3}\right) = 0\end{aligned}$$

so $x = \sqrt[3]{500}$. Using the first derivative test, $\dfrac{dL^2}{dx} < 0$ for $0 < x < \sqrt[3]{500}$ and $\dfrac{dL^2}{dx} > 0$ for $x > \sqrt[3]{500}$, we see that L^2 and hence L is minimized when $x = \sqrt[3]{500}$. In this case $L = (5+\sqrt[3]{500})\sqrt{1+100/500^{2/3}} \approx 20.81$ ft.

32. As seen in Figure 4.8.23, let the height and width of the box be x and the length y. Then $y + 4x = 108$, and we want to maximize $V(x) = x^2y = x^2(108-4x)$. Solving $V'(x) = 216x - 12x^2 = 0$ we obtain the critical numbers 0 and 18. Since $V''(18) = -216 < 0$, the volume is maximized when the width and height are 18 in, and the length is 36 in.

33. As seen in Figure 4.8.24, let r be the radius and h the height of the cylinder. Then, using similar triangles, we have $\dfrac{h}{12} = \dfrac{8-r}{8}$ or $h = 12 - \dfrac{12r}{8} = 12 - \dfrac{3r}{2}$. We want to maximize $V(r) = \pi r^2 h = 12\pi r^2 - \dfrac{3\pi r^3}{2}$ on $[0,8]$. Solving $V'(r) = 24\pi r - \dfrac{9\pi r^2}{2} = 0$ we obtain the critical numbers 0 and 16/3. Comparing $V(0)$, $V(16/3)$, and $V(8)$ we see that the volume is maximized when $r = 16/3$ and $h = 4$.

34. As seen in Figure 4.8.25, let L be the length of the line segment touching both outer walls and the inner corner. Let x be the length of the segment from one outer wall to the inner corner. Using similar triangles, we have



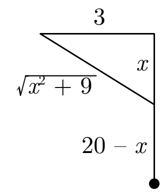
$$\frac{L-x}{8} = \frac{x}{\sqrt{x^2-64}}.$$

Solving for L we obtain $L(x) = x + \dfrac{8x}{\sqrt{8x^2-64}}$, which is to be minimized. Solving $L'(x) = 1 - \dfrac{512}{(x^2-64)^{3/2}} = 0$ we obtain the positive critical number $8\sqrt{2}$. Since $L''(8\sqrt{2}) > 0$, the length is minimized when $x = 8\sqrt{2}$ ft. Thus, the maximum length of a board that will fit around the corner is $L(8\sqrt{2}) = 16\sqrt{2}$ ft.

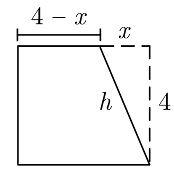
35. As seen in Figure 4.8.26, let r be the radius and h the height of the can. Then $\pi r^2 h = 32$ and $h = 32/\pi r^2$. We want to minimize $A(r) = 2\pi r^2 + 2\pi r h = 2\pi r^2 + 64/r$ on $(0,\infty)$. Solving $A'(r) = 4\pi r - 64/r^2 = 0$ we obtain the critical number $\sqrt[3]{16/\pi}$. Since $A''(r) > 0$ for $r > 0$, the surface area of the can is minimized when $r = \sqrt[3]{16/\pi}$ in and $h = 2\sqrt[3]{16/\pi}$ in.

36. From Problem 35, the volume of the can remains the same and so does the expression for h in terms of r. However, the area that we want to minimize is $A(r) = 2(4r^2) + 2\pi r h = 8r^2 + 64/r$ on $(0,\infty)$. Solving $A'(r) = 16r - 64/r^2 = 0$ we obtain the critical number $\sqrt[3]{4}$. Since $A''(r) > 0$ for $r > 0$, the metal used for the can (including waste) is minimized when $r = \sqrt[3]{4}$ in and $h = 8\sqrt[3]{4}/\pi$ in.

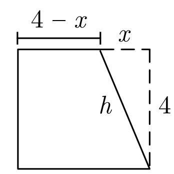
37. Let x be the distance on land across from the island to the point where the bird first intersects the shore. We want to minimize $T(x) = \dfrac{1}{6}\sqrt{x^2+9} + \dfrac{1}{10}(20-x)$ on $[0, 20]$. Solving $T'(x) = \dfrac{1}{6}x/\sqrt{x^2+9} - \dfrac{1}{10} = 0$ we obtain the critical number $9/4$. Since $T'(0) = -1/10 < 0$ and $T'(4) = 1/30 > 0$, the time is minimized when $x = 9/4$ km. Therefore, the bird should fly over water to a point on land $20 - 9/4 = 17.75$ km from the nest.



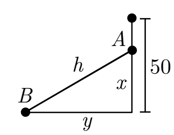
38. Let x be the distance along the opposite bank to which the pipeline is run. Let h be the length of pipe across the swamp. Then $h^2 = x^2 + 16$. We want to minimize $C(x) = 25h + 20(4-x) = 25\sqrt{x^2+16} + 20(4-x)$ on $[0, 4]$, where the cost is measured in units of \$1000. Solving $C'(x) = 25x/\sqrt{x^2+16} - 20 = 0$ we obtain the critical number $16/3$. Comparing $C(0) = 180$, $C(16/3) = 160$, and $C(4) = 100$, we see that the cost is minimized when $x = 4$ mi.



39. Let x be the distance along the opposite bank to which the pipeline is run. Let h be the length of pipe across the swamp. Then $h^2 = x^2+16$. We want to minimize $C(x) = 2h + 4 - x = 2\sqrt{x^2+16} + 4 - x$ on $[0, 4]$, where the cost is measured in units of \$1000. Solving $C'(x) = 2x/\sqrt{x^2+16} - 1 = 0$ we obtain the critical number $4/\sqrt{3}$. Comparing $C(0) = 12$, $C(4/\sqrt{3}) = 4\sqrt{3} + 4$, and $C(4) = 8\sqrt{2}$, we see that the cost is minimized when $x = 4\sqrt{3} + 4$ mi.



40. Let h be the distance between the ships at time t. Then $h^2 = x^2+y^2$ where $50 - x = 20t$ and $y = 10t$. We will maximize $h^2 = (50-20t)^2 + (10t)^2 = 500t^2 - 2000t + 2500$. Solving $dh^2/dt = 1000t - 2000 = 0$ we obtain the critical number 2. Since $d^2h^2/dt^2 = 1000 > 0$, the distance is minimized at 2 AM.



41. As seen in Figure 4.8.30, let r be the radius of the cylinder and h the length of the cylinder without the hemispherical ends. The volume of the container is $\pi r^2 h + \dfrac{4}{3}\pi r^3 = 30\pi$. Solving for h we obtain $h = \dfrac{30}{r^2} - \dfrac{4r}{3}$. We want to minimize

$$C(r) = 2\pi r h + \frac{3}{2}(4\pi r^2) = 2\pi r\left(\frac{30}{r^2} - \frac{4r}{3}\right) + 6\pi r^2 = \frac{60\pi}{r} - \frac{10\pi r^2}{3}.$$

Setting $C'(r) = 0$ we have

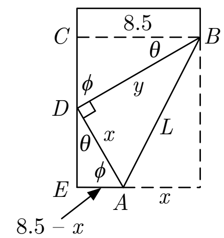
$$C'(r) = -\frac{60\pi}{r^2} + \frac{20\pi}{3}r = 0; \qquad \frac{-3}{r^2} + \frac{r}{3} = 0; \qquad \frac{r^3-9}{3r^2} = 0;$$

so $r = 9^{1/3}$. Since $C''(r) = 120\pi/r^3 + 20\pi/3$ and $C''(9^{1/3}) = 20\pi\left(\dfrac{6}{9} + \dfrac{1}{3}\right) > 0$, the cost is minimized when $r = 9^{1/3}$ and $h = 30/9^{2/3} - 4(9^{1/3})/3 = 2(9^{1/3})$.

42. Let x be the height and y the width of the printed portion. Then $xy = 32$ and $y = 32/x$. We want to minimize $A(x) = (x+2)(y+4) = 40 + 4x + 64/x$ on $(0, \infty)$. Solving $A'(x) =$

$4 - 64/x^2 = 0$ we obtain the critical number 4. Since $A''(4) > 0$, the area is minimized when $x = 4$ and $y = 8$. The page should then be 6 in high and 12 in wide.

43. Label the figure as shown. Note that $\triangle ADE$ and $\triangle DBC$ are similar and that $\triangle ABD$ is a right triangle. Using similarity, we have

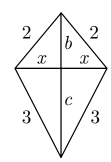


$$\frac{y}{8.5} = \frac{x}{\sqrt{x^2 - (8.5 - x)^2}} \qquad \text{or} \qquad y = \frac{\sqrt{8.5}x}{\sqrt{2x - 8.5}} = \frac{\sqrt{4.25}x}{\sqrt{x - 4.25}}.$$

We want to minimize $L^2 = x^2 + y^2 = x^2 + \dfrac{4.25x^2}{x - 4.25}$ on $(4, 8]$.

Solving $\dfrac{dL^2}{dx} = 2x + \dfrac{4.25x^2 - 36.125x}{(x - 4.25)^2} = \dfrac{2x^2(x - 6.375)}{(x - 4.25)^2} = 0$, we obtain the critical number 6.375. Since $\left.\dfrac{dL^2}{dx}\right|_{x=6} = -8.8163 < 0$ and $\left.\dfrac{dL^2}{dx}\right|_{x=7} = 8.0992 > 0$, we see that the length is minimized when the width of the fold is 6.375 inches.

44. We see from the figure that $x^2 + b^2 = 4$ and $x^2 + c^2 = 9$. The area of the kite is $A = xb + xc$. We want to maximize $A(x) = x(\sqrt{4 - x^2} + \sqrt{9 - x^2})$. Setting



$$\begin{aligned}
A'(x) &= x\left(-\frac{x}{\sqrt{4 - x^2}} - \frac{x}{\sqrt{9 - x^2}}\right) + \sqrt{4 - x^2} + \sqrt{9 - x^2} \\
&= \frac{-x^2(\sqrt{9 - x^2} + \sqrt{4 - x^2}) + (4 - x^2)\sqrt{9 - x^2} + (9 - x^2)\sqrt{4 - x^2}}{\sqrt{4 - x^2}\sqrt{9 - x^2}} \\
&= \frac{(4 - 2x^2)\sqrt{9 - x^2} + (9 - 2x^2)\sqrt{4 - x^2}}{\sqrt{4 - x^2}\sqrt{9 - x^2}} = 0
\end{aligned}$$

we obtain

$$\begin{aligned}
(4 - 2x^2)\sqrt{9 - x^2} &= -(9 - 2x^2)\sqrt{4 - x^2} \\
(16 - 16x^2 + 4x^4)(9 - x^2) &= (81 - 36x^2 + 4x^4)(4 - x^2) \\
144 - 160x^2 + 52x^4 - 4x^6 &= 324 - 225x^2 + 52x^4 - 4x^6 \\
65x^2 - 180 &= 0 \\
5(13x^2 - 36) &= 0.
\end{aligned}$$

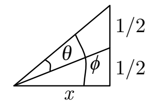
Thus, $x = 6/\sqrt{13}$, $b = \sqrt{4 - 36/13} = 4/\sqrt{13}$, and $c = \sqrt{9 - 36/13} = 9/\sqrt{13}$. The crossbars have lengths $2x = 12/\sqrt{13} \approx 3.33$ ft and $b + c = \sqrt{13} \approx 3.61$ ft.

45. We want to maximize

$$\begin{aligned}
A(\theta) &= (a\sin\theta + b\cos\theta)(a\cos\theta + b\sin\theta) \\
&= (a^2 + b^2)\sin\theta\cos\theta + ab = \frac{1}{2}(a^2 + b^2)\sin 2\theta + ab
\end{aligned}$$

on $(0, \pi/2)$. Solving $A'(\theta) = (a^2 + b^2)\cos 2\theta = 0$ we obtain the critical number $\pi/4$. Since $A''(\pi/4) < 0$, the area is maximized when $\theta = \pi/4$. The circumscribed rectangle is a square whose side is $a\sin\pi/4 + b\cos\pi/4 = (a + b)/\sqrt{2}$.

46. As seen in Figure 4.8.35, let x be the distance from the person to the pedestal. Then $\tan\phi = 1/x$ and $\tan(\phi-\theta) = 1/2x$. Since $\tan(\phi-\theta) = \dfrac{\tan\phi - \tan\theta}{1+\tan\phi\tan\theta}$, we have



$$\frac{1}{2x} = \frac{1/x - \tan\theta}{1+\dfrac{1}{x}\tan\theta} = \frac{1-x\tan\theta}{x+\tan\theta}.$$

Solving for $\tan\theta$ we obtain $\tan\theta = \dfrac{x}{2x^2+1}$. To maximize θ, it will suffice to maximize $\tan\theta$ since $\tan\theta$ is an increasing function on $[0,\pi/2)$. Solving $\dfrac{d}{dx}\tan\theta = \dfrac{1-2x^2}{(2x^2+1)^2} = 0$, we obtain the critical number $\sqrt{2}/2$. Since $\dfrac{d^2}{dx^2}\tan\theta\Big|_{x=\sqrt{2}/2} = -\sqrt{2}/2 < 0$, the angle is maximized when $x = \sqrt{2}/2$ m.

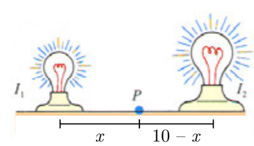
47. As seen in Figure 4.8.36, let x be the width of the beam, y the height, and d the diameter of the log and diagonal of the wooden beam (shown by the dotted line). From the Pythagorean Theorem, we have $d^2 = x^2 + y^2$. We want to maximize $S(x) = xy^2 = x(d^2 - x^2)$. Solving $S'(x) = d^2 - 3x^2 = 0$ we obtain the critical number $d/\sqrt{3}$. Since $S''(d/\sqrt{3}) = -6d/\sqrt{3} < 0$, the strength is maximized when the length is $d/\sqrt{3}$ and the width is $d\sqrt{2/3}$.

48. From Problem 38 in Part C of Chapter 1 in Review we have $S(\theta) = 25\pi\csc\theta - \dfrac{50}{3}\pi\cot\theta + 40$. Setting

$$S'(\theta) = -25\pi\csc\theta\cot\theta + \frac{50}{3}\pi\csc^2\theta = 25\pi\csc\theta\left(\frac{2}{3}\csc\theta - \cot\theta\right) = 0$$

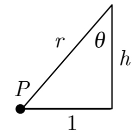
we obtain $\dfrac{2}{3}\csc\theta = \cot\theta$ or $\cos\theta = \dfrac{2}{3}$ and $\theta \approx 0.84$ radian $\approx 48.19°$. This critical number can be seen to be a relative minimum by the first derivative test. The minimum surface area is $S(0.84) \approx 98.54$ ft^2.

49. Let x be the distance from P to the bulb with intensity I_1. We want to minimize $E(x) = \dfrac{125}{x^2} + \dfrac{216}{(10-x)^2}$ on $(0,10)$. Setting $E'(x) = -\dfrac{250}{x^3} + \dfrac{432}{(10-x)^3} = 0$, we have $\dfrac{(10-x)^3}{x^3} = \dfrac{432}{250} = \dfrac{216}{125}$ or $\dfrac{10-x}{x} = \dfrac{6}{5}$.



Thus, $50/11$ is a critical number, and by the first derivative test, we see that the total illuminance will be a minimum at $50/11$ m from the bulb with intensity $I_1 = 125$.

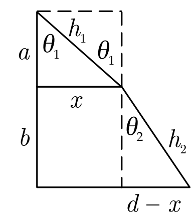
50. Letting r be the distance from the light to P, we have $r = \csc\theta$. We want to maximize $E(\theta) = 100\cos\theta/\csc^2\theta = 100\sin^2\theta\cos\theta = 100\cos\theta - 100\cos^3\theta$ for $\theta > 0$. Solving $E'(\theta) = -100\sin\theta + 300\cos^2\theta\sin\theta = (100\sin\theta)(3\cos^2\theta - 1) = 0$ we obtain a positive critical number θ_c when $\cos\theta_c = 1/\sqrt{3}$. To apply the first derivative test, we note that $100\sin\theta > 0$ for any θ near θ_c. Now for $\theta < \theta_c$, $\cos\theta > \cos\theta_c$ and $3\cos^2\theta - 1 > 3\cos^2\theta_c - 1 = 0$, so $E'(\theta) > 0$.



Similarly, for $\theta > \theta_c$ it can be shown that $E'(\theta) < 0$. Therefore E has a relative maximum at θ_c. The corresponding height is

$$h = r\cos\theta_c = \csc\theta_c\cos\theta_c = \frac{\cos\theta_c}{\sin\theta_c} = \frac{1/\sqrt{3}}{\sqrt{1-(1/\sqrt{3})^2}} = 1/\sqrt{2}\text{ m.}$$

51. Let x be the point on the x-axis where the light crosses from one medium to the other. We want to minimize



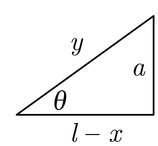
$$T(x) = \frac{h_1}{c_1} + \frac{h_2}{c_2} = \frac{\sqrt{x^2+a^2}}{c_1} + \frac{\sqrt{(d-x)^2+b^2}}{c_2}.$$

To find the critical numbers, we set

$$T'(x) = \frac{x}{c_1\sqrt{x^2+a^2}} - \frac{d-x}{c_2\sqrt{(d-x)^2+b^2}} = 0.$$

Then $\dfrac{x/\sqrt{x^2+a^2}}{c_1} = \dfrac{(d-x)\sqrt{(d-x)^2+b^2}}{c_2}$, and since $\sin\theta_1 = \dfrac{x}{h_1}$ and $\sin\theta_2 = \dfrac{d-x}{h_2}$, we have $\sin\dfrac{\theta_1}{c_1} = \sin\dfrac{\theta_2}{c_2}$ at the critical number. To see that the time is actually minimized at this point, we compute the second derivative $T''(x) = \dfrac{a^2}{c_1(x^2+a^2)^{3/2}} + \dfrac{b^2}{c^2[(d-x)^2+b^2]^{3/2}}$. Since $T''(x) > 0$ for all x, we do have a minimum at the critical number.

52. From the figure we see that $y = a\csc\theta$ and $l - x = y\cos\theta = a\csc\theta\cos\theta = a\cot\theta$. Thus $x = l - a\cot\theta$. We want to minimize



$$R(\theta) = \frac{kx}{r_1^4} + \frac{ky}{r_2^4} = \frac{k}{r_1^4}(l - a\cot\theta) + \frac{k}{r_2^4}(a\csc\theta).$$

Setting $R'(\theta) = \dfrac{ak}{r_1^4}(\csc^2\theta) - \dfrac{ak}{r_2^4}(\csc\theta\cot\theta) = 0$ we obtain $\dfrac{\csc\theta}{r_1^4} = \dfrac{\cot\theta}{r_2^4}$ or $\dfrac{r_2^4}{r_1^4} = \cos\theta$. To see that this value of θ minimizes r, we compute

$$\begin{aligned}
R''(\theta) &= ak\left(-\frac{2\csc^2\theta\cot\theta}{r_1^4} + \frac{\csc^3\theta + \csc\theta\cot^2\theta}{r_2^4}\right)\\
&= ak\csc\theta\left(\frac{\csc^2\theta + \cot^2\theta}{r_2^4} - \frac{2\csc\theta\cot\theta}{r_1^4}\right)\\
&= \frac{ak\csc\theta}{r_2^4}\left(\csc^2\theta + \cot^2\theta - \frac{2r_2^4}{r_1^4}\csc\theta\cot\theta\right)\\
&= \frac{ak\csc\theta}{r_2^4}(\csc^2\theta + \cot^2\theta - 2\cos\theta\csc\theta\cot\theta)\\
&= \frac{ak\csc\theta}{r_2^4}(\csc^2\theta + \cot^2\theta - 2\cot^2\theta) = \frac{ak\csc\theta}{r_2^4}(\csc^2\theta - \cot^2\theta) = \frac{ak\csc\theta}{r_2^4}.
\end{aligned}$$

Thus, $R''(\theta) > 0$ for $0 < \theta < \pi$ and the resistance is minimum when $\cos\theta = r_2^4/r_1^4$.

53. $U'(x) = -24/x^{13} + 6/x^7 = 6(x^6 - 4)/x^{13}$. Solving $U'(x) = 0$ we obtain the critical numbers $\pm\sqrt[3]{2}$. From the first derivative test, we see that relative minima occur at both of these points. Thus, the minimum potential energy is $U(\sqrt[3]{2}) = -1/8$.

x		$-\sqrt[3]{2}$		0		$\sqrt[3]{2}$	
f	$\searrow$		$\nearrow$		$\searrow$		$\nearrow$
f'	$-$	0	$+$	undefined	$-$	0	$+$

54. $y' = \tan\theta_0 - \left(\dfrac{g}{v_0^2\cos^2\theta_0}\right)x$. Solving $y' = 0$ we obtain the critical number $\dfrac{v_0^2\tan\theta_0\cos^2\theta_0}{g} = \dfrac{v_0^2\sin\theta_0\cos\theta_0}{g}$. Since $y'' = -\dfrac{g}{v_0^2\cos^2\theta_0} < 0$, the height is maximum when $x = \dfrac{v_0^2\sin\theta_0\cos\theta_0}{g}$. The maximum height is

$$\begin{aligned} h &= \tan\theta_0\frac{v_0^2\sin\theta_0\cos\theta_0}{g} - \frac{g}{2v_0^2\cos^2\theta_0}\left(\frac{v_0^2\sin\theta_0\cos\theta_0}{g}\right)^2 \\ &= \frac{v_0^2\sin^2\theta_0}{g} - \frac{v_0^2\sin^2\theta_0}{2g} = \frac{v_0^2\sin^2\theta_0}{2g}. \end{aligned}$$

55. (a) We want to maximize

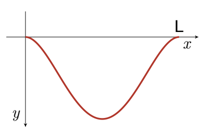
$$y(x) = \frac{w_0}{24EI}(L^2x^2 - 2Lx^3 + x^4) = \frac{w_0}{24EI}x^2(x-L)^2.$$

Now

$$y'(x) = \frac{w_0}{24EI}\left[2x^2(x-L) + 2x(x-L)^2\right] = \frac{w_0}{12EI}x(x-L)(2x-L).$$

Solving $y'(x) = 0$ we obtain the critical numbers 0, $L/2$, and L. Using the first derivative test we see that $y(L/2) = w_0L^4/384EI$.

(b)



56. (a) The tree attains its maximum height H when $h'(d) = a - 2bd = 0$ or $d = a/2b$. Thus $D = a/2b$ and the maximum height is $H = h(D) = 137 + aD - bD^2$. Substituting $a = 2bD$ into the above equation, we obtain $H = 137 + 2bD^2 - bD^2 = 137 + bD^2$. Then $b = \dfrac{H - 137}{D^2}$ and $a = \dfrac{2D(H-137)}{D^2} = \dfrac{2(H-137)}{D}$. Thus

$$h(d) = 137 + 2\left(\frac{H-137}{D}\right)d - \frac{H-137}{D^2}d^2.$$

(b) Letting $H = 1500$, $D = 800$, and $h = 1000$ we obtain

$$1000 = 137 + 2\left(\frac{1500-137}{80}\right)d - \frac{1500-137}{6400}d^2 = 137 + \frac{1363}{40}d - \frac{1363}{6400}d^2$$

or

$$\frac{1363}{6400}d^2 - \frac{1363}{40}d + 863 \approx 0.213d^2 - 34.075d + 863 = 0.$$

Using the quadratic formula, we find $d \approx 31.55$ cm and $d \approx 128.45$ cm. Assuming $d < 0.8$ m when $h = 10$ m we conclude that the diameter of the tree was approximately 31.55 cm.

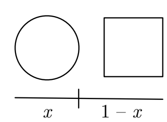
57. We want to minimize $m(x) = \dfrac{\pi\rho M^{2/3}}{2K^{2/3}}\left[\dfrac{2-x^2}{(1-x^4)^{2/3}}\right]$. Solving

$$m'(x) = -\frac{\pi\rho M^{2/3}x}{3K^{2/3}}\left[\frac{x^4-8x^2+3}{(1-x^4)^{5/3}}\right] = 0$$

we obtain the critical numbers 0, $\sqrt{4-\sqrt{13}}$, and $\sqrt{4+\sqrt{13}}$. Since $x = r/R$ where $0 < r < R$, we must have x in the interval $(0, 1)$. Thus, the only appropriate critical number is $x = \sqrt{4-\sqrt{13}} \approx 0.63$. Use the first derivative test with $x = 0.6$ and 0.7 to show that the mass m is minimized for $x = r/R \approx 0.63$ or $r \approx 0.63R$.

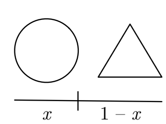
58. We want to maximize $P(I) = \dfrac{100I}{I^2+I+4}$ on $[0, \infty)$. Solving $P'(I) = \dfrac{100(4-I^2)}{(I^2+I+4)^2} = 0$ we obtain the critical number 2. Since $P'(1) > 0$ and $P'(3) < 0$, we see by the first derivative test that P is largest for $I = 2$.

59. Let x be the length of wire formed into a circle and $1-x$ the length formed into a square. Since x is the circumference of the circle, its radius is $x/2\pi$. We want to maximize $A(x) = \pi\left(\dfrac{x}{2\pi}\right)^2 + \left(\dfrac{1-x}{4}\right)^2 = \dfrac{x^2}{4\pi} + \dfrac{(1-x)^2}{16}$ on $[0, 1]$.



Solving $A'(x) = \dfrac{x}{2\pi} - \dfrac{1-x}{8} = 0$ we obtain the critical number $\dfrac{\pi}{4+\pi}$. Comparing $A(0) = 1/6$, $A\left(\dfrac{\pi}{4+\pi}\right) = \dfrac{1}{4(4+\pi)}$, and $A(1) = 1/4\pi$, we see that the maximum area is obtained when the entire wire is formed into a circle of radius $1/2\pi$ m.

60. Let x be the length of wire formed into a circle and $1-x$ the length formed into a triangle. Since x is the circumference of the circle, its radius is $x/2\pi$. The perimeter of the triangle is $1-x$, so each side is $\dfrac{1-x}{3}$ and the area is $\left(\dfrac{1-x}{6}\right)^2\sqrt{3}$. We want to minimize and maximize



$$A(x) = \pi\left(\frac{x}{2\pi}\right)^2 + \left(\frac{1-x}{6}\right)^2\sqrt{3} = \frac{x^2}{4\pi} + \left(\frac{1-x}{6}\right)^2\sqrt{3}$$

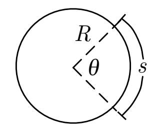
on $[0, 1]$. Solving $A'(x) = \dfrac{x}{2\pi} - \left(\dfrac{1-x}{18}\right)\sqrt{3} = 0$ we obtain the critical number $\dfrac{\pi\sqrt{3}}{9+\pi\sqrt{3}}$. Comparing $A(0) \approx 0.0481$, $A\left(\dfrac{\pi\sqrt{3}}{9+\pi\sqrt{3}}\right) \approx 0.0299$, and $A(1) \approx 0.0796$, we see that the sum of the areas is minimized when $x = \dfrac{\pi\sqrt{3}}{9+\pi\sqrt{3}}$ and maximized when $x = 1$, or when the entire wire is formed into a circle.

61. (a) By the Pythagorean Theorem, $R^2 = r^2 + h^2$ or $r^2 = R^2 - h^2$. We want to maximize $V(h) = \dfrac{1}{3}\pi r^2 h = \dfrac{1}{3}\pi(R^2 - h^2)h$ on $[0, R]$. Solving $V'(h) = \dfrac{1}{3}\pi\left(R^2 - \dfrac{1}{3}h^2\right) = 0$ we obtain the critical numbers $h = \pm R/\sqrt{3}$. Since $V(0) = V(R) = 0$ and $V''(R/\sqrt{3}) = -\dfrac{2}{9}R/\sqrt{3} < 0$, the volume is maximized when $h = r/\sqrt{3}$ or

$$r = \sqrt{R^2 - h^2} = \sqrt{R^2 - R^2/3} = \sqrt{2/3}R.$$

(b) The maximum volume is $V = \dfrac{1}{3}\pi\left(\dfrac{2}{3}R^2\right)R/\sqrt{3} = \dfrac{2}{9}\pi R^3/\sqrt{3}$.

(c) The circumference of the circular piece of paper is $2\pi R$. The circumference of the base of the cone is $2\pi r = 2\pi\sqrt{2/3}R$. Thus, $s = 2\pi R - 2\pi r = 2\pi R(1 - \sqrt{2/3})$ and $\theta = s/R = 2\pi(1 - \sqrt{2/3}) \approx 1.15$ radians.

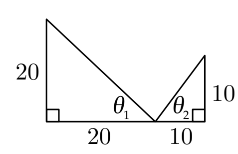


62. From Figure 4.8.45, by the Pythagorean theorem $L^2 = x^2 + y^2$, so $y^2 = L^2 - x^2$, where L is a fixed number. The circumference of the base of the cylinder is $y = 2\pi r$, so $r = y/2\pi$. Thus, we want to maximize the volume of the cylinder

$$V(x) = \pi r^2 x = \pi\left(\frac{y}{2\pi}\right)^2 x = \frac{xy^2}{4\pi} = \frac{1}{4\pi}x(L^2 - x^2)$$

on $[0, L]$. Solving $V'(x) = \dfrac{L^2 - 3x^2}{4\pi} = 0$ we obtain the critical number $L/\sqrt{3}$. Since $V''(x) = -6x/4\pi < 0$ for $x > 0$, the volume is maximized when the cylinder has a height of $L/\sqrt{3}$ and a radius of $\dfrac{\sqrt{2/3}L}{2\pi}$.

63. Problem 27 showed that the optimal amount of wire (the least amount) is used when it is attached 10 feet from the 10-foot flagpole. From the figure, this means that the right triangles formed by the flagpoles, the wire, and the ground are isosceles right triangles, and therefore similar. Thus, the non-right angles θ_1 and θ_2 of each triangle are the same.



64. Using the unknown coordinates of P, the slope of the tangent line is $m = -2x_0$. The equation of the tangent line at P is then $y - (1 - x_0^2) = -2x_0(x - x_0)$. The x-intercept is

then $(\dfrac{x_0^2+1}{2x_0}, 0)$; the y-intercept is $(0, 1+x_0^2)$. The area of the triangle as a function of x_0 is $A(x_0) = \dfrac{1}{4}\left[\dfrac{(x_0^2+1)^2}{x_0}\right]$. Solving $A'(x_0) = \dfrac{1}{4}\left[\dfrac{(x_0^2+1)(3x_0^2-1)}{x_0^2}\right] = 0$ we obtain the critical number $1/\sqrt{3}$. The first or second derivative test shows that there is a relative minimum of the area function at this number. Now, using $y = 1 - x^2$ we find the coordinates of P are $(1/\sqrt{3}, 2/3)$. The slope at the point is $m = -2/\sqrt{3}$. An equation of the tangent line is

$$y - \frac{2}{3} = -\frac{2}{\sqrt{3}}\left(x - \frac{1}{\sqrt{3}}\right) \qquad \text{or} \qquad y = -\frac{2}{\sqrt{3}}x + \frac{4}{3}.$$

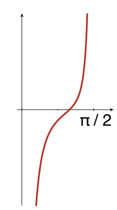
65. The total time it takes the swimmer to reach C from A is $T = \dfrac{\sqrt{x^2+1}}{3} + \dfrac{\sqrt{x^2-8x+17}}{2}$ on $[0, 4]$. Differentiating then solving for 0 gives

$$\frac{dT}{dx} = \frac{x}{3\sqrt{x^2+1}} + \frac{x-4}{2\sqrt{x^2-8x+17}} = 0$$
$$2x\sqrt{x^2-8x+17} = -3(x-4)\sqrt{x^2+1}$$

With CAS help, the foregoing equation has only one real root in the interval $[0, 4]$, namely $x \approx 3.176$. Now, $T(0) \approx 2.395$, $T(3.176) \approx 1.758$, and $T(4) \approx 1.874$. Therefore, to minimize her time in the race, she should swim from point A to point B about 3.18 miles down the beach from the point of the beach closest to A, then proceed directly to C.

66. (a) $L(\theta) = 10\csc\theta + 2\sec\theta$

(b) $L'(\theta) = -10\csc\theta\cot\theta + 2\sec\theta\tan\theta$

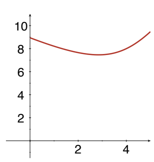


(c) We see from the graph that $L'(\theta) < 0$ for $0 < \theta < \theta_c$ and $L'(\theta) > 0$ for $\theta_c < \theta < L/2$. By the first derivative test, $L(\theta)$ has a relative minimum at $\theta = \theta_c$.

(d) With CAS help to numerically solve $L'(\theta) = 0$ we find $\theta_c \approx 1.04$. (Actually, it is easily seen that $\theta_c = \tan^{-1}\sqrt[3]{5}$.) Then $L(\theta_c) \approx 15.55$.

67. (a) Let x be the length of the cable AB. Then $L(x) = x + 2\sqrt{(4-x)^2+4}$.

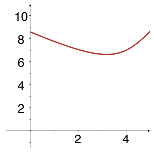
(b)



(c) Solving $L'(x) = 1 + -2(4-x)/\sqrt{(4-x)^2+4} = 0$ we obtain the critical numbers $4 \pm \dfrac{2}{3}\sqrt{3}$. Since $4 + \dfrac{2}{3}\sqrt{3} > 4$, we use $x = 4 - \dfrac{2}{3}\sqrt{3}$. We know from the graph that this gives a minimum.

(d) Let x be the length of the cable AB. Then $L(x) = x + \sqrt{1+(4-x)^2} + \sqrt{4+(4-x)^2}$.

(e)



(f) From the graph in part (e) we estimate that $L(x)$ is minimized when $x \approx 3.2$. (Using a numerical procedure to solve $L'(x) = 0$ gives $x \approx 3.1955$.)

68. (a) From the figure, the distance $\bar{B}$ between the transmitter at (x_t, y_t) and the point (x, y) is $\sqrt{(x - x_t)^2 + (y - y_t)^2}$, and the distance $\bar{A}$ between the second transmitter at $(x_i, 0)$ and (x, y) is $\sqrt{(x - x_i)^2 + y^2}$. Writing y in terms of x, we have $y = \sqrt{r^2 - x^2}$, and so the distances are:

$$\begin{aligned}
\bar{B} &= \sqrt{(x - x_t)^2 + \left(\sqrt{r^2 - x^2} - y_t\right)^2} \\
&= \sqrt{x^2 - 2x_t x + x_t^2 + (r^2 - x^2) - 2y_t\sqrt{r^2 - x^2} + y_t^2} \\
&= \sqrt{r^2 - 2x_t x + x_t^2 - 2y_t\sqrt{r^2 - x^2} + y_t^2} \\
\bar{A} &= \sqrt{(x - x_i)^2 + (r^2 - x^2)} = \sqrt{x^2 - 2x_i x + x_i^2 + r^2 - x^2} = \sqrt{r^2 - 2x_i x + x_i^2}
\end{aligned}$$

Let $S_p(x)$ and $S_s(x)$ be the primary and secondary signal strengths at (x, y), respectively. These functions are therefore

$$\begin{aligned}
S_p(x) &= \frac{1}{r^2 - 2x_t x + x_t^2 - 2y_t\sqrt{r^2 - x^2} + y_t^2} \\
S_s(x) &= \frac{1}{r^2 - 2x_i x + x_i^2}
\end{aligned}$$

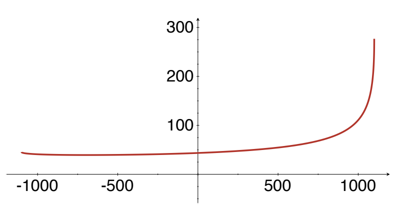
and the signal to noise ratio $R(x)$ is

$$R(x) = \frac{S_p(x)}{S_s(x)} = \frac{r^2 - 2x_i x + x_i^2}{r^2 - 2x_t x + x_t^2 - 2y_t\sqrt{r^2 - x^2} + y_t^2}.$$

(b) Converting to meters, we have $x_t = 760$, $y_t = -560$, $r = 1100$, and $x_i = 12000$. Substituting, we have

$$R(x) = \frac{1100^2 - 2(12000)x + 12000^2}{1100^2 - 2(760)x + 760^2 - 2(-560)\sqrt{1100^2 - x^2} + (-560)^2}$$

and from the resulting CAS-generated graph below, the domain of $R(x)$ appears to be $[-1100, 1100]$, with a range of approximately $(40, 275)$.



(c) Still using the graph, the minimum ratio R appears to occur at $x \approx -700$. This yields $R \approx 39.3653$, which is well above the FAA's minimum threshold.

(d) Using a CAS to differentiate $R(x)$ and finding the root of $R'(x) = 0$, we get $x = -\dfrac{220(30409\sqrt{1740266645} - 153341760)}{353629009} \approx -693.799$, $R(x) \approx 39.3649$, which compare favorably with our estimates from the graph.

(e) Using $x = -693.799$, we get $y = \sqrt{1100^2 - 693.799^2} \approx 853.60585$ and the point on C is approximately $(-693.799, 853.60585)$.

(f) The assumption that (x, y) is in the top half plane when (x_t, y_t) was in the lower half plane is correct because the points on C in the top half plane are farther from (x_t, y_t) than in the lower half plane, with the range of distances from $(x_i, 0)$ remaining the same, thus resulting in lower signal to noise ratios.

(g) According to *Mathematica*, there are two possible expressions for x that result in the minimum interference. The full expressions are too long to reproduce in a practical manner, but suffice it to say that they certainly justify the existence of CAS for functions such as these, especially in terms of multiple symbolic constants.

(h) When $y_t = 0$, we get $R(x) = \dfrac{r^2 - 2x_i x + x_i^2}{r^2 - 2x_t x + x_t^2}$. Since x must be on the circle C, then we seek an extremum on the closed interval $[-r, r]$. By Theorem 4.3.3, candidate extrema occur at $x = -r$, $x = r$, or at the critical number $x = \dfrac{r^2 + x_t^2}{2x_t}$, if this value is in $[-r, r]$.

4.9 Linearization and Differentials

1. Using $f(9) = 3$, $f'(x) = 1/2\sqrt{x}$, and $f'(9) = 1/6$, the tangent line to the graph is $y - 3 = (1/6)(x - 9)$. The linearization of $f(x)$ at $a = 9$ is $L(x) = \dfrac{1}{6}(x - 9) + 3$.

2. Using $f(1) = 1$, $f'(x) = -2x^{-3}$, and $f'(1) = -2$, the tangent line to the graph is $y - 1 = -2(x - 1)$. The linearization of $f(x)$ at $a = 1$ is $L(x) = -2x + 3$.

3. Using $f(\pi/4) = 1$, $f'(x) = \sec^2 x$, and $f'(\pi/4) = 2$, the tangent line to the graph is $y - 1 = 2(x - \pi/4)$. The linearization of $f(x)$ at $a = \pi/4$ is $L(x) = 2(x - \pi/4) + 1$.

4. Using $f(\pi/2) = 0$, $f'(x) = -\sin x$, and $f'(\pi/2) = -1$, the tangent line to the graph is $y - 0 = -1(x - \pi/2)$. The linearization of $f(x)$ at $a = \pi/2$ is $L(x) = \pi/2 - x$.

5. Using $f(1) = 0$, $f'(x) = 1/x$, and $f'(1) = 1$, the tangent line to the graph is $y - 0 = 1(x - 1)$. The linearization of $f(x)$ at $a = 1$ is $L(x) = x - 1$.

6. Using $f(2) = 11$, $f'(x) = 5 + e^{x-2}$, and $f'(2) = 6$, the tangent line to the graph is $y - 11 = 6(x - 2)$. The linearization of $f(x)$ at $a = 2$ is $L(x) = 6x - 1$.

7. Using $f(3) = 2$, $f'(x) = 1/2\sqrt{1 + x}$, and $f'(3) = 1/4$, the tangent line to the graph is $y - 2 = (1/4)(x - 3)$. The linearization of $f(x)$ at $a = 3$ is $L(x) = \dfrac{1}{4}(x - 3) + 2$.

8. Using $f(6) = 1/3$, $f'(x) = -\dfrac{1}{2(3+x)^{3/2}}$, and $f'(6) = -1/54$, the tangent line to the graph is $y - 1/3 = (-1/54)(x-6)$. The linearization of $f(x)$ at $a = 6$ is $L(x) = -\dfrac{1}{54}(x-6) + \dfrac{1}{3}$.

9. Using $f(0) = 1$, $f'(x) = e^x$, and $f'(0) = 1$, the tangent line to the graph is $y - 1 = 1(x-0)$, yielding $L(x) = x + 1$ or $e^x \approx 1 + x$ whenever x is close to 0.

10. Using $f(0) = 0$, $f'(x) = \sec^2 x$, and $f'(0) = 1$, the tangent line to the graph is $y - 0 = 1(x-0)$, yielding $L(x) = x$ or $\tan x \approx x$ whenever x is close to 0.

11. Using $f(0) = 1$, $f'(x) = 10(1+x)^9$, and $f'(0) = 10$, the tangent line to the graph is $y - 1 = 10(x-0)$, yielding $L(x) = 10x + 1$ or $(1+x)^{10} \approx 1 + 10x$ whenever x is close to 0.

12. Using $f(0) = 1$, $f'(x) = -6(1+2x)^{-4}$, and $f'(0) = -6$, the tangent line to the graph is $y - 1 = -6(x-0)$, yielding $L(x) = 1 - 6x$ or $(1+2x)^{-3} \approx 1 - 6x$ whenever x is close to 0.

13. Using $f(0) = 1$, $f'(x) = -1/2\sqrt{1-x}$, and $f'(0) = -1/2$, the tangent line to the graph is $y - 1 = -\dfrac{1}{2}(x-0)$, yielding $L(x) = 1 - \dfrac{1}{2}x$ or $\sqrt{1-x} \approx 1 - \dfrac{1}{2}x$ whenever x is close to 0.

14. Using $f(0) = 2$, $f'(x) = \dfrac{2x+1}{2\sqrt{x^2+x+4}}$, and $f'(0) = 1/4$, the tangent line to the graph is $y - 2 = \dfrac{1}{4}(x-0)$, yielding $L(x) = \dfrac{1}{4}x + 2$ or $\sqrt{x^2+x+4} \approx 2 + \dfrac{1}{4}x$ whenever x is close to 0.

15. Using $f(0) = 1/3$, $f'(x) = -\dfrac{1}{(3+x)^2}$, and $f'(0) = -1/9$, the tangent line to the graph is $y - 1/3 = -\dfrac{1}{9}(x-0)$, yielding $L(x) = -\dfrac{1}{9}x + \dfrac{1}{3}$ or $\dfrac{1}{3+x} \approx \dfrac{1}{3} - \dfrac{1}{9}x$ whenever x is close to 0.

16. Using $f(0) = 1$, $f'(x) = -\dfrac{4}{3(1-4x)^{2/3}}$, and $f'(0) = -4/3$, the tangent line to the graph is $y - 1 = -\dfrac{4}{3}(x-0)$, yielding $L(x) = 1 - \dfrac{4}{3}x$ or $\sqrt[3]{1-4x} \approx 1 - \dfrac{4}{3}x$ whenever x is close to 0.

17. From Problem 2 we have $\dfrac{1}{x^2} \approx -2x + 3$ whenever x is close to 1. Thus,

$$(1.01)^{-2} = f(1.01) \approx -2(1.01) + 3 = 0.98.$$

18. From Problem 1 we have $\sqrt{x} \approx \dfrac{1}{6}(x-9) + 3$ whenever x is close to 9. Thus,

$$\sqrt{9.05} = f(9.05) \approx \frac{1}{6}(9.05 - 9) + 3 = \frac{361}{120}.$$

19. From Problem 6 we have $5x + e^{x-2} \approx 6x - 1$ whenever x is close to 2. Thus,

$$10.5 + e^{0.1} = f(2.1) \approx 6(2.1) - 1 = 11.6.$$

20. From Problem 5 we have $\ln x \approx x - 1$ whenever x is close to 1. Thus,
$$\ln 0.98 = f(0.98) \approx 0.98 - 1 = -0.02.$$

21. From Problem 12 we have $(1 + 2x)^{-3} \approx 1 - 6x$ whenever x is close to 0. Thus,
$$(1.1)^{-3} = f(0.05) \approx 1 - 6(0.05) = 0.7.$$

22. From Problem 11 we have $(1 + x)^{10} \approx 1 + 10x$ whenever x is close to 0. Thus,
$$(1.02)^{10} = f(0.02) \approx 1 + 10(0.02) = 1.2.$$

23. From Problem 16 we have $\sqrt[3]{1 - 4x} \approx 1 - \frac{4}{3}x$ whenever x is close to 0. Thus,
$$(0.88)^{1/3} = f(0.03) \approx 1 - \frac{4}{3}(0.03) = 0.96.$$

24. From Problem 14 we have $\sqrt{x^2 + x + 4} \approx 2 + \frac{1}{4}x$ whenever x is close to 0. Thus,
$$\sqrt{4.11} = f(0.1) \approx 2 + \frac{1}{4}(0.1) = 2.025.$$

25. To find an approximation for $(1.8)^5$ we choose $f(x) = x^5$; $a = 2$. Using $f(2) = 32$, $f'(x) = 5x^4$, and $f'(2) = 80$, the tangent line to the graph is $y - 32 = 80(x - 2)$. The linearization of $f(x)$ at $a = 2$ is $L(x) = 80x - 128$. Thus,
$$(1.8)^5 = f(1.8) \approx 80(1.8) - 128 = 16.$$

26. To find an approximation for $(7.9)^{2/3}$ we choose $f(x) = x^{2/3}$; $a = 8$. Using $f(8) = 4$, $f'(x) = 2/3\sqrt[3]{x}$, and $f'(8) = 1/3$, the tangent line to the graph is $y - 4 = (1/3)(x - 8)$. The linearization of $f(x)$ at $a = 8$ is $L(x) = \frac{1}{3}(x - 8) + 4$. Thus,
$$(7.9)^{2/3} = f(7.9) \approx \frac{1}{3}(7.9 - 8) + 4 = \frac{121}{30}.$$

27. To find an approximation for $\frac{(0.9)^4}{(0.9) + 1}$ we choose $f(x) = \frac{x^4}{x + 1}$; $a = 1$. Using $f(1) = 1/2$, $f'(x) = \frac{4x^3(x + 1) - x^4}{(x + 1)^2} = \frac{x^3(3x + 4)}{(x + 1)^2}$, and $f'(1) = 7/4$, the tangent line to the graph is $y - \frac{1}{2} = \frac{7}{4}(x - 1)$. The linearization of $f(x)$ at $a = 1$ is $L(x) = \frac{7}{4}(x - 1) + \frac{1}{2}$. Thus,
$$\frac{(0.9)^4}{(0.9) + 1} = f(0.9) \approx \frac{7}{4}(0.9 - 1) + \frac{1}{2} = \frac{13}{40} = 0.325.$$

28. To find an approximation for $(1.1)^3+6(1.1)^2$ we choose $f(x) = x^3+6x^2$; $a = 1$. Using $f(1) = 7$, $f'(x) = 3x^2 + 12x$, and $f'(1) = 15$, the tangent line to the graph is $y - 7 = 15(x - 1)$. The linearization of $f(x)$ at $a = 1$ is $L(x) = 15x - 8$. Thus,
$$(1.1)^3 + 6(1.1)^2 = f(1.1) \approx 15(1.1) - 8 = 8.5.$$

29. To find an approximation for $\cos(\pi/2-0.4)$ we choose $f(x) = \cos x$; $a = \pi/2$. Using $f(\pi/2) = 0$, $f'(x) = -\sin x$, and $f'(\pi/2) = -1$, the tangent line to the graph is $y - 0 = -1(x - \pi/2)$. The linearization of $f(x)$ at $a = \pi/2$ is $L(x) = -x + \pi/2$. Thus,
$$\cos(\pi/2 - 0.4) = f(\pi/2 - 0.4) \approx -(\pi/2 - 0.4) + \pi/2 = 0.4.$$

30. To find an approximation for $\sin 1^\circ$ we choose $f(x) = \sin x$; $a = 0$. Using $f(0) = 0$, $f'(x) = \cos x$, and $f'(0) = 1$, the tangent line to the graph is $y - 0 = 1(x - 0)$. The linearization of $f(x)$ at $a = 0$ is $L(x) = x$. Thus,
$$\sin 1^\circ = \sin(\pi/180) = f(\pi/180) \approx \pi/180.$$

31. To find an approximation for $\sin 33^\circ$ we choose $f(x) = \sin(x+\pi/6)$; $a = 0$. Using $f(0) = 1/2$, $f'(x) = \cos(x + \pi/6)$, and $f'(0) = \dfrac{\sqrt{3}}{2}$, the tangent line to the graph is $y - \dfrac{1}{2} = \dfrac{\sqrt{3}}{2}(x - 0)$. The linearization of $f(x)$ at $a = 0$ is $L(x) = \dfrac{\sqrt{3}}{2}x + \dfrac{1}{2}$. Thus,
$$\sin 33^\circ = \sin(\pi/6 + 3\pi/180) = f(\pi/60) \approx \frac{\sqrt{3}}{2}\left(\frac{\pi}{60}\right) + \frac{1}{2} = \frac{\sqrt{3}\pi + 60}{120}.$$

32. To find an approximation for $\tan\left(\dfrac{\pi}{4} + 0.1\right)$ we choose $f(x) = \tan x$; $a = \dfrac{\pi}{4}$. Using $f(\pi/4) = 1$, $f'(x) = \sec^2 x$, and $f'(\pi/4) = 2$, the tangent line to the graph is $y - 1 = 2(x - \pi/4)$. The linearization of $f(x)$ at $a = \dfrac{\pi}{4}$ is $L(x) = 2(x - \pi/4) + 1$. Thus,
$$\tan\left(\frac{\pi}{4} + 0.1\right) = f(\pi/4 + 0.1) \approx 2[(\pi/4 + 0.1) - \pi/4] + 1 = 1.2.$$

33. According to the graph, $f(1) = 4$ and $f'(1) = 2$, so the tangent line to the graph is $y - 4 = 2(x-1)$. The linearization of $f(x)$ at $a = 1$ is $L(x) = 2x+2$. Thus, $f(1.04) \approx 2(1.04)+2 = 4.08$.

34. According to the graph, $f(-2) = 5$ and $f'(-2) = -1/2$, so the tangent line to the graph is $y - 5 = -\dfrac{1}{2}[x - (-2)]$. The linearization of $f(x)$ at $a = -2$ is $L(x) = 4 - x/2$. Thus, $f(-1.98) \approx 4 - (-1.98)/2 = 4.99$.

35. $\Delta y = (x + \Delta x)^2 + 1 - (x^2 + 1) = 2x\Delta x + (\Delta x)^2$; $\quad dy = 2x\ dx$

36. $\Delta y = 3(x+\Delta x)^2 - 5(x+\Delta x) + 6 - (3x^2 - 5x + 6) = 6x\Delta x - 5\Delta x + 3(\Delta x)^2$; $\quad dy = (6x - 5)dx$

37. $$\begin{aligned}\Delta y &= (x + \Delta x + 1)^2 - (x + 1)^2 = (x + \Delta x)^2 + 2(x + \Delta x) + 1 - (x + 1)^2\\ &= x^2 + 2x\Delta x + (\Delta x)^2 + 2x + 2\Delta x + 1 - x^2 - 2x - 1 = 2x\Delta x + 2\Delta x + (\Delta x)^2\end{aligned}$$
$dy = 2(x + 1)dx$

38. $\Delta y = (x+\Delta x)^3 - x^3 = 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3; \quad dy = 3x^2\ dx$

39. $\Delta y = \dfrac{3(x+\Delta x)+1}{x+\Delta x} - \dfrac{3x+1}{x} = \dfrac{3x(x+\Delta x)+x-[3x(x+\Delta x)+(x+\Delta x)]}{x(x+\Delta x)} = -\dfrac{\Delta x}{x(x+\Delta x)}$

$dy = \dfrac{x(3)-(3x+1)}{x^2}\ dx = -\dfrac{dx}{x^2}$

40. $\Delta y = \dfrac{1}{(x+\Delta x)^2} - \dfrac{1}{x^2} = \dfrac{x^2-[x^2+2\Delta x+(\Delta x)^2]}{x^2(x+\Delta x)^2} = -\dfrac{2\Delta x+(\Delta x)^2}{x^2(x+\Delta x)^2}; \quad dy = -\dfrac{2}{x^3}dx$

41. $\Delta y = \sin(x+\Delta x) - \sin x = \sin x\cos\Delta x + \cos x\sin\Delta x - \sin x; \quad dy = \cos x\ dx$

42. $\Delta y = -4\cos 2(x+\Delta x) + 4\cos 2x = 4\cos 2x - 4\cos(2x+2\Delta x); \quad dy = 8\sin 2x\ dx$

43. $\Delta y = 5(x+\Delta x)^2 - 5x^2 = 10x\Delta x + 5(\Delta x)^2$

$\Delta y|_{x=2} = 20\Delta x + 5(\Delta x)^2;$

$dy = 10x\ dx; \quad dy|_{x=2} = 20\ dx$

(Recall that $dx = \Delta x$.)

x	Δx	Δy	dy	$\Delta y - dy$
2	1.00	25.0000	20.0	5.0000
2	0.50	11.2500	10.0	1.2500
2	0.10	2.0500	2.0	0.0500
2	0.01	0.2005	0.2	0.0005

44. $\Delta y = \dfrac{1}{x+\Delta x} - \dfrac{1}{x} - \dfrac{\Delta x}{x(x+\Delta x)}$

$\Delta y|_{x=2} = -\dfrac{\Delta x}{2(2+\Delta x)};$

$dy = -\dfrac{1}{x^2}\ dx; \quad dy|_{x=2} = -\dfrac{dx}{4}$

(Recall that $dx = \Delta x$.)

x	Δx	Δy	dy	$\Delta y - dy$
2	1.00	−0.1667	−0.2500	0.0833
2	0.50	−0.1000	−0.1250	0.0250
2	0.10	−0.0238	−0.0250	0.0012
2	0.01	−0.0025	−0.0025	0.0000

45. (a) $df = (8x+5)\ dx$. When $x = 4$ and $dx = 0.03$, $df = [8(4)+5](0.03) = 1.11$.
(b) When $x = 3$ and $dx = -0.1$, $df = [8(3)+5](-0.1) = -2.9$.

46. (a) $f'(x) = 3x^2 + 6x; \quad f'(1) = 9$. Since $f(1) = 4$ the equation of the tangent line is $y - 4 = 9(x-1)$ or $y = 9x - 5$.
(b) Letting $x = 1.02$ in the equation of the tangent line, we obtain $y = 4.18$.
(c) Identifying $x = 1$ and $dx = 0.02$ we have $f(1.02) = f(1+0.02) \approx f(1) + f'(1)(0.02) = 4 + 9(0.02) = 4.18$. This is exactly the same as the value in part (b). This is to be expected since formula (3) in effect uses the tangent line to approximate the function value.

47. (a) $A(4) = 16\pi$ cm; $\quad A(5) = 25\pi$ cm; $\quad A(5) - A(4) = 9\pi$ cm.
(b) $dA = 2\pi r\ dr$. When $r = 4$ cm and $dr = 1$ cm, $dA = 8\pi$ cm.

48. $R(r) = klr^{-4}; \quad dR = -4klr^{-5}\ dr$. When $r = 0.2$ mm and $dR = 0.1$ mm, the approximate change in R is $dR = -4kl(0.2)^{-5}(0.1) = -1250$ kl/mm^4.

49. The exact volume of the cover is

$$\Delta V = \frac{4}{3}\pi(r+t)^3 - \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(3r^2t + 3rt^2 + t^3).$$

$dV = 4\pi r^2 dr$. When $dr = t$, the approximate volume of the cover is $4\pi r^2 t$. If $r = 0.8$ in and $t = 0.04$ in, then the approximate volume of the cover is $4\pi(0.8)^2(0.04) = 0.1024\pi$ in^3.

50. $V = \pi r^2 h = 150\pi r^2$; $\quad dV = 300\pi r\ dr$. If $r = 2$ cm and $dr = 0.25$ cm, then the approximate volume is $dV = 300\pi(2)(0.25) = 150\pi \approx 471.24$ cm^3.

51. $A = s^2$; $\quad dA = 2s\ ds$. Letting $s = 10$ cm and $ds = 0.3$ cm, we obtain $dA = 2(10)(0.3) = 6$ cm^2. The maximum error in the area is ± 6 cm^2. The approximate relative error is $\pm 6/10^2 = \pm 0.06$ cm^2 and the approximate percentage error is $\pm 6\%$.

52. The volume of a cylinder is $V = \pi r^2 h$ or $V = 5\pi r^2$ when the height is 5 m. The approximate error in the volume is $dV = 10\pi r\ dr$ or $10\pi(8)(\pm 0.25) = \pm 20\pi \approx \pm 62.8318$ m^2 when $r = 8$ m and $dr = \pm 0.25$ m. The approximate volume of the tank is $V = 5\pi(8^2) = 320\pi$, so the approximate relative error is $\pm 20\pi/320\pi = \pm 0.0625$ m^3, and the approximate percentage error is $\pm 6.25\%$.

53. $P = cV^{-\gamma}$; $\quad dP = -\gamma c V^{-\gamma-1} dV$. The approximate relative error in P is

$$\frac{dP}{P} = \frac{-\gamma c V^{-\gamma-1} dV}{cV^{-\gamma}} = -\gamma \frac{dV}{V},$$

so the approximate relative error in P is proportional to the approximate relative error in V.

54. $R = v_0^2(\sin 2\theta)g^{-1}$; $\quad dR = -v_0^2(\sin 2\theta)g^{-2}dg$. The approximate relative error in R is

$$\frac{dR}{R} = \frac{-v_0^2(\sin 2\theta)g^{-2}dg}{v_0^2(\sin 2\theta)g^{-1}} = -\frac{dg}{g},$$

so the approximate relative error in R is proportional to the approximate relative error in g.

55. For $v_0 = 256$ ft/s, $\theta = 45°$, and $g = 32$ ft/s^2, the range is $R = \dfrac{256^2}{32}\sin 90° = 2048$ ft. The approximate change in R with respect to v_0 is $dV = \dfrac{2v_0}{g}\sin 2\theta dv_0$. For $v_0 = 256$ ft/s, $g = 32$ ft/s^2, $\theta = 45°$, and $dv_0 = 10$ ft/s, the approximate change in the range is $dv = \dfrac{2(256)}{32}(1)(10) = 160$ ft.

56. (a) Solving $km_1m_2/r^2 = m_1 g$ for g we obtain $g = km_2/r^2$.

(b) From (a), $g = km_2 r^{-2}$ and

$$dg = -2km_2 r^{-3} dr = \frac{-2km_2 dr}{r^3} = \frac{-2km_2}{r^2}\frac{dr}{r} = -2g\frac{dr}{r}.$$

Thus, $dg/g = -2\ dr/r$.

(c) Letting $g = 9.8$ m/s^2, $r = 6400$ km $\times$ 100 m/km, and $dr = 16$ km $\times$ 100 m/km we have from (b)

$$\frac{dg}{9.8} = \frac{-2(16 \times 1000)}{6400 \times 1000} = -\frac{1}{200},$$

or $dg = -9.8/200$ m/s^2 $= -0.049$ m/s^2. Thus the approximate value of g at $r = 16$ km is $g + dg = 9.8 - 0.049 \approx 9.75$ m/s^2.

57. (a) Setting $g' = 978.0318(53.024 \times 10^{-4} \times 2\sin\theta\cos\theta - 5.9 \times 10^{-6} \times 4\sin 2\theta\cos 2\theta) = 0$ we obtain $\sin 2\theta(53.024 \times 10^{-4} - 5.9 \times 10^{-6} \times 4\cos 2\theta = 0$. From $\sin 2\theta = 0$ we find $\theta = 0°$ or $90°$, and since $4\cos 2\theta = 53.024 \times 10^{-4}/5.9 \times 10^{-6} \times 4 > 1$, we see that these are the only critical numbers. By inspection of g we find that g is minimum on the equator ($\theta = 0°$) and maximum at the poles ($\theta = 90°$).

(b) $g(60°) \approx 981.9169$ cm/s^2

(c) $dg = 978.0318\sin 2\theta(53.024 \times 10^{-4} - 5.9 \times 10^{-6} \times 4\cos 2\theta)d\theta$.
Using $\theta = \pi/3$ and $d\theta = \pi/180$ we find $dg \approx 0.07856$ cm/s^2.

58. $T(4) = 2\pi\sqrt{4/9.8} \approx 4.0142$ s; $\quad T(5) = 2\pi\sqrt{5/9.8} \approx 4.4880$ s.

The change in period is $T(5) - T(4) \approx 0.4738$ s. From $T = \dfrac{2\pi}{\sqrt{g}}L^{1/2}$ we obtain $dT = \dfrac{\pi}{\sqrt{g}}L^{-1/2}dL$. For $g = 9.8$ ft/s^2, $L = 4$ m, and $dL = 1$ m, we have $dT = \dfrac{\pi}{\sqrt{9.8}}4^{-1/2}(1) \approx 0.5018$ s.

59. Writing $T = 2\pi\sqrt{L}g^{-1/2}$, we have $dT = -\pi\sqrt{L}g^{-3/2}dg$. For $L = 4$ m, $g = 9.8$ m/s^2, and $dg = -0.05$ m/s^2, we obtain the approximate change $dT = -\pi\sqrt{4}(9.8)^{-3/2}(-0.05) \approx 0.0102$ s.

60. (a) From Figure 4.9.9 we see that $\tan\dfrac{\theta}{2} = \dfrac{1/2}{D}$, so that $D = \dfrac{1}{2}\cot\dfrac{\theta}{2}$.

(b) Using $\dfrac{1}{2}^{\circ} = \dfrac{1}{2}\left(\dfrac{\pi}{180}\right) = \dfrac{\pi}{360}$ radian, $D\left(\dfrac{1}{2}^{\circ}\right) = D\left(\dfrac{\pi}{360}\right) = \dfrac{1}{2}\cot\dfrac{\pi}{720} \approx \dfrac{1}{2}(229.18) = 114.59$ ft.

(c) Let x and y represent the distances cars A and B, respectively, have traveled from some initial point. Then $y = x + D$. Since 2 seconds $= \dfrac{1}{30}^{\circ}$ we are given $\dfrac{dx}{dt} = 30$ mi/h $= 30\left(\dfrac{5280}{3600}\right)$ ft/s $= 44$ ft/s and $\dfrac{d\theta}{dt} = -\dfrac{1}{30}^{\circ}$/s $= -\dfrac{1}{30}\left(\dfrac{\pi}{180}\right) = -\dfrac{\pi}{5400}$ radian/s. Then car B is moving at a rate of

$$\begin{aligned}\frac{dy}{dt} &= \frac{dx}{dt} + \frac{dD}{dt} = 44 - \frac{1}{4}\left(\csc^2\frac{\theta}{2}\right)\frac{d\theta}{dt} = 44 - \frac{1}{4}\left(\csc^2\frac{\pi}{720}\right)\left(-\frac{\pi}{5400}\right) \\ &\approx 51.64 \text{ ft/s} \approx 35.21 \text{ mi/h}.\end{aligned}$$

(d) From $dD = -\dfrac{1}{4}\csc^2\dfrac{\theta}{2}d\theta$ we obtain

$$\frac{dD}{d} = \frac{-\csc^2\dfrac{\theta}{2}d\theta/4}{\cot\dfrac{\theta}{2}/2} = -\frac{d\theta}{2\sin^2\dfrac{\theta}{2}\cot\dfrac{\theta}{2}} = -\frac{d\theta}{2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}} = -\frac{d\theta}{\sin\theta}.$$

Letting $\theta = \dfrac{1}{2}^\circ$ and $d\theta = \pm\dfrac{1}{60}\left(\dfrac{\pi}{180}\right)$ we find $\dfrac{dD}{D} = \pm\dfrac{\pi/10800}{\sin(\pi/360)} \approx 0.033 = 3.3\%.$

61. By the definition in (2), $L(x) = f(a) + f'(a)(x-a)$. Since $p(a) = f(a)$ and $p'(a) = f'(a)$, we can rewrite $L(x) = p(a) + p'(a)(x-a) = (c_1a + c_0) + (c_1)(x-a)$. Simplifying, c_1a cancels out, and we get $L(x) = c_1x + c_0$, which is $p(x)$.

62. The linearization of $\cos x$ at $a = 0$ is $L(x) = \cos 0 - (\sin 0)(x-0) = 1$. Thus, $\cos x \approx L(x) = 1$ for small values of x.

63. If $f''(x) > 0$ for all x in some open interval containing a, then that interval is concave up. Thus, the tangent at a will lie below $f(x)$ for all x within that interval, and so $L(x)$ will underestimate $f(x)$ for x near a.

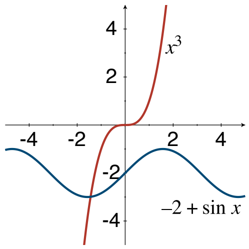
64. If $(c, f(c))$ is a point of inflection for the graph of $y = f(x)$, then the graph of $f(x)$ changes concavity at c. If $f(x)$ changes from concave up to concave down at $x = c$, then $L(x) < f(x)$ or $f(x) - L(x) > 0$ for $x < c$ and $L(x) > f(x)$ or $f(x) - L(x) < 0$ for $x > c$. If $f(x)$ changes from concave down to concave up at $x = c$, then $L(x) > f(x)$ or $f(x) - L(x) < 0$ for $x < c$ and $L(x) < f(x)$ or $f(x) - L(x) > 0$ for $x > c$. In either case, the graph of $f(x) - L(x)$ crosses the x-axis (i.e., changes sign) at $x = c$.

65.
$$\begin{aligned} \Delta A &= (x+\Delta x)^2 - x^2 = 2x\Delta x + (\Delta x)^2 \\ dA &= 2x\ dx = 2x\Delta x \text{ (Recall that } dx = \Delta x.) \\ \Delta A - dA &= [2x\Delta x + (\Delta x)^2] - 2x\Delta x = (\Delta x)^2 \end{aligned}$$

Thus, ΔA is the combined area of the beige and green regions, dA is the combined area of the beige regions only, and $\Delta A - dA$ is the area of the green region.

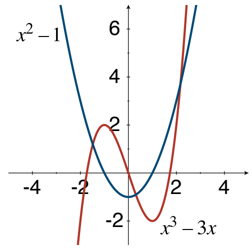
4.10 Newton's Method

1.



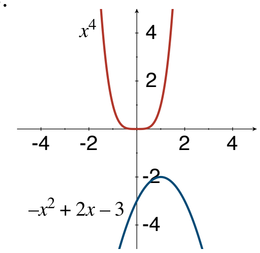
The equation has one real root.

2.



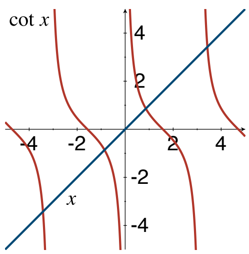
The equation has three real roots.

3.



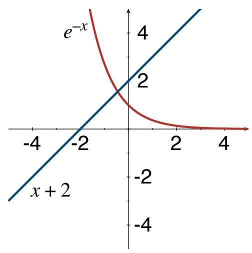
The equation has no real roots.

4.



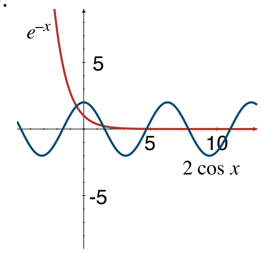
The equation has infinitely many real roots.

5.



The equation has one real root.

6.



The equation has infinitely many real roots.

7. Let $f(x) = x^2 - 10$. Then $f'(x) = 2x$ and

$$x_{n+1} = x_n - \frac{x_n^2 - 10}{2x_n} = \frac{x_n^2 + 10}{2x_n}.$$

Choosing $x_0 = 3$ we obtain $x_1 \approx 3.1667$, $x_2 \approx 3.1623$, $x_3 \approx 3.1623$. Thus, $\sqrt{10} \approx 3.1623$.

8. Let $x = 1+\sqrt{5}$. Then $x-1 = \sqrt{5}$ and $x^2-2x+1 = 5$. We use the function $f(x) = x^2-2x-4$. Then $f'(x) = 2x-2$ and

$$x_{n+1} = x_n - \frac{x_n^2 - 2x_n - 4}{2x_n - 2} = \frac{x_n^2 + 4}{2x_n - 2}.$$

Choosing $x_0 = 3$ we obtain $x_1 \approx 3.2500$, $x_2 \approx 3.2361$, $x_3 \approx 3.2361$. Thus, $1 + \sqrt{5} \approx 3.2361$.

9. Let $f(x) = x^3 - 4$. Then $f'(x) = 3x^2$ and

$$x_{n+1} = x_n - \frac{x_n^3 - 4}{3x_n^2} = \frac{2x_n^3 + 4}{3x_n^2}.$$

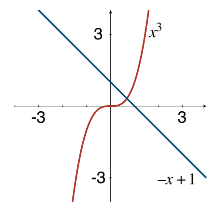
Choosing $x_0 = 1$ we obtain $x_1 \approx 2.0000$, $x_2 \approx 1.6667$, $x_3 \approx 1.5911$, $x_4 \approx 1.5874$, $x_5 \approx 1.5874$. Thus, $\sqrt[3]{4} \approx 1.5874$.

10. Let $f(x) = x^5 - 2$. Then $f'(x) = 5x^4$ and

$$x_{n+1} = x_n - \frac{x_n^5 - 2}{5x_n^4} = \frac{4x_n^5 + 2}{5x_n^4}.$$

Choosing $x_0 = 1$ we obtain $x_1 \approx 1.2000$, $x_2 \approx 1.1529$, $x_3 \approx 1.1487$, $x_4 \approx 1.1487$. Thus, $\sqrt[5]{2} \approx 1.1487$.

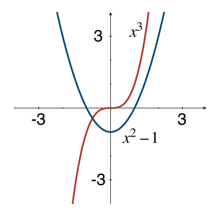
11. Let $f(x) = x^3 + x - 1$. Then $f'(x) = 3x^2 + 1$ and



$$x_{n+1} = x_n - \frac{x_n^3 + x_n - 1}{3x_n^2 + 1} = \frac{2x_n^3 + 1}{3x_n^2 + 1}.$$

From the graph we see that $f(x)$ has a single root near $x_0 = 1$. Then $x_1 \approx 0.7500$, $x_2 \approx 0.6860$, $x_3 \approx 0.6823$, $x_4 \approx 0.6823$, and the only real root is approximately 0.6823.

12. Let $f(x) = x^3 - x^2 + 1$. Then $f'(x) = 3x^2 - 2x$ and

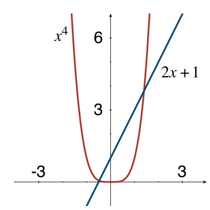


$$x_{n+1} = x_n - \frac{x_n^3 - x_n^2 + 1}{3x_n^2 - 2x_n} = \frac{2x_n^3 - x_n^2 - 1}{3x_n^2 - 2x_n}.$$

From the graph we see that $f(x)$ has a single root near $x_0 = -1$. Then $x_1 \approx -0.8000$, $x_2 \approx -0.7568$, $x_3 \approx -0.7549$, $x_4 \approx -0.7549$, and the only real root is approximately -0.7549.

13. From the quadratic formula, $x^2 = \dfrac{-1 \pm \sqrt{1+12}}{2} = \dfrac{-1 \pm \sqrt{13}}{2}$. Since x^2 must be positive for x real, we have $x = \pm\sqrt{\dfrac{-1 + \sqrt{13}}{2}} \approx \pm 1.1414$. Newton's Method is not necessary.

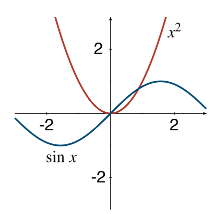
14. Let $f(x) = x^4 - 2x - 1$. Then $f'(x) = 4x^3 - 2$ and



$$x_{n+1} = x_n - \frac{x_n^4 - 2x_n - 1}{4x_n^3 - 2} = \frac{3x_n^4 + 1}{4x_n^3 - 2}.$$

From the graph we see that $f(x)$ has two real roots. Choosing $x_0 = -1$ and then $x_0 = 1$, we obtain $x_1 \approx -0.6667$, $x_2 \approx -0.5000$, $x_3 \approx -0.4750$, $x_4 \approx -0.4746$, $x_5 \approx -0.4746$, and $x_1 \approx 2.0000$, $x_2 \approx 1.6333$, $x_3 \approx 1.4486$, $x_4 \approx 1.3988$, $x_5 \approx 1.3954$, $x_6 \approx 1.3953$, $x_7 \approx 1.3953$. Thus, the two real roots are approximately -0.4746 and 1.3953.

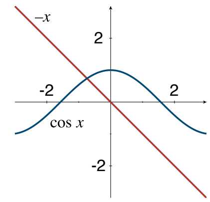
15. Let $f(x) = x^2 - \sin x$. Then $f'(x) = 2x - \cos x$ and



$$x_{n+1} = x_n - \frac{x_n^2 + \sin x_n}{2x_n - \cos x_n} = \frac{x_n^2 - x_n\cos x_n + \sin x_n}{2x_n - \cos x_n}.$$

From the graph we see that $f(x)$ has one root at $x = 0$ and another one near $x_0 = 1$. Then $x_1 \approx 0.8914$, $x_2 \approx 0.8770$, $x_3 \approx 0.8767$, $x_4 \approx 0.8767$. Thus, the two real roots are 0 and approximately 0.8767.

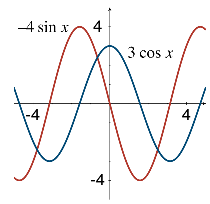
16. Let $f(x) = x + \cos x$. Then $f'(x) = 1 - \sin x$ and



$$x_{n+1} = x_n - \frac{x_n + \cos x_n}{1 - \sin x_n} = \frac{-x_n \sin x_n - \cos x_n}{1 - \sin x_n}.$$

From the graph we see that $f(x)$ has a single root near $x_0 = -1$. Then $x_1 \approx -0.7504$, $x_2 \approx -0.7391$, $x_3 \approx -0.7391$, and the only real root is approximately -0.7391.

17. $f'(x) = -3\sin x + 4\cos x$ and



$$x_{n+1} = x_n - \frac{3\cos x_n + 4\sin x_n}{-3\sin x_n + 4\cos x_n} = \frac{(4x_n - 3)\cos x_n - (3x_n + 4)\sin x_n}{4\cos x_n - 3\sin x_n}.$$

From the graphs of $3\cos x$ and $-4\sin x$ we see that the first positive root of $f(x)$ is near $x_0 = 2$. Then $x_1 \approx 2.5438$, $x_2 \approx 2.4981$, $x_3 \approx 2.4981$. Thus, the smallest positive x-intercept is approximately 2.4981.

18. Let $F(x) = x^5 + x^2 - 4$. Then $F'(x) = 5x^4 + 2x$ and

$$x_{n+1} = x_n - \frac{x_n^5 + x_n^2 - 4}{5x_n^4 + 2x_n} = \frac{4x_n^5 + x_n^2 + 4}{5x_n^4 + 2x_n}.$$

Since $F(x) < 0$ for $0 \le x \le 1$, we try $x_0 = 1$. Then $x_1 \approx 1.2857$, $x_2 \approx 1.2139$, $x_3 \approx 1.2057$, $x_4 \approx 1.2056$, $x_5 \approx 1.2056$, and the smallest positive number for which $x^5 + x^2 = 4$ is approximately 1.2056.

19. We want to solve $\dfrac{60x^2 - x^3}{16000} = 0.01$ or $x^3 - 60x^2 + 160 = 0$. Let $f(x) = x^3 - 60x^2 + 160$. Then $f'(x) = 3x^2 - 120x$ and

$$x_{n+1} = x_n - \frac{x_n^3 - 60x_n^2 + 160}{3x_n^2 - 120x_n} = \frac{2x_n^3 - 60x_n^2 - 160}{3x_n^2 - 120x_n}.$$

Choosing $x_0 = 1$ we obtain $x_1 \approx 1.8632$, $x_2 \approx 1.6670$, $x_3 \approx 1.6560$, $x_4 \approx 1.6560$. Thus, $x \approx 1.6560$ ft.

20. We want to solve $10 = 35r^{2/3}$ or $10^3 = (35)^3 r^2$. Let $f(r) = 42.875r^2 - 1$. Then $f'(r) = 85.75r$ and

$$r_{n+1} = r_n - \frac{42.875r_n^2 - 1}{85.75r_n} = \frac{42.875r_n^2 + 1}{85.75r_n}.$$

Choosing $r_0 = 1$ we obtain $r_1 \approx 0.5117$, $r_2 \approx 0.2786$, $r_3 \approx 0.1812$, $r_4 \approx 0.1550$, $r_5 \approx 0.1527$, $r_6 \approx 0.1527$. Thus, $r \approx 0.1527$ m and $d \approx 0.3054$ m.

21. We want to solve $\sin\theta = \dfrac{1}{1.5} = \dfrac{2}{3}$ or $3\sin\theta = 2$. Let $f(\theta) = 3\sin\theta - 2$. Then $f'(\theta) = 3\cos\theta$ and

$$\theta_{n+1} = \theta_n - \frac{3\sin\theta_n - 2}{3\cos\theta_n} = \frac{3\theta_n\cos\theta_n - 3\sin\theta_n + 2}{3\cos\theta_n}.$$

Choosing $\theta_0 = 0$ we obtain $\theta_1 \approx 0.6667$, $\theta_2 \approx 0.7281$, $\theta_3 \approx 0.7297$, $\theta_4 \approx 0.7297$. Thus, $\theta \approx 0.7297$ radian.

22. We want to solve $404 = 400 + \dfrac{8d^2}{1200} - \dfrac{32d^4}{320,000,000}$ or $1 = \dfrac{d^2}{600} - \dfrac{d^4}{40,000,000}$. Let $f(d) = 2.5 \times 10^{-8}d^4 - 1.667 \times 10^{-3}d^2 + 1$. Then $f'(d) = 10^{-7}d^3 - 3.333 \times 10^{-3}d$ and

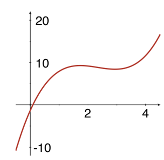
$$d_{n+1} = d_n - \frac{2.5 \times 10^{-8}d_n^4 - 1.667 \times 10^{-3}d_n^2 + 1}{10^{-7}d_n^3 - 3.333 \times 10^{-3}d_n} = \frac{7.5 \times 10^{-8}d_n^4 - 1.667 \times 10^{-3}d_n^2 - 1}{10^{-7}d_n^3 - 3.333 \times 10^{-3}d_n}.$$

Choosing $d_0 = 20$ we obtain $d_1 \approx 25.1219$, $d_2 \approx 24.6125$, $d_3 \approx 24.6074$, $d_4 \approx 24.6074$. Thus $d \approx 24.6$ ft.

23. (a) The volume of water displaced is $V_w = 3(7)(2) = 42$ ft^3. The volume of steel in the tub is $V_s = 42 - (3-2t)(7-2t)(2-t) = 4t^3 - 28t^2 + 61t$. Since the weight of water displaced is equal to the weight of the tub,

$$62.4(42) = 490(4t^3 - 28t^2 + 61t) \quad \text{or} \quad f(t) = t^3 - 7t^2 + \frac{61}{4}t - \frac{1638}{1225} = 0.$$

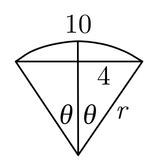
(b) Then $f'(t) = 3t^2 - 14t + 61/4$ and



$$t_{n+1} = t_n - \frac{f(t)}{f'(t)}.$$

From the graph we see that $f(t)$ has its only root near $t_0 = 0$. Then $t_1 \approx 0.0877$, $t_2 \approx 0.0915$, $t_3 \approx 0.0915$, and $t \approx 0.915$ ft.

24. From the figure we see that $\sin\theta = \dfrac{4}{r}$ and $\dfrac{10}{2\pi r} = \dfrac{2\theta}{2\pi}$ or $\sin\theta = 4$ and $\theta = \dfrac{5}{r}$. Then $r \sin\dfrac{5}{r} = 4$ and we identify $f(r) = r\sin\dfrac{5}{r} - 4$. Using $f'(\theta) = \sin\dfrac{5}{r} - \dfrac{5}{r}\cos\dfrac{5}{r}$ we have



$$r_{n+1} = r_n - \frac{r_n \sin\dfrac{5}{r_n} - 4}{\sin\dfrac{5}{r_n} - \dfrac{5}{r_n}\cos\dfrac{5}{r_n}}.$$

Clearly $r > 4$ so we choose $r_0 = 4$. Then $r_1 \approx 4.3678$, $r_2 \approx 4.4196$, $r_3 \approx 4.4205$, $r_4 \approx 4.4205$, and $r \approx 4.4205$ ft.

25. (a) Since θ subtends an arc of length $L/2$ on a circle of radius R, we have $L/2 = R\theta$ and $R = L/2\theta$. From Figure 4.10.11 we see that

$$\sin\theta = \frac{(L-l)/2}{R} = \frac{(L-l)/2}{L/2\theta} = \left(1 - \frac{l}{L}\right)\theta \quad \text{and} \quad \cos\theta = \frac{R-h}{R}.$$

Then

$$\begin{aligned} h = R(1-\cos\theta) &= \frac{L}{2\theta}\left(1 - \sqrt{1-\sin^2\theta}\right) = \frac{L}{2\theta}\left[1 - \sqrt{1-(1-l/L)^2\theta^2}\right] \\ &= \frac{L}{2\theta}\frac{1 - \left[1-(1-l/L)^2\theta^2\right]}{1 + \sqrt{1-(1-l/L)^2\theta^2}} = \frac{L(1-l/L)^2\theta}{2\left[1+\sqrt{1-(1-l/L)^2\theta^2}\right]}. \end{aligned}$$

(b) Setting $L = 5280$ and $l = 1$ we have $\sin\theta = \left(1 - \frac{1}{5280}\right)\theta$ or $f(\theta) = \sin\theta - \frac{5279}{5280}\theta = 0$. The formula for Newton's Method is

$$\theta_{n+1} = \theta_n - \frac{f(\theta_n)}{f'(\theta_n)} = \theta_n - \frac{\sin\theta_n - \frac{5279}{5280}\theta_n}{\cos\theta_n - \frac{5279}{5280}} = \frac{\theta_n\cos\theta_n - \sin\theta_n}{\cos\theta_n - 5279/5280}.$$

Taking $\theta_0 = 0.1$ we obtain $\theta_1 \approx 0.069282$, $\theta_2 \approx 0.050143$, $\theta_3 \approx 0.039358$, $\theta_4 \approx 0.034732$, $\theta_5 \approx 0.033754$, $\theta_6 \approx 0.033711$, $\theta_7 \approx 0.033711$, so $\theta \approx 0.033711$ and $h \approx 44.494$ ft.

(c) From $\sin\theta = (1 - l/L)\theta \approx \theta - \theta^3/6$ we obtain $l/L \approx \theta^2/6$ and $\theta \approx \sqrt{6l/L}$. Then

$$h \approx \frac{L\theta}{4} \approx \frac{L\sqrt{6l/L}}{4} \approx \sqrt{\frac{3lL}{8}}.$$

Setting $l = 1$ and $L = 5280$ we find $h \approx 44.4972$, which is very close to the result obtained in (b).

26. The volume of the sphere is $\frac{4}{3}\pi(2)^3 = \frac{32}{3}\pi$ and the volume of the rod is $15\pi r^2 + \frac{2}{3}\pi r^3$. Thus, $15\pi r^2 + \frac{2}{3}\pi r^3 = \frac{32}{3}\pi$ or $2r^3 + 45r^2 - 32 = 0$. From $f(r) = 2r^3 + 45r^2 - 32$ and $f'(r) = 6r^2 + 90r$ we have

$$r_{n+1} = r_n - \frac{2r_n^3 + 45r_n^2 - 32}{6r_n^2 + 90r_n}.$$

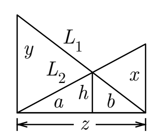
Taking $r_0 = 1$ we obtain $r_1 \approx 0.8437$, $r_2 \approx 0.8283$, $r_3 \approx 0.8282$, $r_4 \approx 0.8282$ and $r \approx 0.8282$ ft.

27. Setting $M = 4m$ we have $4mg\frac{r}{2}\sin\theta - mgr\theta = 0$ or $2\sin\theta - \theta = 0$. Setting $f(\theta) = 2\sin\theta - \theta$ we obtain $f'(\theta) = 2\cos\theta - 1$ and

$$\theta_{n+1} = \theta_n - \frac{2\sin\theta_n - \theta_n}{2\cos\theta_n - 1} = \frac{2\theta_n\cos\theta_n - 2\sin\theta_n}{2\cos\theta_n - 1}.$$

Taking $\theta_0 = 2$ we find $\theta_1 \approx 1.9010$, $\theta_2 \approx 1.8955$, $\theta_3 \approx 1.8955$, and $\theta \approx 1.8955$ radians.

28. (a) Using similar triangles we obtain $\frac{a}{x} = \frac{h}{x}$ and $\frac{b}{z} = \frac{h}{y}$. Adding, we find



$$\frac{a+b}{z} = \frac{h}{x} + \frac{h}{y}, \quad \frac{z}{z} = \frac{hy + hx}{xy}, \quad xy = hy + hx, \quad (x-h)y = hx,$$

so $y = \frac{hx}{x-h}$. Using the Pythagorean Theorem we obtain $z^2 + x^2 = L_2^2$ and $z^2 + y^2 = L_1^2$. Subtracting, we have

$$x^2 - y^2 = L_2^2 - L_1^2$$
$$x^2 - \left(\frac{hx}{x-h}\right)^2 + L_1^2 - L_2^2 = 0$$
$$x^4 - 2hx^3 + (L_1^2 - L_2^2)x^2 - 2h(L_1^2 - L_2^2)x + h^2(L_1^2 - L_2^2) = 0.$$

(b) Letting $h = 10$, $L_1 = 40$, and $L_2 = 30$ we have $f(x) = x^4 - 20x^3 + 700x^2 - 1,400x + 70,000$ and $f'(x) = 4x^3 - 60x^2 + 1,400x - 14,000$. Then

$$x_{n+1} = x_n - \frac{x_n^4 - 20x_n^3 + 700x_n^2 - 1,400x_n + 70,000}{4x_n^3 - 60x_n^2 + 1,400x_n - 14,000}.$$

Since $x > h = 10$, we choose $x_0 = 11$. Then $x_1 = -10.043$, $x_2 \approx -1.8771$, $x_3 \approx 3.9863$, $x_4 \approx 6.6511$, $x_5 \approx 7.2874$, $x_6 \approx 7.3299$, $x_7 \approx 7.3301$, $x_8 \approx 7.330$, and $x = 7.3301$. While this is a root of $f(x)$, it is too small to be a solution to the problem. Trying $x_0 = 12$ we obtain $x_1 \approx 22.2836$, $x_2 \approx 18.1499$, $x_3 \approx 15.8728$, $x_4 \approx 15.0336$, $x_5 \approx 14.9119$, $x_6 \approx 14.9094$, $x_7 \approx 14.9094$, and $x \approx 14.9094$ ft.

(c) From $z^2 + x^2 = L_2^2$ we find $z = \sqrt{L_2^2 - x^2} \approx 26.0329$ ft.

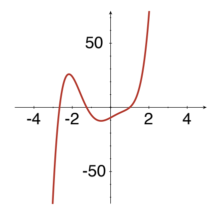
29. $f(x) = 2x^5 + 3x^4 - 7x^3 + 2x^2 + 8x - 8$; $\quad f'(x) = 10x^4 + 12x^3 - 21x^2 + 4x + 8$

$x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$x_0 = -1$, $x_1 \approx -1.3158$, $x_2 \approx -1.2517$, $x_3 \approx -1.2494$, $x_4 \approx -1.2494$

$x_0 = -3$, $x_1 \approx -2.7679$, $x_2 \approx -2.6776$, $x_3 \approx -2.6641$, $x_4 \approx -2.6638$, $x_5 \approx -2.6638$

$x_0 = 1$, $x_1 = 1$



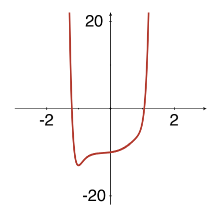
30. $f(x) = 4x^{12} + x^{11} - 4x^8 + 3x^3 - 2x^2 + x - 10$

$f'(x) = 48x^{11} + 11x^{10} - 32x^7 + 9x^2 - 4x + 1$

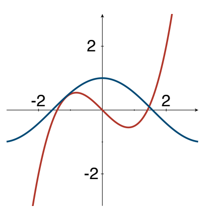
$x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$x_0 = -1.2$, $x_1 \approx -1.2531$, $x_2 \approx -1.2416$, $x_3 \approx -1.2408$, $x_4 \approx -1.2408$

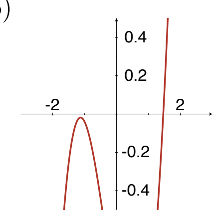
$x_0 = 1$, $x_1 \approx 1.2121$, $x_2 \approx 1.1468$, $x_3 \approx 1.1128$, $x_4 \approx 1.1047$, $x_5 \approx 1.1044$, $x_6 \approx 1.1044$



31. (a)



(b)



(c) The number of roots appears to be two in part (a), but the "zoomed in" graph in part (b) shows there is only one root.

(d) $y = 0.5x^3 - x - \cos x$

$y' = 1.5x^2 - 1 + \sin x$

$x_{n+1} = x_n - \dfrac{0.5x_n^3 - x_n - \cos x_n}{1.5x_n^2 - 1 + \sin x_n}$

$x_0 = 1.5$, $x_1 \approx 1.4654$, $x_2 \approx 1.4645$, $x_3 \approx 1.4645$

32. Since $f(x_0) = -f(x_1)$ and $f'(x_0) = f'(x_1)$, $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} = x_0 + \dfrac{f(x_1)}{f'(x_1)}$ and $\dfrac{f(x_1)}{f'(x_1)} = x_1 - x_0$. Then $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} = x_1 - (x_1 - x_0) = x_0$.

33. $f'(x) = \begin{cases} \dfrac{1}{2\sqrt{4-x}}, & x < 4 \\ \dfrac{1}{2\sqrt{x-4}}, & x > 4 \end{cases}$

Since $f'(4)$ does not exist, Newton's Method will fail for $x_0 = 4$. For any choice of $x_0 < 4$,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - (2x_0 - 8) = 8 - x_0.$$

Since $x_1 = 8 - x_0 > 4$, $f(x_1) = \sqrt{x_1 - 4} = \sqrt{8 - x_0 - 4} = \sqrt{4 - x_0} = -f(x_0)$, and $f'(x_1) = \dfrac{1}{2\sqrt{x_1 - 4}} = \dfrac{1}{2\sqrt{8 - x_0 - 4}} = \dfrac{1}{2\sqrt{4 - x_0}} = f'(x_0)$. By Problem 32, then, $x_2 = x_0$. The same result is obtained for any choice of $x_0 > 4$. Newton's Method will therefore yield the sequence of iterates $x_0, x_1, x_0, x_1, \ldots$, and thus fail to converge.

Chapter 4 in Review

A. True/False

1. False; the function may not be differentiable, or $f'(x)$ may be 0 for some x on the interval.
2. False; for $f(x) = x^3$, $f'(0) = 0$, but $f(x)$ has no extremum at 0.
3. False; this is only true when the velocity is positive.
4. True
5. True
6. False; the concavity must change around c. Consider $f(x) = x^4$ at $x = 0$.
7. False; $f(x)$ need not be differentiable at c.
8. False; $f''(x)$ need not exist at c.
9. True
10. False; if the extremum occurs at an endpoint of the interval, it cannot be a relative extremum.
11. True
12. False; $x = 1$ is not in the domain of $\sqrt{x^2 - 2x}$.
13. True
14. False; consider $f(x) = -x^2$, $f'(x) = -2x$, $f''(x) = -2$ on $(-\infty, 0)$.
15. False; this is an indeterminate form.
16. False; see Problem 55, Section 4.5.
17. True

18. False; an expression with this form will have limit 0.

19. False; let a be ∞, $f(x) = x^2$ and $g(x) = e^x$.

20. False; L'Hôpital's Rule uses the quotient of the derivatives, rather than the derivative of the quotient.

B. Fill in the Blanks

1. Velocity

2. One, since the second derivative is linear and can have at most one root.

3. $f(x) = x^{1/3}$

4. Let x and $8 - x$ be the two numbers. We want to maximize $S(x) = x^2 + (8 - x)^2$ on $[0, 8)$. Solving $S'(x) = 2x - 2(8 - x) = 4x - 16 = 0$ we obtain the critical number 4. Comparing $S(0) = S(8) = 64$ and $S(4) = 32$ we see that the sum is maximized when the two numbers are 0 and 8.

5. 0

6. 0

7. 2

8. $y - 13 = 5(x - 1)$ or $L(x) = 5x + 8$; $f(1.1) \approx 13.5$

9. $\Delta y = (x + \Delta x)^2 - (x + \Delta x) - (x^2 - x) = (2x - 1)\Delta x + (\Delta x)^2$

10. $dy = (-x^3e^{-x} + 3x^2e^{-x})dx = (3x^2 - x^3)e^{-x}dx$

C. Exercises

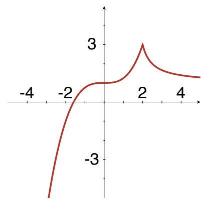
1. Solving $f'(x) = 3x^2 - 75 = 0$, we obtain the critical numbers -5 and 5, neither of which is in the interval $[-3, 4]$. Comparing $f(-3) = 348$ and $f(4) = -86$, we see that the absolute maximum is 348 and the absolute minimum is -86.

2. Solving $f'(x) = 8x + 1/x^2 = 0$, we obtain the critical number $-1/2$ which is not in the interval $[1/4, 1]$. Comparing $f(1/4) = -15/4$ and $f(1) = 3$, we see that the absolute maximum is 3 and the absolute minimum is $-15/4$.

3. Solving $f'(x) = x(x + 8)/(x + 4)^2 = 0$, we obtain the critical numbers -8 and 0. Comparing $f(-1) = 1/3$, $f(0) = 0$, and $f(3) = 9/7$, we see that the absolute maximum is 9/7 and the absolute minimum is 0.

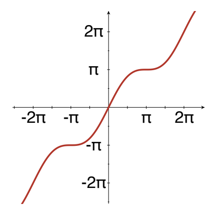
4. Solving $f'(x) = \dfrac{1}{2}(2x - 3)/\sqrt{x^2 - 3x + 5} = 0$, we obtain the critical number 3/2. Comparing $f(1) = \sqrt{3}$, $f(3/2) = \sqrt{11}/2$, and $f(3) = \sqrt{5}$, we see that the absolute maximum is $\sqrt{5}$ and the absolute minimum is $\sqrt{11}/2$.

5.

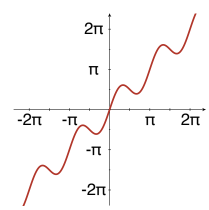


6. $f(x) = x + \sin x$; $f'(x) = 1 + \cos x$; $f''(x) = -\sin x$. Solving $f'(x) = 0$, we obtain the critical numbers $(2k+1)\pi$, for k an integer. Solving $f''(x) = 0$ we obtain the values $k\pi$, for k an integer.

x	$-\pi$		0		π		2π
f	$-\pi$	$\nearrow$	0	$\nearrow$	π	$\nearrow$	2π
f'	0	$+$	$+$	$+$	0	$+$	$+$
f''	0	$+$	0	$-$	0	$+$	0



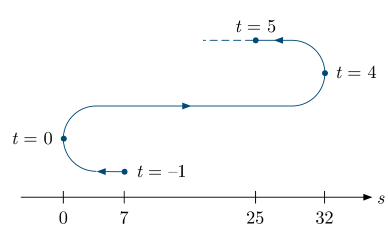
$g(x) = x + \sin 2x$; $g'(x) = 1 + 2\cos 2x$; $g''(x) = -4\sin 2x$. Solving $g'(x) = 0$ we obtain the critical numbers $\dfrac{\pi}{3} + \pi k$ and $\dfrac{2\pi}{3} + \pi k$, for k an integer. Solving $g''(x) = 0$ we obtain the values $k\pi$, for k an integer.



x	$-\frac{2\pi}{3}$		$-\frac{\pi}{2}$		$-\frac{\pi}{3}$		0		$\frac{\pi}{3}$		$\frac{\pi}{2}$		$\frac{2\pi}{3}$		π		$\frac{4\pi}{3}$
g	-1.2		$-\frac{\pi}{2}$		0.2		0		1.9		$\frac{\pi}{2}$		1.2		π		5.1
g'	0	$-$	$-$	$-$	0	$+$	$+$	$+$	0	$-$	$-$	$-$	0	$+$	$+$	$+$	0
g''	$-$	$-$	0	$+$	$+$	$+$	0	$-$	$-$	$-$	0	$+$	$+$	$+$	0	$-$	$-$

7. $v(t) = -3t^2 + 12t$. Solving $v(t) = 0$ we obtain $t = 0,\ 4$. To find the maximum velocity, we solve $v'(t) = -6t + 12 = 0$ and obtain $t = 2$. Comparing $v(-1) = -15$, $v(2) = 12$, and $v(5) = -15$, we see that the maximum velocity is 12. Since speed is the absolute value of velocity, the maximum speed on the interval is 15 when $t = -1$ and $t = 5$.

t	-1		0		4		5
s	7	$\leftarrow$	0	$\rightarrow$	32	$\leftarrow$	25
v		$-$	0	$+$	0	$-$	



8. Solving $s'(t) = -9.8t + 14.7 = 0$ we obtain $t = 3/2$. Since $s''(t) = -9.8 < 0$, the maximum height is $s(3/2) = 60.025$. Solving $s(t) = -4.9(t^2 - 3t - 10) = -4.9(t-5)(t+2) = 0$, we obtain $t = -2$ and 5. When the projectile strikes the ground at 5 seconds, the speed is $|s'(5)| = 34.3$ m/s.

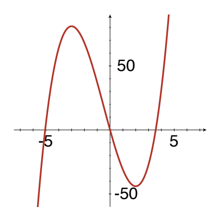
9. (a) $$\begin{aligned} f'(x) &= (x-a)^2[(x-b)^2 g'(x) + 2(x-b)g(x)] + 2(x-a)(x-b)^2 g(x) \\ &= (x-a)(x-b)[(x-a)(x-b)g'(x) + 2(x-a)g(x) + 2(x-b)g(x)] \end{aligned}$$
We see immediately that $f(a) = f(b) = 0$, so by Rolle's Theorem there exists c in (a, b) such that $f'(c) = 0$.

   (b) If $g(x) = C$ then $g'(x) = 0$ and $f'(x) = (x-a)(x-b)[2C(x-a+x-b)] = 2C(x-a)(x-b)(2x-a-b)$. Solving $f'(x) = 0$ we obtain the critical numbers a, b, and $\dfrac{a+b}{2}$.

10. $f'(x) = 1/3x^{2/3}$ is not defined at $x = 0$, so $f(x)$ is not differentiable on $[-1, 8]$. Solving $1/3x^{2/3} = [8^{1/3} - (-1)^{1/3}]/[8 - (-1)] = \dfrac{1}{3}$, we obtain $x = \pm 1$. There is no problem with the conclusion of a theorem applying, even though the hypotheses are not satisfied, unless the theorem is of the "if and only if" type.

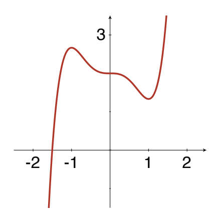
11. Solving $f'(x) = 6x^2 + 6x - 36 = 6(x+3)(x-2) = 0$ we obtain the critical numbers -3 and 2. The relative maximum is $(-3, 81)$ and the relative minimum is $(2, -44)$.

x		-3		2	
f	↗	81	↘	-44	↗
f'	$+$	0	$-$	0	$+$



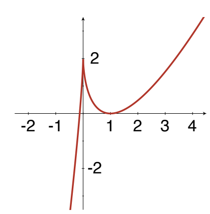
12. Solving $f'(x) = 5x^4 - 5x^2 = 5x^2(x-1)(x+1) = 0$ we obtain the critical numbers -1, 0, and 1. The relative maximum is $(-1, 8/3)$ and the relative minimum is $(1, 4/3)$.

x		-1		0		1	
f	↗	$8/3$	↘	2	↘	$4/3$	↗
f'	$+$	0	$-$	0	$-$	0	$+$



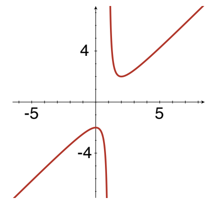
13. Solving $f'(x) = 4 - 4x^{-1/3} = 4(x^{1/3} - 1)/x^{1/3} = 0$ we obtain the critical number 1. Since $f'(0)$ does not exist, 0 is also a critical number. The relative maximum is $(0, 2)$ and the relative minimum is $(1, 0)$.

x		0		1	
f	↗	2	↘	0	↗
f'	$+$	undefined	$-$	0	$+$



14. Solving $f'(x) = x(x-2)/(x-1)^2 = 0$ we obtain the critical numbers 0 and 2. The relative maximum is $(0, -2)$ and the relative minimum is $(2, 2)$.

x		0		1		2	
f	$\nearrow$	-2	$\searrow$	undefined	$\searrow$	2	$\nearrow$
f'	$+$	0	$-$	undefined	$-$	0	$+$



15. Solving $f'(x) = 4x^3 + 24x^2 + 36x = 4x(x+3)^2 = 0$ we obtain the critical numbers -3 and 0. Solving $f''(x) = 12x^2 + 48x + 36 = 12(x+1)(x+3) = 0$ we obtain $x = -1$ and -3. The relative minimum is $(0, 0)$. The inflection points are $(-3, 27)$ and $(-1, 11)$.

x		-3		-1		0	
f	$\searrow$	27	$\searrow$	11	$\searrow$	0	$\nearrow$
f'	$-$	0	$-$	$-$	$-$	0	$+$
f''	$+$	0	$-$	0	$+$	$+$	$+$

16. Solving $f'(x) = 6x^5 - 12x^3 = 6x^3(x^2 - 2) = 0$ we obtain the critical numbers $-\sqrt{2}$, 0, and $\sqrt{2}$. Solving $f''(x) = 30x^4 - 36x^2 = 6x^2(5x^2 - 6) = 0$ we obtain the values $-\sqrt{6/5}$, 0, and $\sqrt{6/5}$. The relative maximum is $(0, 5)$. The relative minima are $(-\sqrt{2}, 1)$ and $(\sqrt{2}, 1)$. The inflection points are $(-\sqrt{6/5}, 301/125)$ and $(\sqrt{6/5}, 301/125)$.

x		$-\sqrt{2}$		$-\sqrt{6/5}$		0		$\sqrt{6/5}$		$\sqrt{2}$	
f	$\searrow$	1	$\nearrow$	301/125	$\nearrow$	5	$\searrow$	301/125	$\searrow$	1	$\nearrow$
f'	$-$	0	$+$	$+$	$+$	0	$-$	$-$	$-$	0	$+$
f''	$+$	$+$	$+$	0	$-$	0	$-$	0	$+$	$+$	$+$

17. Since $f'(x) = -\dfrac{1}{3}(x-3)^{-2/3}$, we see that the only critical number is 3. $f''(x) = \dfrac{2}{9}(x-3)^{-5/3}$ is undefined at $x = 3$. There are no relative extrema. The inflection point is $(3, 10)$.

x		3	
f		10	
f'	$-$	undefined	$-$
f''	$-$	undefined	$+$

18. The domain of $f(x)$ is $[1, \infty)$. Solving $f'(x) = (x-1)^{3/2}\left(\dfrac{7}{2}x - 1\right) = 0$ we obtain the critical number 1. Solving $f''(x) = (x-1)^{1/2}\left(\dfrac{35}{4}x - 5\right) = 0$ we obtain the value 1. Since neither $f'(x)$ nor $f''(x)$ is 0 in $(1, \infty)$, the function has no relative extrema or inflection points.

19. (c), (d)

20. (d), (e), (f)

21. (c), (d), (e)

22. (d), (f)

23. (a), (c)

24. (a), (b), (d)

25. $f'(x) = (x-a)[(x-b)+(x-c)]+(x-b)(x-c) = (x-a)(x-b)+(x-a)(x-c)+(x-b)(x-c)$
$f''(x) = x-a+x-b+x-a+x-c+x-b+x-c = 6x-2(a+b+c)$
Solving $f'(x) = 0$ we obtain $x = \dfrac{1}{3}(a+b+c)$. Since $f''(\dfrac{1}{6}(a+b+c)) = -(a+b+c)$ and $f''(a+b+c) = 4(a+b+c)$, we see that $f''(x)$ has opposite signs on either side of $\dfrac{1}{3}(a+b+c)$. Thus, the graph of $f(x)$ has a point of inflection at $\dfrac{1}{3}(a+b+c)$.

26. Let b be the base of the triangle, h the altitude, and A the area. Then $A = \dfrac{1}{2}bh$ and $dA/dt = \dfrac{1}{2}b\ dh/dt + \dfrac{1}{2}h\ db/dt$. Given $dA/dt = 15$, $db/dt = -\dfrac{1}{2}$, $h = 8$, and $b = 6$, we have $15 = \dfrac{1}{2}(6)dh/dt + \dfrac{1}{2}(8)\left(-\dfrac{1}{2}\right) = 3\ dh/dt - 2$, so $dh/dt = \dfrac{17}{3}$ in//min.

27. Assume that the center of the circle is at the origin so that $x^2+y^2 = r^2$. In the first quadrant a vertex of the square is $(x,y) = (x,x)$. Therefore $2x^2 = r^2$, $A = (2x)(2x)$, $A(r) = 2r^2$, and $\dfrac{dA}{dt} = 4r\dfrac{dr}{dt}$. At the instant when $r = 2$,

$$\frac{dA}{dt} = 4(2)(4) = 32 \text{ in}^2/\text{min}.$$

28. The rate of change of the volume of water is $\dfrac{dV}{dt} = \dfrac{1}{10} - \dfrac{1}{5} = -\dfrac{1}{10}$ m^3/min. Since this is negative, the volume of the water, and hence the height, is decreasing. We want to find dh/dt when $h = 5$ m. Differentiating the volume with respect to time gives $\dfrac{dV}{dt} = (20\pi h - \pi h^2)\dfrac{dh}{dt}$, and so

$$\frac{dh}{dt} = \frac{1}{20\pi h - \pi h^2}\frac{dV}{dt} \quad \text{and} \quad \left.\frac{dh}{dt}\right|_{h=5} = \frac{1}{75\pi}\left(-\frac{1}{10}\right) = -\frac{1}{750\pi} \text{ m/min}.$$

29. $$B'(x) = \frac{1}{2}\mu_0 r_0^2 I\left\{-\frac{3}{2}\left[r_0^2 + \left(x+\frac{r_0}{2}\right)^2\right]^{-5/2}\left[2\left(x+\frac{r_0}{2}\right)\right]\right.$$
$$\left.-\frac{3}{2}\left[r_0^2 + \left(x-\frac{r_0}{2}\right)^2\right]^{-5/2}\left[2\left(x-\frac{r_0}{2}\right)\right]\right\}$$
$$= -\frac{3}{2}\mu_0 r_0^2 I\left\{\frac{x+r_0/2}{\left[r_0^2+\left(x+\frac{r_0}{2}\right)^2\right]^{5/2}} + \frac{x-r_0/2}{\left[r_0^2+\left(x-\frac{r_0}{2}\right)^2\right]^{5/2}}\right\}$$

We see that $B'(0) = 0$. To apply the first derivative test, we compute

$$B'\left(-\frac{r_0}{4}\right) = -\frac{384\mu_0 I}{r_0^2}\left(\frac{1}{17^{5/2}} - \frac{3}{25^{5/2}}\right) \quad \text{and} \quad B'\left(\frac{r_0}{4}\right) = -\frac{384\mu_0 I}{r_0^2}\left(\frac{3}{25^{5/2}} - \frac{1}{17^{5/2}}\right).$$

Since $B'\left(-\dfrac{r_0}{4}\right) > 0$ and $B'\left(\dfrac{r_0}{4}\right) < 0$, B has a maximum at $x = 0$.

30. We want to maximize $P(R) = RE^2/(r+R)^2$. Solving

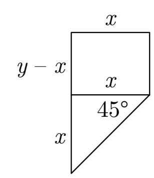
$$P'(R) = \frac{(r+R)^2E^2 - RE^2[2(r+R)]}{(r+R)^4} = \frac{(r-R)E^2}{(r+R)^3} = 0,$$

we obtain the critical number $R = r$. Since $P'(R) > 0$ for $R < r$ and $P'(R) < 0$ for $R > r$, we see from the first derivative test that the power dissipated is maximum when $R = r$.

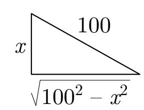
31. $\dfrac{dx}{dy} = \dfrac{h-2y}{\sqrt{y(h-y)}}$. Solving $\dfrac{dx}{dy} = 0$ we obtain the critical number $h/2$. Since $0 \le y \le h$, we compare $z(0) = 0$, $x(h/2) = h$, $x(h) = 0$, and observe that the maximum distance of h ft is obtained for $y = h/2$.

32. We are given $2r + s = 60$ and we want to maximize $A(r) = \dfrac{1}{2}rs = \dfrac{1}{2}r(60-2r) = 30r - r^2$. Solving $A'(r) = 30 - 2r = 0$ we obtain the critical number 15. Since $A''(15) = -2 < 0$, the maximum area is $A(15) = 225$ cm^2.

33. Since 585 feet of fence is available, $y = 585 - x$. We want to maximize $A(x) = x(y-x) + \dfrac{1}{2}x^2 = x(585-2x) + \dfrac{1}{2}x^2 = 585x - \dfrac{3}{2}x^2$. Solving $A'(x) = 585 - 3x$ we obtain the critical number 195. Since $A''(x) = -3 < 0$, the maximum area is obtained when $x = 195$ ft and $y = 390$ ft. The maximum area is $A(195) = 57037.5$ ft^2.



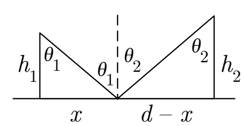
34. Let x and $\sqrt{100^2 - x^2}$ be the lengths along the walls. We want to maximize $A(x) = \dfrac{1}{2}x\sqrt{100^2 - x^2}$ on $[0, 100]$. Solving



$$A'(x) = -\frac{x^2}{2\sqrt{100^2 - x^2}} + \frac{\sqrt{100^2 - x^2}}{2} = \frac{100^2 - 2x^2}{2\sqrt{100^2 - x^2}} = 0$$

we obtain the critical number $50\sqrt{2}$. Comparing $A(0) = 0$, $A(50\sqrt{2}) = 2500$, and $A(100) = 0$, we see that the area is maximized when the length along each wall is $50\sqrt{2}$ m.

35. We want to minimize $T(x) = \dfrac{\sqrt{x^2 + h_1^2}}{c} + \dfrac{\sqrt{(d-x)^2 + h_2^2}}{c}$ on $[0, d]$. Setting



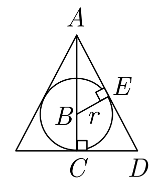
$$T'(x) = \frac{1}{c}\left[\frac{x}{\sqrt{x^2 + h_1^2}} - \frac{d-x}{\sqrt{(d-x)^2 + h_2^2}}\right] = 0,$$

we obtain $x\sqrt{(d-x)^2 + h_2^2} = (d-x)\sqrt{x^2 + h_1^2}$ or $x^2[(d-x)^2 + h_2^2] = (d-x)^2(x^2 + h_1^2)$. Simplifying, we have $\dfrac{d-x}{h_2} = \dfrac{x}{h_1}$ or $\tan\theta_2 = \tan\theta_1$. Solving for x, we find $x = \dfrac{h_1 d}{h_1 + h_2}$. Since

$$T''(x) = \frac{1}{c}\left\{\frac{h_1^2}{(x^2 + h_1^2)^{3/2}} + \frac{h_2^2}{[(d-x)^2 + h_2^2]^{3/2}}\right\} > 0,$$

the time is minimized when $\tan\theta_2 = \tan\theta_1$.

36. Note that $\triangle AEB$ is similar to $\triangle ACD$. Let $h = \overline{AC}$ and $R = \overline{CD}$. Then



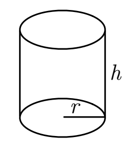
$$\frac{R}{r} = \frac{h}{AE} = \frac{h}{\sqrt{(h-r)^2 - r^2}} = \frac{h}{\sqrt{r^2 - 2rh}} \quad \text{and} \quad R = \frac{rh}{\sqrt{h^2 - 2rh}}.$$

We want to minimize $V(h) = \dfrac{1}{3}\pi R^2 h = \dfrac{1}{3}\pi \dfrac{r^2h^3}{h^2 - 2rh}$ on $(2r, \infty)$. Solving

$$V'(h) = \frac{1}{3}\pi r^2 h \frac{h - 4r}{(h-2r)^2} = 0,$$

we obtain $h = 4r$. Since $V'(3r) < 0$ and $V'(5r) > 0$, we have by the first derivative test that the volume is minimized when $h = 4r$ and $R = \sqrt{2}r$.

37. Let r be the radius and h the height. Then $\pi r^2 h = 100$ and $h = 100/\pi r^2$. We want to minimize $C(r) = 3\pi r^2 + \pi r^2 + 2\pi r h = 4\pi r^2 + 200/r$. Solving $C'(r) = 8\pi r - 200/r^2 = 0$ we obtain the critical number $\sqrt[3]{25/\pi}$. Since $C''(\sqrt[3]{25/\pi}) > 0$, the cost is minimized when $r = \sqrt[3]{25/\pi}$ and $h = 4\sqrt[3]{25/\pi}$. (In this case $h = 4r$.)



38. Let x be the side of the square cut-out. We want to maximize $V(x) = (15-2x)\left(\dfrac{30-2x}{2}\right)x = 225x - 45x^2 + 2x^3$ on $[0, 7.5]$. Solving $V'(x) = 225 - 90x + 6x^2 = 0$ we obtain the critical numbers $\dfrac{15 \pm 5\sqrt{3}}{2}$. Only $\dfrac{15 - 5\sqrt{3}}{2} \approx 3.17$ is in $[0, 7.5]$, so comparing $V(0) = V(7.5) = 0$ and $V(3.17) \approx 324.76$ we see that the maximum volume of $324.76\ \text{in}^3$ is obtained when the square cut-out has sides approximately 3.17 in. The dimensions of the box are approximately 8.66 in by 11.83 in by 3.17 in.

39. $$\lim_{x\to\sqrt{3}} \frac{\sqrt{3} - \tan(\pi/x^2)}{x - \sqrt{3}} \stackrel{h}{=} \lim_{x\to\sqrt{3}} \frac{[\sec^2(\pi/x^2)](2\pi/x^3)}{1} = \frac{8\pi}{3\sqrt{3}}$$

40. $$\lim_{\theta\to 0} \frac{10\theta - 5\sin 2\theta}{10\theta - 2\sin 5\theta} \stackrel{h}{=} \lim_{\theta\to 0} \frac{10 - 10\cos 2\theta}{10 - 10\cos 5\theta} \stackrel{h}{=} \lim_{\theta\to 0} \frac{20\sin 2\theta}{50\sin 5\theta} \stackrel{h}{=} \lim_{\theta\to 0} \frac{40\cos 2\theta}{250\cos 5\theta} = \frac{4}{25}$$

41. $$\lim_{x\to\infty} x\left(\cos\frac{1}{x} - e^{2/x}\right) = \lim_{x\to\infty} \frac{\cos(1/x) - e^{2/x}}{1/x} \stackrel{h}{=} \lim_{x\to\infty} \frac{[\sin(1/x)](1/x^2) - e^{2/x}(-2/x^2)}{-1/x^2}$$
$$= \lim_{x\to\infty} \frac{\sin(1/x) + 2e^{2/x}}{-1} = -2$$

42. $$\lim_{y\to 0}\left[\frac{1}{y} - \frac{1}{\ln(y+1)}\right] = \lim_{y\to 0} \frac{\ln(y+1) - y}{y\ln(y+1)} \stackrel{h}{=} \lim_{y\to 0} \frac{1/(y+1) - 1}{y/(y+1) + \ln(y+1)}$$
$$= \lim_{y\to 0} \frac{1 - (y+1)}{y + (y+1)\ln(y+1)} \stackrel{h}{=} \lim_{y\to 0} \frac{-1}{1 + 1 + \ln(y+1)} = -\frac{1}{2}$$

43. $$\lim_{t\to 0} \frac{(\sin t)^2}{\sin t^2} \stackrel{h}{=} \lim_{t\to 0} \frac{2\sin t\cos t}{2t\cos t^2} = \lim_{t\to 0} \frac{\sin 2t}{2t\cos t^2} \stackrel{h}{=} \lim_{t\to 0} \frac{2\cos 2t}{-4t^2\sin t^2 + 2\cos t^2} = 1$$

44. $$\lim_{x\to 0} \frac{\tan 5x}{e^{3x/2} - e^{-x/2}} \stackrel{h}{=} \lim_{x\to 0} \frac{5\sec^2 5x}{(3/2)e^{3x/2} + (1/2)e^{-x/2}} = \frac{5}{3/2 + 1/2} = \frac{5}{2}$$

45. Set $y = (3x)^{-1/\ln x}$. Then $\ln y = -\dfrac{\ln 3x}{\ln x}$ and $\lim\limits_{x\to 0^+}\left(-\dfrac{\ln 3x}{\ln x}\right) \overset{h}{=} \lim\limits_{x\to 0^+}\dfrac{-1/x}{1/x} = -1$. Thus, $\lim\limits_{x\to 0^+}(3x)^{-1/\ln x} = e^{-1}$.

46. Set $y = (2x+e^{3x})^{4/x}$. Then $\ln y = \dfrac{4\ln(2x+e^{3x})}{x}$ and

$$\lim_{x\to 0}\frac{4\ln(2x+e^{3x})}{x} \overset{h}{=} \lim_{x\to 0}\frac{4(2+3e^{3x})/(2x+e^{3x})}{1} = \lim_{x\to 0}\frac{4(2+3e^{3x})}{2x+e^{3x}} = 20.$$

Thus, $\lim\limits_{x\to 0}(2x+e^{3x})^{4/x} = e^{20}$.

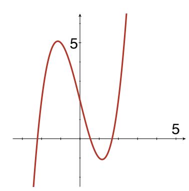
47. $$\lim_{x\to\infty}\ln\left(\frac{x+e^{2x}}{1+e^{4x}}\right) = \ln\left(\lim_{x\to\infty}\frac{x+e^{2x}}{1+e^{4x}}\right) \overset{h}{=} \ln\left(\lim_{x\to\infty}\frac{1+2e^{2x}}{4e^{4x}}\right) \overset{h}{=} \ln\left(\lim_{x\to\infty}\frac{4e^{2x}}{16e^{4x}}\right)$$
$$= \ln\left(\lim_{x\to\infty}\frac{1}{4e^{2x}}\right) = \ln 0 = -\infty$$

The limit does not exist.

48. $$\lim_{x\to 0^+}x(\ln x)^2 = \lim_{x\to 0^+}\frac{(\ln x)^2}{1/x} \overset{h}{=} \lim_{x\to 0^+}\frac{(2\ln x)/x}{-1/x^2} = \lim_{x\to 0^+}\frac{2\ln x}{-1/x} \overset{h}{=} \lim_{x\to 0^+}\frac{2/x}{1/x^2} = \lim_{x\to 0^+}2x = 0$$

49. Let $f(x) = x^3 - 4x + 2$. Then $f'(x) = 3x^2 - 4$ and

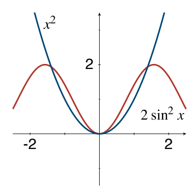
$$x_{n+1} = x_n - \frac{x_n^3 - 4x_n + 2}{3x_n^2 - 4} = \frac{2x_n^3 - 2}{3x_n^2 - 4}.$$



From the graph we see that $f(x)$ has its largest positive root near $x_0 = 2$. Then $x_1 \approx 1.7500$, $x_2 \approx 1.6807$, $x_3 \approx 1.6752$, $x_4 \approx 1.6751$, $x_5 \approx 1.6751$, and the largest positive root is approximately 1.6751.

50. Write the equation as $2\sin^2 x = x^2$ and let $f(x) = 2\sin^2 x - x^2$. Then $f'(x) = 4\sin x\cos x - 2x$ and

$$x_{n+1} = x_n - \frac{2\sin^2 x_n - x_n^2}{4\sin x_n\cos x_n - 2x_n}.$$



From the graph we see that $f(x)$ has its smallest positive root near $x_0 = 1$. Then $x_1 \approx 3.2940$, $x_2 \approx 1.4896$, $x_3 \approx 1.4022$, $x_4 \approx 1.3917$, $x_5 \approx 1.3916$, $x_6 \approx 1.3916$, and the smallest positive root is approximately 1.3916.

Chapter 5

Integrals

5.1 The Indefinite Integral

1. $\int 3\,dx = 3x + C$

2. $\int (\pi^2 - 1)\,dx = (\pi^2 - 1)x + C$

3. $\int x^5\,dx = \frac{1}{6}x^6 + C$

4. $\int 5x^{1/4}\,dx = 4x^{5/4} + C$

5. $\int \frac{dx}{\sqrt[3]{x}} = \int x^{-1/3}\,dx = \frac{3}{2}x^{2/3} + C$

6. $\int \sqrt[3]{x^2}\,dx = \int x^{2/3}\,dx = \frac{3}{5}x^{5/3} + C$

7. $\int (1 - t^{-0.52})\,dt = t - \frac{1}{0.48}t^{0.48} + C$

8. $\int 10w\sqrt{w}\,dw = \int 10w^{3/2}\,dw = 4w^{5/2} + C$

9. $\int (3x^2 + 2x - 1)\,dx = x^3 + x^2 - x + C$

10. $\int \left(2\sqrt{t} - t - \frac{9}{t^2}\right)\,dt = \int (2t^{1/2} - t - 9t^{-2})\,dt = \frac{4}{3}t^{3/2} - \frac{1}{2}t^2 + 9t^{-1} + C$

11. $\int \sqrt{x}(x^2 - 2)\,dx = \int (x^{5/2} - 2x^{1/2})\,dx = \frac{2}{7}x^{7/2} - \frac{4}{3}x^{3/2} + C$

12. $\displaystyle\int \left(\frac{5}{\sqrt[3]{s^2}} + \frac{2}{\sqrt{s^3}}\right) ds = \int (5s^{-2/3} + 2s^{-3/2})\, ds = 15s^{1/3} - 4s^{-1/2} + C$

13. $\displaystyle\int (4x+1)^2\, dx = \int (16x^2 + 8x + 1)\, dx = \frac{16}{3}x^3 + 4x^2 + x + C$

14. $\displaystyle\int (\sqrt{x} - 1)^2\, dx = \int (x - 2x^{1/2} + 1)\, dx = \frac{1}{2}x^2 - \frac{4}{3}x^{3/2} + x + C$

15. $\displaystyle\int (4w-1)^3\, dw = \int (64w^3 - 48w^2 + 12w - 1)\, dw = 16w^4 - 16w^3 + 6w^2 - w + C$

16. $\displaystyle\int (5u-1)(3u^2+2)\, du = \int (15u^3 - 3u^2 + 10u - 2)\, du = \frac{15}{4}u^4 - u^3 + 5u^2 - 2u + C$

17. $\displaystyle\int \frac{r^2 - 10r + 4}{r^3}\, dr = \int (r^{-1} - 10r^{-2} + 4r^{-3})\, dr = \ln|r| + 10r^{-1} - 2r^{-2} + C$

18. $\displaystyle\int \frac{(x+1)^2}{\sqrt{x}}\, dx = \int (x^{3/2} + 2x^{1/2} + x^{-1/2})\, dx = \frac{2}{5}x^{5/2} + \frac{4}{3}x^{3/2} + 2x^{1/2} + C$

19. $\displaystyle\int \frac{x^{-1} - x^{-2} + x^{-3}}{x^2}\, dx = \int (x^{-3} - x^{-4} + x^{-5})\, dx = -\frac{1}{2}x^{-2} + \frac{1}{3}x^{-3} - \frac{1}{4}x^{-4} + C$

20. $$\begin{aligned}\int \frac{t^3 - 8t + 1}{(2t)^4}\, dt &= \frac{1}{16}\int (t^{-1} - 8t^{-3} + t^{-4})\, dt = \frac{1}{16}\left(\ln|t| + 4t^{-2} - \frac{1}{3}t^{-3}\right) + C \\ &= \frac{\ln|t|}{16} + \frac{1}{4}t^{-2} - \frac{1}{48}t^{-3} + C\end{aligned}$$

21. $\displaystyle\int (4\sin x - 1 + 8x^{-5})\, dx = -4\cos x - x - 2x^{-4} + C$

22. $\displaystyle\int (-3\cos x + 4\sec^2 x)\, dx = -3\sin x + 4\tan x + C$

23. $\displaystyle\int \csc x(\csc x - \cot x)\, dx = \int (\csc^2 x - \csc x \cot x)\, dx = -\cot x + \csc x + C$

24. $\displaystyle\int \frac{\sin t}{\cos^2 t}\, dt = \int \tan t\ \sec t\ dt = \sec t + C$

25. $\displaystyle\int \frac{2 + 3\sin^2 x}{\sin^2 x}\, dx = \int (2\csc^2 x + 3)\, dx = -2\cot x + 3x + C$

26. $\displaystyle\int \left(40 - \frac{2}{\sec\theta}\right) d\theta = \int (40 - 2\cos\theta)\, d\theta = 40\theta - 2\sin\theta + C$

27. $\displaystyle\int (8x + 1 - 9e^x)\, dx = 4x^2 + x - 9e^x + C$

28. $\displaystyle\int (15x^{-1} - 4\sinh x)\, dx = 15\ln x - 4\cosh x + C$

29. $\displaystyle\int \frac{2x^3 - x^2 + 2x + 4}{1 + x^2}\,dx = \int \left(2x - 1 + \frac{5}{x^2 + 1}\right) dx = x^2 - x + 5\tan^{-1} x + C$

30. $\displaystyle\int \frac{x^6}{1 + x^2}\,dx = -\int \left(x^4 - x^2 + 1 - \frac{1}{x^2 + 1}\right) dx = \frac{1}{5}x^5 - \frac{1}{3}x^3 + x - \tan^{-1} x + C$

31. $\displaystyle\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx = \tan x - x + C$

32. $\displaystyle\int \cos^2 \frac{x}{2}\,dx = \int \frac{1}{2}(1 + \cos x)\,dx = \int \left(\frac{1}{2} + \frac{\cos x}{2}\right) dx = \frac{1}{2}x + \frac{\sin x}{2} + C$

33. $\displaystyle\frac{d}{dx}(\sqrt{2x + 1} + C) = \frac{2}{2\sqrt{2x + 1}} = \frac{1}{\sqrt{2x + 1}}$

34. $\displaystyle\frac{d}{dx}\left[\frac{1}{40}(2x^2 - 4x)^{10} + C\right] = \frac{1}{4}(2x^2 - 4x)^9(4x - 4) = (2x^2 - 4x)^9(x - 1)$

35. $\displaystyle\frac{d}{dx}\left(\frac{1}{4}\sin 4x + C\right) = \frac{4}{4}\cos 4x = \cos 4x$

36. $\displaystyle\frac{d}{dx}\left(\frac{1}{2}\sin^2 x + C\right) = \frac{2}{2}\sin x \cos x = \sin x \cos x$

37. $\displaystyle\frac{d}{dx}\left(-\frac{1}{2}\cos x^2 + C\right) = \frac{2x}{2}\sin x^2 = x \sin x^2$

38. $\displaystyle\frac{d}{dx}\left(-\frac{1}{2\sin^2 x} + C\right) = -\frac{(2\sin^2 x) \cdot 0 - 1 \cdot (4 \sin x \cos x)}{4\sin^4 x} = \frac{\cos x}{\sin^3 x}$

39. $\displaystyle\frac{d}{dx}(x \ln x - x + C) = x\left(\frac{1}{x}\right) + \ln x - 1 = \ln x$

40. $\displaystyle\frac{d}{dx}(xe^x - e^x + C) = xe^x + e^x - e^x = xe^x$

41. $\displaystyle\frac{d}{dx}\int (x^2 - 4x + 5)\,dx = \frac{d}{dx}\left(\frac{1}{3}x^3 - 2x^2 + 5x + C\right) = x^2 - 4x + 5$

42. $\displaystyle\int \frac{d}{dx}(x^2 - 4x + 5)\,dx = \int (2x - 4)\,dx = x^2 - 4x + C$

43. $\displaystyle y = \int (6x^2 + 9)\,dx = 2x^3 + 9x + C$

44. $\displaystyle y = \int (10x + 3x^{1/2})\,dx = 5x^2 + 2x^{3/2} + C$

45. $\displaystyle y = \int x^{-2}\,dx = -x^{-1} + C = -\frac{1}{x} + C$

46. $y = \int \frac{(2+x)^2}{x^5} = \int (4x^{-5} + 4x^{-4} + x^{-3})\,dx = -x^{-4} - \frac{4}{3}x^{-3} - \frac{1}{2}x^{-2} + C$

47. $y = \int (1 - 2x + \sin x)\,dx = x - x^2 - \cos x + C$

48. $y = \int \sec^2 x\,dx = \tan x + C$

49. We have $f(x) = \int (2x-1)\,dx = x^2 - x + C$. Solving $3 = f(2) = 4 - 2 + C = 2 + C$ we obtain $C = 1$. Thus $f(x) = x^2 - x + 1$.

50. We have $f(x) = \int x^{-1/2}\,dx = 2x^{1/2} + C$. Solving $1 = f(9) = 2\sqrt{9} + C = 6 + C$ we obtain $C = -5$. Thus $f(x) = 2\sqrt{x} - 5$.

51. $f'(x) = \int 2x\,dx = x^2 + C_1$; $\quad f(x) = \int (x^2 + C_1)\,dx = \frac{1}{3}x^3 + C_1x + C_2$

52. We have $f'(x) = \int 6\,dx = 6x + C$. Solving $2 = f'(-1) = -6 + C$ we obtain $C = 8$. Then $f'(x) = 6x+8$ and $f(x) = \int (6x+8)\,dx = 3x^2+8x+C$. Solving $0 = f(-1) = 3-8+C = -5+C$ we obtain $C = 5$. Thus $f(x) = 3x^2 + 8x + 5$.

53. We have $f'(x) = \int (12x^2 + 2)\,dx = 4x^3 + 2x + C$. Solving $3 = f'(1) = 6 + C$ we obtain $C = -3$. Then $f'(x) = 4x^3 + 2x - 3$ and $f(x) = \int (4x^3 + 2x - 3)\,dx = x^4 + x^2 - 3x + C$. Solving $1 = f(1) = -1 + C$ we obtain $C = 2$. Thus $f(x) = x^4 + x^2 - 3x + 2$.

54. $f(x) = a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$

55. G is an antiderivative of f. In other words, since $G'(x) = f(x)$, f is the slope function for G. Observe where G is increasing, and the graph of f is always positive. Also, G appears to have no relative extrema on the interval shown, and correspondingly the graph of f does not cross the x-axis.

56. F is an antiderivative of f. In other words, since $F'(x) = f(x)$, f is the slope function for F. Observe where the tangent lines to the graph of F have positive (negative) slope, the graph of f is positive (negative). Also, the graph of F has two relative extrema and the graph of f correspondingly crosses the x-axis.

57. $y = \int \left(\frac{\omega^2}{g}x\right)dx = \frac{\omega^2}{2g}x^2 + C$. From Figure 5.1.5 we see that $y(0) = 0$. Thus, $0 = y(0) = C$, and $y = \frac{\omega^2 x^2}{2g}$.

58. We have $f'(x) = \int \left(\frac{qL}{2EI}x - \frac{q}{2EI}x^2\right)dx = \frac{qL}{4EI}x^2 - \frac{q}{6EI}x^3 + C$. Solving $0 = f'\left(\frac{L}{2}\right) = \frac{qL^3}{16EI} - \frac{qL^3}{48EI} + C$ we obtain $C = -\frac{qL^3}{24EI}$. Then $f'(x) = \frac{qL}{4EI}x^2 - \frac{q}{6EI}x^3 - \frac{qL^3}{24EI}$ and $f(x) = \int \left(\frac{qL}{4EI}x^2 - \frac{q}{6EI}x^3 - \frac{qL^3}{24EI}\right)dx = \frac{qL}{12EI}x^3 - \frac{q}{24EI}x^4 - \frac{qL^3}{24EI}x + C$. Solving $0 = f(0) = C$ we obtain $C = 0$. Thus $f(x) = \frac{q}{24EI}(2Lx^3 - x^4 - L^3x)$.

59. $\dfrac{d}{dx}(\ln|\ln x| + C) = \dfrac{1}{\ln x}\left(\dfrac{1}{x}\right) = \dfrac{1}{x\ln x}$

60. $\dfrac{d}{dx}(x^2e^x - 2xe^x + 2e^x + C) = x^2e^x + 2xe^x - 2xe^x - 2e^x + 2e^x = x^2e^x$

61. Since $f'(x) = x^2$, $f(x) = \int x^2\,dx = \dfrac{1}{3}x^3 + C$. Since $y = 4x + 7$ is a tangent line to the graph of f, then $4x + 7 = \dfrac{1}{3}x^3 + C$ at some point on f. In addition, the slope at this point is $4 = f'(x) = x^2$, so $x = \pm 2$. Thus, $4(\pm 2) + 7 = \dfrac{1}{3}(\pm 2)^3 + C$, so $C = 37/3$ or $5/3$. Thus, $f(x) = \dfrac{1}{3}x^3 + \dfrac{37}{3}$ or $f(x) = \dfrac{1}{3}x^3 + \dfrac{5}{3}$.

62. $e^{4\int dx/x} = e^{4\ln|x|+C} = e^{\ln x^4}e^C = C_1e^{\ln x^4} = C_1x^4$

63. $\dfrac{d}{dx}\left[\dfrac{1}{4}(x+1)^4 + C\right] = (x+1)^3$

$\dfrac{d}{dx}\left(\dfrac{1}{4}x^4 + x^3 + \dfrac{3}{2}x^2 + x + C\right) = x^3 + 3x^2 + 3x + 1 = (x+1)^3$

Thus, both results are correct.

64. Since $\dfrac{d}{dx}\sin\pi x = \pi\cos\pi x$, the antiderivative F of $\cos\pi x$ would be of the form $\dfrac{1}{\pi}\sin\pi x + C$. Solving $F(3/2) = 0 = \dfrac{1}{\pi}\sin\dfrac{3\pi}{2} + C$ we obtain $C = \dfrac{1}{\pi}$. Thus, $F(x) = \dfrac{1}{\pi}\sin\pi x + \dfrac{1}{\pi}$.

5.2 Integration by the u-Substitution

1. $$\int\sqrt{1-4x}\,dx = -\frac{1}{4}\int(1-4x)^{1/2}(-4\,dx) \qquad \boxed{u = 1-4x,\ du = -4\,dx}$$
$$= -\frac{1}{4}\int u^{1/2}du = \frac{1}{6}u^{3/2} + C = -\frac{1}{6}(1-4x)^{3/2} + C$$

2. $$\int(8x+2)^{1/3}\,dx = \frac{1}{8}\int(8x+2)^{1/3}(8\,dx) \qquad \boxed{u = 8x+2,\ du = 8\,dx}$$
$$= \frac{1}{8}\int u^{1/3}du = \frac{3}{32}u^{4/3} + C = \frac{3}{32}(8x+2)^{4/3} + C$$

3. $$\int\frac{1}{(5x+1)^3}\,dx = \frac{1}{5}\int(5x+1)^{-3}(5\,dx) \qquad \boxed{u = 5x+1,\ du = 5\,dx}$$
$$= \frac{1}{5}\int u^{-3}\,du = -\frac{1}{10}u^{-2} + C = -\frac{1}{10(5x+1)^2} + C$$

4. $$\int(7-x)^{49}\,dx = -\int(7-x)^{49}(-dx) \qquad \boxed{u = 7-x,\ du = -dx}$$
$$= -\int u^{49}du = -\frac{1}{50}u^{50} + C = -\frac{1}{50}(7-x)^{50} + C$$

5. $$\int x\sqrt{x^2+4}\,dx = \frac{1}{2}\int \sqrt{x^2+4}\ (2x\,dx) \qquad \boxed{u = x^2+4,\ du = 2x\,dx}$$
$$= \frac{1}{2}\int u^{1/2}du = \frac{1}{3}u^{3/2} + C = \frac{1}{3}(x^2+4)^{3/2} + C$$

6. $$\int \frac{t}{\sqrt[3]{t^2+9}}\,dt = \frac{1}{2}\int (t^2+9)^{-1/3}(2t\,dt) \qquad \boxed{u = t^2+9,\ dt = 2t\,dt}$$
$$= \frac{1}{2}\int u^{-1/3}du = \frac{3}{4}u^{2/3} + C = \frac{3}{4}(t^2+9)^{2/3} + C$$

7. $$\int \sin^5 3x \cos 3x\,dx = \frac{1}{3}\int (\sin^5 3x)(3\cos 3x\,dx) \qquad \boxed{u = \sin 3x,\ du = 3\cos 3x\,dx}$$
$$= \frac{1}{3}\int u^5 du = \frac{1}{18}u^6 + C = \frac{1}{18}\sin^6 3x + C$$

8. $$\int \sin 2\theta \cos^4 2\theta\,d\theta = -\frac{1}{2}\int (\cos^4 2\theta)(-2\sin 2\theta\,d\theta) \qquad \boxed{u = \cos 2\theta,\ du = -2\sin 2\theta\,d\theta}$$
$$= -\frac{1}{2}\int u^4 du = -\frac{1}{10}u^5 + C = -\frac{1}{10}\cos^5 2\theta + C$$

9. $$\int \tan^2 2x \sec^2 2x\,dx = \frac{1}{2}\int (\tan^2 2x)(2\sec^2 2x\,dx) \qquad \boxed{u = \tan 2x,\ du = 2\sec^2 2x\,dx}$$
$$= \frac{1}{2}\int u^2\,du = \frac{1}{6}u^3 + C = \frac{1}{6}\tan^3 2x + C$$

10. $$\int \sqrt{\tan x}\sec^2 x\,dx = \int (\tan x)^{1/2}(\sec^2 x\,dx) \qquad \boxed{u = \tan x,\ du = \sec^2 x\,dx}$$
$$= \int u^{1/2}du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(\tan x)^{3/2} + C$$

11. $$\int \sin 4x\,dx = \frac{1}{4}\int (\sin 4x)(4\,dx) = -\frac{1}{4}\cos 4x + C$$

12. $$\int 5\cos\frac{x}{2}\,dx = 10\int \left(\cos\frac{x}{2}\right)\left(\frac{dx}{2}\right) = 10\sin\frac{x}{2} + C$$

13. $$\int (\sqrt{2t} - \cos 6t)\,dt = \sqrt{2}\int t^{1/2}dt - \frac{1}{6}\int (\cos 6t)(6\,dt) = \frac{2\sqrt{2}}{3}t^{3/2} - \frac{1}{6}\sin 6t + C$$
$$= \frac{1}{3}(2t)^{3/2} - \frac{1}{6}\sin 6t + C$$

14. $$\int \sin(2-3x)\,dx = -\frac{1}{3}\int \sin(2-3x)(-3\,dx) = \frac{1}{3}\cos(2-3x) + C$$

15. $$\int x\sin x^2\,dx = \frac{1}{2}\int (\sin x^2)(2x\,dx) = -\frac{1}{2}\cos x^2 + C$$

16. $$\int \frac{\cos(1/x)}{x^2}\,dx = -\int [\cos(1/x)](-dx/x^2) = -\sin(1/x) + C$$

17. $$\int x^2 \sec^2 x^3\,dx = \frac{1}{3}\int (\sec^2 x^3)(3x^2\,dx) = \frac{1}{3}\tan x^3 + C$$

18. $$\int \csc^2(0.1x)\,dx = \frac{1}{0.1}\int (\csc^2 0.1x)(0.1\,dx) = -10\cot(0.1x) + C$$

19. $$\int \frac{\csc\sqrt{x}\cot\sqrt{x}}{\sqrt{x}}dx = 2\int \frac{\csc\sqrt{x}\cot\sqrt{x}}{2\sqrt{x}}dx \qquad \boxed{u=\sqrt{x},\ du = dx/2\sqrt{x}}$$
$$= 2\int \csc u\cot u\,du = -2\csc u + C = -2\csc\sqrt{x} + C$$

20. $$\int \tan 5v \sec 5v\,dv = \frac{1}{5}\sec 5v + C$$

21. $$\int \frac{1}{7x+3}\,dx = \frac{1}{7}\int \frac{1}{7x+3}(7\,dx) \qquad \boxed{u = 7x+3,\ du = 7\,dx}$$
$$= \frac{1}{7}\int \frac{1}{u}du = \frac{1}{7}\ln|u| + C = \frac{1}{7}\ln|7x+3| + C$$

22. $$\int (5x+6)^{-1}\,dx = \frac{1}{5}\int \frac{1}{5x+6}(5\,dx) \qquad \boxed{u = 5x+6,\ du = 5\,dx}$$
$$= \frac{1}{5}\int \frac{1}{u}du = \frac{1}{5}\ln|u| + C = \frac{1}{5}\ln|5x+6| + C$$

23. $$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{2x\,dx}{x^2+1} = \frac{1}{2}\ln(x^2+1) + C$$

24. $$\int \frac{x^2}{5x^3+8}\,dx = \frac{1}{15}\int \frac{15x^2\,dx}{5x^3+8} = \frac{1}{15}\ln|5x^3+8| + C$$

25. $$\int \frac{x}{x+1}\,dx = \int \frac{x+1-1}{x+1}\,dx = \int dx - \int \frac{dx}{x+1} = x - \ln|x+1| + C$$

26. $$\int \frac{(x+3)^2}{x+2}\,dx = \int \frac{x^2+6x+9}{x+2}\,dx = \int \left(x+4+\frac{1}{x+2}\right)\,dx = \frac{1}{2}x^2 + 4x + \ln|x+2| + C$$

27. $$\int \frac{1}{x\ln x}\,dx = \frac{1}{\ln x}\left(\frac{1}{x}\,dx\right) \qquad \boxed{u = \ln x,\ du = \frac{1}{x}\,dx}$$
$$= \int \frac{1}{u}du = \ln|u| + C = \ln|\ln x| + C$$

28. $$\int \frac{1-\sin\theta}{\theta+\cos\theta}\,d\theta = \int \frac{1}{\theta+\cos\theta}[(1-\sin\theta)\,d\theta] \qquad \boxed{u = \theta+\cos\theta,\ du = (1-\sin\theta)\,d\theta}$$
$$= \int \frac{1}{u}du = \ln|u| + C = \ln|\theta+\cos\theta| + C$$

29. $$\int \frac{\sin(\ln x)}{x}\,dx = \int \sin(\ln x)\left(\frac{1}{x}\,dx\right) \qquad \boxed{u = \ln x,\ du = \frac{1}{x}\,dx}$$
$$= \int \sin u\,du = -\cos u + C = -\cos(\ln x) + C$$

30. $$\int \frac{1}{x(\ln x)^2}\,dx = \int \frac{1}{(\ln x)^2}\left(\frac{1}{x}\,dx\right) \qquad \boxed{u = \ln x,\ du = \frac{1}{x}\,dx}$$
$$= \int \frac{1}{u^2}du = -\frac{1}{u} + C = -\frac{1}{\ln x} + C$$

31. $$\int e^{10x}\,dx = \frac{1}{10}\int e^{10x}(10\,dx) = \frac{1}{10}e^{10x} + C$$

32. $$\int \frac{1}{e^{4x}}\,dx = -\frac{1}{4}\int e^{-4x}(-4\,dx) = -\frac{1}{4}e^{-4x} + C$$

33. $$\int x^2 e^{-2x^3}\,dx = -\frac{1}{6}\int e^{-2x^3}(-6x^2\,dx) = -\frac{1}{6}e^{-2x^3} + C$$

34. $$\int \frac{e^{1/x^3}}{x^4}\,dx = -\frac{1}{3}\int e^{x^{-3}}(-3x^{-4}\,dx) = -\frac{1}{3}e^{1/x^3} + C$$

35. $$\int \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx = -2\int e^{-\sqrt{x}}\left(-\frac{1}{2\sqrt{x}}\,dx\right) = -2e^{-\sqrt{x}} + C$$

36. $$\int \sqrt{e^x}\,dx = \int e^{x/2}\,dx = 2e^{x/2} + C$$

37. $$\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx = \int \frac{1}{e^x + e^{-x}}[(e^x - e^{-x})\,dx] \qquad \boxed{u = e^x + e^{-x},\ du = (e^x - e^{-x})\,dx}$$
$$= \int \frac{1}{u}\,du = \ln|u| + C = \ln(e^x + e^{-x}) + C$$

38. $$\int e^{3x}\sqrt{1 + 2e^{3x}}\,dx = \frac{1}{6}\int (1 + 2e^{3x})^{1/2}(6e^{3x}\,dx) \qquad \boxed{u = 1 + 2e^{3x},\ du = 6e^{3x}\,dx}$$
$$= \frac{1}{6}\int u^{1/2}du = \frac{1}{9}u^{3/2} + C = \frac{1}{9}\sqrt{(1 + 2e^{3x})^3} + C$$

39. $$\int \frac{1}{\sqrt{5 - x^2}}\,dx = \sin^{-1}\frac{x}{\sqrt{5}} + C$$

40. $$\int \frac{1}{\sqrt{9 - 16x^2}}\,dx = \frac{1}{4}\int \frac{1}{\sqrt{9 - (4x)^2}}(4\,dx) = \frac{1}{4}\sin^{-1}\frac{4x}{3} + C$$

41. $$\int \frac{1}{1 + 25x^2}\,dx = \frac{1}{5}\int \frac{1}{1 + (5x)^2}(5\,dx) = \frac{1}{5}\tan^{-1}5x + C$$

42. $$\int \frac{1}{2 + 9x^2}\,dx = \frac{1}{9}\int \frac{1}{2/9 + x^2}\,dx = \frac{1}{9}\left(\frac{1}{\sqrt{2/9}}\tan^{-1}\frac{x}{\sqrt{2/9}}\right) + C = \frac{1}{3\sqrt{2}}\tan^{-1}\frac{3x}{\sqrt{2}} + C$$

43. $$\int \frac{e^x}{1 + e^{2x}}\,dx = \int \frac{1}{1 + (e^x)^2}(e^x dx) \qquad \boxed{u = e^x,\ du = e^x dx}$$
$$= \int \frac{1}{1 + u^2}\,du = \tan^{-1}u + C = \tan^{-1}e^x + C$$

44. $$\int \frac{\theta}{\sqrt{1-\theta^4}}\,d\theta = \frac{1}{2}\int \frac{1}{\sqrt{1-(\theta^2)^2}}(2\theta\,d\theta) = \frac{1}{2}\sin^{-1}\theta^2 + C$$

45. $$\int \frac{2x-3}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{1-x^2}}(2x\,dx) - \int \frac{3}{\sqrt{1-x^2}}dx \qquad \boxed{u = x^2,\ du = 2x\,dx}$$
$$= \int \frac{1}{\sqrt{1-u}}\,du - 3\int \frac{1}{\sqrt{1-x^2}}\,dx = -2(1-u)^{1/2} - 3\sin^{-1}x + C$$
$$= -2(1-x^2)^{1/2} - 3\sin^{-1}x + C$$

46. $$\int \frac{x-8}{x^2+2}\,dx = \frac{1}{2}\int \frac{1}{x^2+2}(2x\,dx) - \int \frac{8}{x^2+2}\,dx \qquad \boxed{u = x^2+2,\ du = 2x\,dx}$$
$$= \frac{1}{2}\int \frac{1}{u}\,du - 8\int \frac{1}{(\sqrt{2})^2+x^2}\,dx = \frac{1}{2}\ln|u| - 8\left(\frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}\right) + C$$
$$= \frac{1}{2}\ln(x^2+2) - 4\sqrt{2}\tan^{-1}\frac{x}{\sqrt{2}} + C$$

47. $$\int \frac{\tan^{-1}x}{1+x^2}\,dx = \int (\tan^{-1}x)\left(\frac{1}{1+x^2}\,dx\right) \qquad \boxed{u = \tan^{-1}x,\ du = \frac{1}{1+x^2}\,dx}$$
$$= \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}(\tan^{-1}x)^2 + C$$

48. $$\int \sqrt{\frac{\sin^{-1}x}{1-x^2}}\,dx = \int (\sin^{-1}x)^{1/2}\left(\frac{1}{\sqrt{1-x^2}}\,dx\right) \qquad \boxed{u = \sin^{-1}x,\ du = \frac{1}{\sqrt{1-x^2}}\,dx}$$
$$= \int u^{1/2}du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(\sin^{-1}x)^{3/2} + C$$

49. $$\int \tan 5x\,dx = \frac{1}{5}\int (\tan 5x)(5\,dx) = -\frac{1}{5}\ln|\cos 5x| + C$$

50. $$\int e^x \cot e^x\,dx = \int (\cot e^x)(e^x\,dx) \qquad \boxed{u = e^x,\ du = e^x\,dx}$$
$$= \int \cot u\,du = \ln|\sin u| + C = \ln|\sin e^x| + C$$

51. $$\int \sin^2 x\,dx = \int \frac{1}{2}(1-\cos 2x)\,dx = \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + C$$

52. $$\int \cos^2 \pi x\,dx = \int \frac{1}{2}(1+\cos 2\pi x)\,dx = \frac{1}{2}\left(x + \frac{1}{2\pi}\sin 2\pi x\right) + C$$

53. $$\int \cos^2 4x\,dx = \int \frac{1}{2}(1+\cos 8x)\,dx = \frac{1}{2}\left(x + \frac{1}{8}\sin 8x\right) + C$$

54. $$\int \sin^2 \frac{3}{2}x\,dx = \int \frac{1}{2}(1-\cos 3x)\,dx = \frac{1}{2}\left(x - \frac{1}{3}\sin 3x\right) + C$$

55. $$\int (3-2\sin x)^2\,dx = \int (9-12\sin x + 4\sin^2 x)\,dx = 9x + 12\cos x + 4\int \frac{1}{2}(1-\cos 2x)\,dx$$
$$= 9x + 12\cos x + 2\left(x - \frac{1}{2}\sin 2x\right) + C = 11x + 12\cos x - \sin 2x + C$$

56. $$\int (1+\cos 2x)^2\,dx = \int (1 + 2\cos 2x + \cos^2 2x)\,dx = x + \sin 2x + \int \frac{1}{2}(1+\cos 4x)\,dx$$
$$= x + \sin 2x + \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right) + C = \frac{3}{2}x + \sin 2x + \frac{1}{8}\sin 4x + C$$

57. $$y = \int \sqrt[3]{1-x}\,dx = -\int (1-x)^{1/3}(-dx) = -\frac{3}{4}(1-x)^{4/3} + C$$

58. $$y = \int \frac{(1-\tan x)^5}{\cos^2 x}\,dx = -\int (1-\tan x)^5(-\sec^2 x\,dx) = -\frac{1}{6}(1-\tan x)^6 + C$$

59. We have $f(x) = \int (1-6\sin 3x)\,dx = x + 2\cos 3x + C$. Solving $-1 = f(\pi) = \pi + 2\cos 3\pi + C = \pi - 2 + C$ we obtain $C = 1-\pi$. Thus $f(x) = x + 2\cos 3x + 1 - \pi$.

60. We have $f'(x) = \int (1+2x)^5\,dx = \frac{1}{12}(1+2x)^6 + C$. Solving $0 = f'(0) = \frac{1}{12} + C$ we obtain $C = -\frac{1}{12}$. Then

$$f(x) = \int \left[\frac{1}{12}(1+2x)^6 - \frac{1}{12}\right]dx = \frac{1}{12}\int [(1+2x)^6 - 1]\,dx = \frac{1}{12}\left[\frac{1}{14}(1+2x)^7 - x\right] + C.$$

Solving $0 = f(0) = \frac{1}{12}\left(\frac{1}{14}\right) + C$ we obtain $C = -\frac{1}{168}$. Thus $f(x) = \frac{1}{168}(1+2x)^7 - \frac{1}{12}x - \frac{1}{168}$.

61. (a) $$\int \sin x\cos x\,dx = \int \sin x(\cos x\,dx) \qquad \boxed{u = \sin x,\ du = \cos x\,dx}$$
$$= \int u\,du = \frac{1}{2}u^2 + C_1 = \frac{1}{2}\sin^2 x + C_1$$

(b) $$\int \sin x\cos x\,dx = -\int \cos x(-\sin x\,dx) \qquad \boxed{u = \cos x,\ du = -\sin x\,dx}$$
$$= -\int u\,du = -\frac{1}{2}u^2 + C_2 = -\frac{1}{2}\cos^2 x + C_2$$

(c) $$\int \sin x\cos x\,dx = \frac{1}{2}\int \sin 2x\,dx = -\frac{1}{4}\cos 2x + C_3$$

62. (a) $$\frac{d}{dx}\left(\frac{1}{2}\sin^2 x + C_1\right) = \sin x\cos x$$
$$\frac{d}{dx}\left(-\frac{1}{2}\cos^2 x + C_2\right) = \cos x\sin x$$
$$\frac{d}{dx}\left(-\frac{1}{4}\cos 2x + C_3\right) = \frac{1}{2}\sin 2x = \sin x\cos x$$

(b) $\dfrac{1}{2}\sin^2 x + C_1 = \dfrac{1}{2}(1-\cos^2 x) + C_1 = -\dfrac{1}{2}\cos^2 x + \left(C_1 + \dfrac{1}{2}\right) = -\dfrac{1}{2}\cos^2 x + C_2$

(c)
$$\begin{aligned}
\int \sin x\cos x\,dx + \int \sin x\cos x\,dx &= \frac{1}{2}\sin^2 x + C_1 - \frac{1}{2}\cos^2 x + C_2\\
2\int \sin x\cos x\,dx &= -\frac{1}{2}(\cos^2 x - \sin^2 x) + (C_1 + C_2)\\
\int \sin x\cos x\,dx &= -\frac{1}{4}(\cos^2 x - \sin^2 x) + \frac{1}{2}(C_1 + C_2)\\
&= -\frac{1}{4}\cos 2x + C_3
\end{aligned}$$

63. (a) From the given derivative, we have $t(s) = \sqrt{\dfrac{L}{g}}\sin^{-1}\left(\dfrac{s}{s_C}\right) + C$. Solving $t(0) = 0$, we obtain $C = 0$.

(b) $t(s_C) = \sqrt{\dfrac{L}{g}}\sin^{-1}\left(\dfrac{s_C}{s_C}\right) = \sqrt{\dfrac{L}{g}}\sin^{-1} 1 = \dfrac{\pi}{2}\sqrt{\dfrac{L}{g}}$

(c) By symmetry, $T = 4t(s_C) = 4\left(\dfrac{\pi}{2}\sqrt{\dfrac{L}{g}}\right) = 2\pi\sqrt{\dfrac{L}{g}}$.

64.
$$\begin{aligned}
y &= \int \cos^3 x\,dx = \int \cos^2 x\cos x\,dx = \int (1-\sin^2 x)\cos x\,dx\\
&= \int \cos x - \int (\sin^2 x)(\cos x\,dx) \qquad \boxed{u = \sin x,\ du = \cos x\,dx}\\
&= \int \cos x - \int u^2\,du = \sin x - \frac{1}{3}u^3 + C = \sin x - \frac{1}{3}\sin^3 x + C
\end{aligned}$$

Solving $f(\pi/2) = 0 = 1 - \dfrac{1}{3} + C = \dfrac{2}{3} + C$ we obtain $C = -\dfrac{2}{3}$. Thus $f(x) = \sin x - \dfrac{1}{3}\sin^3 x - \dfrac{2}{3}$.

65.
$$\begin{aligned}
\int \cos^4 x\,dx &= \int (\cos^2 x)^2\,dx = \int \left[\frac{1}{2}(1+\cos 2x)\right]^2 dx = \frac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x)\,dx\\
&= \frac{1}{4}x + \frac{1}{4}\sin 2x + \frac{1}{4}\int \frac{1}{2}(1+\cos 4x)\,dx = \frac{1}{4}x + \frac{1}{4}\sin 2x + \frac{1}{8}\int (1+\cos 4x)\,dx\\
&= \frac{1}{4}x + \frac{1}{4}\sin 2x + \frac{1}{8}x + \frac{1}{32}\sin 4x + C = \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C
\end{aligned}$$

66.
$$\begin{aligned}
\int \sin^4 x\,dx &= \int (\sin^2 x)^2\,dx = \int \left[\frac{1}{2}(1-\cos 2x)\right]^2 dx = \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)\,dx\\
&= \frac{1}{4}x - \frac{1}{4}\sin 2x + \frac{1}{4}\int \frac{1}{2}(1+\cos 4x)\,dx = \frac{1}{4}x - \frac{1}{4}\sin 2x + \frac{1}{8}\int (1+\cos 4x)\,dx\\
&= \frac{1}{4}x - \frac{1}{4}\sin 2x + \frac{1}{8}x + \frac{1}{32}\sin 4x + C = \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C
\end{aligned}$$

67.
$$\begin{aligned}
\int \frac{1}{x\sqrt{x^4-16}}\,dx &= \int \frac{1}{2x^2\sqrt{x^4-16}}(2x\,dx) \qquad \boxed{u = x^2,\ du = 2x\,dx}\\
&= \frac{1}{2}\int \frac{1}{u\sqrt{u^2-4^2}}\,du = \frac{1}{4}\sec^{-1}\left|\frac{u}{4}\right| + C = \frac{1}{4}\sec^{-1}\frac{x^2}{4} + C
\end{aligned}$$

68. $$\int \frac{e^{2x}}{e^x+1}\,dx = \int \left(e^x - \frac{e^x}{e^x+1}\right)dx$$
$$= \int e^x\,dx - \int \frac{1}{e^x+1}(e^x\,dx) \qquad \boxed{u = e^x+1,\ du = e^x\,dx}$$
$$= e^x - \int \frac{1}{u}\,du = e^x - \ln|u| + C = e^x - \ln(e^x+1) + C$$

69. $$\int \frac{1}{1-\cos x}\,dx = \int \frac{1}{1-\cos x}\left(\frac{1+\cos x}{1+\cos x}\right)dx = \int \frac{1+\cos x}{1-\cos^2 x}\,dx = \int \frac{1+\cos x}{\sin^2 x}\,dx$$
$$= \int \left[\frac{1}{\sin^2 x} + \frac{\cos x}{(\sin x)(\sin x)}\right]dx = \int (\csc^2 x + \csc x \cot x)\,dx$$
$$= -\cot x - \csc x + C$$

70. $$\int \frac{1}{1+\sin 2x}\,dx = \int \frac{1}{1+\sin 2x}\left(\frac{1-\sin 2x}{1-\sin 2x}\right)dx = \int \frac{1-\sin 2x}{1-\sin^2 2x}\,dx = \int \frac{1-\sin 2x}{\cos^2 2x}\,dx$$
$$= \int \left[\frac{1}{\cos^2 2x} - \frac{\sin 2x}{(\cos 2x)(\cos 2x)}\right]dx = \int (\sec^2 2x - \sec 2x \tan 2x)\,dx$$
$$= \frac{1}{2}\tan 2x - \frac{1}{2}\sec 2x + C$$

71. $$\int f'(8x)\,dx = \frac{1}{8}\int f'(8x)(8\,dx) \qquad \boxed{u = 8x,\ du = 8\,dx}$$
$$= \frac{1}{8}\int f'(u)\,du = \frac{1}{8}f(u) + C = \frac{1}{8}f(8x) + C$$

72. $$\int xf'(5x^2)\,dx = \frac{1}{10}\int f'(5x^2)(10x\,dx) \qquad \boxed{u = 5x^2,\ du = 10x\,dx}$$
$$= \frac{1}{10}\int f'(u)\,du = \frac{1}{10}f(u) + C = \frac{1}{10}f(5x^2) + C$$

73. $$\int \sqrt{f(2x)}f'(2x)\,dx = \frac{1}{2}\int [f(2x)]^{1/2}[2f'(2x)\,dx] \qquad \boxed{u = f(2x),\ du = 2f'(2x)\,dx}$$
$$= \frac{1}{2}\int u^{1/2}du = \frac{1}{3}u^{3/2} + C = \frac{1}{3}[f(2x)]^{3/2} + C$$

74. $$\int \frac{f'(3x+1)}{f(3x+1)}\,dx = \frac{1}{3}\int \frac{1}{f(3x+1)}[3f'(3x+1)\,dx] \qquad \boxed{u = f(3x+1),\ du = 3f'(3x+1)\,dx}$$
$$= \frac{1}{3}\int \frac{1}{u}\,du = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|f(3x+1)| + C$$

75. For any f, $$\int f''(4x)\,dx = \frac{1}{4}\int f''(4x)(4\,dx) \qquad \boxed{u = 4x,\ du = 4\,dx}$$
$$= \frac{1}{4}\int f''(u)\,du = \frac{1}{4}f'(u) + C = \frac{1}{4}f'(4x) + C$$

Given $f(x) = \sqrt{x^4+1} = (x^4+1)^{1/2}$, we have $f'(x) = 2x^3(x^4+1)^{-1/2}$. Thus,

$$\int f''(4x)\,dx = \frac{1}{4}f'(4x) + C = \frac{1}{4}\{2(4x)^3[(4x)^4+1]^{-1/2}\} + C = \frac{32x^3}{\sqrt{256x^4+1}} + C.$$

To check this, take the derivative of the above function, yielding $\dfrac{96x^2}{\sqrt{256x^4+1}} - \dfrac{16384x^6}{\sqrt{(256x^4+1)^3}}$, which should be the same as $f''(4x)$. Since $f''(x) = \dfrac{6x^2}{\sqrt{x^4+1}} - \dfrac{4x^6}{\sqrt{(x^4+1)^3}}$, we have $f''(4x) = \dfrac{6(4x)^2}{\sqrt{(4x)^4+1}} - \dfrac{4(4x)^6}{\sqrt{[(4x)^4+1]^3}} = \dfrac{96x^2}{\sqrt{256x^4+1}} - \dfrac{16384x^6}{\sqrt{(256x^4+1)^3}}$.

76. First evaluating $\displaystyle\int \sec^2 3x\,dx$, we get

$$\begin{aligned}\int \sec^2 3x\,dx &= \frac{1}{3}\int (\sec^2 3x)(3\,dx) \qquad \boxed{u = 3x,\ du = 3\,dx}\\ &= \frac{1}{3}\int \sec^2 u\,du = \frac{1}{3}\tan u + C = \frac{1}{3}\tan 3x + C\end{aligned}$$

Next, evaluating $\displaystyle\int\left(\int \sec^2 3x\,dx\right)dx = \int\left(\frac{1}{3}\tan 3x + C\right)dx$, we get

$$\begin{aligned}\int\left(\frac{1}{3}\tan 3x + C\right)dx &= (Cx + C_1) + \frac{1}{3}\int \tan 3x\,dx\\ &= (Cx + C_1) + \frac{1}{9}\int (\tan 3x)(3\,dx) \qquad \boxed{u = 3x,\ du = 3\,dx}\\ &= (Cx + C_1) + \frac{1}{9}\int \tan u\,du = (Cx + C_1) - \frac{1}{9}\ln|\cos u| + C_2\\ &= Cx - \frac{1}{9}\ln|\cos 3x| + C_3.\end{aligned}$$

5.3 The Area Problem

1. $3 + 6 + 9 + 12 + 15$

2. $-1 + 1 + 3 + 5 + 7$

3. $2 + 2 + 8/3 + 4$

4. $\dfrac{3}{10} + \dfrac{9}{100} + \dfrac{27}{1000} + \dfrac{81}{10,000}$

5. $-\dfrac{1}{7} + \dfrac{1}{9} - \dfrac{1}{11} + \dfrac{1}{13} - \dfrac{1}{15} + \dfrac{1}{17} - \dfrac{1}{19} + \dfrac{1}{21} - \dfrac{1}{23} + \dfrac{1}{25}$

6. $1 - \dfrac{1}{4} + \dfrac{1}{9} - \dfrac{1}{16} + \dfrac{1}{25} - \dfrac{1}{36} + \dfrac{1}{49} - \dfrac{1}{64} + \dfrac{1}{81} - \dfrac{1}{100}$

7. $0 + 3 + 8 + 15$

8. $1 + 4 + 9 + 16 + 25$

9. $-1 + 1 - 1 + 1 - 1$

10. $1 + 0 - \dfrac{1}{3} + 0 + \dfrac{1}{5}$

11. $\displaystyle\sum_{k=1}^{7}(2k+1)$

12. $\displaystyle\sum_{k=1}^{6} 2^k$

13. $\displaystyle\sum_{k=1}^{13}(3k-2)$

14. $\displaystyle\sum_{k=1}^{10}(4k-2)$

15. $\displaystyle\sum_{k=1}^{5}\frac{(-1)^{k+1}}{k}$

16. $\displaystyle\sum_{k=1}^{5}\frac{(-1)^k k}{k+1}$

17. $\displaystyle\sum_{k=1}^{8} 6$

18. $\displaystyle\sum_{k=1}^{9}\sqrt{k}$

19. $\displaystyle\sum_{k=1}^{4}\frac{(-1)^{k+1}}{k^2}\cos\frac{k\pi}{p}x$

20. $\displaystyle\sum_{k=1}^{5}\frac{(-1)^{k+1}f^{(k)}(1)}{2k-1}(x-1)^k$

21. $\displaystyle\sum_{k=1}^{20} 2k = 2\sum_{k=1}^{20} k = 2\left(\frac{20\cdot 21}{2}\right) = 420$

22. $\displaystyle\sum_{k=0}^{50}(-3k) = -3\sum_{k=1}^{50} k = -3\left(\frac{50\cdot 51}{2}\right) = -3825$

23. $\displaystyle\sum_{k=1}^{10}(k+1) = \sum_{k=1}^{10} k + \sum_{k=1}^{10} 1 = \frac{10\cdot 11}{2} + 10\cdot 1 = 65$

24. $\displaystyle\sum_{k=1}^{1000}(2k-1) = 2\sum_{k=1}^{1000} k - \sum_{k=1}^{1000} 1 = 2\left(\frac{1000\cdot 1001}{2}\right) - 1000\cdot 1 = 1,000,000$

25. $\displaystyle\sum_{k=1}^{6}(k^2+3) = \sum_{k=1}^{6} k^2 + \sum_{k=1}^{6} 3 = \frac{6\cdot 7\cdot 13}{6} + 6\cdot 3 = 109$

26. $\displaystyle\sum_{k=1}^{5}(6k^2-k) = 6\sum_{k=1}^{5} k^2 - \sum_{k=1}^{5} k = 6\left(\frac{5\cdot 6\cdot 11}{6}\right) - \frac{5\cdot 6}{2} = 315$

27. $\displaystyle\sum_{p=0}^{10}(p^3+4) = 0 + 4 + \sum_{p=1}^{10} p^3 + \sum_{p=1}^{10} 4 = 4 + \frac{10^2\cdot 11^2}{4} + 10\cdot 4 = 3069$

28. $\displaystyle\sum_{i=1}^{10}(2i^3-5i+3) = 2\sum_{i=1}^{10} i^3 - 5\sum_{i=1}^{10} i + \sum_{i=1}^{10} 3 = 2\left(\frac{10^2\cdot 11^2}{4}\right) - 5\left(\frac{10\cdot 11}{2}\right) + 10\cdot 3 = 5805$

29. Using $\Delta x = \dfrac{6-0}{n} = \dfrac{6}{n}$ and $f\left(a + k\dfrac{b-a}{n}\right) = f\left(\dfrac{6k}{n}\right) = \dfrac{6k}{n}$ we have

$$A = \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{6k}{n}\right)\frac{6}{n} = \lim_{n\to\infty}\frac{36}{n^2}\sum_{k=1}^{n} k = \lim_{n\to\infty}\frac{36}{n^2}\cdot\frac{n(n+1)}{2} = \lim_{n\to\infty} 18\left(1+\frac{1}{n}\right) = 18.$$

30. Using $\Delta x = \dfrac{3-1}{n} = \dfrac{2}{n}$ and $f\left(a + k\dfrac{b-a}{n}\right) = f\left(1 + \dfrac{2k}{n}\right) = 2 + \dfrac{4k}{n}$ we have

$$\begin{aligned} A &= \lim_{n\to\infty}\sum_{k=1}^{n}\left(2+\frac{4k}{n}\right)\frac{2}{n} = \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{4}{n}+\frac{8k}{n^2}\right) = \lim_{n\to\infty}\left(\frac{4}{n}\sum_{k=1}^{n} 1 + \frac{8}{n^2}\sum_{k=1}^{n} k\right) \\ &= \lim_{n\to\infty}\left[\frac{4}{n}\cdot n + \frac{8}{n^2}\cdot\frac{n(n+1)}{2}\right] = \lim_{n\to\infty}\left[4 + 4\left(1+\frac{1}{n}\right)\right] = 8. \end{aligned}$$

31. Using $\Delta x = \dfrac{4}{n}$ and $f\left(a + k\dfrac{b-a}{n}\right) = 3 + \dfrac{8k}{n}$ we have

$$\begin{aligned} A &= \lim_{n\to\infty}\sum_{k=1}^{n}\left(3+\frac{8k}{n}\right)\frac{4}{n} = \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{12}{n}+\frac{32k}{n^2}\right) = \lim_{n\to\infty}\left(\frac{12}{n}\sum_{k=1}^{n} 1 + \frac{32k}{n^2}\sum_{k=1}^{n} k\right) \\ &= \lim_{n\to\infty}\left[\frac{12}{n}\cdot n + \frac{32k}{n^2}\cdot\frac{n(n+1)}{2}\right] = \lim_{n\to\infty}\left[12 + 16\left(1+\frac{1}{n}\right)\right] = 28. \end{aligned}$$

32. Using $\Delta x = \dfrac{2}{n}$ and $f\left(a + k\dfrac{b-a}{n}\right) = \dfrac{6k}{n}$ we have

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{6k}{n}\right)\frac{2}{n} = \lim_{n\to\infty} \frac{12}{n^2}\sum_{k=1}^{n} k = \lim_{n\to\infty} \frac{12}{n^2}\cdot\frac{n(n+1)}{2} = \lim_{n\to\infty} 6\left(1+\frac{1}{n}\right) = 6.$$

33. Using $\Delta x = \dfrac{2}{n}$ and $f\left(a + k\dfrac{b-a}{n}\right) = \dfrac{4k^2}{n^2}$ we have

$$\begin{aligned} A &= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{4k^2}{n^2}\cdot\frac{2}{n}\right) = \lim_{n\to\infty} \frac{8}{n^3}\sum_{k=1}^{n} k^2 = \lim_{n\to\infty} \left[\frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right] \\ &= \lim_{n\to\infty} \frac{4}{3}\left(2+\frac{3}{n}+\frac{1}{n^2}\right) = \frac{8}{3}. \end{aligned}$$

34. Using $\Delta x = \dfrac{3}{n}$ and $f\left(a + k\dfrac{b-a}{n}\right) = 4 - \dfrac{12k}{n} + \dfrac{9k^2}{n^2}$ we have

$$\begin{aligned} A &= \lim_{n\to\infty} \sum_{k=1}^{n} \left(4 - \frac{12k}{n} + \frac{9k^2}{n^2}\right)\frac{3}{n} = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{12}{n} - \frac{36}{n^2} + \frac{27k^2}{n^3}\right) \\ &= \lim_{n\to\infty} \left(\frac{12}{n}\sum_{k=1}^{n} 1 - \frac{36}{n^2}\sum_{k=1}^{n} k + \frac{27}{n^3}\sum_{k=1}^{n} k^2\right) \\ &= \lim_{n\to\infty} \left[\frac{12}{n}\cdot n - \frac{36}{n^2}\cdot\frac{n(n+1)}{2} + \frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right] \\ &= \lim_{n\to\infty} \left[12 - 18\left(1+\frac{1}{n}\right) + \frac{9}{2}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)\right] = 12 - 18 + 9 = 3. \end{aligned}$$

35. Using $\Delta x = \dfrac{2}{n}$ and $f\left(a + k\dfrac{b-a}{n}\right) = \dfrac{4k}{n} - \dfrac{4k^2}{n^2}$ we have

$$\begin{aligned} A &= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{4k}{n} - \frac{4k^2}{n^2}\right)\frac{2}{n} = \lim_{n\to\infty} \left(\frac{8}{n^2}\sum_{k=1}^{n} k - \frac{8}{n^3}\sum_{k=1}^{n} k^2\right) \\ &= \lim_{n\to\infty} \left[4\left(1+\frac{1}{n}\right) - \frac{4}{3}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)\right] = 4 - \frac{8}{3} = \frac{4}{3}. \end{aligned}$$

36. Using $\Delta x = \dfrac{2}{n}$ and $f\left(a + k\dfrac{b-a}{n}\right) = f\left(-3 + \dfrac{2k}{n}\right) = 21 - \dfrac{24k}{n} + \dfrac{8k^2}{n^2}$ we have

$$\begin{aligned} A &= \lim_{n\to\infty} \sum_{k=1}^{n} \left(21 - \frac{24k}{n} - \frac{8k^2}{n^2}\right)\frac{2}{n} = \lim_{n\to\infty} \left(\frac{42}{n}\sum_{k=1}^{n} 1 - \frac{48}{n^2}\sum_{k=1}^{n} k + \frac{16}{n^3}\sum_{k=1}^{n} k^2\right) \\ &= \lim_{n\to\infty} \left[42 - 24\left(1+\frac{1}{n}\right) + \frac{8}{3}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)\right] = 42 - 24 + \frac{16}{3} = \frac{70}{3}. \end{aligned}$$

37. Using $\Delta x = \dfrac{1}{n}$ and $f\left(a + k\dfrac{b-a}{n}\right) = 3 + \dfrac{4k}{n} + \dfrac{k^2}{n^2}$ we have

$$\begin{aligned} A &= \lim_{n\to\infty} \sum_{k=1}^{n} \left(3 + \frac{4k}{n} + \frac{k^2}{n^2}\right)\frac{1}{n} = \lim_{n\to\infty}\left(\frac{3}{n}\sum_{k=1}^{n} 1 + \frac{4}{n^2}\sum_{k=1}^{n} k + \frac{1}{n^3}\sum_{k=1}^{n} k^2\right) \\ &= \lim_{n\to\infty}\left[3 + 2\left(1 + \frac{1}{n}\right) + \frac{1}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)\right] = 3 + 2 + \frac{1}{3} = \frac{16}{3}. \end{aligned}$$

38. Using $\Delta x = \dfrac{2}{n}$ and $f\left(a + k\dfrac{b-a}{n}\right) = \dfrac{4k^2}{n^2} - \dfrac{4k}{n} + 1$ we have

$$\begin{aligned} A &= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{4k^2}{n^2} - \frac{4k}{n} + 1\right)\frac{2}{n} = \lim_{n\to\infty}\left(\frac{8}{n^3}\sum_{k=1}^{n} k^2 - \frac{8}{n^2}\sum_{k=1}^{n} k + \frac{2}{n}\sum_{k=1}^{n} 1\right) \\ &= \lim_{n\to\infty}\left[\frac{4}{3}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) - 4\left(1 + \frac{1}{n}\right) + 2\right] = \frac{8}{3} - 4 + 2 = \frac{2}{3}. \end{aligned}$$

39. Using $\Delta x = \dfrac{1}{n}$ and $f\left(a + k\dfrac{b-a}{n}\right) = \dfrac{k^3}{n^3}$ we have

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{k^3}{n^3} \cdot \frac{1}{n}\right) = \lim_{n\to\infty} \frac{1}{n^4}\sum_{k=1}^{n} k^3 = \lim_{n\to\infty} \frac{1}{4}\left(1 + \frac{2}{n} + \frac{1}{n^2}\right) = \frac{1}{4}.$$

40. Using $\Delta x = \dfrac{2}{n}$ and $f\left(a + k\dfrac{b-a}{n}\right) = \dfrac{8k^3}{n^3} - \dfrac{12k^2}{n^2} + 4$ we have

$$\begin{aligned} A &= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{8k^3}{n^3} - \frac{12k^2}{n^2} + 4\right)\frac{2}{n} = \lim_{n\to\infty}\left(\frac{16}{n^4}\sum_{k=1}^{n} k^3 - \frac{24}{n^3}\sum_{k=1}^{n} k^2 + \frac{8}{n}\sum_{k=1}^{n} 1\right) \\ &= \lim_{n\to\infty}\left[4\left(1 + \frac{2}{n} + \frac{1}{n^2}\right) - 4\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) + 8\right] = 4 - 8 + 8 = 4. \end{aligned}$$

41. Let $A = A_1 + A_2$ where A_1 is the area under $f(x) = 2$ on $[0, 1)$ and A_2 is the area under $f(x) = x + 1$ on $[1, 4]$. For A_1, we have $\Delta x = \dfrac{1}{n}$, $f\left(a + k\dfrac{b-a}{n}\right) = 2$, and

$$A_1 = \lim_{n\to\infty} \sum_{k=1}^{n} \left(2 \cdot \frac{1}{n}\right) = \lim_{n\to\infty} \frac{2}{n}\sum_{k=1}^{n} 1 = \lim_{n\to\infty} \frac{2n}{n} = 2.$$

For A_2, we have $\Delta x = \dfrac{3}{n}$, $f\left(a + k\dfrac{b-a}{n}\right) = 2 + \dfrac{3k}{n}$, and

$$\begin{aligned} A_2 &= \lim_{n\to\infty} \sum_{k=1}^{n} \left(2 + \frac{3k}{n}\right)\frac{3}{n} = \lim_{n\to\infty}\left(\frac{6}{n}\sum_{k=1}^{n} 1 + \frac{9}{n^2}\sum_{k=1}^{n} k\right) \\ &= \lim_{n\to\infty}\left[\frac{6n}{n} + \frac{9}{2}\left(1 + \frac{1}{n}\right)\right] = 6 + \frac{9}{2} = \frac{21}{2}. \end{aligned}$$

Then $A = 2 + \dfrac{21}{2} = \dfrac{25}{2}$.

42. Let $A = A_1 + A_2$ where A_1 is the area under $f(x) = -x+1$ on $[0,1)$ and A_2 is the area under $f(x) = x+2$ on $[1,3]$. For A_1, we have $\Delta x = \dfrac{1}{n}$, $f\left(a + k\dfrac{b-a}{n}\right) = 1 - \dfrac{k}{n}$, and

$$A_1 = \lim_{n\to\infty} \sum_{k=1}^{n} \left(1 - \frac{k}{n}\right)\frac{1}{n} = \lim_{n\to\infty} \left(\frac{1}{n}\sum_{k=1}^{n} 1 - \frac{1}{n^2}\sum_{k=1}^{n} k\right) = \lim_{n\to\infty} \left[1 - \frac{1}{2}\left(1 + \frac{1}{n}\right)\right] = \frac{1}{2}.$$

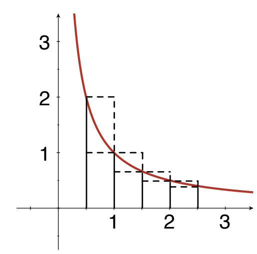
For A_2, we have $\Delta x = \dfrac{2}{n}$, $f\left(a + k\dfrac{b-a}{n}\right) = 3 + \dfrac{2k}{n}$, and

$$A_2 = \lim_{n\to\infty} \sum_{k=1}^{n} \left(3 - \frac{2k}{n}\right)\frac{2}{n} = \lim_{n\to\infty} \left(\frac{6}{n}\sum_{k=1}^{n} 1 + \frac{4}{n^2}\sum_{k=1}^{n} k\right) = \lim_{n\to\infty} \left[6 + 2\left(1 + \frac{1}{n}\right)\right] = 10.$$

Then $A = \dfrac{1}{2} + 8 = \dfrac{17}{2}$.

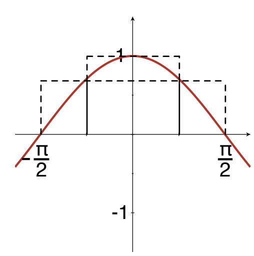
43. $A_R = 1 \cdot \dfrac{1}{2} + \dfrac{2}{3} \cdot \dfrac{1}{2} + \dfrac{1}{2} \cdot \dfrac{1}{2} + \dfrac{2}{5} \cdot \dfrac{1}{2} = \dfrac{77}{60}$

$A_L = 2 \cdot \dfrac{1}{2} + 1 \cdot \dfrac{1}{2} + \dfrac{2}{3} \cdot \dfrac{1}{2} + \dfrac{1}{2} \cdot \dfrac{1}{2} = \dfrac{25}{12}$



44. $A_R = \dfrac{\sqrt{2}}{2} \cdot \dfrac{\pi}{4} + 1 \cdot \dfrac{\pi}{4} + \dfrac{\sqrt{2}}{2} \cdot \dfrac{\pi}{4} + 0 \cdot \dfrac{\pi}{4} = \dfrac{(1+\sqrt{2})\pi}{4}$

$A_L = 0 \cdot \dfrac{\pi}{4} + \dfrac{\sqrt{2}}{2} \cdot \dfrac{\pi}{4} + 1 \cdot \dfrac{\pi}{4} + \dfrac{\sqrt{2}}{2} \cdot \dfrac{\pi}{4} = \dfrac{(1+\sqrt{2})\pi}{4}$



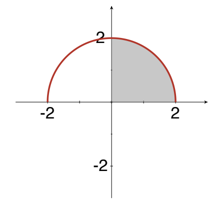
45. Using $\Delta x = \dfrac{2-(-1)}{n} = \dfrac{3}{n}$ and $x_k^* = -1 + (k-1)\dfrac{3}{n}$ we obtain

$$\begin{aligned}
A &= \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x = \lim_{n\to\infty}\sum_{k=1}^{n}\left\{4 - \left[-1 + (k-1)\frac{3}{n}\right]^2\right\}\frac{3}{n} \\
&= \lim_{n\to\infty}\frac{3}{n}\sum_{k=1}^{n}\left[3 + 6\frac{k-1}{n} - 9\frac{(k-1)^2}{n^2}\right] \\
&= \lim_{n\to\infty}\frac{3}{n}\left[3\sum_{k=1}^{n}1 + \frac{6}{n}\sum_{k=1}^{n}(k-1) - \frac{9}{n^2}\sum_{k=1}^{n}(k^2-2k+1)\right] \\
&= \lim_{n\to\infty}\frac{3}{n}\left[3\sum_{k=1}^{n}1 + \frac{6}{n}\sum_{k=1}^{n}k - \frac{6}{n}\sum_{k=1}^{n}1 - \frac{9}{n^2}\sum_{k=1}^{n}k^2 + \frac{18}{n^2}\sum_{k=1}^{n}k - \frac{9}{n^2}\sum_{k=1}^{n}1\right] \\
&= \lim_{n\to\infty}\left[\frac{9}{n}n + \frac{18}{n^2}\frac{n(n+1)}{2} - \frac{18}{n^2}n - \frac{27}{n^3}\frac{n(n+1)(2n+1)}{6} + \frac{54}{n^3}\frac{n(n+1)}{2} - \frac{27}{n^3}n\right] \\
&= \lim_{n\to\infty}\left[9 + 9\left(1+\frac{1}{n}\right) - \frac{18}{n} - \frac{9}{2}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) + \frac{27}{n}\left(1+\frac{1}{n}\right) - \frac{27}{n^2}\right] \\
&= 9 + 9 - 0 - 9 + 0 - 0 = 9.
\end{aligned}$$

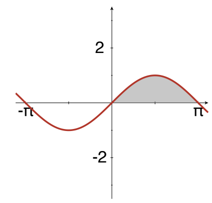
46. Using $\Delta x = \dfrac{2-(-1)}{n} = \dfrac{3}{n}$ and $x_k^* = -1 + \dfrac{2k-1}{2}\dfrac{3}{n}$ we obtain

$$\begin{aligned}
A &= \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x = \lim_{n\to\infty}\sum_{k=1}^{n}\left[4 - \left(-1 + \frac{2k-1}{2}\frac{3}{n}\right)^2\right]\frac{3}{n} \\
&= \lim_{n\to\infty}\frac{3}{n}\sum_{k=1}^{n}\left[3 + 3\frac{2k-1}{n} - \frac{9}{4}\frac{(2k-1)^2}{n^2}\right] \\
&= \lim_{n\to\infty}\frac{3}{n}\left[3\sum_{k=1}^{n}1 + \frac{3}{n}\sum_{k=1}^{n}(2k-1) - \frac{9}{4n^2}\sum_{k=1}^{n}(4k^2-4k+1)\right] \\
&= \lim_{n\to\infty}\frac{3}{n}\left[3\sum_{k=1}^{n}1 + \frac{6}{n}\sum_{k=1}^{n}k - \frac{3}{n}\sum_{k=1}^{n}1 - \frac{9}{n^2}\sum_{k=1}^{n}k^2 + \frac{9}{n^2}\sum_{k=1}^{n}k - \frac{9}{4n^2}\sum_{k=1}^{n}1\right] \\
&= \lim_{n\to\infty}\left[\frac{9}{n}n + \frac{18}{n^2}\frac{n(n+1)}{2} - \frac{9}{n^2}n - \frac{27}{n^3}\frac{n(n+1)(2n+1)}{6} + \frac{27}{n^3}\frac{n(n+1)}{2} - \frac{9}{4n^3}n\right] \\
&= \lim_{n\to\infty}\left[9 + 9\left(1+\frac{1}{n}\right) - \frac{9}{n} - \frac{9}{2}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) + \frac{27}{2n}\left(1+\frac{1}{n}\right) - \frac{9}{4n^2}\right] \\
&= 9 + 9 - 0 - 9 + 0 - 0 = 9.
\end{aligned}$$

47. Identify $b - a = 2$. Taking $a = 0$, we have $f\left(a + k\dfrac{b-a}{n}\right) = f\left(\dfrac{2k}{n}\right) = \sqrt{4 - \left(\dfrac{2k}{n}\right)^2}$. Then A is the area under $f(x) = \sqrt{4 - x^2}$ from $x = 0$ to $x = 2$.



48. Identify $b - a = \pi$. Taking $a = 0$, we have $f\left(a + k\dfrac{b-a}{n}\right) = f\left(\dfrac{k\pi}{n}\right) = \sin\dfrac{k\pi}{n}$. Then A is the area under $f(x) = \sin x$ from $x = 0$ to $x = \pi$.



49. $0.11111111 = \dfrac{1}{10} + \dfrac{1}{10^2} + \cdots + \dfrac{1}{10^8} = \displaystyle\sum_{k=1}^{8} \frac{1}{10^k}$

50. $0.3737373737 = \dfrac{37}{100} + \dfrac{37}{100^2} + \cdots + \dfrac{37}{100^5} = 37\left(\dfrac{1}{100} + \dfrac{1}{100^2} + \cdots + \dfrac{1}{100^5}\right) = 37\displaystyle\sum_{k=1}^{5} \frac{1}{100^k}$

51. $\displaystyle\sum_{k=21}^{60} k^2 = \sum_{k=1}^{60} k^2 - \sum_{k=1}^{20} k^2 = \frac{60 \cdot 61 \cdot 121}{6} - \frac{20 \cdot 21 \cdot 41}{6} = 70,940$

52. $\displaystyle\sum_{k=0}^{6}(k+6); \qquad \sum_{k=1}^{7}(k+5); \qquad \sum_{k=2}^{8}(k+4)$

53. $0 = \displaystyle\sum_{k=1}^{n} x_k - \sum_{k=1}^{n} \overline{x} = \sum_{k=1}^{n} x_k - n\overline{x}; \quad \overline{x} = \frac{1}{n}\sum_{k=1}^{n} x_k$

54. (a) $$\begin{aligned}\sum_{k=1}^{n}[f(k) - f(k-1)] &= [f(1) - f(0)] + [f(2) - f(1)] + [f(3) - f(2)] + \cdots \\ &\quad + [f(n-1) - f(n-2)] + [f(n) - f(n-1)] \\ &= f(n) - f(0)\end{aligned}$$

(b) $f(k) = \sqrt{k}; \quad \displaystyle\sum_{k=1}^{400}(\sqrt{k} - \sqrt{k-1}) = \sqrt{400} - \sqrt{0} = 20$

55. (a) Identifying $f(k) = (k+1)^2$ in part (a) of Problem 54, we have $\displaystyle\sum_{k=1}^{n}[(k+1)^2 - k^2] = (n+1)^2 - 1^2 = n^2 + 2n$.

(b) $\displaystyle\sum_{k=1}^{n}[(k+1)^2 - k^2] = \sum_{k=1}^{n}(2k+1) = \sum_{k=1}^{n} 2k + \sum_{k=1}^{n} 1 = 2\sum_{k=1}^{n} k + n$

(c) Comparing the results of (a) and (b), we find that equating them leads to summation formula (*ii*):

$$2\sum_{k=1}^{n} k + n = n^2 + 2n; \qquad \sum_{k=1}^{n} k = \frac{n^2+2n-n}{2} = \frac{n^2+n}{2} = \frac{n(n+1)}{2}$$

Using $f(k) = (k+1)^3$ similarly to (a), we obtain

$$\sum_{k=1}^{n}[(k+1)^3 - k^3] = (n+1)^3 - 1^3 = n^3 + 3n^2 + 3n.$$

Analogously for (b), we also have

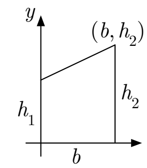
$$\sum_{k=1}^{n}[(k+1)^3 - k^3] = \sum_{k=1}^{n}(3k^2+3k+1) = 3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + n.$$

Combining these, we obtain

$$\begin{aligned}
3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + n &= n^3 + 3n^2 + 3n \\
3\sum_{k=1}^{n} k^2 + \frac{3n(n+1)}{2} + n &= n^3 + 3n^2 + 3n \\
3\sum_{k=1}^{n} k^2 &= n^3 + 3n^2 + 2n - \frac{3n^2+3n}{2} \\
\sum_{k=1}^{n} k^2 &= \frac{2n^3+6n^2+4n-3n^2-3n}{6} = \frac{2n^3+3n^2+n}{6} \\
&= \frac{n(2n^2+3n+1)}{6} = \frac{n(n+1)(2n+1)}{6}.
\end{aligned}$$

56. The pattern illustrated in Figure 5.3.9 indicates that the summation of cubes is the square of the summation of the numbers being cubed. That is: $\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2$. Expanding the summation, we get $\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2 = \left[\frac{n(n+1)}{2}\right]^2 = \frac{n^2(n+1)^2}{4}$.

57. The equation of the line through $(0, h_1)$ and (b, h_2) is $f(x) = \dfrac{h_2 - h_1}{b}x + h_1$. Using $\Delta x = \dfrac{b}{n}$ and $f\left(a + k\dfrac{b-a}{n}\right) = f\left(\dfrac{kb}{n}\right) = \dfrac{k(h_2 - h_1)}{n} + h_1$ we find



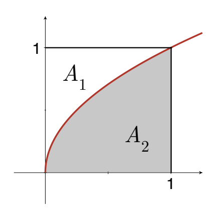
$$A = \lim_{n\to\infty} \sum_{k=1}^{n} \left[\frac{k(h_1 - h_2)}{n} + h_1\right] \frac{b}{n} = \lim_{n\to\infty} \left[\frac{b}{n^2}(h_2 - h_1)\sum_{k=1}^{n} k + \frac{bh_1}{n}\sum_{k=1}^{n} 1\right]$$
$$= \lim_{n\to\infty} \left[\frac{b(h_2 - h_1)}{2}\left(1 + \frac{1}{n}\right) + \frac{bh_1 n}{n}\right] = \frac{b(h_2 - h_1)}{2} + bh_1$$
$$= \frac{bh_2 - bh_2 + 2bh_1}{2} = \left(\frac{h_1 + h_2}{2}\right) b.$$

58. Since the total number of cans is 136 and there is one additional can per row, we have $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2} = 136$. Solving for n will yield the number of cans in the bottom row, so we have $n^2 + n - 272 = 0$ and $(n - 16)(n + 17) = 0$, yielding $n = 16$ or $n = -17$. Thus, there are 16 cans in the bottom row.

59. Using $\Delta x = \dfrac{4}{n}$ and $f\left(a + k\dfrac{b-a}{n}\right) = \dfrac{128k}{n} - \dfrac{384k^2}{n^2} + \dfrac{512k^3}{n^3} - \dfrac{256k^4}{n^4}$ we have

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{128k}{n} - \frac{384k^2}{n^2} + \frac{512k^3}{n^3} - \frac{256k^4}{n^4}\right) \frac{4}{n}$$
$$= \lim_{n\to\infty} \left(\frac{512}{n^2}\sum_{k=1}^{n} k - \frac{1536}{n^3}\sum_{k=1}^{n} k^2 + \frac{2048}{n^4}\sum_{k=1}^{n} k^3 - \frac{1024}{n^5}\sum_{k=1}^{n} k^4\right)$$
$$= \lim_{n\to\infty} \left[256\left(1 + \frac{1}{n}\right) - 256\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) + 512\left(1 + \frac{2}{n} + \frac{1}{n^2}\right) - \frac{512}{15}\left(6 + \frac{15}{n} + \frac{10}{n^2} - \frac{1}{n^4}\right)\right]$$
$$= 256 - 512 + 512 - \frac{1024}{5} = \frac{256}{5}.$$

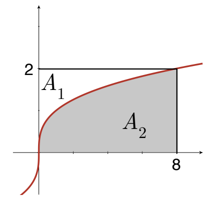
60. We note that $A_2 = 1 - A_1$ where A_1 is the area under $y = x^2$ from 0 to 1. Using $\Delta x = \dfrac{1}{n}$ and $f\left(a + k\dfrac{b-a}{n}\right) = \dfrac{k^2}{n^2}$ we find



$$A_1 = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{k^2}{n^2}\frac{1}{n}\right) = \lim_{n\to\infty} \frac{1}{n^3}\sum_{k=1}^{n} k^2 = \lim_{n\to\infty} \frac{1}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) = \frac{1}{3}.$$

Thus, $A_2 = 1 - \dfrac{1}{3} = \dfrac{2}{3}$.

61. We note that $A_2 = 16 - A_1$ where A_1 is the area under $y = x^3$ from 0 to 2. Using $\Delta x = \dfrac{2}{n}$ and $f\left(a + k\dfrac{b-a}{n}\right) = \dfrac{8k^3}{n^3}$ we find



$$A_1 = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{8k^3}{n^3}\frac{2}{n}\right) = \lim_{n\to\infty} \frac{16}{n^4} \sum_{k=1}^{n} k^3 = \lim_{n\to\infty} 4\left(1 + \frac{2}{n} + \frac{1}{n^2}\right) = 4.$$

Thus, $A_2 = 16 - 4 = 12$.

62. (a) Using $\Delta x = \dfrac{x_0}{n}$ and $f\left(a + k\dfrac{b-a}{n}\right) = f\left(\dfrac{kx_0}{n}\right) = a\dfrac{k^2x_0^2}{n^2} + b\dfrac{kx_0}{n} + c$ we have

$$\begin{aligned} A &= \lim_{n\to\infty} \sum_{k=1}^{n} \left(a\frac{k^2x_0^2}{n^2} + b\frac{kx_0}{n} + c\right)\frac{x_0}{n} = \lim_{n\to\infty}\left(a\frac{x_0^3}{n^3}\sum_{k=1}^{n} k^2 + b\frac{x_0^2}{n^2}\sum_{k=1}^{n} k + c\frac{x_0}{n}\sum_{k=1}^{n} 1\right) \\ &= \lim_{n\to\infty}\left[a\frac{x_0^3}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) + b\frac{x_0^2}{2}\left(1 + \frac{1}{n}\right) + cx_0\right] = a\frac{x_0^3}{3} + b\frac{x_0^2}{2} + cx_0. \end{aligned}$$

(b) Let A_1 be the area under the graph on $[0, 2]$ and A_2 the area under the graph on $[0, 5]$. Then the area under the graph on $[2, 5]$ is

$$\begin{aligned} A = A_2 - A_1 &= \left(6\frac{5^3}{3} + 2\frac{5^2}{2} + 1.5\right) - \left(6\frac{2^3}{3} + 2\frac{2^2}{2} + 1.2\right) \\ &= (250 + 25 + 5) - (16 + 4 + 2) = 258. \end{aligned}$$

63. By (8) of this section,

$$\begin{aligned} A &= \lim_{n\to\infty} \sum_{k=1}^{n} f\left(0 + [k-1]\frac{1}{n}\right)\frac{1}{n} = \lim_{n\to\infty} \sum_{k=1}^{n} e^{(k-1)/n} \cdot \frac{1}{n} \\ &= \lim_{n\to\infty} \frac{1}{n}\left[1 + e^{1/n} + e^{2/n} + \cdots + e^{(n-1)/n}\right] \\ &= \lim_{n\to\infty} \frac{1}{n}\left[1 + e^{1/n} + (e^{1/n})^2 + \cdots + (e^{1/n})^{n-1}\right]. \end{aligned}$$

Using $a = 1$, $r = e^{1/n}$, we obtain

$$A = \lim_{n\to\infty} \frac{1}{n} \cdot 1 \cdot \left[\frac{1 - (e^{1/n})^n}{1 - e^{1/n}}\right] = (1 - e)\lim_{n\to\infty} \frac{1}{n(1 - e^{1/n})}$$

Now, $\displaystyle\lim_{n\to\infty} n(1 - e^{1/n}) = \lim_{n\to\infty} \frac{1 - e^{1/n}}{1/n}$ (form $\infty \cdot 0$)

$$\overset{h}{=} \lim_{n\to\infty} \frac{-e^{1/n}(-1/n^2)}{-1/n^2} = -\lim_{n\to\infty} e^{1/n} = -1,$$

so $A = (1 - e)\left(\dfrac{1}{-1}\right) = e - 1$.

64. $1+3+5+\cdots+2n-1=\sum_{k=1}^{n}(2k-1)=2\sum_{k=1}^{n}k-\sum_{k=1}^{n}1=n(n+1)-n=n^2$

The total distance moved is thus proportional to $1+3+5+\cdots+2n-1=n^2$.

5.4 The Definite Integral

1. From $\Delta x_1=1$, $\Delta x_2=2/3$, $\Delta x_3=2/3$, and $\Delta x_4=2/3$ we see that the norm of the partition is $\|P\|=1$. Using $f(x_1^*)=5/2$, $f(x_2^*)=5$, $f(x_3^*)=7$, and $f(x_4^*)=9$ we compute the Riemann sum
$$\sum_{k=1}^{4}f(x_k^*)\Delta x_k=\frac{5}{2}(1)+5\left(\frac{2}{3}\right)+7\left(\frac{2}{3}\right)+9\left(\frac{2}{3}\right)=\frac{33}{2}.$$

2. From $\Delta x_1=1$, $\Delta x_2=1/2$, $\Delta x_3=1$, $\Delta x_4=5/2$, and $\Delta x_5=2$ we see that the norm of the partition is $\|P\|=5/2$. Using $f(x_1^*)=-11/2$, $f(x_2^*)=-9/2$, $f(x_3^*)=-4$, $f(x_4^*)=-2$, and $f(x_5^*)=0$ we compute the Riemann sum
$$\sum_{k=1}^{5}f(x_k^*)\Delta x_k=\left(-\frac{11}{2}\right)(1)+\left(-\frac{9}{2}\right)\left(\frac{1}{2}\right)+(-4)(1)+(-2)\left(\frac{5}{2}\right)+0(2)=-\frac{67}{4}.$$

3. From $\Delta x_1=3/4$, $\Delta x_2=1/2$, $\Delta x_3=1/2$, and $\Delta x_4=1/4$ we see that the norm of the partition is $\|P\|=3/4$. Using $f(x_1^*)=9/16$, $f(x_2^*)=0$, $f(x_3^*)=1/4$, and $f(x_4^*)=49/64$ we compute the Riemann sum
$$\sum_{k=1}^{4}f(x_k^*)\Delta x_k=\frac{9}{15}\left(\frac{3}{4}\right)+0\left(\frac{1}{2}\right)+\frac{1}{4}\left(\frac{1}{2}\right)+\frac{49}{64}\left(\frac{1}{4}\right)=\frac{189}{256}.$$

4. From $\Delta x_1=1/2$, $\Delta x_2=1$, and $\Delta x_3=1/2$ we see that the norm of the partition is $\|P\|=1$. Using $f(x_1^*)=41/16$, $f(x_2^*)=65/16$, and $f(x_3^*)=10$ we compute the Riemann sum
$$\sum_{k=1}^{3}f(x_k^*)\Delta x_k=\frac{41}{16}\left(\frac{1}{2}\right)+\frac{65}{16}(1)+10\left(\frac{1}{2}\right)=\frac{331}{32}.$$

5. From $\Delta x_1=\pi$, $\Delta x_2=\pi/2$, and $\Delta x_3=\pi/2$ we see that the norm of the partition is $\|P\|=\pi$. Using $f(x_1^*)=1$, $f(x_2^*)=-1/2$, and $f(x_3^*)=-\sqrt{2}/2$ we compute the Riemann sum
$$\sum_{k=1}^{3}f(x_k^*)\Delta x_k=1(\pi)+\left(-\frac{1}{2}\right)\left(\frac{\pi}{2}\right)+\left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\pi}{2}\right)=\frac{(3-\sqrt{2})\pi}{4}.$$

6. From $\Delta x_1=\pi/4$, $\Delta x_2=\pi/4$, $\Delta x_3=\pi/3$, and $\Delta x_4=\pi/6$ we see that the norm of the partition is $\|P\|=\pi/3$. Using $f(x_1^*)=1/2$, $f(x_2^*)=\sqrt{3}/2$, $f(x_3^*)=\sqrt{2}/2$, and $f(x_4^*)=1/2$ we compute the Riemann sum
$$\sum_{k=1}^{4}f(x_k^*)\Delta x_k=\frac{1}{2}\left(\frac{\pi}{24}\right)+\frac{\sqrt{3}}{2}\left(\frac{\pi}{4}\right)+\frac{\sqrt{2}}{2}\left(\frac{\pi}{3}\right)+\frac{1}{2}\left(\frac{\pi}{6}\right)=\frac{(5+3\sqrt{3}+4\sqrt{2})\pi}{24}.$$

7. We have $\Delta x_k = 1$ and $x_k^* = k$ for $k = 1, 2, 3, 4, 5$. Using $f(x_1^*) = -1$, $f(x_2^*) = 0$, $f(x_3^*) = 1$, $f(x_4^*) = 2$, and $f(x_5^*) = 3$ we compute the Riemann sum

$$\sum_{k=1}^{5} f(x_k^*)\Delta x_n = -1(1) + 0(1) + 1(1) + 2(1) + 3(1) = 5.$$

8. We have $\Delta x_k = 1/3$ and $x_k^* = \dfrac{k-1}{3}$ for $k = 1, 2, 3$. Using $f(x_1^*) = 1$, $f(x_2^*) = 7/9$, and $f(x_3^*) = 7/9$ we compute the Riemann sum

$$\sum_{k=1}^{3} f(x_k^*)\Delta x_n = 1\left(\frac{1}{3}\right) + \frac{7}{9}\left(\frac{1}{3}\right) + \frac{7}{9}\left(\frac{1}{3}\right) = \frac{23}{27}.$$

9. $\displaystyle\int_{-2}^{4} \sqrt{9+x^2}\,dx$

10. $\displaystyle\int_{0}^{\pi/4} \tan x\,dx$

11. Identify $a = 0$ and $b = 2$. Then

$$\left(1+\frac{2k}{n}\right)\frac{2}{n} = \left[f\left(a+k\frac{b-a}{n}\right)\right]\frac{b-a}{n} \quad \text{and} \quad f\left(a+k\frac{b-a}{n}\right) = f\left(\frac{2k}{n}\right) = 1+\frac{2k}{n}.$$

Taking $f(x) = x+1$ we have $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n}\left(1+\frac{2k}{n}\right)\frac{2}{n} = \int_0^2 (x+1)\,dx.$

12. Identify $a = 1$ and $b = 4$. Then

$$\left(1+\frac{3k}{n}\right)^3\frac{3}{n} = \left[f\left(a+k\frac{b-a}{n}\right)\right]\frac{b-a}{n} \quad \text{and} \quad f\left(a+k\frac{b-a}{n}\right) = f\left(1+\frac{3k}{n}\right) = \left(1+\frac{3k}{n}\right)^3.$$

Taking $f(x) = x^3$ we have $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n}\left(1+\frac{3k}{n}\right)^3\frac{3}{n} = \int_1^4 x^3\,dx.$

13. Using $\dfrac{b-a}{n} = \dfrac{4}{n}$ and $f\left(a+k\dfrac{b-a}{n}\right) = -3+\dfrac{4k}{n}$ we have

$$\begin{aligned}\int_{-3}^{1} x\,dx &= \lim_{n\to\infty}\sum_{k=1}^{n}\left(-3+\frac{4k}{n}\right)\frac{4}{n} = \lim_{n\to\infty}\left(-\frac{12}{n}\sum_{k=1}^{n}1 + \frac{16}{n^2}\sum_{k=1}^{n}k\right)\\ &= \lim_{n\to\infty}\left[-\frac{12n}{n} + 8\left(1+\frac{1}{n}\right)\right] = -12+8 = -4.\end{aligned}$$

14. Using $\dfrac{b-a}{n} = \dfrac{3}{n}$ and $f\left(a+k\dfrac{b-a}{n}\right) = \dfrac{3k}{n}$ we have

$$\int_0^3 x\,dx = \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{3k}{n}\cdot\frac{3}{n}\right) = \lim_{n\to\infty}\frac{9}{n^2}\sum_{k=1}^{n}k = \lim_{n\to\infty}\left[\frac{9}{2}\left(1+\frac{1}{n}\right)\right] = \frac{9}{2}.$$

15. Using $\dfrac{b-a}{n} = \dfrac{1}{n}$ and $f\left(a + k\dfrac{b-a}{n}\right) = \left(1+\dfrac{k}{n}\right)^2 - \left(1+\dfrac{k}{n}\right) = \dfrac{k}{n} + \dfrac{k^2}{n^2}$ we have

$$\begin{aligned}\int_1^2 (x^2 - x)\,dx &= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{k}{n} + \frac{k^2}{n^2}\right)\frac{1}{n} = \lim_{n\to\infty}\left(\frac{1}{n^2}\sum_{k=1}^{n} k + \frac{1}{n^3}\sum_{k=1}^{n} k^2\right)\\ &= \lim_{n\to\infty}\left[\frac{1}{2}\left(1+\frac{1}{n}\right) + \frac{1}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)\right] = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}.\end{aligned}$$

16. Using $\dfrac{b-a}{n} = \dfrac{5}{n}$ and $f\left(a + k\dfrac{b-a}{n}\right) = \left(-2+\dfrac{5k}{n}\right)^2 - 4 = -\dfrac{20k}{n} + \dfrac{25k^2}{n^2}$ we have

$$\begin{aligned}\int_{-2}^3 (x^2 - 4)\,dx &= \lim_{n\to\infty} \sum_{k=1}^{n} \left(-\frac{20k}{n} + \frac{25k^2}{n^2}\right)\frac{5}{n} = \lim_{n\to\infty}\left(-\frac{100}{n^2}\sum_{k=1}^{n} k + \frac{125}{n^3}\sum_{k=1}^{n} k^2\right)\\ &= \lim_{n\to\infty}\left[-\frac{100}{2}\left(1+\frac{1}{n}\right) + \frac{125}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)\right] = -50 + \frac{125}{3} = -\frac{25}{3}.\end{aligned}$$

17. Using $\dfrac{b-a}{n} = \dfrac{1}{n}$ and $f\left(a + k\dfrac{b-a}{n}\right) = \dfrac{k^3}{n^3} - 1$ we have

$$\begin{aligned}\int_0^1 (x^3 - 1)\,dx &= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{k^3}{n^3} - 1\right)\frac{1}{n} = \lim_{n\to\infty}\left(\frac{1}{n^4}\sum_{k=1}^{n} k^3 - \frac{1}{n}\sum_{k=1}^{n} 1\right)\\ &= \lim_{n\to\infty}\left[\frac{1}{4}\left(1+\frac{2}{n} + \frac{1}{n^2}\right) - \frac{n}{n}\right] = \frac{1}{4} - 1 = -\frac{3}{4}.\end{aligned}$$

18. Using $\dfrac{b-a}{n} = \dfrac{2}{n}$ and $f\left(a + k\dfrac{b-a}{n}\right) = 3 - \dfrac{8k^3}{n^3}$ we have

$$\begin{aligned}\int_0^2 (3 - x^3)\,dx &= \lim_{n\to\infty} \sum_{k=1}^{n} \left(3 - \frac{8k^3}{n^3}\right)\frac{2}{n} = \lim_{n\to\infty}\left(\frac{6}{n}\sum_{k=1}^{n} 1 - \frac{16}{n^4}\sum_{k=1}^{n} k^3\right)\\ &= \lim_{n\to\infty}\left[6 - 4\left(1+\frac{2}{n} + \frac{1}{n^2}\right)\right] = 6 - 4 = 2.\end{aligned}$$

19. Using $f\left(a + k\dfrac{b-a}{n}\right) = a + \dfrac{k(b-a)}{n}$ we have

$$\begin{aligned}\int_a^b x\,dx &= \lim_{n\to\infty} \sum_{k=1}^{n} \left[a + \frac{k(b-a)}{n}\right]\frac{b-a}{n} = \lim_{n\to\infty}\left[\frac{a(b-a)}{n}\sum_{k=1}^{n} 1 + \frac{(b-a)^2}{n^2}\sum_{k=1}^{n} k\right]\\ &= \lim_{n\to\infty}\left[\frac{a(b-a)n}{n} + \frac{(b-a)^2}{2}\left(1+\frac{1}{n}\right)\right] = a(b-a) + \frac{(b-a)^2}{2}\\ &= \frac{b-a}{2}(2a + b - a) = \frac{b-a}{2}(b+a) = \frac{b^2-a^2}{2}.\end{aligned}$$

20. Using $f\left(a+k\dfrac{b-a}{n}\right)=\left[a+\dfrac{k(b-a)}{n}\right]^2=a^2+\dfrac{2ka(b-a)}{n}+\dfrac{k^2(b-a)^2}{n^2}$ we have

$$\begin{aligned}\int_a^b x^2\,dx &= \lim_{n\to\infty}\sum_{k=1}^{n}\left[a^2+\frac{2ka(b-a)}{n}+\frac{k^2(b-a)^2}{n^2}\right]\frac{b-a}{n}\\ &= \lim_{n\to\infty}\left[\frac{a^2(b-a)}{n}\sum_{k=1}^{n}1+\frac{2a(b-a)^2}{n^2}\sum_{k=1}^{n}k+\frac{(b-a)^3}{n^3}\sum_{k=1}^{n}k^2\right]\\ &= \lim_{n\to\infty}\left[\frac{a^2(b-a)n}{n}+\frac{2a(b-a)^2}{2}\left(1+\frac{1}{n}\right)+\frac{(b-a)^3}{6}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)\right]\\ &= a^2(b-a)+a(b-a)^2+\frac{(b-a)^3}{3}=\frac{b-a}{3}[3a^2+3a(b-a)+(b-a)^2]\\ &= \frac{b-a}{3}(b^2+ab+a^2)=\frac{b^3-a^3}{3}.\end{aligned}$$

21. $\displaystyle\int_{-1}^{3} x\,dx=\frac{1}{2}[3^2-(-1)^2]=4$

22. $\displaystyle\int_{-1}^{3} x^2\,dx=\frac{1}{3}[3^3-(-1)^3]=\frac{28}{3}$

23. $\displaystyle\int_{3}^{6} 4\,dx=4(6-3)=12$

24. $\displaystyle\int_{-2}^{5} (-2)\,dx=-2(5+2)=-14$

25. $\displaystyle\int_{4}^{-2} \frac{1}{2}\,dx=\frac{1}{2}(-2-4)=-3$

26. $\displaystyle\int_{5}^{5} 10x^4\,dx=0$

27. $\displaystyle-\int_{3}^{-1} 10x\,dx=10\int_{-1}^{3}x\,dx=10(4)=40$

28. $\displaystyle\int_{-1}^{3}(3x+1)\,dx=\int_{-1}^{3}3x\,dx+\int_{-1}^{3}1\,dx=3(4)+1[3-(-1)]=16$

29. $\displaystyle\int_{3}^{-1}t^2\,dt=-\int_{-1}^{3}t^2\,dt=-\frac{28}{3}$

30. $\displaystyle\int_{-1}^{3}(3x^2-5)\,dx=\int_{-1}^{3}3x^2\,dx-\int_{-1}^{3}5\,dx=3\left(\frac{28}{3}\right)-5[3-(-1)]=8$

31. $\displaystyle\int_{-1}^{3}(-3x^2+4x-5)\,dx = -3\int_{-1}^{3}x^2\,dx + 4\int_{-1}^{3}x\,dx - \int_{-1}^{3}5\,dx$

$$= -3\frac{28}{3} + 4(4) - 5[3-(-1)] = -32$$

32. $\displaystyle\int_{-1}^{3}6x(x-1)\,dx = \int_{-1}^{3}(6x^2-6x)\,dx = 6\int_{-1}^{3}x^2\,dx - 6\int_{-1}^{3}x\,dx = 6\left(\frac{28}{3}\right) - 6(4) = 32$

33. $\displaystyle\int_{-1}^{0}x^2\,dx + \int_{0}^{3}x^2\,dx = \int_{-1}^{3}x^2\,dx = \frac{28}{3}$

34. $\displaystyle\int_{-1}^{1.2}2t\,dt - \int_{3}^{1.2}2t\,dt = \int_{-1}^{1.2}2t\,dt + \int_{1.2}^{3}2t\,dt = \int_{-1}^{3}2t\,dt = 2\int_{-1}^{3}t\,dt = 2(4) = 8$

35. $\displaystyle\int_{0}^{4}x\,dx + \int_{0}^{4}(9-x)\,dx = \int_{0}^{4}[x+(9-x)]\,dx = \int_{0}^{4}9\,dx = 9(4-0) = 36$

36. $\displaystyle\int_{-1}^{0}t^2\,dt + \int_{0}^{2}x^2\,dx + \int_{2}^{3}u^2\,du = \int_{-1}^{0}x^2\,dx + \int_{0}^{2}x^2\,dx + \int_{2}^{3}x^2\,dx = \int_{-1}^{3}x^2\,dx = \frac{28}{3}$

37. $\displaystyle\int_{0}^{3}x^3\,dx + \int_{3}^{0}t^3\,dt = \int_{0}^{3}x^3\,dx - \int_{0}^{3}x^3\,dx = 0$

38. $\displaystyle\int_{-1}^{-1}5x\,dx - \int_{3}^{-1}(x-4)\,dx = 0 + \int_{-1}^{3}(x-4)\,dx = \int_{-1}^{3}x\,dx - \int_{-1}^{3}4\,dx$

$$= 4 - 4[3-(-1)] = -12$$

39. $\displaystyle\int_{2}^{5}f(x)\,dx = \int_{0}^{5}f(x)\,dx - \int_{0}^{2}f(x)\,dx = 8.5 - 6 = 2.5$

40. $\displaystyle\int_{1}^{3}f(x)\,dx = \int_{1}^{4}f(x)\,dx - \int_{3}^{4}f(x)\,dx = 2.4 - (-1.7) = 4.1$

41. $\displaystyle\int_{-1}^{2}[2f(x)+g(x)]\,dx = \int_{-1}^{2}2f(x)\,dx + \int_{-1}^{2}g(x)\,dx = 2\int_{-1}^{2}f(x)\,dx + \frac{1}{3}(3)\int_{-1}^{2}g(x)\,dx$

$$= 2(3.4) + \frac{1}{3}\int_{-1}^{2}3g(x)\,dx = 6.8 + \frac{1}{3}(12.6) = 6.8 + 4.2 = 11$$

42. Since $\displaystyle\int_{-2}^{2}[f(x)-5g(x)]\,dx = 24$, we have

$$\begin{aligned}
\int_{-2}^{2}f(x)\,dx - \int_{-2}^{2}5g(x)\,dx &= 24\\
-5\int_{-2}^{2}g(x)\,dx &= 24 - \int_{-2}^{2}f(x)\,dx\\
\int_{-2}^{2}g(x)\,dx &= -\frac{1}{5}\left[24 + \int_{2}^{-2}f(x)\,dx\right] = -\frac{24+14}{5} = -\frac{38}{5}.
\end{aligned}$$

43. (a) $\displaystyle\int_a^b f(x)\,dx = -2.5$

(b) $\displaystyle\int_b^c f(x)\,dx = 3.9$

(c) $\displaystyle\int_c^d f(x)\,dx = -1.2$

(d) $\displaystyle\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx = -2.5 + 3.9 = 1.4$

(e) $\displaystyle\int_b^d f(x)\,dx = \int_b^c f(x)\,dx + \int_c^d f(x)\,dx = 3.9 - 1.2 = 2.7$

(f) $\displaystyle\int_a^d f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx + \int_c^d f(x)\,dx = -2.5 + 3.9 - 1.2 = 0.2$

44. (a) $\displaystyle\int_a^b f(x)\,dx = 6.8$

(b) $\displaystyle\int_b^c f(x)\,dx = -7.3$

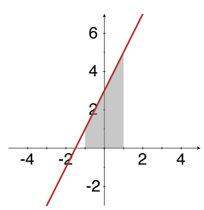
(c) $\displaystyle\int_c^d f(x)\,dx = 9.2$

(d) $\displaystyle\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx = 6.8 - 7.3 = -0.5$

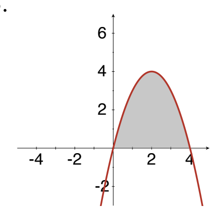
(e) $\displaystyle\int_b^d f(x)\,dx = \int_b^c f(x)\,dx + \int_c^d f(x)\,dx = -7.3 + 9.2 = 1.9$

(f) $\displaystyle\int_a^d f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx + \int_c^d f(x)\,dx = 6.8 - 7.3 + 9.2 = 8.7$

45.



46.



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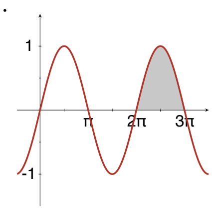
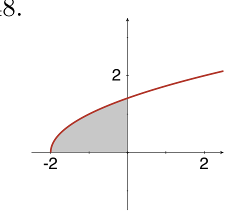
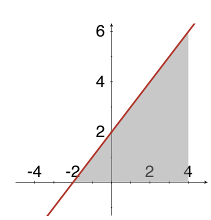


48.



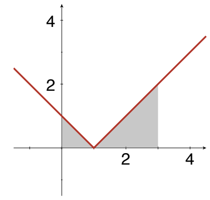
49. From the figure, we see that the area under the graph is a triangle with a base and height of 6. Thus, the area from geometry is

$$A = \frac{bh}{2} = \frac{6(6)}{2} = 18.$$



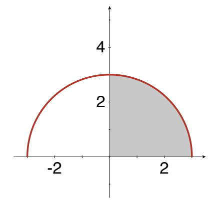
50. From the figure, we see that the area under the graph consists of two triangles; one has a base and height of 1 while the other has a base and height of 2. Thus, the area from geometry is

$$A = \frac{b_1h_1}{2} + \frac{b_2h_2}{2} = \frac{1(1)}{2} + \frac{2(2)}{2} = \frac{5}{2}.$$



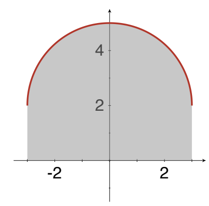
51. From the figure, we see that the area under the graph consists of one-fourth of a circle of radius 3. Thus, the area from geometry is

$$A = \frac{\pi r^2}{4} = \frac{\pi(3)^2}{4} = \frac{9\pi}{4}.$$



52. From the figure, we see that the area under the graph consists of a semicircle of radius 3 above a rectangle of width 6 and height 2. Thus, the area from geometry is

$$A = \frac{\pi r^2}{2} + wh = \frac{\pi(3)^2}{2} + 6(2) = \frac{9\pi}{2} + 12.$$



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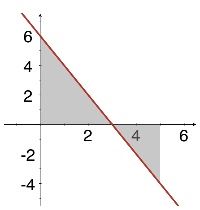

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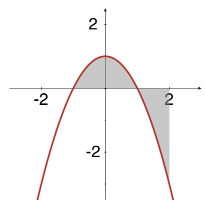

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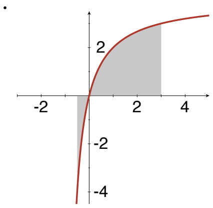

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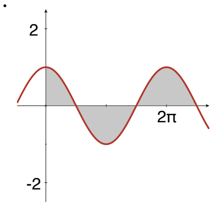
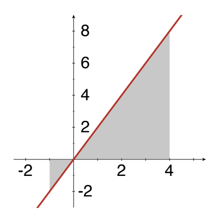


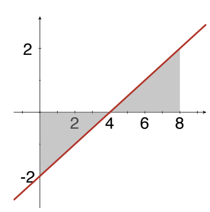
57. From the figure, we see that the net signed area under the graph is the area of a triangle with a base of 1 and a height of 2 subtracted from the area of a triangle with a base of 4 and a height of 8. Thus, the net signed area from geometry is

$$A = \frac{b_1h_1}{2} - \frac{b_2h_2}{2} = \frac{4(8)}{2} - \frac{1(2)}{2} = 15.$$



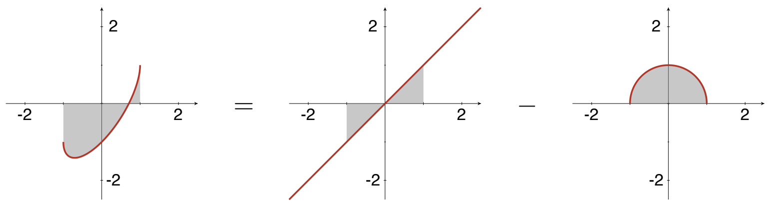
58. From the figure, we see that the net signed area under the graph is the area of a triangle with a base of 4 and a height of 2 subtracted from the area of a triangle with a base of 4 and a height of 2. Thus, the net signed area from geometry is

$$A = \frac{b_1h_1}{2} - \frac{b_2h_2}{2} = \frac{4(2)}{2} - \frac{4(2)}{2} = 0.$$



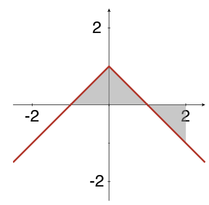
59. $\int_{-1}^{1}(x-\sqrt{1-x^2})\,dx$ can be rewritten as $\int_{-1}^{1}x\,dx-\int_{-1}^{1}\sqrt{1-x^2}\,dx$, so the net signed area of the graph below left is the same as the difference between the net signed areas of the graphs below right. This difference, in turn, is the area of a semicircle of radius 1 subtracted from the net signed area of two triangles with bases and heights of 1. From geometry, this is

$$A=\left(\frac{b_1h_1}{2}-\frac{b_2h_2}{2}\right)-\frac{\pi r^2}{2}=\left[\frac{1(1)}{2}-\frac{1(1)}{2}\right]-\frac{\pi(1)^2}{2}=-\frac{\pi}{2}.$$



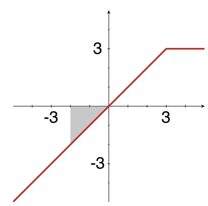
60. From the figure, we see that the net signed area under the graph is the area of a triangle with a base of 1 and a height of 1 subtracted from the area of a triangle with a base of 2 and a height of 1. Thus, the net signed area from geometry is

$$A=\frac{b_1h_1}{2}-\frac{b_2h_2}{2}=\frac{2(1)}{2}-\frac{1(1)}{2}=\frac{1}{2}.$$



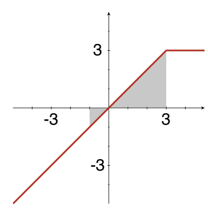
61. From the figure, we see that the net signed area under the graph is the negative of the area of a triangle with a base of 2 and a height of 2. Thus, the net signed area from geometry is

$$A=-\frac{bh}{2}=-\frac{2(2)}{2}=-2.$$



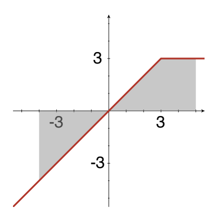
62. From the figure, we see that the net signed area under the graph is the area of a triangle with a base of 1 and a height of 1 subtracted from the area of a triangle with a base of 3 and a height of 3. Thus, the net signed area from geometry is

$$A=\frac{b_1h_1}{2}-\frac{b_2h_2}{2}=\frac{3(3)}{2}-\frac{1(1)}{2}=4.$$

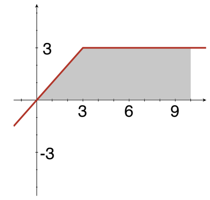


63. From the figure, we see that the net signed area under the graph is the area of a triangle with a base of 4 and a height of 4 subtracted from the sum of the areas of a triangle with a base of 3 and a height of 3, and a rectangle of width 2 and height 3. Thus, the net signed area from geometry is

$$A=(wh+\frac{b_1h_1}{2})-\frac{b_2h_2}{2}=[2(3)+\frac{3(3)}{2}]-\frac{4(4)}{2}=\frac{5}{2}=2.5.$$



64. From the figure, we see that the net signed area under the graph is the sum of the areas of a triangle with a base of 3 and a height of 3, and a rectangle of width 7 and height 3. Thus, the net signed area from geometry is



$$A = wh + \frac{bh}{2} = 7(3) + \frac{3(3)}{2} = \frac{51}{2} = 25.5.$$

65. For $-1 \le x \le 0$, $e^x \le 1$ and $e^{-x} \ge 1$. Then $e^{-x} \ge e^x$ on $[-1, 0]$ and by Theorem 5.4.7(i) we have $\displaystyle\int_{-1}^{0} e^x\,dx \le \int_{-1}^{0} e^{-x}\,dx$.

66. For $0 \le x \le \pi/4$, $\tan x \le 1$. Then $\sin x/\cos x \le 1$ and $\sin x \le \cos x$. Thus, by Theorem 5.4.7(i) we have

$$\int_0^{\pi/4} \cos x\,dx \ge \int_0^{\pi/4} \sin x\,dx, \qquad \int_0^{\pi/4} \cos x\,dx - \int_0^{\pi/4} \sin x\,dx \ge 0,$$

and $\displaystyle\int_0^{\pi/4} (\cos x - \sin x)\,dx \ge 0$.

67. Letting $f(x) = (x^3+1)^{1/2}$ we have $f'(x) = \dfrac{3}{2}x^2(x^3+1)^{-1/2}$. For $0 \le x \le 1$, $f'(x) \ge 0$ and $f(0) \le f(x) \le f(1)$. Since $f(0) = 1$ and $f(1) = \sqrt{2} < 1.42$, we identify $m = 1$ and $M = 1.42$. Then by Theorem 5.4.7(ii)

$$1(1-0) \le \int_0^1 (x^3+1)^{1/2}\,dx \le 1.42(1-0) \qquad \text{and} \qquad 1 \le \int_0^1 (x^3+1)^{1/2}\,dx \le 1.42.$$

68. Letting $f(x) = x^2 - 2x$ we have $f'(x) = 2x - 2$ and $f''(x) = 2$. Solving $f'(x) = 0$ we obtain the critical number 1, and since $f''(x) > 0$ for all x, the graph of f is concave up with the absolute minimum at $x = 1$. Since $f(0) = f(2) = 0$, we identify $m = -1$ and $M = 0$. Then by Theorem 5.4.7(ii)

$$-1(2-0) \le \int_0^2 (x^2-2x)\,dx \le 0(2-0) \qquad \text{and} \qquad -2 \le \int_0^2 (x^2-2x)\,dx \le 0.$$

69. On $[0, 1]$, $x^2 - x^3 = x^2(1-x) \ge 0$, so $x^2 \ge x^3$. Thus by Theorem 5.4.7(i), $\displaystyle\int_0^1 x^2\,dx \ge \int_0^1 x^3\,dx$.

70. On $[0, 1]$, $x^2 - x = x(x-1) \le 0$, so $x^2 \le x$. Thus, $\sqrt{4+x^2} \le \sqrt{4+x}$, and by Theorem 5.4.7(i), $\displaystyle\int_0^1 \sqrt{4+x^2}\,dx \le \int_0^1 \sqrt{4+x}\,dx$.

71. Since $f^2(x) \ge 0$ on $[a, b]$, by (12), $\displaystyle\int_a^b f^2(x)\,dx \ge 0$.

72. We will use the fact that any interval with nonzero length contains both rational and irrational numbers. Let P be a partition of $[-1, 1]$. Then $\displaystyle\int_{-1}^{1} f(x)\,dx = \lim_{\|P\|\to 0}\sum_{k=1}^{n} f(x_k^*)\Delta x_k$. First, choosing each x_k^* to be rational, we obtain $\displaystyle\int_{-1}^{1} f(x)\,dx = \lim_{\|P\|\to 0}\sum_{k=1}^{n} 0\cdot\Delta x_k = 0$. Then, choosing each x_k^* to be irrational, we obtain $\displaystyle\int_{-1}^{1} f(x)\,dx = \lim_{\|P\|\to 0}\sum_{k=1}^{n} 1\cdot\Delta x_k = 1-(-1) = 2$. Since $0 \neq 2$, $\displaystyle\int_{-1}^{1} f(x)\,dx$ does not exist.

73. Using $\Delta x = \dfrac{k^2}{n^2} - \dfrac{(k-1)^2}{n^2} = \dfrac{2k-1}{n^2}$ we have

$$\begin{aligned}
\int_0^1 \sqrt{x}\,dx &= \lim_{n\to\infty}\sum_{k=1}^{n}\sqrt{\frac{k^2}{n^2}}\left(\frac{2k-1}{n^2}\right) = \lim_{n\to\infty}\frac{1}{n^3}\sum_{k=1}^{n}(2k^2-k)\\
&= \lim_{n\to\infty}\left(\frac{2}{n^3}\sum_{k=1}^{n}k^2 - \frac{1}{n^3}\sum_{k=1}^{n}k\right) = \lim_{n\to\infty}\left[\frac{2}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} - \frac{1}{n^3}\cdot\frac{n(n+1)}{2}\right]\\
&= \lim_{n\to\infty}\left[\frac{1}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) - \frac{1}{2n}\left(1+\frac{1}{n}\right)\right] = \frac{2}{3} - 0 = \frac{2}{3}.
\end{aligned}$$

74. The midpoint of the kth subinterval is $k\left(\dfrac{\pi}{2n}\right) - \dfrac{1}{2}\left(\dfrac{\pi}{2n}\right) = \dfrac{(2k-1)\pi}{4n}$. Then

$$\begin{aligned}
\int_0^{\pi/2}\cos x\,dx &= \lim_{n\to\infty}\sum_{k=1}^{n}\left\{\cos\left[\frac{(2k-1)\pi}{4n}\right]\right\}\frac{\pi}{2n}\\
&= \lim_{n\to\infty}\frac{\pi}{2n}\left\{\cos\left(\frac{\pi}{4n}\right) + \cos\left(3\frac{\pi}{4n}\right) + \cdots + \cos\left[(2n-1)\frac{\pi}{4n}\right]\right\}\\
&= \lim_{n\to\infty}\frac{\pi}{2n}\left[\frac{\sin\left(2n\cdot\frac{\pi}{4n}\right)}{2\sin\left(\frac{\pi}{4n}\right)}\right] = \frac{\pi}{4}\lim_{n\to\infty}\frac{\sin\left(\frac{\pi}{2}\right)}{n\sin\left(\frac{\pi}{4n}\right)}\\
&= \frac{\pi}{4}\lim_{n\to\infty}\frac{1}{n\sin\left(\frac{\pi}{4n}\right)} = \frac{\pi}{4}\cdot\frac{4}{\pi} = 1.
\end{aligned}$$

5.5 Fundamental Theorem of Calculus

1. $\displaystyle\int_3^7 dx = x\big]_3^7 = 7 - 3 = 4$

2. $\displaystyle\int_2^{10}(-4)\,dx = -4x\big]_2^{10} = -40-(-8) = -32$

3. $\displaystyle\int_{-1}^{2}(2x+3)\,dx = (x^2+3x)\big]_{-1}^{2} = 10-(-2) = 12$

4. $\displaystyle\int_{-5}^{4} t^2\,dt = \frac{1}{3}t^3\Big]_{-5}^{4} = \frac{1}{3}[64-(-125)] = 63$

5. $\displaystyle\int_{1}^{3} (6x^2-4x+5)\,dx = (2x^3-2x^2+5x)\big]_1^3 = 51-5 = 46$

6. $\displaystyle\int_{-2}^{1} (12x^5-36)\,dx = (2x^6-36x)\big]_{-2}^{1} = -34-200 = -234$

7. $\displaystyle\int_{0}^{\pi/2} \sin x\,dx = -\cos x]_0^{\pi/2} = 0-(-1) = 1$

8. $\displaystyle\int_{-\pi/3}^{\pi/4} \cos\theta d\theta = \sin\theta]_{-\pi/3}^{\pi/4} = \frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{2}+\sqrt{3}}{2}$

9. $\displaystyle\int_{\pi/4}^{\pi/2} \cos 3t\,dt = \frac{1}{3}\sin 3t\Big]_{\pi/4}^{\pi/2} = \frac{1}{3}\sin\frac{3\pi}{2} - \frac{1}{3}\sin\frac{3\pi}{4} = -\frac{1}{3} - \frac{\sqrt{2}}{6} = -\frac{2+\sqrt{2}}{6}$

10. $\displaystyle\int_{1/2}^{1} \sin 2\pi x\,dx = -\frac{1}{2\pi}\cos 2\pi x\Big]_{1/2}^{1} = -\frac{1}{2\pi}(\cos 2\pi - \cos\pi) = -\frac{1}{\pi}$

11. $\displaystyle\int_{1/2}^{3/4} \frac{1}{u^2}\,du = -\frac{1}{u}\Big]_{1/2}^{3/4} = -\frac{4}{3} - (-2) = \frac{2}{3}$

12. $\displaystyle\int_{-3}^{-1} \frac{2}{x}\,dx = 2\ln|x|]_{-3}^{-1} = 0 - 2\ln 3 = -\ln 9$

13. $\displaystyle\int_{-1}^{1} e^x\,dx = e^x]_{-1}^{1} = e - \frac{1}{e}$

14. $\displaystyle\int_{0}^{2} (2x-3e^x)\,dx = x^2-3e^x\big]_0^2 = (4-3e^2)-(-3) = 7-3e^2$

15. $\displaystyle\int_{0}^{2} x(1-x)\,dx = \int_0^2 (x-x^2)\,dx = \left(\frac{1}{2}x^2 - \frac{1}{3}x^3\right)\Big]_0^2 = 2 - \frac{8}{3} - 0 = -\frac{2}{3}$

16. $\displaystyle\int_{3}^{2} x(x-2)(x+2)\,dx = -\int_2^3 (x^3-4x)\,dx = -\left(\frac{1}{4}x^3 - 2x^2\right)\Big]_2^3 = -\frac{25}{4}$

17. $$\int_{-1}^{1} (7x^3-2x^2+5x-4)\,dx = \left(\frac{7}{4}x^4 - \frac{2}{3}x^3 + \frac{5}{2}x^2 - 4x\right)\Big]_{-1}^{1}$$
$$= \left(\frac{7}{4} - \frac{2}{3} + \frac{5}{2} - 4\right) - \left(\frac{7}{4} + \frac{2}{3} + \frac{5}{2} + 4\right) = -\frac{28}{3}$$

18. $\displaystyle\int_{-3}^{-1} (x^2-4x+8)\,dx = \left(\frac{1}{3}x^3 - 2x^2 + 8x\right)\Big]_{-3}^{-1} = \left(-\frac{1}{3} - 2 - 8\right) - (-9-18-24) = \frac{122}{3}$

19. $\displaystyle\int_1^4 \frac{x-1}{\sqrt{x}}\,dx = \int_1^4 (x^{1/2} - x^{-1/2})\,dx = \left(\frac{2}{3}x^{3/2} - 2x^{1/2}\right)\Big]_1^4 = \left(\frac{16}{3} - 4\right) - \left(\frac{2}{3} - 2\right) = \frac{8}{3}$

20. $\displaystyle\int_2^4 \frac{x^2+8}{x^2}\,dx = \int_2^4 (1 + 8x^{-2})\,dx = (x - 8x^{-1})\big]_2^4 - \left(x - \frac{8}{x}\right)\Big]_2^4 = 2 - (-2) = 4$

21. $\displaystyle\int_1^{\sqrt{3}} \frac{1}{1+x^2}\,dx = \tan^{-1} x\big]_1^{\sqrt{3}} = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$

22. $\displaystyle\int_0^{1/4} \frac{1}{\sqrt{1-4x^2}}\,dx = \frac{1}{2}\int_0^{1/4} \frac{2}{\sqrt{1-(2x)^2}}\,dx = \frac{1}{2}\left(\sin^{-1} 2x\big]_0^{1/4}\right) = \frac{1}{2}\left(\frac{\pi}{6} - 0\right) = \frac{\pi}{12}$

23. $\displaystyle\int_{-4}^{12} \sqrt{z+4}\,dz$ $\quad \boxed{u = z+4,\ du = dz}$

$$= \int_0^{16} u^{1/2}\,du = \frac{2}{3}u^{3/2}\Big]_0^{16} = \frac{128}{3}$$

24. $\displaystyle\int_0^{7/2} (2x+1)^{-1/3}\,dx$ $\quad \boxed{u = 2x+1,\ du = 2\,dx}$

$$= \frac{1}{2}\int_1^8 u^{-1/3}\,du = \frac{1}{2}\left(\frac{3}{2}u^{2/3}\right)\Big]_1^8 = 3 - \frac{3}{4} = \frac{9}{4}$$

25. $\displaystyle\int_0^3 \frac{x}{\sqrt{x^2+16}}\,dx$ $\quad \boxed{u = x^2+16,\ du = 2x\,dx}$

$$= \frac{1}{2}\int_{16}^{25} \frac{1}{\sqrt{u}}\,du = \frac{1}{2}\int_{16}^{25} u^{-1/2}\,du = \sqrt{u}\big]_{16}^{25} = 5 - 4 = 1$$

26. $\displaystyle\int_{-2}^1 \frac{t}{(t^2+1)^2}\,dt$ $\quad \boxed{u = t^2+1,\ du = 2t\,dt}$

$$= \int_5^2 \frac{1/2}{u^2}\,du = -\frac{1}{2}\int_2^5 u^{-2}\,du = \frac{1}{2}u^{-1}\Big]_2^5 = \frac{1}{2}\left(\frac{1}{5} - \frac{1}{2}\right) = -\frac{3}{20}$$

27. $\displaystyle\int_{1/2}^1 \left(1 + \frac{1}{x}\right)^2 \frac{1}{x^2}\,dx$ $\quad \boxed{u = 1 + \frac{1}{x},\ du = -\frac{1}{x^2}\,dx}$

$$= -\int_3^2 u^3\,du = \int_2^3 u^3\,du = \frac{1}{4}u^4\Big]_2^3 = \frac{81}{4} - 4 = \frac{65}{4}$$

28. $\displaystyle\int_1^4 \frac{\sqrt[3]{1+4\sqrt{x}}}{\sqrt{x}}\,dx$ $\quad \boxed{u = 1 + 4\sqrt{x},\ du = \frac{2}{\sqrt{x}}\,dx}$

$$= \frac{1}{2}\int_5^9 u^{1/3}\,du = \frac{1}{2}\left(\frac{3}{4}u^{4/3}\right)\Big]_5^9 = \frac{3}{8}(9^{4/3} - 5^{4/3})$$

29. $\displaystyle\int_0^1 \frac{x+1}{\sqrt{x^2+2x+3}}\,dx \qquad \boxed{u = x^2+2x+3,\ du = 2(x+1)\,dx}$

$$= \frac{1}{2}\int_3^6 \frac{1}{\sqrt{u}}\,du = \frac{1}{2}\int_3^6 u^{-1/2}\,du = \sqrt{u}\Big]_3^6 = \sqrt{6}-\sqrt{3}$$

30. $\displaystyle\int_{-1}^1 \frac{u^3+u}{(u^4+2u^2+1)^5}\,du \qquad \boxed{z = u^4+2u^2+1,\ du = 4(u^3+u)\,du}$

$$= \frac{1}{4}\int_4^4 \frac{1}{z^5}\,dz = 0$$

31. $\displaystyle\int_0^{\pi/8} \sec^2 2x\,dx \qquad \boxed{u = 2x,\ du = 2\,dx}$

$$= \frac{1}{2}\int_0^{\pi/4} \sec^2 u\,du = \frac{1}{2}\tan u\Big]_0^{\pi/4} = \frac{1}{2}$$

32. $\displaystyle\int_{\sqrt{\pi/4}}^{\sqrt{\pi/2}} x\csc x^2 \cot x^2\,dx \qquad \boxed{u = x^2,\ du = 2x\,dx}$

$$= \frac{1}{2}\int_{\pi/4}^{\pi/2} \csc u \cot u\,du = -\frac{1}{2}\csc u\Big]_{\pi/4}^{\pi/2} = -\frac{1}{2}(1-\sqrt{2}) = \frac{\sqrt{2}-1}{2}$$

33. $\displaystyle\int_{-1/2}^{3/2} (x-\cos\pi x)\,dx = \left(\frac{1}{2}x^2 - \frac{1}{\pi}\sin\pi x\right)\Big]_{-1/2}^{3/2} = \left[\frac{9}{8}-\left(-\frac{1}{\pi}\right)\right] - \left[\frac{1}{8}-\left(-\frac{1}{\pi}\right)\right] = 1$

34. $\displaystyle\int_1^4 \frac{\cos\sqrt{x}}{2\sqrt{x}}\,dx \qquad \boxed{u = \sqrt{x},\ du = \frac{1}{2\sqrt{x}}\,dx}$

$$= \int_1^2 \cos u\,du = \sin u\Big]_1^2 = \sin 2 - \sin 1$$

35. $\displaystyle\int_0^{\pi/2} \sqrt{\cos x}\sin x\,dx \qquad \boxed{u = \cos x,\ du = -\sin x\,dx}$

$$= -\int_1^0 \sqrt{u}\,du = \int_0^1 u^{1/2}\,du = \frac{2}{3}u^{3/2}\Big]_0^1 = \frac{2}{3}$$

36. $\displaystyle\int_{\pi/6}^{\pi/3} \sin x\cos x\,dx \qquad \boxed{u = \sin x,\ du = \cos x\,dx}$

$$= \int_{1/2}^{\sqrt{3}/2} u\,du = \frac{1}{2}u^2\Big]_{1/2}^{\sqrt{3}/2} = \frac{3}{8}-\frac{1}{8} = \frac{1}{4}$$

37. $\displaystyle\int_{\pi/6}^{\pi/2} \frac{1+\cos\theta}{(\theta+\sin\theta)^2}\,d\theta \qquad \boxed{u = \theta+\sin\theta,\ du = (1+\cos\theta)\,d\theta}$

$$= \int_{(\pi+3)/6}^{(\pi+2)/2} u^{-2}\,du = -\frac{1}{u}\Big]_{(\pi+3)/6}^{(\pi+2)/2} = -\frac{2}{\pi+2}+\frac{6}{\pi+3} = \frac{4\pi+6}{(\pi+3)(\pi+2)}$$

38. $$\begin{aligned}\int_{-\pi/4}^{\pi/4}(\sec x+\tan x)^2\,dx &= \int_{-\pi/4}^{\pi/4}(\sec^2 x+2\sec x\tan x+\tan^2 x)\,dx\\ &= \int_{-\pi/4}^{\pi/4}(\sec^2 x+2\sec x\tan x+\sec^2 x-1)\,dx\\ &= \int_{-\pi/4}^{\pi/4}(2\sec^2 x+2\sec x\tan x-1)\,dx\\ &= (2\tan x+2\sec x-x)\big]_{-\pi/4}^{\pi/4}\\ &= \left(2+2\sqrt{2}-\frac{\pi}{4}\right)-\left(-2+2\sqrt{2}+\frac{\pi}{4}\right)=\frac{8-\pi}{2}\end{aligned}$$

39. $$\begin{aligned}\int_0^{3/4}\sin^2\pi x\,dx &= \int_0^{3/4}\frac{1}{2}(1-\cos 2\pi x)\,dx=\left(\frac{1}{2}x-\frac{1}{4\pi}\sin 2\pi x\right)\Big]_0^{3/4}\\ &= \frac{3}{8}-\frac{1}{4\pi}\sin\frac{3\pi}{2}=\frac{3}{8}+\frac{1}{4\pi}\end{aligned}$$

40. Using the fact that $f(x)=\cos^2 x$ is even, we have

$$\begin{aligned}\int_{-\pi/2}^{\pi/2}\cos^2 x\,dx=2\int_0^{\pi/2}\cos^2 x\,dx &= 2\int_0^{\pi/2}\frac{1}{2}(1+\cos 2x)\,dx\\ &= \left(x+\frac{1}{2}\sin 2x\right)\Big]_0^{\pi/2}=\left(\frac{\pi}{2}+0\right)-0=\frac{\pi}{2}.\end{aligned}$$

41. $$\int_1^5\frac{1}{1+2x}\,dx \qquad \boxed{u=1+2x,\ du=2\,dx}$$
$$=\frac{1}{2}\int_3^{11}\frac{1}{u}\,du=\frac{1}{2}\ln|u|\Big]_3^{11}=\frac{1}{2}(\ln 11-\ln 3)$$

42. Since $f(x)=\tan x$ is an odd function on $[-1,1]$, we have $\displaystyle\int_{-1}^{1}\tan x\,dx=0$.

43. $$\frac{d}{dx}\int_0^x te^t\,dt=xe^x$$

44. $$\frac{d}{dx}\int_1^x \ln t\,dt=\ln x$$

45. $$\frac{d}{dt}\int_2^t(3x^2-2x)^6\,dx=(3t^2-2t)^6$$

46. $$\frac{d}{dx}\int_x^9\sqrt[3]{u^2+2}\,du=-\sqrt[3]{x^2+2}$$

47. $\dfrac{d}{dx}\displaystyle\int_3^{6x-1}\sqrt{4t+9}\,dt \qquad \boxed{u=6x-1,\ du=6\,dx}$

$$= \frac{d}{du}\left(\int_3^u \sqrt{4t+9}\,dt\right)\frac{du}{dx} = \sqrt{4u+9}\ \frac{du}{dx}$$
$$= \sqrt{4(6x-1)+9}\cdot 6 = 6\sqrt{24x+5}$$

48. $\dfrac{d}{dx}\displaystyle\int_\pi^{\sqrt{x}}\sin t^2\,dt \qquad \boxed{u=\sqrt{x}=x^{1/2},\ du=\frac{1}{2}x^{-1/2}\,dx}$

$$= \frac{d}{du}\left(\int_\pi^u \sin t^2\,dt\right)\frac{du}{dx} = \sin u^2\ \frac{du}{dx} = \sin x\cdot\left(\frac{1}{2}x^{-1/2}\right) = \frac{1}{2\sqrt{x}}\sin x$$

49. $F'(x) = \dfrac{d}{dx}\left(\displaystyle\int_{3x}^0 \frac{1}{t^3+1}\,dt + \int_0^{x^2}\frac{1}{t^3+1}\,dt\right) = \dfrac{d}{dx}\left(-\displaystyle\int_0^{3x}\frac{1}{t^3+1}\,dt + \int_0^{x^2}\frac{1}{t^3+1}\,dt\right)$

$$\boxed{u=3x,\ du=3\,dx;\ z=x^2,\ dz=2x\,dx}$$
$$= \frac{d}{du}\left(-\int_0^u \frac{1}{t^3+1}\,dt\right)\frac{du}{dx} + \frac{d}{dz}\left(\int_0^z \frac{1}{t^3+1}\,dt\right)\frac{dz}{dx}$$
$$= -\frac{1}{(3x)^3+1}(3) + \frac{1}{(x^2)^3+1}(2x) = \frac{2x}{x^6+1} - \frac{3}{27x^3+1}$$

50. $F'(x) = \dfrac{d}{dx}\left(\displaystyle\int_{\sin x}^0 \sqrt{t^2+1}\,dt + \int_0^{5x}\sqrt{t^2+1}\,dt\right)$

$$= \frac{d}{dx}\left(-\int_0^{\sin x}\sqrt{t^2+1}\,dt + \int_0^{5x}\sqrt{t^2+1}\,dt\right)$$
$$\boxed{u=\sin x,\ du=\cos x\,dx;\ z=5x,\ dz=5\,dx}$$
$$= \frac{d}{du}\left(-\int_0^u \sqrt{t^2+1}\,dt\right)\frac{du}{dx} + \frac{d}{dz}\left(\int_0^z \sqrt{t^2+1}\,dt\right)\frac{dz}{dx}$$
$$= -\sqrt{\sin^2 x+1}(\cos x) + \sqrt{(5x)^2+1}(5) = 5\sqrt{25x^2+1} - \cos x\sqrt{\sin^2 x+1}$$

51. $\dfrac{d}{dx}\displaystyle\int_1^x (6t^2-8t+5)\,dt = \dfrac{d}{dx}\ (2t^3-4t^2+5t)\Big]_1^x$

$$= \frac{d}{dx}[(2x^3-4x^2+5x)-(2-4+5)] = 6x^2-8x+5$$

52. $\dfrac{d}{dt}\displaystyle\int_\pi^t \sin\frac{x}{3}\,dx = \dfrac{d}{dt}\left(-3\cos\dfrac{x}{3}\right)\Big]_\pi^t = \dfrac{d}{dt}\left[-3\cos\dfrac{t}{3} - \left(-3\cos\dfrac{\pi}{3}\right)\right] = \sin\dfrac{t}{3}$

53. (a) $f(1) = \displaystyle\int_1^1 \ln(2t+1)\,dt = 0$

(b) $f'(x) = \ln(2x+1)$, so $f'(1) = \ln[2(1)+1] = \ln 3$.

(c) $f''(x) = \dfrac{2}{2x+1}$, so $f''(1) = \dfrac{2}{2(1)+1} = \dfrac{2}{3}$.

(d) $f'''(x) = -\dfrac{4}{(2x+1)^2}$, so $f'''(1) = -\dfrac{4}{[2(1)+1]^2} = -\dfrac{4}{9}$.

54. (a) $G(x^2) = \displaystyle\int_a^{x^2} f(t)\,dt$

(b) $\dfrac{d}{dx}G(x^2) = \dfrac{d}{dx}\displaystyle\int_a^{x^2} f(t)\,dt = 2xf(x^2)$

(c) $G(x^3+2x) = \displaystyle\int_a^{x^3+2x} f(t)\,dt$

(d) $\dfrac{d}{dx}G(x^3+2x) = \dfrac{d}{dx}\displaystyle\int_a^{x^3+2x} f(t)\,dt = (3x^2+2)f(x^3+2x)$

55. $$\int_{-1}^{2} f(x)\,dx = \int_{-1}^{0} f(x)\,dx + \int_0^2 f(x)\,dx = \int_{-1}^{0} -x\,dx + \int_0^2 x^2\,dx = -\frac{1}{2}x^2\Big]_{-1}^{0} + \frac{1}{3}x^3\Big]_0^2$$
$$= -\left(0 - \frac{1}{2}\right) + \frac{1}{3}(8-0) = \frac{19}{6}$$

56. $$\int_{-1}^{2} f(x)\,dx = \int_{-1}^{0} f(x)\,dx + \int_0^2 f(x)\,dx = \int_{-1}^{0} (2x+3)\,dx + \int_0^2 3\,dx$$
$$= (x^2+3x)\Big]_{-1}^{0} + 3x]_0^2 = [0-(-2)] + 6 = 8$$

57. $$\int_0^3 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^3 f(x)\,dx = \int_0^2 4\,dx + \int_2^3 dx = 4x]_0^2 + x]_2^3$$
$$= (8-0) + (3-2) = 9$$

58. $$\int_0^{\pi} f(x)\,dx = \int_0^{\pi/2} f(x)\,dx + \int_{\pi/2}^{\pi} f(x)\,dx = \int_0^{\pi/2} \sin x\,dx + \int_{\pi/2}^{\pi} \cos x\,dx$$
$$= -\cos x]_0^{\pi/2} + \sin x]_{\pi/2}^{\pi} = -(0-1) + (0-1) = 0$$

59. Using the fact that $f(x)$ is an even function on $[-2,2]$, we have
$$\int_{-2}^{2} f(x)\,dx = 2\int_0^2 f(x)\,dx = 2\left[\int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx\right] = 2\left(\int_0^1 4\,dx + \int_1^2 x^2\,dx\right)$$
$$= 2\left(4x]_0^1 + \frac{1}{3}x^3\Big]_1^2\right) = 2\left[(4-0) + \frac{1}{3}(8-1)\right] = \frac{38}{3}.$$

60. $$\int_0^4 \lfloor x\rfloor\,dx = \int_0^1 \lfloor x\rfloor\,dx + \int_1^2 \lfloor x\rfloor\,dx + \int_2^3 \lfloor x\rfloor\,dx + \int_3^4 \lfloor x\rfloor\,dx$$
$$= \int_0^1 0\,dx + \int_1^2 1\,dx + \int_2^3 2\,dx + \int_3^4 3\,dx = x]_1^2 + 2x]_2^3 + 3x]_3^4$$
$$= (2-1) + (6-4) + (12-9) = 6$$

61. $\displaystyle\int_{-3}^{1} |x|\,dx = \int_{-3}^{0} -x\,dx + \int_{0}^{1} x\,dx = -\frac{1}{2}x^2\Big]_{-3}^{0} + \frac{1}{2}x^2\Big]_{0}^{1} = \frac{9}{2} + \frac{1}{2} = 5$

62. $\displaystyle\int_{0}^{4} |2x-6|\,dx = \int_{0}^{3} -(2x-6)\,dx + \int_{3}^{4}(2x-6)\,dx = (-x^2+6x)\big]_0^3 + (x^2-6x)\big]_3^4$
$\displaystyle\quad = (9-0) + [(-8)-(-9)] = 10$

63. $\displaystyle\int_{-8}^{3} \sqrt{|x|+1}\,dx = \int_{-8}^{0} \sqrt{-x+1}\,dx + \int_{0}^{3} \sqrt{x+1}\,dx = -\frac{2}{3}(1-x)^{3/2}\Big]_{-8}^{0} + \frac{2}{3}(x+1)^{3/2}\Big]_{0}^{3}$
$\displaystyle\quad = -\frac{2}{3}(1-27) + \frac{2}{3}(8-1) = 22$

64. $\displaystyle\int_{0}^{2} |x^2-1|\,dx = \int_{0}^{1} -(x^2-1)\,dx + \int_{1}^{2}(x^2-1)\,dx = \left(-\frac{1}{3}x^3+x\right)\Big]_0^1 + \left(\frac{1}{3}x^3-x\right)\Big]_1^2$
$\displaystyle\quad = \left(\frac{2}{3}-0\right) + \left[\frac{2}{3} - \left(-\frac{2}{3}\right)\right] = 2$

65. Using the fact that $f(x) = |\sin x|$ is an even function on $[-\pi,\pi]$ and $\sin x > 0$ for $0 \le x \le \pi$,

$$\int_{-\pi}^{\pi} |\sin x|\,dx = 2\int_{0}^{\pi} |\sin x|\,dx = 2\int_{0}^{\pi} \sin x\,dx = -2\cos x\big]_0^{\pi} = -2(-1-1) = 4.$$

66. $\displaystyle\int_{0}^{\pi} |\cos x|\,dx = \int_{0}^{\pi/2} \cos x\,dx + \int_{\pi/2}^{\pi}(-\cos x)\,dx = \sin x\big]_0^{\pi/2} - \sin x\big]_{\pi/2}^{\pi}$
$\displaystyle\quad = (1-0) + (0-1) = 2$

67. $\displaystyle\int_{1/2}^{e} \frac{(\ln 2t)^5}{t}\,dt \qquad \boxed{u = \ln 2t,\ du = \frac{1}{t}\,dt;\ u(1/2) = 0,\ u(e) = 1+\ln 2}$
$\displaystyle\quad = \int_{0}^{1+\ln 2} u^5\,du = \frac{1}{6}u^6\Big]_0^{1+\ln 2} = \frac{1}{6}[(1+\ln 2)^6 - 0] = \frac{(1+\ln 2)^6}{6} \approx 3.9266$

68. (Ask Scott for analytic value of arctan sqrt(2)/2, if any)

(While we're at it, 5.2.63 and 64 include definite integral notation, seemingly before it is used, so ask about their placement also)

(One last set of questions: 84 and [misnumbered] 86 seem to be open-ended; 86 has a few typos in addition to the numbering)

(And, of course, don't forget to ask about whatever is blank)

$$\int_{\sqrt{2}/2}^{1} \frac{1}{(\tan^{-1}x)(1+x^2)}\,dx$$

$$\boxed{u = \tan^{-1}x,\ du = \frac{1}{1+x^2}\,dx;\ u(\sqrt{2}/2) = \tan^{-1}(\sqrt{2}/2),\ u(1) = \frac{\pi}{4}}$$

$$= \int_{\tan^{-1}(\sqrt{2}/2)}^{\pi/4} \frac{1}{u}\,du = \ln|u|\big]_{\tan^{-1}\sqrt{2}/2}^{\pi/4} = \ln\frac{\pi}{4} - \ln\left|\tan^{-1}\frac{\sqrt{2}}{2}\right| \approx 0.2438$$

69. $\displaystyle\int_0^1 \frac{e^{-2x}}{e^{-2x}+1}\,dx$ $\quad \boxed{u = e^{-2x}+1,\ du = -2e^{-2x};\ u(0) = 2,\ u(1) = 1+e^{-2}}$

$$= -\frac{1}{2}\int_2^{1+e^{-2}} \frac{1}{u}\,du = -\frac{1}{2}\ln|u|\Big]_2^{1+e^{-2}} = -\frac{1}{2}[\ln(1+e^{-2}) - \ln 2] \approx 0.2831$$

70. $\displaystyle\int_0^{1/\sqrt{2}} \frac{x}{\sqrt{1-x^4}}\,dx$ $\quad \boxed{u = x^2,\ du = 2x\,dx;\ u(0) = 0,\ u(1/\sqrt{2}) = 1/2}$

$$= \frac{1}{2}\int_0^{1/2} \frac{1}{\sqrt{1-u^2}}\,du = \frac{1}{2}\sin^{-1} u\Big]_0^{1/2} = \frac{1}{2}\left(\frac{\pi}{6} - 0\right) = \frac{\pi}{12} \approx 0.2618$$

71. (a) Since $\operatorname{erf}'(x) = \dfrac{2}{\sqrt{\pi}}e^{-x^2} > 0$, $\operatorname{erf}(x)$ is increasing for all x.

(b) The derivative of $y = e^{x^2}[1+\sqrt{\pi}\operatorname{erf}(x)]$ is $\dfrac{dy}{dx} = 2 + 2e^{x^2}x[1+\sqrt{\pi}\operatorname{erf}(x)]$, so

$$\frac{dy}{dx} - 2xy = 2 + 2e^{x^2}x[1+\sqrt{\pi}\operatorname{erf}(x)] - 2xe^{x^2}[1+\sqrt{\pi}\operatorname{erf}(x)] = 2$$

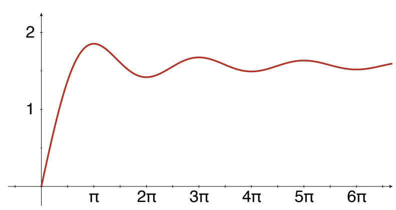
Also, $y(0) = e^0[1+\sqrt{\pi}\operatorname{erf}(0)] = 1 + \sqrt{\pi}\cdot 0 = 1$.

72. (a) $\operatorname{Si}(x) = \displaystyle\int_0^x \frac{\sin t}{t}\,dt$, $\operatorname{Si}'(x) = \dfrac{\sin x}{x}$, and so $\operatorname{Si}'(x) = 0$ for $x = n\pi$, $n = 1, 2, \ldots.$ The first four positive critical numbers are then π, 2π, 3π, and 4π. Now, $\operatorname{Si}''(x) = \dfrac{x\cos x - \sin x}{x^2}$, therefore

$$\operatorname{Si}''(\pi) = -\frac{1}{\pi} < 0, \quad \operatorname{Si}''(2\pi) = \frac{1}{2\pi} > 0, \quad \operatorname{Si}''(3\pi) = -\frac{1}{3\pi} < 0, \quad \operatorname{Si}''(4\pi) = \frac{1}{4\pi} > 0$$

shows that there are relative maxima at $x = \pi$ and $x = 3\pi$ and relative minima at $x = 2\pi$ and $x = 4\pi$.

(b)



73. $\displaystyle\lim_{\|P\|\to 0}\sum_{k=1}^{n}(2x_k^* + 5)\Delta x_k = \int_{-1}^{3}(2x+5)\,dx = (x^2+5x)\Big]_{-1}^{3} = 24 - (-4) = 28$

74. $\displaystyle\lim_{\|P\|\to 0}\sum_{k=1}^{n}\cos\frac{x_k^*}{4}\Delta x_k = \int_0^{2\pi}\cos\frac{x}{4}\,dx = 4\sin\frac{x}{4}\Big]_0^{2\pi} = 4$

75. Letting $\Delta x_k = \pi/n$ we have

$$\lim_{n\to\infty}\frac{\pi}{n}\sum_{k=1}^{n}\sin x_k^* = \lim_{n\to\infty}\sum_{k=1}^{n}(\sin x_k^*)\Delta x_k = \int_0^{\pi}\sin x\,dx = -\cos x]_0^{\pi} = -(-1-1) = 2.$$

76. Letting $\Delta x_k = 2/n$ we have

$$\lim_{n\to\infty} \frac{2}{n}\sum_{k=1}^{n} x_k^* = \lim_{n\to\infty}\sum_{k=1}^{n} x_k^*\Delta x_k = \int_{-1}^{1} x\,dx = \frac{1}{2}x^2\Big]_{-1}^{1} = \frac{1}{2}(1-1) = 0.$$

77. $\displaystyle\int_{-1}^{2}\left\{\int_{1}^{x} 12t^2\,dt\right\}dx = \int_{-1}^{2}\left(4t^3\Big]_1^x\right)dx = \int_{-1}^{2}(4x^3-4)\,dx = (x^4-4x)\Big]_{-1}^{2} = 8-5 = 3$

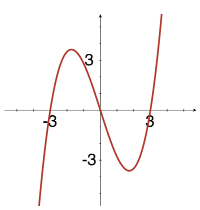
78. $$\int_0^{\pi/2}\left\{\int_0^t \sin x\,dx\right\}dt = \int_0^{\pi/2}\left(-\cos x\Big]_0^t\right)dt = \int_0^{\pi/2}(-\cos t+1)\,dt$$
$$= (-\sin t + t)\Big]_0^{\pi/2} = \left(-1+\frac{\pi}{2}\right) - 0 = \frac{\pi-2}{2}$$

79. Since $f(x)$ is even, $f(-x) = f(x)$. Then

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx \qquad \boxed{t=-x,\ dt=-dx}$$
$$= \int_a^0 f(-t)(-dt) + \int_0^a f(x)\,dx = -\int_a^0 f(t)\,dt + \int_0^a f(x)\,dx$$
$$= \int_0^a f(t)\,dt + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx.$$

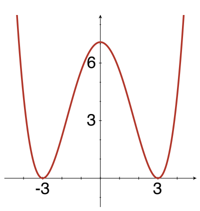
80. (a) Since f is odd and continuous at $x=0$, $f(0)=0$.

(b)



(c) $F(-3) = \displaystyle\int_{-3}^{-3} f(t)\,dt = 0;\ F(3) = \int_{-3}^{3} f(t)\,dt = 0$ since f is odd.

(d)



(e) Since $F'(x) = f(x)$, critical numbers occur at $x=-3$, $x=0$, and $x-3$. Solving $F''(x) = f'(x) = 0$ we see that points of inflection occur at $x=-2$ and $x=2$.

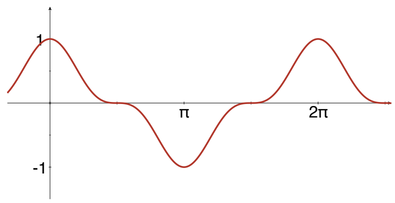
81. The reasoning is flawed at the point that $\sin t$ is substituted with $\sqrt{1-\cos^2 t}$. The use of the square root loses $\sin t$'s sign changes.

82. (a) $\dfrac{d}{dx}\, x \displaystyle\int_1^{2x} \sqrt{t^3+7}\,dt$ $\quad \boxed{u = 2x,\ du = 2\,dx}$

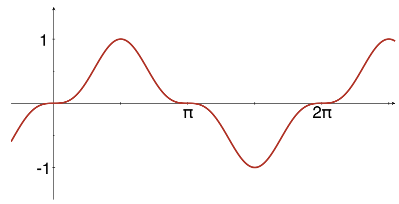
$$\begin{aligned}
&= \frac{d}{dx}\, x\left[\frac{d}{du}\left(\int_1^u \sqrt{t^3+7}\,dt\right)\frac{du}{dx}\right] = \frac{d}{dx}\, x\left(\sqrt{u^3+7}\,\frac{du}{dx}\right)\\
&= \frac{d}{dx}\, x\left[2\sqrt{(2x)^3+7}\right] = \frac{d}{dx}\, 2x\sqrt{8x^3+7}\\
&= 2x\left[\frac{1}{2}(8x^3+7)^{-1/2}(24x^2)\right] + 2\sqrt{8x^3+7} = \frac{40x^3+14}{\sqrt{8x^3+7}}
\end{aligned}$$

(b) $\dfrac{d}{dx}\, x \displaystyle\int_1^4 \sqrt{t^3+7}\,dt = \int_1^4 \sqrt{t^3+7}\,dt$, since $\displaystyle\int_1^4 \sqrt{t^3+7}\,dt$ is a constant.

83. (a)



$f(x) = \cos^3 x$



$f(x) = \sin^3 x$

(b) $\displaystyle\int_0^{2\pi} \cos^3 x\,dx = 0; \qquad \int_0^{2\pi} \sin^3 x\,dx = 0$

84. As this project's exact results may vary for every "run" of the exercise, no exact solution is given. In general, the student should see the empirical probability n/N approach the area of the region as the number of random points increases.

85. (a) At time n the radius of the circle is $r_0 + cu$ and the area is $A(u) = \pi(r_0+cu)^2$. Then

$$\begin{aligned}
\frac{RT}{Pv} &= \int_0^t \frac{k\pi(r_0+cu)^2}{V_0}\,du = \frac{K\pi}{cV_0}\int_0^t (r_0+cu)^2(c\,du)\\
&= \frac{K\pi}{cV_0}\,\frac{1}{3}(r_0+cu)^3\Big]_0^t = \frac{K\pi}{3cV_0}\left[(r_0+ct)^3 - r_0^3\right]\\
\frac{3cV_0RT}{PKv\pi} &= (r_0+ct)^3 - r_0^3\\
(r_0+ct)^3 &= \frac{3cV_0RT}{PKv\pi} + r_0^3\\
r_0 + ct &= \sqrt[3]{\frac{3cV_0RT}{PKv\pi} + r_0^3}\\
t &= \frac{1}{c}\sqrt[3]{\frac{3cV_0RT}{PKv\pi} + r_0^3} - \frac{r_0}{c}.
\end{aligned}$$

(b) Substituting $RT/Pv = 1.9\times10^6$, $K = 0.01\times10^{-3}$, $c = 0.01$, $r_0 = 100$, and $V_0 = 10,000$, we find $t \approx 2,617,695$ seconds, or $t \approx 30$ days and 7 hours.

(c) The final area is $A(2,617,695) = \pi[100+0.01(2,617,695)]^2 \approx 2.169\times10^9\ \text{m}^2 = 2169\ \text{km}^2$.

86. Since this exercise involves a research report, no solution is given. The need for the definite integral $\int_0^{\theta_0} \sec x\, dx$ can be found in the derivation of the projection, whose key properties are that it is *conformal* (i.e., angle-preserving) and that it represent lines of constant course as straight segments.

Chapter 5 in Review

A. True/False

1. False. Consider $f(x) = x^3 + x^2 + 1$.
2. True
3. True
4. True
5. True
6. False. Continuity implies integrability, but not necessarily the other way around. Consider the discontinuous function

$$f(x) = \begin{cases} 0, & x \neq 1 \\ 1, & x = 1 \end{cases}$$

which is integrable on $[0, 2]$ by (15) in Section 5.4.

7. True, since no portion of the graph of $y = x - x^3$ lies below the x-axis on $[0, 1]$.
8. False. This is only true when no portion of the graph lies below the x-axis.
9. False. Consider the partition $\left\{0, \frac{1}{n}, \frac{1}{n-1}, \ldots, \frac{1}{2}, 1\right\}$ of $\{0, 1\}$.
10. True
11. True
12. True
13. False. $\int \sin x\, dx = -\cos x + C$.
14. True
15. True
16. True

B. Fill in the Blanks

1. $f(x)$
2. $x^2 + C$
3. $\dfrac{\ln x}{x}$
4. $\sqrt{6}$
5. $-f(g(x))g'(x)\,dx$
6. $\dfrac{1}{2e^x\sqrt{x}} - \dfrac{5}{e^{25x^2}}$
7. $\displaystyle\sum_{k=1}^{5} \frac{k}{2k+1}$
8. 3480
9. $\displaystyle\int_5^{17}$
10. 4; 3
11. 5/2
12. regular
13. $\displaystyle\int_0^4 \sqrt{x}\,dx;\qquad \frac{2}{3}x^{3/2}\Big|_0^4 = \frac{16}{3}$
14. -4
15. $$\int_{-1}^{1}\left\{\int_0^x e^{-t}\,dt\right\}dx = \int_{-1}^{1}\left(-e^{-t}\big]_0^x\right)\,dx = -\int_{-1}^{1}(e^{-x}-1)\,dx$$
$$= -(-e^{-x}-x)\big]_{-1}^{1} = -[(-e^{-1}-1)-(-e^{1}+1)] = \frac{1}{e} - e + 2$$
$$\int_{-1}^{1}\frac{d}{dx}\left\{\int_0^x e^{-t}\,dt\right\}dx = \int_{-1}^{1} e^{-x}\,dx = -e^{-x}\big]_{-1}^{1} = e - \frac{1}{e}$$
16. 4/3

C. Exercises

1. $\displaystyle\int_{-1}^{1}(4x^3-6x^2+2x-1)\,dx = (x^4-2x^3+x^2-x)\big]_{-1}^{1} = -1-5 = -6$
2. $\displaystyle\int_1^9 \frac{6}{\sqrt{x}}\,dx = 12x^{1/2}\Big]_1^9 = 36-12 = 24$

3. $\displaystyle\int (5t+1)^{100}\,dt = \frac{1}{505}(5t+1)^{101} + C$

4. $\displaystyle\int w^2\sqrt{3w^3+1}\,dw \qquad \boxed{u = 3w^3+1,\ du = 9w^2\,dw}$

$$= \frac{1}{9}\int \sqrt{u}\,du = \frac{1}{9}\int u^{1/2}\,du = \frac{2}{27}u^{3/2} + C = \frac{2}{27}\sqrt{(3w^3+1)^3} + C$$

5. $\displaystyle\int_0^{\pi/4} (\sin 2x - 5\cos 4x)\,dx = \left(-\frac{1}{2}\cos 2x - \frac{5}{4}\sin 4x\right)\Big]_0^{\pi/4} = 0 - \left(-\frac{1}{2}\right) = \frac{1}{2}$

6. $\displaystyle\int_{\pi^2/9}^{\pi^2} \frac{\sin\sqrt{z}}{\sqrt{z}}\,dz \qquad \boxed{u = \sqrt{z},\ du = \frac{1}{2\sqrt{z}}\,dz}$

$$= \int_{\pi/3}^{\pi} 2\sin u\,du = -2\cos u\Big]_{\pi/3}^{\pi} = -2\left(-1 - \frac{1}{2}\right) = 3$$

7. $\displaystyle\int_4^4 (-2x^2 + x^{1/2})\,dx = 0$

8. $\displaystyle\int_{-\pi/4}^{\pi/4} dx + \int_{-\pi/4}^{\pi/4} \tan^2 x\,dx = \int_{-\pi/4}^{\pi/4} (1+\tan^2 x)\,dx = \int_{-\pi/4}^{\pi/4} \sec^2 x\,dx$

$$= \tan x\Big]_{-\pi/4}^{\pi/4} = 1 - (-1) = 2$$

9. $\displaystyle\int \cot^6 8x\csc^2 8x\,dx \qquad \boxed{u = \cot 8x,\ du = -8\csc^2 x\,dx}$

$$= -\frac{1}{8}\int u^6\,du = -\frac{1}{56}u^7 + C = -\frac{1}{56}\cot^7 8x + C$$

10. $\displaystyle\int \csc 3x\cot 3x\,dx = -\frac{1}{3}\csc 3x + C$

11. $\displaystyle\int (4x^2 - 16x + 7)^4(x-2)\,dx = \frac{1}{8}\int (4x^2-16x+7)^4[8(x-2)\,dx]$

$$\boxed{u = 4x^2 - 16x + 7,\ du = 8(x-2)\,dx}$$

$$= \frac{1}{8}\int u^4\,du = \frac{1}{40}u^5 + C = \frac{1}{40}(4x^2-16x+7)^5 + C$$

12. $\displaystyle\int (x^2+2x-10)^{2/3}(5x+5)\,dx = \frac{5}{2}\int (x^2+2x-10)^{2/3}[2(x+1)\,dx]$

$$\boxed{u = x^2+2x-10,\ du = 2(x+1)\,dx}$$

$$= \frac{5}{2}\int u^{2/3}\,du = \frac{3}{2}u^{5/3} + C = \frac{3}{2}(x^2+2x-10)^{5/3} + C$$

13. $\displaystyle\int \frac{x^2+1}{\sqrt[3]{x^3+3x-16}}\,dx = \frac{1}{3}\int (x^3+3x-16)^{-1/3}[3(x^2+1)\,dx]$

$$\boxed{u = x^3+3x-16,\ du = 3(x^2+1)\,dx}$$

$$= \frac{1}{3}\int u^{-1/3}\,du = \frac{1}{2}u^{2/3} + C = \frac{1}{2}(x^3+3x-16)^{2/3} + C$$

14. $\displaystyle\int \frac{x^2+1}{x^3+3x-16}\,dx = \frac{1}{3}\int \frac{1}{x^3+3x-16}[3(x^2+1)\,dx]$

$$\boxed{u = x^3+3x-16,\ du = 3(x^2+1)\,dx}$$

$$= \frac{1}{3}\int \frac{1}{u}\,du = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|x^3+3x-16| + C$$

15. $\displaystyle\int_0^4 \frac{x}{16+x^2}\,dx \qquad \boxed{u = x^2+16,\ du = 2x\,dx}$

$$= \frac{1}{2}\int_{16}^{32} \frac{1}{u}\,du = \frac{1}{2}\ln|u|\Big]_{16}^{32} = \frac{1}{2}(\ln 32 - \ln 16) = \frac{1}{2}\ln\frac{32}{16} = \frac{1}{2}\ln 2$$

16. $\displaystyle\int_0^4 \frac{1}{16+x^2}\,dx = \frac{1}{4}\tan^{-1}\frac{x}{4}\Big]_0^4 = \frac{1}{4}(\tan^{-1}1 - \tan^{-1}0) = \frac{\pi}{16}$

17. $\displaystyle\int_0^2 \frac{1}{\sqrt{16-x^2}}\,dx = \sin^{-1}\frac{x}{4}\Big]_0^2 = \sin^{-1}\frac{1}{2} - \sin^{-1}0 = \frac{\pi}{6}$

18. $\displaystyle\int_0^2 \frac{x}{\sqrt{16-x^2}}\,dx \qquad \boxed{u = 16-x^2,\ du = -2x\,dx}$

$$= -\frac{1}{2}\int_{16}^{12} u^{-1/2}\,du = \sqrt{u}\Big]_{12}^{16} = \sqrt{16} - \sqrt{12} = 4 - 2\sqrt{3}$$

19. $\displaystyle\int \tan 10x\,dx = -\frac{1}{10}\ln|\cos 10x| + C$

20. $\displaystyle\int \cot 10x\,dx = \frac{1}{10}\ln|\sin 10x| + C$

21. $\displaystyle\int_0^7 f(x)\,dx = \int_0^5 f(x)\,dx + \int_5^7 f(x)\,dx; \quad 2 = -3 + \int_5^7 f(x)\,dx; \quad \int_5^7 f(x)\,dx = 5$

22. $\displaystyle\int_1^9 f(x)\,dx = \int_1^4 f(x)\,dx + \int_4^9 f(x)\,dx = 2 + (-8) = -6$

23. Since $|x-1| = \begin{cases} -x+1, & 0 \le x < 1 \\ x-1, & 1 \le x \le 3 \end{cases}$, we have

$$\begin{aligned}\int_0^3 (1+|x-1|)\,dx &= \int_0^1 (1+|x-1|)\,dx + \int_1^3 (1+|x-1|)\,dx \\ &= \int_0^1 (1-x+1)\,dx + \int_1^3 (1+x-1)\,dx \\ &= \left(2x - \frac{1}{2}x^2\right)\Big]_0^1 + \frac{1}{2}x^2\Big]_1^3 = \left(\frac{3}{2}-0\right) + \left(\frac{9}{2}-\frac{1}{2}\right) = \frac{11}{2}.\end{aligned}$$

24. $\displaystyle\int_0^1 \frac{d}{dt}\left[\frac{10t^4}{(2t^3+6t+1)^2}\right] dt = \frac{10t^4}{(2t^3+6t+1)^2}\Big]_0^1 = \frac{10}{81}$

25. $\displaystyle\int_{\pi/2}^{\pi/2} \frac{\sin^{10} t}{16t^7+1}\,dt = 0$

26. Since $f(t) = t^5 \sin t^2$ is an odd function, $\displaystyle\int_{-1}^1 t^5 \sin t^2\,dt = 0.$

27. Since $f(x) = \dfrac{1}{1+3x^2}$ is an even function, $\displaystyle\int_{-1}^1 \frac{1}{1+3x^2}\,dx = 2\int_0^1 \frac{1}{1+3x^2}\,dx.$ Therefore

$$\begin{aligned} 2\int_0^1 \frac{1}{1+3x^2}\,dx \qquad & \boxed{u = \sqrt{3}x,\ du = \sqrt{3}\,dx} \\ &= \frac{2}{\sqrt{3}}\int_0^{\sqrt{3}} \frac{1}{1+u^2}\,du = \frac{2}{\sqrt{3}}\tan^{-1} u\Big]_0^{\sqrt{3}} \\ &= \frac{2}{\sqrt{3}}(\tan^{-1}\sqrt{3} - \tan^{-1} 0) = \frac{2}{\sqrt{3}}\left(\frac{\pi}{3}-0\right) = \frac{2\pi}{3\sqrt{3}}.\end{aligned}$$

28. $$\begin{aligned}\int_{-2}^2 f(x)\,dx &= \int_{-2}^0 f(x)\,dx + \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx = \int_{-2}^0 x^3\,dx + \int_0^1 x^2\,dx + \int_1^2 x\,dx \\ &= \frac{1}{4}x^4\Big]_{-2}^0 + \frac{1}{3}x^3\Big]_0^1 + \frac{1}{2}x^2\Big]_1^2 = \frac{1}{4}(0-16) + \frac{1}{3}(1-0) + \frac{1}{2}(4-1) = -\frac{13}{6}\end{aligned}$$

29. $$\begin{aligned}\lim_{n\to\infty} \frac{1+2+3+\cdots+n}{n^2} &= \lim_{n\to\infty}\left(\sum_{k=1}^n k\cdot\frac{1}{n^2}\right) = \lim_{n\to\infty}\left[\frac{n(n+1)}{2}\cdot\frac{1}{n^2}\right] \\ &= \lim_{n\to\infty}\left(\frac{n^2}{2n^2}+\frac{n}{2n^2}\right) = \lim_{n\to\infty}\left(\frac{1}{2}+\frac{1}{2n}\right) = \frac{1}{2}\end{aligned}$$

30. $$\begin{aligned}\lim_{n\to\infty} \frac{1^2+2^2+3^2+\cdots+n^2}{n^3} &= \lim_{n\to\infty}\left(\sum_{k=1}^n k^2\cdot\frac{1}{n^3}\right) = \lim_{n\to\infty}\left[\frac{n(n+1)(2n+1)}{6}\cdot\frac{1}{n^3}\right] \\ &= \lim_{n\to\infty}\left(\frac{2n^3}{6n^3}+\frac{3n^2}{6n^3}+\frac{n}{6n^3}\right) = \lim_{n\to\infty}\left(\frac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2}\right) = \frac{1}{3}\end{aligned}$$

31. Since $\dfrac{dV}{dt} = \dfrac{1}{4}$, $V = \dfrac{1}{4}t + C$. When $t = 0$, $V = \dfrac{1}{2}$ ft^3 and $C = \dfrac{1}{2}$. Thus, $V(t) = \dfrac{1}{4}t + \dfrac{1}{2}$ and $V(8) = \dfrac{5}{2}$ ft^3. The scale at this time will read $\dfrac{5}{2}(62.4) = 156$ lbs. The volume of the bucket is $V = \dfrac{\pi}{3} \cdot 3\left[\left(\dfrac{1}{2}\right)^2 + \dfrac{1}{2}(1) + 1^2\right] = \dfrac{7\pi}{4}$ ft^3. Solving $\dfrac{7\pi}{4} = \dfrac{1}{4}t + \dfrac{1}{2}$ for t, we obtain $t \approx 20$ min.

32. (a) The outer radius of the kth disk (from the top) is $r_k = \dfrac{1.5(k+1)}{2}$ cm, its inner radius is $\dfrac{1.5}{2}$ cm, and its thickness is 1.5 cm. Then its volume is $\pi(1.5)\left[r_k^2 - \left(\dfrac{1.5}{2}\right)^2\right] = \pi(1.5)^3\dfrac{k^2+2k}{4}$. Thus, the total volume is

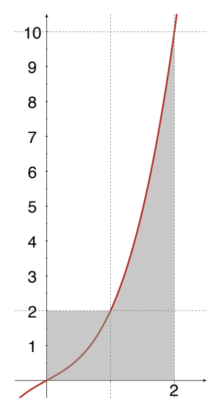
$$\begin{aligned}\frac{\pi}{4}(1.5)^3\sum_{k=1}^{n}(k^2+2k) &= \frac{\pi}{4}(1.5)^3\left[\frac{n(n+1)(2n+1)}{6} + 2\frac{n(n+1)}{2}\right] \\ &= \frac{\pi}{4}(1.5)^3\left[\frac{n(n+1)(2n+7)}{6}\right],\end{aligned}$$

and therefore the value of the gold is

$$14 \times 19.3 \times \frac{\pi}{4}(1.5)^3\left[\frac{n(n+1)(2n+7)}{6}\right] \approx 38\pi n(n+1)(2n+7).$$

(b) For $n = 64$, the value of the gold is approximately $38\pi(64)(64+1)[2(64)+7] = 38\pi(64 \cdot 65 \cdot 135) = 21,340,800\pi \approx \$67,044,100.50$.

33. From the figure we note that $\displaystyle\int_{f(1)}^{f(2)} f^{-1}(x)\,dx = 20 - 2 - \int_1^2 f(x)\,dx$.



Since

$$\begin{aligned}\int_1^2 f(x)\,dx = \int_1^2 (x^3+x)\,dx &= \left(\frac{1}{4}x^4 + \frac{1}{2}x^2\right)\Bigg]_1^2 \\ &= \left(\frac{16}{4} + \frac{4}{2}\right) - \left(\frac{1}{4} + \frac{1}{2}\right) = \frac{21}{4},\end{aligned}$$

we have $\displaystyle\int_{f(1)}^{f(2)} f^{-1}(x)\,dx = 20 - 2 - \frac{21}{4} = \frac{51}{4}$.

Chapter 6

Applications of the Integral

6.1 Rectilinear Motion Revisited

1. $s(t) = \int 6\,dt = 6t + c; \quad 5 = s(2) = 6(2) + c; \quad c = -7; \quad s(t) = 6t - 7$

2. $s(t) = \int (2t+1)\,dt = t^2 + t + c; \quad 0 = s(1) = 1^2 + 1 + c = 2 + c; \quad c = -2;$
   $s(t) = t^2 + t - 2$

3. $s(t) = \int (t^2 - 4t)\,dt = \frac{1}{3}t^3 - 2t^2 + c; \quad 6 = s(3) = -9 + c; \quad c = 15; \quad s(t) = \frac{1}{3}t^3 - 2t^2 + 15$

4. $s(t) = \int \sqrt{4t+5}\,dt = \frac{1}{6}(4t+5)^{3/2} + c; \quad 2 = s(1) = \frac{9}{2} + c; \quad c = -\frac{5}{2};$
   $s(t) = \frac{1}{6}(4t+5)^{2/3} - \frac{5}{2}$

5. $s(t) = \int -10\cos\left(4t + \frac{\pi}{6}\right)\,dt = \frac{5}{2}\sin\left(4t + \frac{\pi}{6}\right) + c; \quad \frac{5}{4} = s(0) = -\frac{5}{2}\left(\frac{1}{2}\right) + c = -\frac{5}{4} + c;$
   $c = \frac{5}{2}; \quad s(t) = -\frac{5}{2}\sin\left(4t + \frac{\pi}{6}\right) + \frac{5}{2}$

6. $s(t) = \int 2\sin 3t\,dt = -\frac{2}{3}\cos 3t + c; \quad 0 = s(\pi) = \frac{2}{3} + c; \quad c = -\frac{2}{3}; \quad s(t) = -\frac{2}{3}\cos 3t - \frac{2}{3}$

7. $v(t) = \int -5\,dt = -5t + c; \quad 4 = v(1) = -5 + c; \quad c = 9; \quad v(t) = -5t + 9;$
   $s(t) = \int (-5t+9)\,dt = -\frac{5}{2}t^2 + 9t + c; \quad 2 = s(1) = \frac{13}{2} + c; \quad c = -\frac{9}{2};$
   $s(t) = -\frac{5}{2}t^2 + 9t - \frac{9}{2}$

8. $v(t) = \int 6t\,dt = 3t^2 + c; \quad 0 = v(2) = 12 + c; \quad c = -12; \quad v(t) = 3t^2 - 12;$
$s(t) = \int (3t^2 - 12)\,dt = t^3 - 12t + c; \quad -5 = s(2) = -16 + c; \quad c = 11;$
$s(t) = t^3 - 12t + 11$

9. $v(t) = \int (3t^2 - 4t + 5)\,dt = t^3 - 2t^2 + 5t + c; \quad -3 = v(0) = c; \quad v(t) = t^3 - 2t^2 + 5t - 3;$
$s(t) = \int (t^3 - 2t^2 + 5t - 3)\,dt = \frac{1}{4}t^3 - \frac{2}{3}t^3 + \frac{5}{2}t^2 - 3t + c; \quad 10 = s(0) = c;$
$s(t) = \frac{1}{4}t^4 - \frac{2}{3}t^3 + \frac{5}{2}t^2 - 3t + 10$

10. $v(t) = \int (t-1)^2\,dt = \frac{1}{3}(t-1)^3 + c; \quad 4 = v(1) = c; \quad c = 4; \quad v(t) = \frac{1}{3}(t-1)^3 + 4;$
$s(t) = \int \left[\frac{1}{3}(t-3)^3 + 4\right]\,dt = \frac{1}{12}(t-1)^4 + 4t + c; \quad 6 = s(1) = 4 + c; \quad c = 2;$
$s(t) = \frac{1}{12}(t-1)^4 + 4t + 2$

11. $v(t) = \int (7t^{1/3} - 1)\,dt = \frac{21}{4}t^{4/3} - t + c; \quad 50 = v(8) = 76 + c; \quad c = -26;$
$v(t) = \frac{21}{4}t^{4/3} - t + 26;$
$s(t) = \int \left(\frac{21}{4}t^{4/3} - t - 26\right)\,dt = \frac{9}{4}t^{7/3} - \frac{1}{2}t^2 - 26t + c; \quad 0 = s(8) = 48 + c; \quad c = -48;$
$s(t) = \frac{9}{4}t^{7/3} - \frac{1}{2}t^2 - 26t - 48$

12. $v(t) = \int 100\cos 5t\,dt = 20\sin 5t + c; \quad -20 = v(\pi/2) = 20 + c; \quad c = -40;$
$v(t) = 20\sin 5t - 40;$
$s(t) = \int (20\sin 5t - 40)\,dt = -4\cos 5t - 40t + c; \quad 15 = s(\pi/2) = -20\pi + c; \quad c = 15 + 20\pi;$
$s(t) = -4\cos 5t - 40t + 15 + 20\pi$

13. $v(t) = 2t - 2 = 2(t-1)$

$$\begin{aligned}\text{dist.} &= \int_0^5 |2(t-1)|\,dt = 2\int_0^1 -(t-1)\,dt + 2\int_1^5 (t-1)\,dt \\ &= 2\left(-\frac{1}{2}t^2 + t\right)\Big]_0^1 + 2\left(\frac{1}{2}t^2 - t\right)\Big]_1^5 = 2\left(\frac{1}{2} - 0\right) + 2\left[\frac{15}{2} - \left(-\frac{1}{2}\right)\right] = 17 \text{ cm}\end{aligned}$$

14. $v(t) = -2t + 4 = -2(t - 2)$

$$\text{dist.} = \int_0^6 |-2(t-2)|\,dt = 2\int_0^2 -(t-2)\,dt + 2\int_2^6 (t-2)\,dt$$
$$= 2\left(-\frac{1}{2}t^2 + 2t\right)\Big]_0^2 + 2\left(\frac{1}{2}t^2 - 2t\right)\Big]_2^6 = 2(2-0) + 2[6-(-2)] = 20 \text{ cm}$$

15. $v(t) = 3t^2 - 6t - 9 = 3(t+1)(t-3)$

$$\text{dist.} = \int_0^4 |13t^2 - 6t - 9|\,dt = 3\int_0^3 -(t^2 - 2t - 3)\,dt + 3\int_3^4 (t^2 - 2t - 3)\,dt$$
$$= 3\left(-\frac{1}{3}t^3 + t^2 + 3t\right)\Big]_0^3 + 3\left(\frac{1}{3}t^3 - t^2 - 3t\right)\Big]_3^4 = 3(9-0) + 3\left[-\frac{20}{3} - (-9)\right] = 34 \text{ cm}$$

16. $v(t) = 4t^3 - 64t = 4t(t+4)(t-4)$

$$\text{dist.} = \int_1^5 |4t^3 - 64t|\,dt = 4\int_1^4 -(t^3 - 16t)\,dt + 4\int_4^5 (t^3 - 16t)\,dt$$
$$= 4\left(-\frac{1}{4}t^4 + 8t^2\right)\Big]_1^4 + 4\left(\frac{1}{4}t^4 - 8t^2\right)\Big]_4^5 = 4\left(64 - \frac{31}{4}\right) + 4\left[-\frac{175}{4} - (-64)\right] = 306 \text{ cm}$$

17. $v(t) = 6\pi\cos\pi t$

$$\text{dist.} = \int_1^3 |6\pi\cos\pi t|\,dt = 6\int_1^{3/2} -\pi\cos\pi t\,dt + 6\int_{3/2}^{5/2} \pi\cos\pi t\,dt + 6\int_{5/2}^3 -\pi\cos\pi t\,dt$$
$$= 6(-\sin\pi t)\Big]_1^{3/2} + 6(\sin\pi t)\Big]_{3/2}^{5/2} + 6(-\sin\pi t)\Big]_{5/2}^{3}$$
$$= 6[-(-1) - 0] + 6[1 - (-1)] + 6[0 - (-1)] = 24 \text{ cm}$$

18. $v(t) = 2(t - 3)$

$$\text{dist.} = \int_2^7 |2(t-3)|\,dt = 2\int_2^3 -(t-3)\,dt + 2\int_3^7 (t-3)\,dt$$
$$= 2\left(-\frac{1}{2}t^2 + 3t\right)\Big]_2^3 + 2\left(\frac{1}{2}t^2 - 3t\right)\Big]_3^7 = 2\left(\frac{9}{2} - 4\right) + 2\left[\frac{7}{2} - \left(-\frac{9}{2}\right)\right] = 17 \text{ cm}$$

19. We first convert mi/h to mi/s: 60 mi/h = 60/3600 mi/s. Then the distance traveled is

$$\int_0^2 \frac{60}{3600}\,dt = \frac{60}{3600}t\Big]_0^2 = \frac{60}{1800} \text{ mi} = \frac{1}{30} \text{ mi} \times 5280 \text{ ft/mi} = 176 \text{ ft.}$$

20. $a(t) = -32$; $v(0) = 0$; $s(0) = 144$; $v(t) = \int -32\,dt = -32t + c$; $0 = v(0) = c$; $v(t) = -32t$;

$s(t) = \int -32t\,dt = -16t^2 + c$; $144 = s(0) = c$; $s(t) = -16t^2 + 144$

To find when the ball hits the ground, we solve $s(t) = -16t^2 + 144 = 0$. This gives $t = \pm 3$. The ball hits the ground in 3 seconds. Its speed at this time is $|v(t)| = |-96| = 96$ ft/s.

21. $a(t) = -32$; $v(0) = 0$; $s(4) = 0$; $v(t) = \displaystyle\int -32\,dt = -32t + c$; $0 = v(0) = c$; $v(t) = -32t$;

$s(t) = \displaystyle\int -32t\,dt = -16t^2 + c$; $0 = s(4) = -256 + c$; $c = 256$; $s(t) = -16t^2 + 256$

The height of the building is $s(0) = 256$ ft.

22. Let the depth of the well be h.

$a(t) = -32$; $v(0) = 0$; $s(0) = h$; $v(t) = \displaystyle\int -32\,dt = -32t + c$; $0 = v(0) = c$; $v(t) = -32t$;

$s(t) = \displaystyle\int -32t\,dt = -16t^2 + c$; $h = s(0) = c$; $s(t) = -16t^2 + h$

If t_r is the time for the rock to hit the water, then $0 = s(t_r) = -16t_r^2 + h$, and $h = 16t_r^2$. Since the speed of sound is 1080 ft/s and the sound is heard after 2 seconds, $h = 1080(2 - t_r)$. Then $16t_r^2 = 1080(2 - t_r)$ or $2t_r^2 + 135t_r - 270 = 0$. Using the quadratic formula to find the positive root, we obtain

$$t_r = \frac{-135 + \sqrt{18,225 + 2,160}}{4} = \frac{-135\sqrt{20,385}}{4} \approx 1.9440 \text{ s}.$$

Then the depth of the well is $h = 1080(2 - t_r) \approx 60.4669$ ft.

23. $a(t) = -9.8$; $v(0) = 24.5$; $s(0) = 0$; $v(t) = \displaystyle\int -9.8\,dt = -9.8t + c$; $24.5 = v(0) = c$;

$v(t) = -9.8t + 24.5$; $s(t) = \displaystyle\int (-9.8t + 24.5)\,dt = -4.9t^2 + 24.5t + c$; $0 = s(0) = c$;

$s(t) = -4.9t^2 + 24.5t$

Solving $v(t) = -9.8t + 24.5 = 0$, we see that the maximum height is attained when $t = 2.5$ seconds. The maximum height is $s(2.5) = 30.625$ m.

24. $a(t) = -3.7$; $v(0) = 24.5$; $s(0) = 0$; $v(t) = \displaystyle\int -3.7\,dt = -3.7t + c$; $24.5 = v(0) = c$;

$v(t) = -3.7t + 24.5$; $s(t) = \displaystyle\int (-3.7t + 24.5)\,dt = -1.85t^2 + 24.5t + c$; $0 = s(0) = c$;

$s(t) = -1.85t^2 + 24.5t$

Solving $v(t) = -3.7t + 24.5 = 0$ we see that the maximum height is attained when $t \approx 6.6216$ seconds. The maximum height is $s(6.6216) \approx 81.1149$ m.

25. $a(t) = -32$; $v(0) = 32$; $s(0) = 384$; $v(t) = \displaystyle\int -32\,dt = -32t + c$; $32 = v(0) = c$;

$v(t) = -32t + 32$; $s(t) = \displaystyle\int (-32t + 32)\,dt = -16t^2 + 32t + c$; $384 = s(0) = c$;

$s(t) = -16t^2 + 32t + 384$

Solving $v(t) = -32t + 32 = 0$ we see that the maximum height is attained when $t = 1$ second. The maximum height is $s(1) = 400$ ft. Setting $s(t) = -16t^2 + 32t + 384 = 0$, we have $t^2 - 2t - 24 = (t - 6)(t + 4) = 0$. Thus, the ball hits the ground at 6 seconds.

26. Setting $s(t) = -16t^2 + 32t + 384 = 256$, we have $t^2 - 2t - 8 = (t-4)(t+2) = 0$. Thus, the ball passes the observer at 4 seconds. At this time $v(4) = -96$ ft/s.

27. $a(t) = -32$; $v(0) = -16$; $s(0) = 102$; $v(t) = \int -32\,dt = -32t + c$; $-16 = v(0) = c$;

 $v(t) = -32t - 16$; $s(t) = \int (-32t - 16)\,dt = -16t^2 - 16t + c$; $102 = s(0) = c$;

 $s(t) = -16t^2 - 16t + 102$

 Solving $s(t) = -16t^2 - 16t + 102 = 6$, we see that the marshmallow hits the person at $t = 2$ seconds. The impact velocity is $v(2) = -80$ ft/s.

28. $a(t) = -32$; $v(0) = 96$; $s(0) = 22$; $v(t) = \int -32\,dt = -32t + c$; $96 = v(0) = c$;

 $v(t) = -32t + 96$; $s(t) = \int (-32t + 96)\,dt = -16t^2 + 96t + c$; $22 = s(0) = c$;

 $s(t) = -16t^2 + 96t + 22$

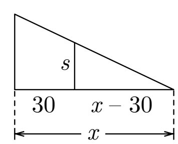
 Solving $s(t) = -16t^2 + 96t + 22 = 102$, we see that the stone hits the culprit at $t = 1$ second (or $t = 5$ seconds if it misses on the way up and hits on its way back down). The impact velocity is $v(1) = 64$ ft/s.

29. We measure upward from the top of the volcano, so that $s(0) = 0$. From $a(t) = g = -1.8$ we obtain $v(t) = -1.8t + v_0$ and $s(t) = -0.9t^2 + v_0 t$. If the rock attains its maximum height at time t_1, then $v(t_1) = 0 = -1.8t + v_0$ and $t_1 = v_0/1.8$. Solving

$$200,000 = -0.9t_1^2 + v_0 t_1 = -0.9\left(\frac{v_0}{1.8}\right)^2 + v_0\left(\frac{v_0}{1.8}\right) = 0.9\left(\frac{v_0}{1.8}\right)^2 = \frac{v_0^2}{3.6}$$

 gives $v_0\sqrt{3.6(200,000)} \approx 848.5$ m/s.

30. (a) Taking $a(t) = -32$, $v(0) = -2$, and $s(0) = 25$, we have $v(t) = -32t - 2$ and $s(t) = -16t^2 - 2t + 25$. Using similar triangles, we obtain $\dfrac{25}{s} = \dfrac{x}{x-30}$, $25(x-30) = sx$ and $x = \dfrac{750}{25-s}$. Then



$$\frac{dx}{dt} = \frac{750}{(25-s)^2}\frac{ds}{dt} = \frac{750}{(25-s)^2}\,v(t) = \frac{750}{(25-s)^2}(-32t-2)$$
$$= -\frac{1500(16t+1)}{(25-s)^2} = -\frac{1500(16t+1)}{(16t^2+2t)^2} = -\frac{375(16t+1)}{t^2(8t+1)^2}.$$

 (b) When $t = 1/2$, $\dfrac{dx}{dt} = -\dfrac{375(8+1)}{\frac{1}{4}(4+1)^2} = -540$ ft/s.

31. From the hint, $a = \dfrac{dv}{dt} = \dfrac{dv}{ds}v$, and integrating with respect to s gives $\int a\,ds = \int \left(v\,\dfrac{dv}{ds}\right)\,ds$. Then $as = \dfrac{1}{2}v^2 + c$, and solving for v we have $v^2 = 2as - 2c$. Since $v = v_0$ when $s = 0$, $v_0^2 = -2c$ and $v^2 = 2as + v_0^2$.

32. Let a be the acceleration due to gravity, $v(0) = v_0$, and $s(0) = 0$.

$v(t) = \int a\,dt = at + c; \quad v_0 = v(0) = c; \quad v(t) = at + v_0;$

$s(t) = \int (at + v_0)\,dt = \frac{1}{2}at^2 + v_0t + c; \quad 0 = s(0) = c; \quad s(t) = \frac{1}{2}at^2 + v_0t$

Solving $s(t) = \frac{1}{2}at^2 + v_0t = 0$, we obtain $t = 0$ and $t = -\frac{2v_0}{a}$. Then $v(-2v_0/a) = -v_0$, and the speed at impact with the ground is the initial velocity v_0.

33. Let a be the acceleration of gravity on the earth, $v(0) = v_0$, and $s(0) = 0$.

$v(t) = \int a\,dt = at + c; \quad v_0 = v(0) = c; \quad v(t) = at + v_0;$

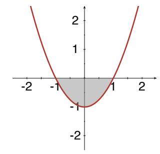
$s(t) = \int (at + v_0)\,dt = \frac{1}{2}at^2 + v_0t + c \quad 0 = s(0) = c; \quad s(t) = \frac{1}{2}at^2 + v_0t$

To find the maximum height reached on earth, we solve $v(t) = at + v_0 = 0$. The maximum height is reached when $t = -v_0/a$ and is $s(-v_0/a) = v_0^2/2a - v_0^2/a = -v_0^2/2a$. On the planet, the acceleration of gravity is $a/2$. Proceeding as on the earth, we obtain $v(t) = \frac{1}{2}at + v_0$, and $s(t) = \frac{1}{4}at^2 + v_0t$. To find the maximum height reached on the planet, we solve $v(t) = \frac{1}{2}at + v_0 = 0$. The maximum height is reached when $t = -2v_0/a$ and is $s(-2v_0/a) = v_0^2/a - 2v_0^2/a = -v_0^2/a$. Thus, the maximum height reached on the planet is twice that reached on earth.

34. Let a be the acceleration due to gravity on earth. Then, with initial velocity $2v_0$, we have $v_e(t) = at + 2v_0$ and $s_e(t) = \frac{1}{2}at^2 + 2v_0t$. On the planet, with acceleration due to gravity $a/2$ and initial velocity v_0, we have $v_p(t) = \frac{1}{2}at + v_0$ and $s_p(t) = \frac{1}{4}at^2 + v_0t$. To find the maximum height reached on earth, we solve $v_e(t) = at + 2v_0 = 0$ and obtain $t = -\frac{2v_0}{a}$. The maximum height is $s_e(-2v_0/a) = \frac{2v_0^2}{a} - \frac{4v_0^2}{a} = -\frac{2v_0^2}{a}$. To find the maximum height reached on the planet, we solve $v_p(t) = \frac{1}{2}at + v_0 = 0$ and obtain $t = -\frac{2v_0}{a}$. The maximum height is $s_p(-2v_0/a) = \frac{v_0^2}{a} - \frac{2v_0^2}{a} = -\frac{v_0^2}{a}$. Thus, the maximum height reached on earth is twice that reached on the planet. We want to find the initial velocity ϑ_0 on the earth so that the maximum height reached on earth is $-\frac{v_0^2}{a}$, the maximum height reached on the planet. With initial velocity ϑ_0, we have $v_e(t) = at + \vartheta_0$ and $s_e(t) = \frac{1}{2}at^2 + \vartheta_0t$. Solving $v_e(t) = at + \vartheta_0 = 0$ we obtain $t = -\frac{\vartheta_0}{a}$. Then, we want $s(-\vartheta_0/a) = \frac{\vartheta_0^2}{2a} - \frac{\vartheta_0^2}{a} = -\frac{\vartheta_0^2}{2a}$ to be equal to $-\frac{v_0^2}{a}$. Solving for ϑ_0 we see that the initial velocity on earth must be $\sqrt{2}v_0$.

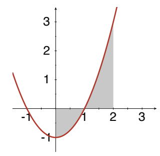
6.2 Area Revisited

1. $A = \displaystyle\int_{-1}^{1} -(x^2-1)\,dx = \left(-\frac{1}{3}x^3 + x\right)\Bigg]_{-1}^{1} = \frac{2}{3} - \left(-\frac{2}{3}\right) = \frac{4}{3}$

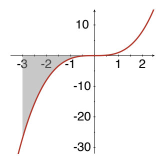


2. $A = \displaystyle\int_{0}^{1} -(x^2-1)\,dx + \int_{1}^{2} (x^2-1)\,dx = \left(-\frac{1}{3}x^3 + x\right)\Bigg]_{0}^{1} + \left(\frac{1}{3}x^3 - x\right)\Bigg]_{1}^{2}$

$= \left(\dfrac{2}{3} - 0\right) + \left[\dfrac{2}{3} - \left(-\dfrac{2}{3}\right)\right] = 2$

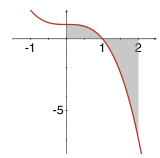


3. $A = \displaystyle\int_{-3}^{0} -x^3\,dx = -\frac{1}{4}x^4\Bigg]_{-3}^{0} = 0 - \left(-\frac{81}{4}\right) = \frac{81}{4}$

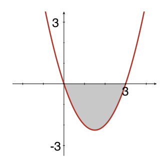


4. $A = \displaystyle\int_{0}^{1} (1-x^3)\,dx + \int_{1}^{2} -(1-x^3)\,dx = \left(x - \frac{1}{4}x^4\right)\Bigg]_{0}^{1} + \left(-x + \frac{1}{4}x^4\right)\Bigg]_{1}^{2}$

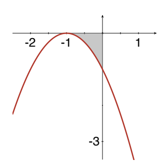
$= \left(\dfrac{3}{4} - 0\right) + \left[2 - \left(-\dfrac{3}{4}\right)\right] = \dfrac{7}{2}$



5. $A = \displaystyle\int_{0}^{3} -(x^2-3x)\,dx = \left(-\frac{1}{3}x^3 - \frac{3}{2}x^2\right)\Bigg]_{0}^{3} = \frac{9}{2} - 0 = \frac{9}{2}$



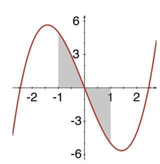
6. $A = \displaystyle\int_{-1}^{0} (x+1)^2\,dx = \frac{1}{3}(x+1)^3\Bigg]_{-1}^{0} = \frac{1}{3} - 0 = \frac{1}{3}$



7. $A = \displaystyle\int_{-1}^{0} (x^3-6x)\,dx + \int_{0}^{1} -(x^3-6x)\,dx$

$= \left(\dfrac{1}{4}x^4 - 3x^2\right)\Bigg]_{-1}^{0} + \left(-\dfrac{1}{4}x^4 - 3x^2\right)\Bigg]_{0}^{1}$

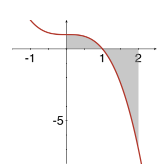
$= \left[0 - \left(-\dfrac{11}{4}\right)\right] + \left(\dfrac{11}{4} - 0\right) = \dfrac{11}{2}$



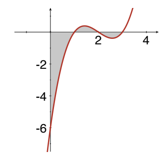
8. $A = \displaystyle\int_{0}^{1} (x^3-3x^2+2)\,dx + \int_{1}^{2} -(x^3-3x^2+2)\,dx$

$= \left(\dfrac{1}{4}x^4 - x^3 + 2x\right)\Bigg]_{0}^{1} + \left(-\dfrac{1}{4}x^4 + x^3 - 2x\right)\Bigg]_{1}^{2}$

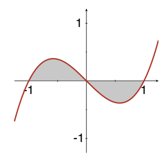
$= \left(\dfrac{5}{4} - 0\right) + \left[0 - \left(-\dfrac{5}{4}\right)\right] = \dfrac{5}{2}$



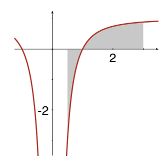
9. $$\begin{aligned} A &= \int_0^1 -(x^3 - 6x^2 + 11x - 6)\,dx + \int_1^2 (x^3 - 6x^2 + 11x - 6)\,dx \\ &\qquad + \int_2^3 -(x^3 - 6x^2 + 11x - 6)\,dx \\ &= \left(-\frac{1}{4}x^4 + 2x^3 - \frac{11}{2}x^2 + 6x\right)\Bigg]_0^1 + \left(\frac{1}{4}x^4 - 2x^3 + \frac{11}{2}x^2 - 6x\right)\Bigg]_1^2 \\ &\qquad + \left(-\frac{1}{4}x^4 + 2x^3 - \frac{11}{2}x^2 + 6x\right)\Bigg]_2^3 \\ &= \left(\frac{9}{4} - 0\right) + \left[(-2) - \left(-\frac{9}{4}\right)\right] + \left(\frac{9}{4} - 2\right) = \frac{11}{4} \end{aligned}$$



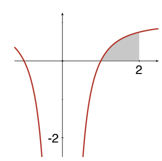
10. $$\begin{aligned} A &= \int_{-1}^0 (x^3 - x)\,dx + \int_0^1 -(x^3 - x)\,dx \\ &= \left(\frac{1}{4}x^4 - \frac{1}{2}x^2\right)\Bigg]_{-1}^0 + \left(-\frac{1}{4}x^4 + \frac{1}{2}x^2\right)\Bigg]_0^1 \\ &= \left[0 - \left(-\frac{1}{4}\right)\right] + \left(\frac{1}{4} - 0\right) = \frac{1}{2} \end{aligned}$$



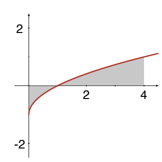
11. $$\begin{aligned} A &= \int_{1/2}^1 -(1 - x^{-2})\,dx + \int_1^3 (1 - x^{-2})\,dx = \left(-x - \frac{1}{x}\right)\Bigg]_{1/2}^1 + \left(x + \frac{1}{x}\right)\Bigg]_1^3 \\ &= \left[-2 - \left(-\frac{5}{2}\right)\right] + \left(\frac{10}{3} - 2\right) = \frac{11}{6} \end{aligned}$$



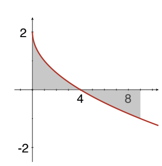
12. $$A = \int_1^2 (1 - x^{-2})\,dx = \left(x + \frac{1}{x}\right)\Bigg]_1^2 = \frac{5}{2} - 2 = \frac{1}{2}$$



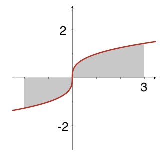
13. $$\begin{aligned} A &= \int_0^1 -(x^{1/2} - 1)\,dx + \int_1^4 (x^{1/2} - 1)\,dx \\ &= \left(-\frac{2}{3}x^{3/2} + x\right)\Bigg]_0^1 + \left(\frac{2}{3}x^{3/2} - x\right)\Bigg]_1^4 = \left(\frac{1}{3} - 0\right) + \left[\frac{4}{3} - \left(-\frac{1}{3}\right)\right] = 2 \end{aligned}$$



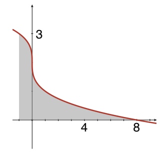
14. $$\begin{aligned} A &= \int_0^4 (2 - x^{1/2})\,dx + \int_4^9 -(2 - x^{1/2})\,dx \\ &= \left(2x - \frac{2}{3}x^{3/2}\right)\Bigg]_0^4 + \left(-2x + \frac{2}{3}x^{3/2}\right)\Bigg]_4^9 = \left(\frac{8}{3} - 0\right) + \left[0 - \left(-\frac{8}{3}\right)\right] = \frac{16}{3} \end{aligned}$$



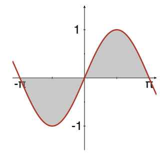
15. $A = \int_{-2}^{0} -x^{1/3}\,dx + \int_{0}^{3} x^{1/3}\,dx = -\frac{3}{4}x^{4/3}\Big]_{-2}^{0} + \frac{3}{4}x^{4/3}\Big]_{0}^{3}$
$= \left[0 + \frac{3}{4}(2^{4/3})\right] + \left[\frac{3}{4}(3^{4/3}) - 0\right] = \frac{3}{4}(2^{4/3} + 3^{4/3})$



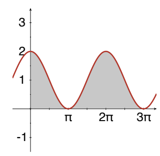
16. $A = \int_{-1}^{8} (2 - x^{1/3})\,dx = \left(2x - \frac{3}{4}x^{4/3}\right)\Big]_{-1}^{8} = 4 - \left(-\frac{11}{4}\right) = \frac{27}{4}$



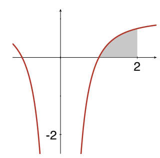
17. $A = \int_{-\pi}^{0} -\sin x\,dx + \int_{0}^{\pi} \sin x\,dx = \cos x\big]_{-\pi}^{0} - \cos x\big]_{0}^{\pi}$
$= [1 - (-1)] - (-1 - 1) = 4$



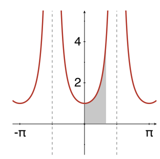
18. $A = \int_{0}^{3\pi} (1 + \cos x)\,dx = (x + \sin x)\big]_{0}^{3\pi} = 3\pi - 0 = 3\pi$



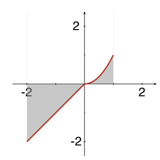
19. $A = \int_{-3\pi/2}^{\pi/2} -(-1 + \sin x)\,dx = (x + \cos x)\big]_{-3\pi/2}^{\pi/2} = \frac{\pi}{2} - \left(-\frac{3\pi}{2}\right) = 2\pi$



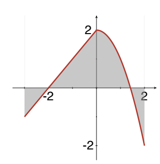
20. $A = \int_{0}^{\pi/3} \sec^2 x\,dx = \tan x\big]_{0}^{\pi/3} = \sqrt{3} - 0 = \sqrt{3}$



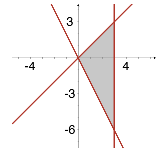
21. $A = \int_{-2}^{0} -x\,dx + \int_{0}^{1} x^2\,dx = -\frac{1}{2}x^2\Big]_{-2}^{0} + \frac{1}{3}x^3\Big]_{0}^{1} = -(0-2) + \left(\frac{1}{3} - 0\right) = \frac{7}{3}$



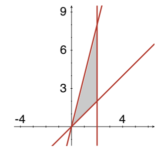
22. $A = \int_{-3}^{-2} -(x+2)\,dx + \int_{-2}^{0} (x+2)\,dx + \int_{0}^{\sqrt{2}} (2 - x^2)\,dx + \int_{\sqrt{2}}^{2} -(2 - x^2)\,dx$
$= -\left(\frac{1}{2}x^2 + 2x\right)\Big]_{-3}^{-2} + \left(\frac{1}{2}x^2 + 2x\right)\Big]_{-2}^{0} + \left(2x - \frac{1}{3}x^3\right)\Big]_{0}^{\sqrt{2}}$
$- \left(2x - \frac{1}{3}x^3\right)\Big]_{\sqrt{2}}^{2}$
$= -\left(-2 + \frac{3}{2}\right) + (0 + 2) + \left(\frac{4}{3}\sqrt{2} - 0\right) - \left(\frac{4}{3} - \frac{4}{3}\sqrt{2}\right) = \frac{7}{6} + \frac{8}{3}\sqrt{2}$



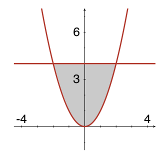
23. $A = \int_0^3 [x - (-2x)]\, dx = \int_0^3 3x\, dx = \frac{3}{2}x^2\Big]_0^3 = \frac{27}{2}$



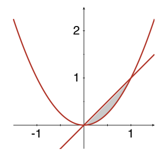
24. $A = \int_0^2 (4x - x)\, dx = \int_0^2 3x\, dx = \frac{3}{2}x^2\Big]_0^2 = 6$



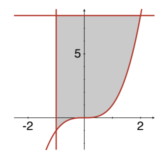
25. $A = \int_{-2}^2 (4 - x^2)\, dx = \left(4x - \frac{1}{3}x^2\right)\Big]_{-2}^2 = \frac{16}{3} - \left(-\frac{16}{3}\right) = \frac{32}{3}$



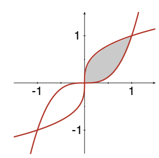
26. $A = \int_0^1 (x - x^2)\, dx = \left(\frac{1}{2}x^2 - \frac{1}{3}x^3\right)\Big]_0^1 = \frac{1}{6}$



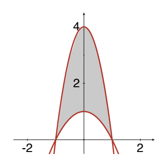
27. $A = \int_{-1}^2 (8 - x^3)\, dx = \left(8x - \frac{1}{4}x^4\right)\Big]_{-1}^2 = 12 - \left(-\frac{33}{4}\right) = \frac{81}{4}$



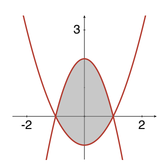
28. $A = \int_0^1 (x^{1/3} - x^3)\, dx = \left(\frac{3}{4}x^{4/3} - \frac{1}{4}x^4\right)\Big]_0^1 = \frac{1}{2}$



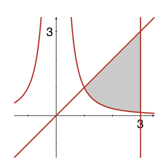
29. $A = \int_{-1}^1 [4(1 - x^2) - (1 - x^2)]\, dx = \int_{-1}^1 (3 - 3x^2)\, dx = (3x - x^3)\big]_{-1}^1$
$= 2 - (-2) = 4$



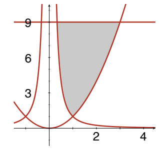
30. $A = \int_{-1}^1 [2(1 - x^2) - (x^2 - 1)]\, dx = \int_{-1}^1 (3 - 3x^2)\, dx = (3x - x^3)\big]_{-1}^1$
$= 2 - (-2) = 4$



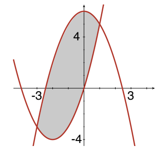
31. $A = \int_1^3 (x - x^{-2})\, dx = \left(\frac{1}{2}x^2 + \frac{1}{x}\right)\Big]_1^3 = \frac{29}{6} - \frac{3}{2} = \frac{10}{3}$



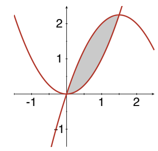
32. $A = \displaystyle\int_1^9 \left(\sqrt{y} - \frac{1}{\sqrt{y}}\right) dy = \int_1^9 (y^{1/2} - y^{-1/2})\, dy = \left(\frac{2}{3}x^{3/2} - 2y^{1/2}\right)\Big]_1^9$
$= 12 - \left(-\dfrac{4}{3}\right) = \dfrac{40}{3}$



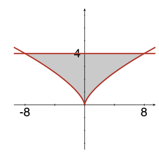
33. $A = \displaystyle\int_{-3}^1 [(-x^2+6) - (x^2+4x)]\, dx = \int_{-3}^1 (6 - 4x - 2x^2)\, dx$
$= \left(6x - 2x^2 - \dfrac{2}{3}x^3\right)\Big]_{-3}^1 = \dfrac{10}{3} - (-18) = \dfrac{64}{3}$



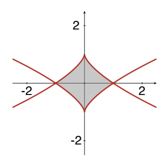
34. $A = \displaystyle\int_0^{3/2} [(-x^2+3x) - x^2]\, dx = \int_0^{3/2} (3x - 2x^2)\, dx$
$= \left(\dfrac{3}{2}x^2 - \dfrac{2}{3}x^3\right)\Big]_0^{3/2} = \dfrac{9}{8}$



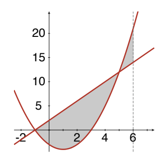
35. $A = \displaystyle\int_{-8}^8 (4 - x^{2/3})\, dx = \left(4x - \frac{3}{5}x^{5/3}\right)\Big]_{-8}^8 = \frac{64}{5} - \left(-\frac{64}{5}\right) = \frac{128}{5}$



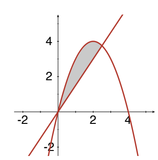
36. $A = \displaystyle\int_{-1}^1 [(1 - x^{2/3}) - (x^{2/3} - 1)]\, dx = \int_{-1}^1 (2 - 2x^{2/3})\, dx$
$= \left(2x - \dfrac{6}{5}x^{5/3}\right)\Big]_{-1}^1 = \dfrac{4}{5} - \left(-\dfrac{4}{5}\right) = \dfrac{8}{5}$



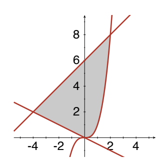
37. $A = \displaystyle\int_{-1}^5 [(2x+2) - (x^2 - 2x - 3)]\, dx + \int_5^6 [(x^2 - 2x - 3) - (2x+2)]\, dx$
$= \displaystyle\int_{-1}^5 (5 + 4x - x^2)\, dx + \int_5^6 (x^2 - 4x - 5)\, dx$
$= \left(5x + 2x^2 + \dfrac{1}{3}x^3\right)\Big]_{-1}^5 + \left(\dfrac{1}{3}x^3 - 2x^2 - 5x\right)\Big]_5^6$
$= \dfrac{100}{3} - \left(-\dfrac{8}{3}\right) + (-30) - \left(-\dfrac{100}{3}\right) = \dfrac{118}{3}$



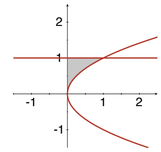
38. $A = \displaystyle\int_0^{5/2} \left[(-x^2+4x) - \frac{3}{2}x\right] dx = \int_0^{5/2} \left(\frac{5}{2}x - x^2\right) dx$
$= \left(\dfrac{5}{4}x^2 - \dfrac{1}{3}x^3\right)\Big]_0^{5/2} = \dfrac{125}{48}$



39. $A = \displaystyle\int_{-4}^0 \left(x + 6 + \frac{1}{2}x\right) dx + \int_0^2 (x + 6 - x^3)\, dx$
$= \left(\dfrac{3}{4}x^2 + 6x\right)\Big]_{-4}^0 + \left(\dfrac{1}{2}x^2 + 6x - \dfrac{1}{4}x^4\right)\Big]_0^2 = (0 + 12) + (10 - 0) = 22$

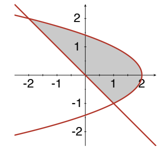


40. $A = \int_0^1 y^2\,dy = \frac{1}{3}y^3\Big]_0^1 = \frac{1}{3}$



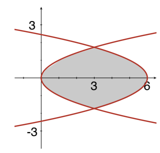
41. $A = \int_{-1}^{2} [(2-y^2)-(-y)]\,dy = \int_{-1}^{2} (2+y-y^2)\,dy$

$= \left(2y + \frac{1}{2}y^2 - \frac{1}{3}y^3\right)\Big]_{-1}^{2} = \frac{10}{3} - \left(-\frac{7}{6}\right) = \frac{9}{2}$



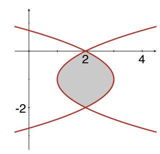
42. $A = \int_{-\sqrt{3}}^{\sqrt{3}} [(6-y^2)-y^2]\,dy = \int_{-\sqrt{3}}^{\sqrt{3}} (6-2y^2)\,dy = \left(6y - \frac{2}{3}y^3\right)\Big]_{-\sqrt{3}}^{\sqrt{3}}$

$= 4\sqrt{3} - (-4\sqrt{3}) = 8\sqrt{3}$



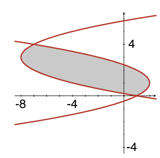
43. $A = \int_{-2}^{0} [(-y^2-2y+2)-(y^2+2y+2)]\,dy = \int_{-2}^{0} (-2y^2-4y)\,dy$

$= \left(-\frac{2}{3}y^3 - 2y^2\right)\Big]_{-2}^{0} = 0 - \left(-\frac{8}{3}\right) = \frac{8}{3}$



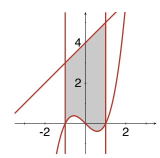
44. $A = \int_0^4 [(-y^2+2y+1)-(y^2-6y+1)]\,dy = \int_0^4 (8y-2y^2)\,dy$

$= \left(4y^2 - \frac{2}{3}y^3\right)\Big]_0^4 = \frac{64}{3}$



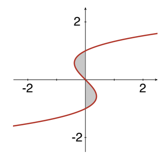
45. $A = \int_{-1}^{1} [(x+4)-(x^3-x)]\,dx = \int_{-1}^{1} (4+2x-x^3)\,dx$

$= \left(4x - x^2 - \frac{1}{4}x^4\right)\Big]_{-1}^{1} = \frac{19}{4} - \left(-\frac{13}{4}\right) = 8$



46. $A = \int_{-1}^{0} (y^3-y)\,dy + \int_0^1 -(y^3-y)\,dy$

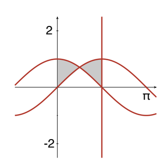
$= \left(\frac{1}{4}y^4 - \frac{1}{2}y^2\right)\Big]_{-1}^{0} + \left(-\frac{1}{4}y^4 + \frac{1}{2}y^2\right)\Big]_0^1 = 0 - \left(-\frac{1}{4}\right) + \frac{1}{4} - 0 = \frac{1}{2}$



47. $A = \int_0^{\pi/4} (\cos x - \sin x)\,dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\,dx$

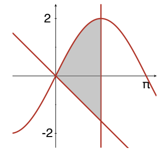
$= (\sin x + \cos x)]_0^{\pi/4} + (-\cos x - \sin x)]_{\pi/4}^{\pi/2}$

$= \sqrt{2} - 1 + (-1) - (-\sqrt{2}) = 2\sqrt{2} - 2$



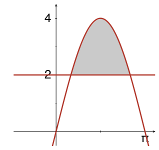
48. $A = \displaystyle\int_0^{\pi/2} [2\sin x - (-x)]\,dx = \int_0^{\pi/2} (2\sin x + x)\,dx = \left(-2\cos x + \frac{1}{2}x^2\right)\Big]_0^{\pi/2}$

$= \dfrac{\pi^2}{8} - (-2) = \dfrac{16+\pi^2}{8}$



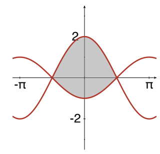
49. $A = \displaystyle\int_{\pi/6}^{5\pi/6} (4\sin x - 2)\,dx = (-4\cos x - 2x)\Big]_{\pi/6}^{5\pi/6}$

$= 2\sqrt{3} - \dfrac{5\pi}{3} - \left(-2\sqrt{3} - \dfrac{\pi}{3}\right) = \dfrac{12\sqrt{3} - 4\pi}{3}$

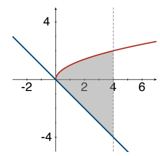


50. $A = \displaystyle\int_{-\pi/2}^{\pi/2} [2\cos x - (-\cos x)]\,dx = \int_{-\pi/2}^{\pi/2} 3\cos x\,dx$

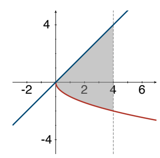
$= 3\sin x\Big]_{-\pi/2}^{\pi/2} = 3 - (-3) = 6$



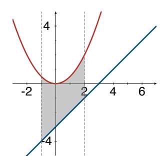
51. Region 1: $y = \sqrt{x}$, $y = -x$, $x = 0$, $x = 4$



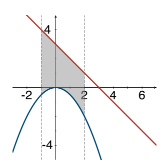
Region 2: $y = -\sqrt{x}$, $y = x$, $x = 0$, $x = 4$



52. Region 1: $y = \dfrac{1}{2}x^2$, $y = x - 3$, $x = -1$, $x = 2$



Region 2: $y = 3 - x$, $y = -\dfrac{1}{2}x^2$, $x = -1$, $x = 2$

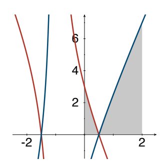


53. $\displaystyle\int_0^2 \left|\frac{3}{x+1} - 4x\right| dx = \int_0^{1/2} \left(\frac{3}{x+1} - 4x\right) dx + \int_{1/2}^2 \left(4x - \frac{3}{x+1}\right) dx$

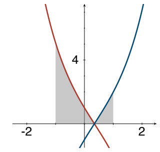
$= \left(3\ln|x+1| - 2x^2\right)\Big]_0^{1/2} + \left(2x^2 - 3\ln|x+1|\right)\Big]_{1/2}^2$

$= 3\ln\dfrac{3}{2} - \dfrac{1}{2} + 8 - 3\ln 3 - \dfrac{1}{2} + 3\ln\dfrac{3}{2}$

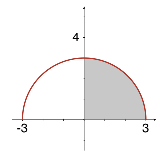
$= 7 + 3\ln\dfrac{3}{4} \approx 6.1370$



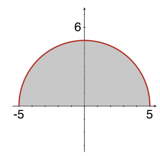
54. $$\begin{aligned}\int_{-1}^{1} \left|e^x - 2e^{-x}\right|\, dx &= \int_{-1}^{\ln\sqrt{2}} \left(2e^{-x} - e^x\right)\, dx + \int_{\ln\sqrt{2}}^{1} \left(e^x - 2e^{-x}\right)\, dx \\ &= \left(-2e^{-x} - e^x\right)\Big]_{-1}^{\ln\sqrt{2}} + \left(e^x + 2e^{-x}\right)\Big]_{\ln\sqrt{2}}^{1} \\ &= -\sqrt{2} - \sqrt{2} + 2e + e^{-1} + e + 2e^{-1} - \sqrt{2} - \sqrt{2} \\ &= 3e + 3e^{-1} - 4\sqrt{2}\end{aligned}$$



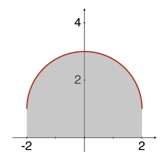
55. $$\int_0^3 \sqrt{9 - x^2}\, dx = \frac{1}{4}\pi(3)^2 = \frac{9\pi}{4}$$



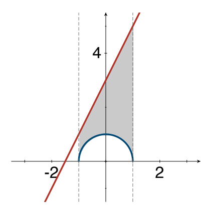
56. $$\int_{-5}^{5} \sqrt{25 - x^2}\, dx = \frac{1}{2}\pi(5)^2 = \frac{25\pi}{2}$$



57. $$\int_{-2}^{2} (1 + \sqrt{4 - x^2})\, dx = \int_{-2}^{2} 1\, dx + \int_{-2}^{2} \sqrt{4 - x^2}\, dx = 4 + \frac{1}{2}\pi(2)^2 = 4 + 2\pi$$

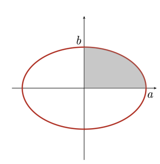


58. $$\begin{aligned}\int_{-1}^{1} (2x + 3 - \sqrt{1 - x^2})\, dx &= \int_{-1}^{1} (2x + 3)\, dx - \int_{-1}^{1} \sqrt{1 - x^2}\, dx \\ &= \left(x^2 + 3x\right)\Big]_{-1}^{1} - \frac{1}{2}\pi(1)^2 \\ &= 1 + 3 - 1 - 3(-1) - \frac{\pi}{2} = 6 - \frac{\pi}{2}\end{aligned}$$



59. The area of the ellipse is four times the area in the first quadrant portion of the ellipse. Thus,

$$A = 4\int_0^a \sqrt{b^2 - b^2x^2/a^2}\, dx = \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\, dx = \frac{4b}{a}\left(\frac{1}{4}\pi a^2\right) = \pi ab.$$

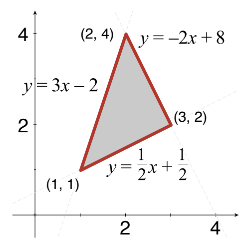


60. $A = \int_1^2 \left[(3x-2) - \left(\frac{1}{2}x + \frac{1}{2}\right)\right] dx + \int_2^3 \left[(-2x+8) - \left(\frac{1}{2}x + \frac{1}{2}\right)\right] dx$

$= \int_1^2 \left(\frac{5}{2}x - \frac{5}{2}\right) dx + \int_2^3 \left(-\frac{5}{2}x + \frac{15}{2}\right) dx$

$= \left(\frac{5}{4}x^2 - \frac{5}{2}x\right)\Big]_1^2 + \left(-\frac{5}{4}x^2 + \frac{15}{2}\right)\Big]_2^3$

$= 0 - \left(-\frac{5}{4}\right) + \frac{45}{4} - 10 = \frac{5}{2}$



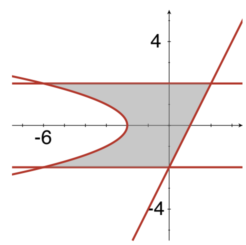
61. $A = \int_{-6}^{-2} (2 - \sqrt{-x-2})\, dx + \int_{-6}^{-2} [-\sqrt{-x-2} - (-2)]\, dx$

$+ \int_{-2}^{0} [2 - (-2)]\, dx + \int_0^2 [2 - (2x-2)]\, dx$

$= 2\int_{-6}^{-2} (2 - \sqrt{-x-2})\, dx + \int_{-2}^{0} 4\, dx + \int_0^2 (4-2x)\, dx$

$= 2\left[2x + \frac{2}{3}(-x-2)^{3/2}\right]\Big|_{-6}^{-2} + 4x\ \Big]_{-2}^{0} + (4x - x^2)\Big]_0^2$

$= 2\left[-4 - \left(-\frac{20}{3}\right)\right] + 8 + 4 - 0 = \frac{52}{3}$



62. $A = \int_{-2}^{2} \left[\left(\frac{1}{2}y + 1\right) - (-y^2 - 2)\right] dy = \int_{-2}^{2} \left(y^2 + \frac{1}{2}y + 3\right) dy = \left(\frac{1}{3}y^3 + \frac{1}{4}y^2 + 3y\right)\Big]_{-2}^{2}$

$= \frac{29}{3} - \left(-\frac{23}{3}\right) = \frac{52}{3}$

63. The area with respect to x is $A_x = \int_0^{\ln 3/2} (e^x - 1)\, dx + \int_{\ln 3/2}^{\ln 2} (2 - e^x)\, dx.$

The area with respect to y is $A_y = \int_1^2 \left(\ln y - \ln \frac{y+1}{2}\right) dy.$

If integration with respect to x is chosen, we get

$$A_x = \int_0^{\ln 3/2} (e^x - 1)\, dx + \int_{\ln 3/2}^{\ln 2} (2 - e^x)\, dx = (e^x - x)\Big]_0^{\ln 3/2} + (2x - e^x)\Big]_{\ln 3/2}^{\ln 2}$$

$$= \frac{3}{2} - \ln\frac{3}{2} - 1 + 2\ln 2 - 2 - 2\ln\frac{3}{2} + \frac{3}{2} = -3\ln 3 + 5\ln 2 \approx 0.1699.$$

If integration with respect to y is chosen, we get

$$A_y = \int_1^2 \left(\ln y - \ln\frac{y+1}{2}\right) dy = \left[y\ln y - y - (y+1)\ln\frac{y+1}{2} + (y+1)\right]\Big|_1^2$$

$$= \left[y\ln y - (y+1)\ln\frac{y+1}{2} + 1\right]\Big|_1^2 = 2\ln 2 - 3\ln\frac{3}{2} + 1 - \ln 1 + 2\ln 1 - 1$$

$$= -3\ln 3 + 5\ln 2 \approx 0.1699$$

(see Problem 5.1.39 for the antiderivative of $\ln x$)

64. Using *Mathematica* the numbers at which the curves intersect are approximately

$$-0.4077767094044803 \quad \text{and} \quad 0.7148059123627778.$$

The area is then

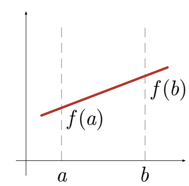
$$A = \int_{-0.4077767094044803}^{0.7148059123627778} (e^x - 4x^2)\,dx \approx 0.801284.$$

65. At $P(x_0, 1/x_0)$ the slope of the line segment is $-1/x_0^2$. The equation of the line through Q and R is then $y = -x/x_0^2 + 2/x_0$. Setting $y = 0$ we see that the x-intercept is $2x_0$. The area is

$$A = \int_0^{2x_0} \left(-\frac{1}{x_0^2}x + \frac{2}{x_0}\right) dx = \left(-\frac{1}{2x_0^2}x^2 + \frac{2}{x_0}x\right)\Bigg]_0^{2x_0} = -2 + 4 = 2,$$

which does not depend on x_0.

66. $$A = \int_a^b (Ax + B)\,dx = \left(\frac{1}{2}Ax^2 + Bx\right)\Bigg]_a^b = \frac{1}{2}Ab^2 + Bb - \left(\frac{1}{2}Aa^2 + Ba\right)$$
$$= \frac{A}{2}(b^2 - a^2) + B(b - a) = \left[\frac{A}{2}(b + a) + B\right](b - a)$$
$$= \frac{Aa + B + Ab + B}{2}(b - a) = \frac{f(a) + f(b)}{2}(b - a)$$



67. By symmetry with respect to the line $y = x$,

$$A = 2\int_0^a (\cos x - x)\,dx = 2\left(\sin x - \frac{1}{2}x^2\right)\Bigg]_0^a = 2\sin a - a^2$$

(Using *Mathematica* it is easily shown that $a \approx 0.739085$.)

68. The areas are the same. In Figure 6.2.16(b), the area of the straight swath of paint is $k(b-a)$. Now, if $y = f(x)$ describes the lower edge of the swath in Figure 6.2.16(a), then an equation for the upper edge is $y = f(x) + k$. The area between the two graphs is then

$$\int_a^b \{[f(x) + k] - f(x)\}\,dx = \int_a^b k\,dx = k(b - a).$$

69. The areas are the same. Let w be the length of the line segments $\overline{AB}$ and $\overline{CD}$, and without loss of generality, let $\overline{AB}$ reside on $y = 0$, with $\overline{CD}$ residing on $y = h$. Thus, in Figure 6.2.17(a), the area of the rectangle is wh. Since Figure 6.2.17(b) describes a parallelogram, the line defined by $\overline{AD'}$ can be written as $x = f(y)$. Thus, the line defined by $\overline{BC'}$ is $x = f(y) + w$. The area of the parallelogram is therefore

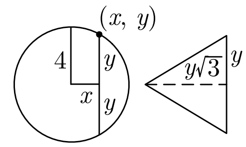
$$\int_0^h \{[f(y) + w] - f(y)\}\,dy = \int_0^h w\,dy = wh.$$

70. This project involves a research report, and thus a preset solution is not applicable. It is noted, however, that Cavalieri's Principle relates directly to the situations presented in Problems 68 and 69.

6.3 Volumes of Solids: Slicing Method

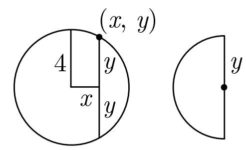
1. $x^2 + y^2 = 16; \quad y = \sqrt{16 - x^2}; \quad A(x) = \sqrt{3}y^2 = \sqrt{3}(16 - x^2)$

$$V = \int_{-4}^{4} \sqrt{3}(16 - x^2)\,dx = \sqrt{3}\left(16x - \frac{1}{3}x^3\right)\Big]_{-4}^{4}$$
$$= \sqrt{3}\left[\frac{128}{3} - \left(-\frac{128}{3}\right)\right] = \frac{256\sqrt{3}}{3}\ \text{ft}^3$$

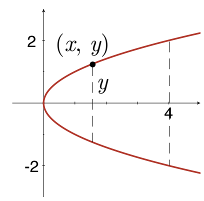


2. $x^2 + y^2 = 16; \quad y = \sqrt{16 - x^2}; \quad A(x) = \frac{1}{2}\pi y^2 = \pi\left(8 - \frac{1}{2}x^2\right)$

$$V = \int_{-4}^{4} \pi\left(8 - \frac{1}{2}x^2\right)\,dx = \pi\left(8x - \frac{1}{6}x^3\right)\Big]_{-4}^{4}$$
$$= \pi\left[\frac{64}{3} - \left(-\frac{64}{3}\right)\right] = \frac{128\pi}{3}\ \text{ft}^3$$

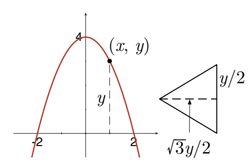


3. $x = y^2; \quad A(x) = 2y(8y) = 16y^2 = 16x; \quad V = \int_0^4 16x\,dx = 8x^2\Big]_0^4 = 128$



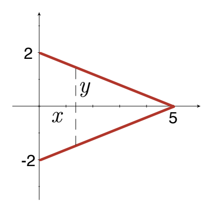
4. $y = 4 - x^2; \quad A(x) = \frac{\sqrt{3}y^2}{4} = \frac{\sqrt{3}}{4}(4 - x^2)^2 = \sqrt{3}\left(4 - 2x^2 + \frac{1}{4}x^4\right)$

$$V = \int_{-2}^{2} \sqrt{3}\left(4 - 2x^2 + \frac{1}{4}x^4\right)\,dx = \sqrt{3}\left(4x - \frac{2}{3}x^3 + \frac{1}{20}x^5\right)\Big]_{-2}^{2}$$
$$= \sqrt{3}\left[\frac{64}{15} - \left(-\frac{64}{15}\right)\right] = \frac{128\sqrt{3}}{15}$$



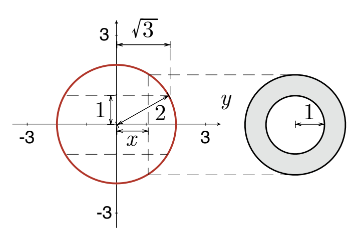
5. $y = -\frac{2}{5}x + 2; \quad A(x) = \frac{1}{2}\pi y^2 = \frac{2\pi}{25}(x - 5)^2$

$$V = \int_0^5 \frac{2\pi}{25}(x - 5)^2\,dx = \frac{2\pi}{75}(x - 5)^2\Big]_0^5 = \frac{10\pi}{3}\ \text{ft}^3$$



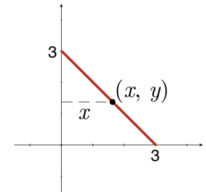
6. $y = \sqrt{4 - x^2}; \quad A(x) = \pi y^2 - \pi(1^2) = \pi(3 - x^2)$

$$V = \int_{-\sqrt{3}}^{\sqrt{3}} \pi(3 - x^2)\,dx = \pi\left(3x - \frac{1}{3}x^3\right)\Big]_{-\sqrt{3}}^{\sqrt{3}}$$
$$= \pi[2\sqrt{3} - (-2\sqrt{3})] = 4\pi\sqrt{3}\ \text{ft}^3$$



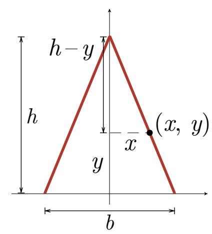
7. $x = -y + 3; \quad A(y) = x^2 = (y-3)^2$

$$V = \int_0^3 (y-3)^2\,dy = \frac{1}{3}(y-3)^3\Big]_0^3 = 9$$



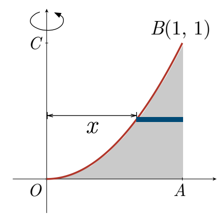
8. Let b denote the length of one side of the square base. Thus, $B = b^2$. Using similar triangles, we have $\dfrac{h}{b} = \dfrac{h-y}{2x}$ and $x = \dfrac{b(h-y)}{2h}$. Thus, $A(y) = (2x)^2 = \dfrac{b^2(h-y)^2}{h^2}$, and

$$V = \int_0^h \frac{b^2(h-y)^2}{h^2}\,dy = \int_0^h \left(b^2 - \frac{2b^2}{h}y + \frac{b^2}{h^2}y^2\right)dy$$
$$= \left(b^2y - \frac{b^2}{h}y^2 + \frac{b^2}{3h^2}y^3\right)\Big]_0^h = b^2h - b^2h + \frac{b^2h}{3} = \frac{1}{3}hB.$$



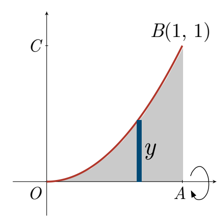
9. $x = \sqrt{y}$

$$V = \pi\int_0^1 [1^2 - (\sqrt{y})^2]\,dy = \pi\int_0^1 (1-y)\,dy = \pi\left(y - \frac{1}{2}y^2\right)\Big]_0^1 = \frac{\pi}{2}$$



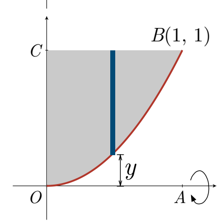
10. $y = x^2$

$$V = \pi\int_0^1 (x^2)^2\,dx = \pi\int_0^1 x^4\,dx = \frac{\pi}{5}x^5\Big]_0^1 = \frac{\pi}{5}$$



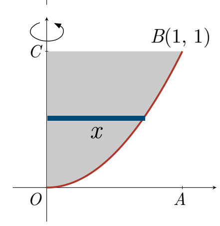
11. $y = x^2$

$$V = \pi\int_0^1 (1^2 - x^4)\,dx = \pi\left(x - \frac{1}{5}x^5\right)\Big]_0^1 = \frac{4\pi}{5}$$



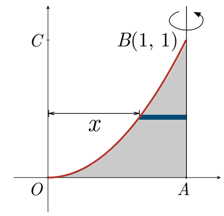
12. $x = \sqrt{y}$

$$V = \pi\int_0^1 (\sqrt{y})^2\,dy = \pi\int_0^1 y\,dy = \frac{\pi}{2}y^2\Big]_0^1 = \frac{\pi}{2}$$



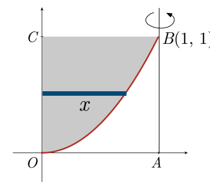
13. $x = \sqrt{y}$

$$V = \pi \int_0^1 (1 - \sqrt{y})^2\, dy = \pi \int_0^1 (1 - 2\sqrt{y} + y)\, dy$$
$$= \pi \left(y - \frac{4}{3} y^{3/2} + \frac{1}{2} y^2 \right) \Big]_0^1 = \frac{\pi}{6}$$



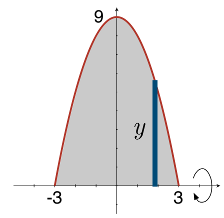
14. $x = \sqrt{y}$

$$V = \pi \int_0^1 [1^2 - (1 - \sqrt{y})^2]\, dy = \pi \int_0^1 (2\sqrt{y} - y)\, dy$$
$$= \pi \left(\frac{4}{3} y^{3/2} - \frac{1}{2} y^2 \right) \Big]_0^1 = \frac{5\pi}{6}$$



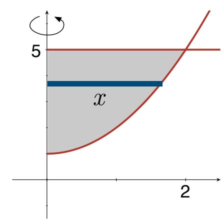
15. $y = 9 - x^2$

$$V = \pi \int_{-3}^3 (9 - x^2)^2\, dx = 2\pi \int_0^3 (81 - 18x^2 + x^4)\, dx$$
$$= 2\pi \left(81x - 6x^3 + \frac{1}{5} x^5 \right) \Big]_0^3 = \frac{1296\pi}{5}$$



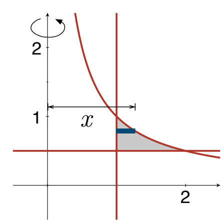
16. $x = \sqrt{y - 1}$

$$V = \pi \int_1^5 (\sqrt{y - 1})^2\, dy = \pi \int_1^5 (y - 1)\, dy$$
$$= \pi \left(\frac{1}{2} y^2 - y \right) \Big]_1^5 = \pi \left[\frac{15}{2} - \left(-\frac{1}{2} \right) \right] = 8\pi$$



17. $x = \dfrac{1}{y}$

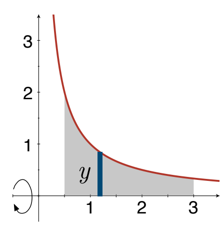
$$V = \pi \int_{1/2}^1 \left[\left(\frac{1}{y} \right)^2 - 1^2 \right] dy = \pi \int_{1/2}^1 (y^{-2} - 1)\, dy$$
$$= \pi \left(-\frac{1}{y} - y \right) \Big]_{1/2}^1 = \pi \left[-2 - \left(-\frac{5}{2} \right) \right] = \frac{\pi}{2}$$



6.3.17

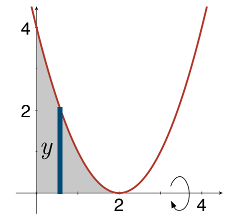
18. $y = \dfrac{1}{x}$

$$V = \pi \int_{1/2}^3 \left(\frac{1}{x} \right)^2 dx = \pi \left(-\frac{1}{x} \right) \Big]_{1/2}^3 = \pi \left[-\frac{1}{3} - (-2) \right] = \frac{5\pi}{3}$$



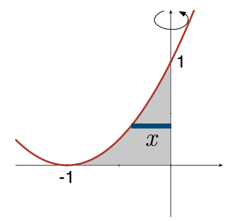
19. $y = (x-2)^2$

$$V = \pi \int_0^2 (x-2)^4\,dx \quad = \frac{\pi}{5}(x-2)^5\Big]_0^2 = \frac{32\pi}{5}$$



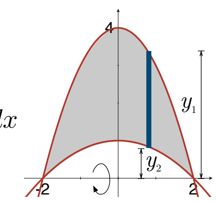
20. $x = \sqrt{y} - 1$

$$V = \pi \int_0^1 (\sqrt{y}-1)^2\,dy = \pi \int_0^1 (y - 2\sqrt{y} + 1)\,dy$$
$$= \pi\left(\frac{1}{2}y^2 - \frac{4}{3}y^{3/2} + y\right)\Bigg]_0^1 = \frac{\pi}{6}$$



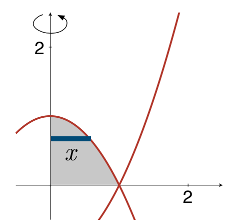
21. $y_1 = 4 - x^2; \quad y_2 = 1 - \frac{1}{4}x^2$

$$V = 2\pi \int_0^2 \left[(4-x^2)^2 - \left(1 - \frac{1}{4}x^2\right)^2\right] dx = 2\pi \int_0^2 \left(15 - \frac{15}{2}x^2 + \frac{15}{16}x^4\right) dx$$
$$= 2\pi\left(15x - \frac{5}{2}x^3 + \frac{3}{16}x^5\right)\Bigg]_0^2 = 32\pi$$



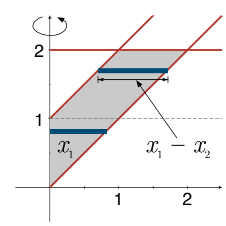
22. $x = \sqrt{1-y}$

$$V = \pi \int_0^1 (\sqrt{1-y})^2\,dy = \pi \int_0^1 (1-y)\,dy = \pi\left(y - \frac{1}{2}y^2\right)\Bigg]_0^1 = \frac{\pi}{2}$$



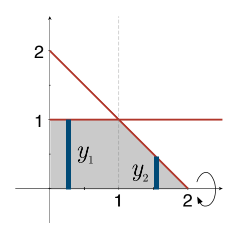
23. $x_1 = y; \quad x_2 = y - 1$

$$V = \pi \int_0^1 y^2\,dy + \pi \int_1^2 [y^2 - (y-1)^2]\,dy$$
$$= \pi\left(\frac{1}{3}y^3\right)\Bigg]_0^1 + \pi(y^2 - y)\Big]_1^2 = \pi\left(\frac{1}{3} + 2\right) = \frac{7\pi}{3}$$

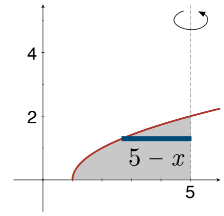


24. $y_1 = 1; \quad y_2 = 2 - x$

$$V = \pi \int_0^1 1^2\,dx + \pi \int_1^2 (2-x)^2\,dx = \pi \int_0^1 dx + \pi \int_1^2 (4 - 4x + x^2)\,dx$$
$$= \pi x]_0^1 + \pi\left(4x - 2x^2 + \frac{1}{3}x^3\right)\Bigg]_1^2 = \pi + \pi\left(\frac{8}{3} - \frac{7}{3}\right) = \frac{4\pi}{3}$$

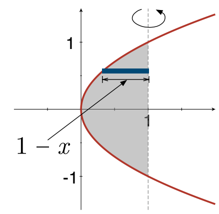


25. $x = y^2 + 1$



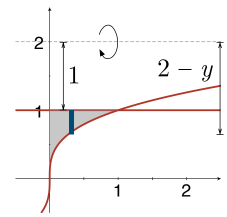
$$V = \pi \int_0^2 [5 - (y^2+1)]^2 \, dy = \pi \int_0^2 (4 - y^2)^2 \, dy$$
$$= \pi \int_0^2 (16 - 8y^2 + y^4) \, dy = \pi \left(16y - \frac{8}{3}y^3 + \frac{1}{5}y^5\right)\Bigg]_0^2 = \frac{256\pi}{15}$$

26. $x = y^2$



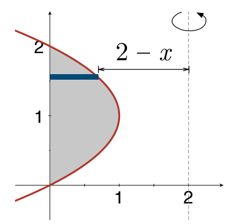
$$V = \pi \int_{-1}^1 (1 - y^2)^2 \, dy = \pi \int_{-1}^1 (1 - 2y^2 + y^4) \, dy$$
$$= \pi \left(y - \frac{2}{3}y^3 + \frac{1}{5}y^5\right)\Bigg]_{-1}^1 = \pi \left[\frac{8}{15} - \left(-\frac{8}{15}\right)\right] = \frac{16\pi}{15}$$

27. $y = x^{1/3}$



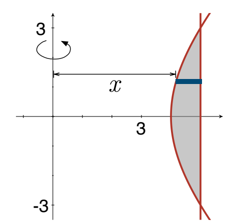
$$V = \pi \int_0^1 [(2 - x^{1/3})^2 - 1^2] \, dx = \pi \int_0^1 (3 - 4x^{1/3} + x^{2/3}) \, dx$$
$$= \pi \left(3x - 3x^{4/3} + \frac{3}{5}x^{5/3}\right)\Bigg]_0^1 = \frac{3\pi}{5}$$

28. $x = -y^2 + 2y$



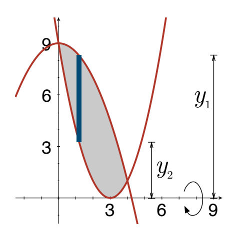
$$V = \pi \int_0^2 \left\{2^2 - \left[2 - (-y^2 + 2y)\right]^2\right\} \, dy = \pi \int_0^2 (8y - 8y^2 + 4y^3 - y^4) \, dy$$
$$= \pi \left(4y^2 - \frac{8}{3}y^3 + y^4 - \frac{1}{5}y^5\right)\Bigg]_0^2 = \frac{64\pi}{15}$$

29. $x = \sqrt{y^2 + 16}$



$$V = \pi \int_{-3}^3 \left[5^2 - \left(\sqrt{y^2+16}\right)^2\right] dy = \pi \int_{-3}^3 (9 - y^2) \, dy$$
$$= \pi \left(9y - \frac{1}{3}y^3\right)\Bigg]_{-3}^3 = \pi[18 - (-18)] = 36\pi$$

30. $y_1 = 9 - \dfrac{1}{2}x^2; \quad y_2 = x^2 - 6x + 9 = (x-3)^2$

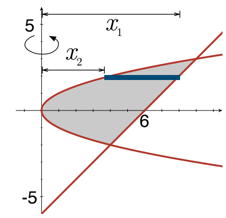


$$V = \pi \int_0^4 \left[\left(9 - \frac{1}{2}x^2\right)^2 - (x-3)^4\right] dx$$
$$= \pi \int_0^4 \left(108x - 63x^2 + 12x^3 - \frac{3}{4}x^4\right) dx$$
$$= \pi \left(54x^2 - 21x^3 + 3x^4 - \frac{3}{20}x^5\right)\Bigg]_0^4 = \frac{672\pi}{5}$$

31. $x_1 = y + 6; \quad x_2 = y^2$

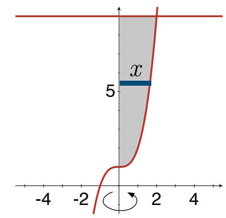
$$V = \pi \int_{-2}^{3} [(y+6)^2 - y^4]\, dy = \pi \int_{-2}^{3} (36 + 12y + y^2 - y^4)\, dy$$

$$= \pi \left(36y + 6y^2 + \frac{1}{3}y^3 - \frac{1}{5}y^5 \right) \Big]_{-2}^{3} = \pi \left[\frac{612}{5} - \left(-\frac{664}{15} \right) \right] = \frac{500\pi}{3}$$



32. $x = (y-1)^{1/3}$

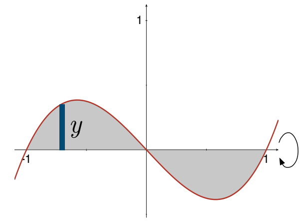
$$V = \pi \int_{1}^{9} (y-1)^{2/3}\, dy = \frac{3\pi}{5}(y-1)^{5/3} \Big]_{1}^{9} = \frac{96\pi}{5}$$



33. $y = x^3 - x$

$$V = \pi \int_{-1}^{1} (x^3 - x)^2\, dx = \pi \int_{-1}^{1} (x^6 - 2x^4 + x^2)\, dx$$

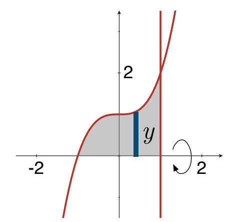
$$= \pi \left(\frac{1}{7}x^7 - \frac{2}{5}x^5 + \frac{1}{3}x^3 \right) \Big]_{-1}^{1} = \pi \left[\frac{8}{105} - \left(-\frac{8}{105} \right) \right] = \frac{16\pi}{105}$$



34. $y = x^3 + 1$

$$V = \pi \int_{-1}^{1} (x^3 + 1)^2\, dx = \pi \int_{-1}^{1} (x^6 + 2x^3 + 1)\, dx$$

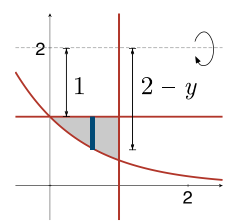
$$= \pi \left(\frac{1}{7}x^7 + \frac{1}{2}x^4 + x \right) \Big]_{-1}^{1} = \pi \left[\frac{23}{14} - \left(-\frac{9}{14} \right) \right] = \frac{16\pi}{7}$$



35. $y = e^{-x}$

$$V = \pi \int_{0}^{1} [(2 - e^{-x})^2 - 1^2]\, dx = \pi \int_{0}^{1} (3 - 4e^{-x} + e^{-2x})\, dx$$

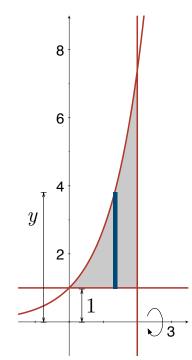
$$= \pi \left(3x + 4e^{-x} - \frac{1}{2}e^{-2x} \right) \Big]_{0}^{1} = \pi \left(4e^{-1} - \frac{1}{2}e^{-2} - \frac{1}{2} \right)$$



36. $y = e^x$

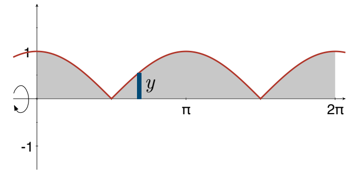
$$V = \pi \int_{0}^{2} [(e^x)^2 - 1^2]\, dx = \pi \int_{0}^{2} (e^{2x} - 1)\, dx$$

$$= \pi \left(\frac{1}{2}e^{2x} - x \right) \Big]_{0}^{2} = \pi \left(\frac{1}{2}e^4 - \frac{5}{2} \right)$$



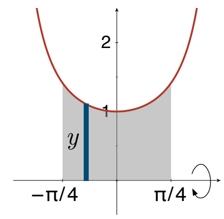
37. $y = |\cos x|$

$$V = \pi \int_0^{2\pi} |\cos x|^2\, dx = \pi \int_0^{2\pi} \frac{1+\cos 2x}{2}\, dx$$
$$= \frac{\pi}{2} \int_0^{2\pi} (1+\cos 2x)\, dx = \frac{\pi}{2}\left(x + \frac{1}{2}\sin 2x\right)\Bigg]_0^{2\pi} = \pi^2$$



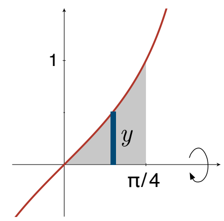
38. $y = \sec x$

$$V = \pi \int_{-\pi/4}^{\pi/4} \sec^2 x\, dx = \pi \tan x\big]_{-\pi/4}^{\pi/4} = \pi[1-(-1)] = 2\pi$$



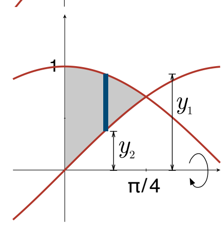
39. $y = \tan x$

$$V = \pi \int_0^{\pi/4} \tan^2 x\, dx = \pi \int_0^{\pi/4} (\sec^2 x - 1)\, dx = \pi(\tan x - x)\big]_0^{\pi/4}$$
$$= \pi\left(1 - \frac{\pi}{4} - 0\right) = \frac{4\pi - \pi^2}{4}$$



40. $y_1 = \cos x; \quad y_2 = \sin x$

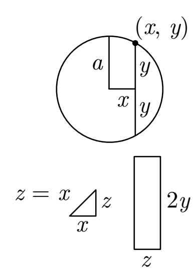
$$V = \pi \int_0^{\pi/4} (\cos^2 x - \sin^2 x)\, dx = \pi \int_0^{\pi/4} \cos 2x\, dx = \frac{\pi}{2}\sin 2x\Big]_0^{\pi/4} = \frac{\pi}{2}$$



41. The volume of the right circular cylinder is $\pi r^2 h$. Placing the center of the red circular cylinder's base in Figure 6.3.19 on the origin, we see that $A = \pi r^2$ for every slice from $y = 0$ to h. Thus, the volume V of the cylinder is

$$V = \int_0^h \pi r^2\, dy = \pi r^2 y\,\Big]_0^h = \pi r^2 h.$$

42. Take the cross-sections to be rectangles perpendicular to the base of the cylinder and parallel to the diameter.

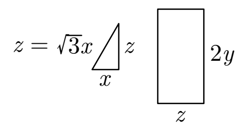


$$x^2 + y^2 = a^2; \quad y = \sqrt{a^2 - x^2}$$

(a) $A(x) = 2yz = (2\sqrt{a^2-x^2})x = 2x\sqrt{a^2-x^2}$

$$V = \int_0^a 2x\sqrt{a^2-x^2}\, dx \qquad \boxed{u = a^2 - x^2,\ du = -2x\, dx}$$
$$= \int_{a^2}^0 -u^{1/2}\, du = -\frac{2}{3}u^{3/2}\Big]_{a^2}^0 = -\frac{2}{3}(0 - a^3) = \frac{2}{3}a^3$$

(b) $A(x) = 2yz = (2\sqrt{a^2 - x^2})\sqrt{3}x = 2\sqrt{3}x\sqrt{a^2 - x^2}$



$$V = \int_0^a 2\sqrt{3}x\sqrt{a^2 - x^2}\,dx \qquad \boxed{u = a^2 - x^2,\ du = -2x\,dx}$$

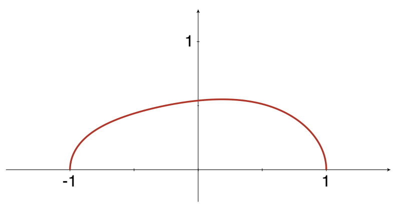
$$= \int_{a^2}^0 -\sqrt{3}u^{1/2}\,du = -\frac{2\sqrt{3}}{3}u^{3/2}\Big]_{a^2}^0 = -\frac{2\sqrt{3}}{3}(0 - a^3) = \frac{2\sqrt{3}}{3}a^3$$

43. (a) Using *Mathematica*, we obtain with the disk method

$$V = \pi \int_{-1}^1 [P(x)]^2(1 - x^2)\,dx = \frac{4\pi}{315}(5a^2 + 9b^2 + 21c^2 + 105d^2 + 18ac + 42bd).$$

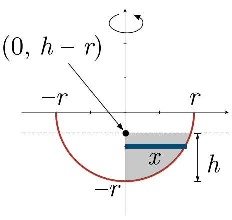
(b) Setting $a = -0.07$, $b = -0.02$, $c = 0.2$, and $d = 0.56$ we obtain $V \approx 1.32$ cubic units.

(c)



(d) Setting $a = -0.06$, $b = 0.04$, $c = 0.1$, and $d = 0.54$ we obtain $V \approx 1.26$ cubic units.

44. (a) Using $x = \sqrt{r^2 - y^2}$ and the disk method, we obtain



$$V = \pi \int_{-r}^{h-r} \left(\sqrt{r^2 - y^2}\right)^2 dy$$

$$= \pi \int_{-r}^{h-r} (r^2 - y^2)\,dy = \pi\left(r^2 y - \frac{1}{3}y^3\right)\Big]_{-r}^{h-r}$$

$$= \pi\left[r^2(h - r) - \frac{1}{3}(h - r)^2 - \left(-r^3 + \frac{1}{3}r^3\right)\right]$$

$$= \pi r^2 h - \frac{1}{3}\pi h^3.$$

(b) The weight of the ball is $\frac{4}{3}\pi r^3 \rho_{\text{ball}}$ and the weight of water displaced is $\frac{\pi}{3}(3rh^2 - h^3)\rho_{\text{water}}$. Using Archimedes' principle and $\frac{\rho_{\text{ball}}}{\rho_{\text{water}}} = 0.4$ we have $\frac{4}{3}\pi(3)^2(0.4) = \frac{\pi}{3}(9h^2 - h^3)$ or $h^3 - 9h^2 + 43.2 = 0$. Solving for h we obtain $h \approx 2.5976$ in.

45. (a) Each eighth of the bicylinder can be sliced into squares whose sides follow the perimeter of a quadrant of the cylinders' base; that is, $x^2 + y^2 = r^2$, one side of the square is $y = \sqrt{r^2 - x^2}$, and its area is $y^2 = r^2 - x^2$. Using symmetry, the volume common to the cylinders is thus

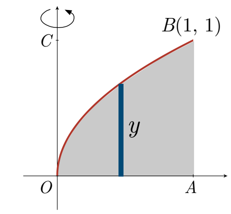
$$V = 8\int_0^r (r^2 - x^2)\,dx = 8\left(r^2 x - \frac{x^3}{3}\right)\Big]_0^r = \frac{16r^3}{3}.$$

(b) This item involves a research report, and thus a preset solution is not applicable.

6.4 Volumes of Solids: Shell Method

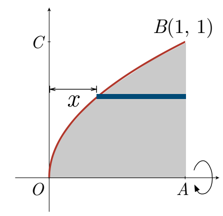
1. $y = \sqrt{x}$

$$V = 2\pi \int_0^1 x\sqrt{x}\,dx = \frac{4\pi}{5}x^{5/2}\bigg]_0^1 = \frac{4\pi}{5}$$



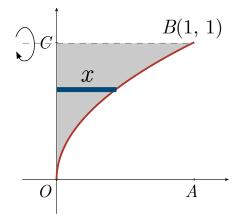
2. $x = y^2$

$$V = 2\pi \int_0^1 y(1-y^2)\,dy = 2\pi\left(\frac{1}{2}y^2 - \frac{1}{4}y^4\right)\bigg]_0^1 = \frac{\pi}{2}$$



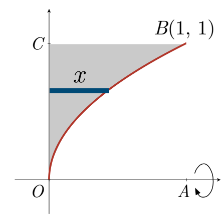
3. $x = y^2$

$$V = 2\pi \int_0^1 (1-y)y^2\,dy = 2\pi\left(\frac{1}{3}y^3 - \frac{1}{4}y^4\right)\bigg]_0^1 = \frac{\pi}{6}$$



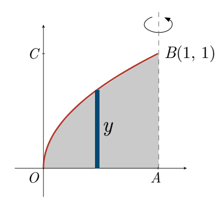
4. $x = y^2$

$$V = 2\pi \int_0^1 y \cdot y^2\,dy = \frac{\pi}{2}y^4\bigg]_0^1 = \frac{\pi}{2}$$



5. $y = \sqrt{x}$

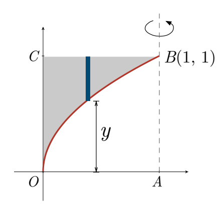
$$V = 2\pi \int_0^1 (1-x)\sqrt{x}\,dx = 2\pi\left(\frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2}\right)\bigg]_0^1 = \frac{8\pi}{15}$$



6. $y = \sqrt{x}$

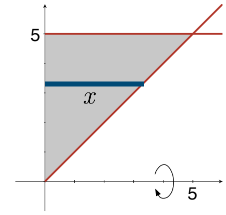
$$V = 2\pi \int_0^1 (1-x)(1-\sqrt{x})\,dx = 2\pi \int_0^1 (1 - \sqrt{x} - x + x^{3/2})\,dx$$

$$= 2\pi\left(x - \frac{2}{3}x^{3/2} - \frac{1}{2}x^2 + \frac{2}{5}x^{5/2}\right)\bigg]_0^1 = \frac{7\pi}{15}$$



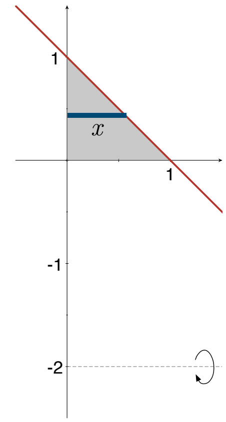
7. $x = y$

$$V = 2\pi \int_0^5 y \cdot y \, dy = \frac{2\pi}{3} y^3 \Big]_0^5 = \frac{250\pi}{3}$$



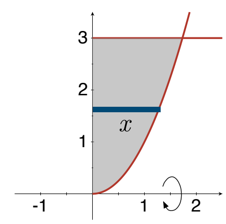
8. $x = 1 - y$

$$V = 2\pi \int_0^1 (y+2)(1-y)\, dy = 2\pi \int_0^1 (2 - y - y^2)\, dy$$
$$= 2\pi \left(2y - \frac{1}{2}y^2 - \frac{1}{3}y^3\right)\Big]_0^1 = \frac{7\pi}{3}$$



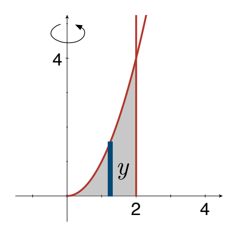
9. $x = \sqrt{y}$

$$V = 2\pi \int_0^3 y\sqrt{y}\, dy = \frac{4\pi}{5} y^{5/2} \Big]_0^3 = \frac{36\pi\sqrt{3}}{5}$$



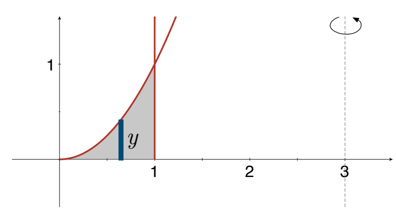
10. $y = x^2$

$$V = 2\pi \int_0^2 x \cdot x^2 \, dx = \frac{\pi}{2} x^4 \Big]_0^2 = 8\pi$$



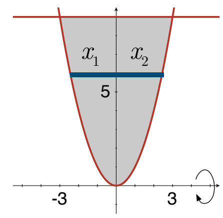
11. $y = x^2$

$$V = 2\pi \int_0^1 (3-x)x^2 \, dx = 2\pi \left(x^3 - \frac{1}{4}x^4\right)\Big]_0^1 = \frac{3\pi}{2}$$



12. $x_1 = \sqrt{y}; \quad x_2 = -\sqrt{y}$

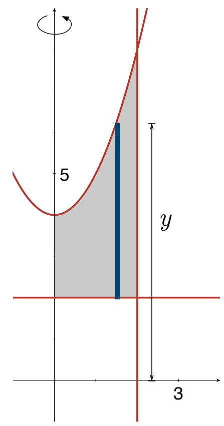
$$V = 2\pi \int_0^9 y[\sqrt{y} - (-\sqrt{y})]\,dy = 2\pi \int_0^9 2y^{3/2}\,dy = \frac{8\pi}{5} y^{5/2} \Big]_0^9 = \frac{1944\pi}{5}$$



13. $y = x^2 + 4$

$$V = 2\pi \int_0^2 x(x^2 + 4 - 2)\,dx = 2\pi \int_0^2 (x^3 + 2x)\,dx$$

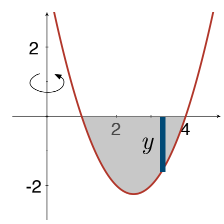
$$= 2\pi \left(\frac{1}{4} x^4 + x^2 \right) \Big]_0^2 = 16\pi$$



14. $y = x^2 - 5x + 4$

$$V = 2\pi \int_1^4 x(-x^2 + 5x - 4)\,dx = 2\pi \int_1^4 (-x^3 + 5x^2 - 4x)\,dx$$

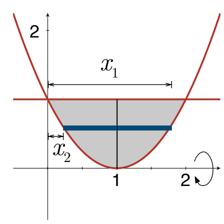
$$= 2\pi \left(-\frac{1}{4} x^4 + \frac{5}{3} x^3 - 2x^2 \right) \Big]_1^4 = 2\pi \left(\frac{32}{3} + \frac{7}{12} \right) = \frac{135\pi}{6}$$



15. $x_1 = 1 + \sqrt{y}; \quad x_2 = 1 - \sqrt{y}$

$$V = 2\pi \int_0^1 y[1 + \sqrt{y} - (1 - \sqrt{y})]\,dy = 2\pi \int_0^1 2y^{3/2}\,dy$$

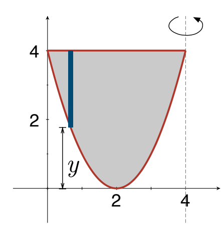
$$= \frac{8\pi}{5} y^{5/2} \Big]_0^1 = \frac{8\pi}{5}$$



16. $y = (x - 2)^2$

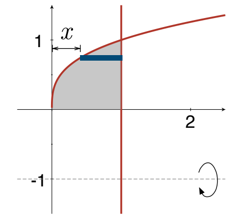
$$V = 2\pi \int_0^4 (4 - x)[4 - (x - 2)^2]\,dx = 2\pi \int_0^4 (x^3 - 8x^2 + 16x)\,dx$$

$$= 2\pi \left(\frac{1}{4} x^4 - \frac{8}{3} x^3 + 8x^2 \right) \Big]_0^4 = \frac{128\pi}{3}$$



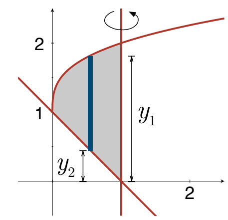
17. $x = y^3$

$$V = 2\pi \int_0^1 (y+1)(1-y^3)\,dy = 2\pi \int_0^1 (1 + y - y^3 - y^4)\,dy$$
$$= 2\pi \left(y + \frac{1}{2}y^2 - \frac{1}{4}y^4 - \frac{1}{5}y^5 \right) \Big]_0^1 = \frac{21\pi}{10}$$



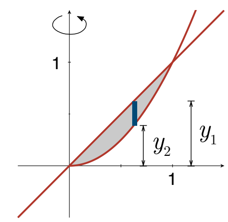
18. $y_1 = x^{1/3} + 1; \quad y_2 = -x + 1$

$$V = 2\pi \int_0^1 (1-x)[x^{1/3} + 1 - (-x+1)]\,dx$$
$$= 2\pi \int_0^1 (x^{1/3} + x - x^{4/3} - x^2)\,dx$$
$$= 2\pi \left(\frac{3}{4}x^{4/3} + \frac{1}{2}x^2 - \frac{3}{7}x^{7/3} - \frac{1}{3}x^3 \right) \Big]_0^1 = \frac{41\pi}{42}$$



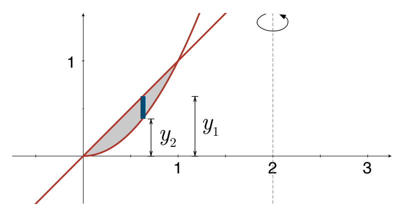
19. $y_1 = x; \quad y_2 = x^2$

$$V = 2\pi \int_0^1 x(x - x^2)\,dx = 2\pi \left(\frac{1}{3}x^3 - \frac{1}{4}x^4 \right) \Big]_0^1 = \frac{\pi}{6}$$



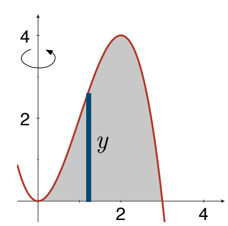
20. $y_1 = x; \quad y_2 = x^2$

$$V = 2\pi \int_0^1 (2-x)(x-x^2)\,dx$$
$$= 2\pi \int_0^1 (2x - 3x^2 + x^3)\,dx$$
$$= 2\pi \left(x^2 - x^3 + \frac{1}{4}x^4 \right) \Big]_0^1 = \frac{\pi}{2}$$



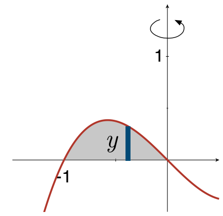
21. $y = -x^3 + 3x^2$

$$V = 2\pi \int_0^3 x(-x^3 + 3x^2)\,dx = 2\pi \int_0^3 (-x^4 + 3x^3)\,dx$$
$$= 2\pi \left(-\frac{1}{5}x^5 + \frac{3}{4}x^4 \right) \Big]_0^3 = \frac{243\pi}{10}$$



22. $y = x^3 - x$

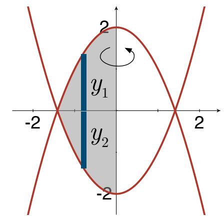
$$V = 2\pi \int_{-1}^{0} -x(x^3 - x)\,dx = 2\pi\left(-\frac{1}{5}x^5 + \frac{1}{3}x^3\right)\Bigg]_{-1}^{0} = \frac{4\pi}{15}$$



23. $y_1 = 2 - x^2; \quad y_2 = x^2 - 2$

$$V = 2\pi \int_{-\sqrt{2}}^{0} -x[2 - x^2 - (x^2 - 2)]\,dx = 2\pi \int_{-\sqrt{2}}^{0} (2x^3 - 4x)\,dx$$

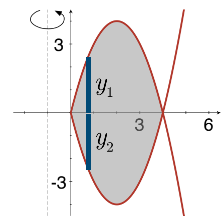
$$= 2\pi\left(\frac{1}{2}x^4 - 2x^2\right)\Bigg]_{-\sqrt{2}}^{0} = 4\pi$$



24. $y_1 = 4x - x^2; \quad y_2 = x^2 - 4x$

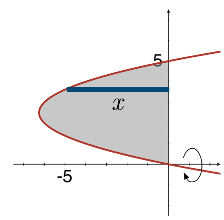
$$V = 2\pi \int_{0}^{4} (x+1)[4x - x^2 - (x^2 - 4x)]\,dx = 2\pi \int_{0}^{4} (8x + 6x^2 - 2x^3)\,dx$$

$$= 2\pi\left(4x^2 + 2x^3 - \frac{1}{2}x^4\right)\Bigg]_{0}^{4} = 128\pi$$



25. $x = y^2 - 5y$

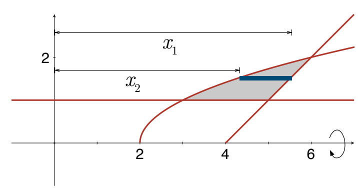
$$V = 2\pi \int_{0}^{5} y(-y^2 + 5y)\,dy = 2\pi\left(-\frac{1}{4}y^4 + \frac{5}{3}y^3\right)\Bigg]_{0}^{5} = \frac{625\pi}{6}$$



26. $x_1 = y + 4; \quad x_2 = y^2 + 2$

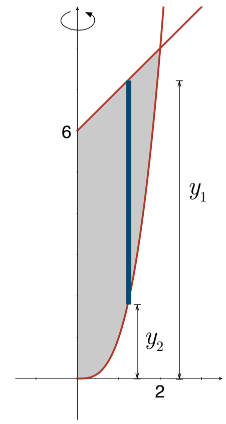
$$V = 2\pi \int_{1}^{2} y[y + 4 - (y^2 + 2)]\,dy = 2\pi \int_{1}^{2} (2y + y^2 - y^3)\,dy$$

$$= 2\pi\left(y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4\right)\Bigg]_{1}^{2} = 2\pi\left(\frac{8}{3} - \frac{13}{12}\right) = \frac{19\pi}{6}$$



27. $y_1 = x + 6; \quad y_2 = x^3$

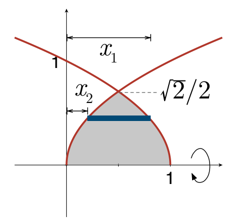
$$V = 2\pi \int_0^2 x(x + 6 - x^3)\, dx = 2\pi \left(\frac{1}{3}x^3 + 3x^2 - \frac{1}{5}x^5 \right) \Bigg]_0^2 = \frac{248\pi}{15}$$



28. $x_1 = 1 - y^2; \quad x_2 = y^2$

$$V = 2\pi \int_0^{\sqrt{2}/2} y(1 - y^2 - y^2)\, dy = 2\pi \int_0^{\sqrt{2}/2} (y - 2y^3)\, dy$$

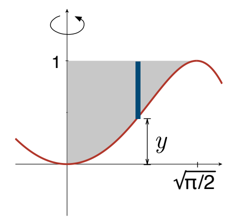
$$= 2\pi \left(\frac{1}{2}y^2 - \frac{1}{2}y^4 \right) \Bigg]_0^{\sqrt{2}/2} = \frac{\pi}{4}$$



29. $y = \sin x^2$

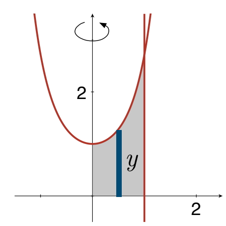
$$V = 2\pi \int_0^{\sqrt{\pi/2}} x(1 - \sin x^2)\, dx = 2\pi \int_0^{\sqrt{\pi/2}} (x - x \sin x^2)\, dx$$

$$= 2\pi \left(\frac{1}{2}x^2 + \frac{1}{2}\cos x^2 \right) \Bigg]_0^{\sqrt{\pi/2}} = 2\pi \left(\frac{\pi}{4} - \frac{1}{2} \right) = \frac{\pi^2 - 2\pi}{2}$$



30. $y = e^{x^2}$

$$V = 2\pi \int_0^1 x(e^{x^2})\, dx = 2\pi \left(\frac{1}{2}e^{x^2} \right) \Bigg]_0^1 = \pi e - \pi$$



31. We use the shell method.

$$V = 2\pi \int_0^r (r - x) \left(\frac{h}{r}x \right) dx = 2\pi \int_0^r \left(hx - \frac{h}{r}x^2 \right) dx = 2\pi \left(\frac{1}{2}hx^2 - \frac{h}{3r}x^3 \right) \Bigg]_0^r = \frac{1}{3}\pi r^2 h$$

32. The equation of the line through (r_1, h) and $(r_2, 0)$ is $x = \dfrac{1}{h}(r_1 - r_2)y + r_2$. We use the disk

method.

$$\begin{aligned}
V &= \pi \int_0^h \left[\frac{1}{h}(r_1 - r_2)y + r_2\right]^2 dy = \pi \int_0^h \left[\frac{1}{h^2}(r_1 - r_2)^2 y^2 + \frac{2}{h} r_2 (r_1 - r_2) y + r_2^2\right] dy \\
&= \pi \left[\frac{1}{3h^2}(r_1 - r_2)^2 y^3 + \frac{1}{h} r_2 (r_1 - r_2) y^2 + r_2^2 y\right]\Bigg|_0^h \\
&= \pi h \left[\frac{1}{3}(r_1^2 - 2r_1 r_2 + r_2^2) + r_2(r_1 - r_2) + r_2^2\right] = \frac{\pi h}{3}(r_1^2 + r_1 r_2 + r_2^2)
\end{aligned}$$

33. We use the disk method.

$$\begin{aligned}
V &= \pi \int_{-r}^{r} (\sqrt{r^2 - y^2})^2 \, dy = \pi \int_{-r}^{r} (r^2 - y^2) \, dy \\
&= \pi \left(r^2 y - \frac{1}{3} y^3\right)\Bigg]_{-r}^{r} = \pi \left[\frac{2}{3} r^3 - \left(-\frac{2}{3} r^3\right)\right] = \frac{4}{3}\pi r^3
\end{aligned}$$

34. The equation of the line is $y = \dfrac{1}{a}\sqrt{r^2 - a^2}\,x$ and the equation of the circle is $y = \sqrt{r^2 - x^2}$. We use the disk method.

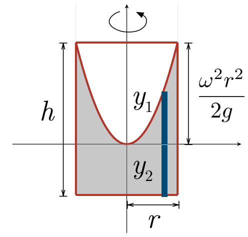
$$\begin{aligned}
V &= \pi \int_a^b \left(\frac{\sqrt{r^2 - a^2}}{a} x\right)^2 dx + \pi \int_a^b \left(\sqrt{r^2 - x^2}\right)^2 dx \\
&= \pi \frac{r^2 - a^2}{a^2} \int_0^a x^2 \, dx + \pi \int_a^b (r^2 - x^2) \, dx = \pi \left(\frac{r^2 - a^2}{a^2}\right) \frac{1}{3} x^3 \Bigg]_0^a + \pi \left(r^2 x - \frac{1}{3} x^3\right)\Bigg]_a^b \\
&= \frac{\pi}{3}(r^2 - a^2)a + \pi \left[\left(br^2 - \frac{1}{3} b^3\right) - \left(ar^2 - \frac{1}{3} a^3\right)\right] = \frac{\pi}{3}(3br^2 - 2ar^2 - b^3)
\end{aligned}$$

35. The equation of the ellipse is $y = b\sqrt{1 - \dfrac{x^2}{a^2}}$. We use the disk method.

$$\begin{aligned}
V &= \pi \int_{-a}^{a} \left(b\sqrt{1 - \frac{x^2}{a^2}}\right)^2 dx = \pi b^2 \int_{-a}^{a} \left(1 - \frac{x^2}{a^2}\right) dx = \pi b^2 \left(x - \frac{1}{3a^2} x^3\right)\Bigg]_{-a}^{a} \\
&= \pi b^2 \left[\frac{2a}{3} - \left(-\frac{2a}{3}\right)\right] = \frac{4\pi a b^2}{3}
\end{aligned}$$

36. The equation of the ellipse is $y = b\sqrt{1 - \dfrac{x^2}{a^2}}$. Since the solid is symmetric with respect to the x-axis, we will find the volume of the upper hemispheroid and multiply by 2. We use the shell method.

$$V = 2\left(2\pi \int_0^a x\sqrt{b^2 - \frac{b^2}{a^2}x^2} \, dx\right) = 4\pi \left(-\frac{a^2}{3b^2}\right)\left(b^2 - \frac{b^2}{a^2}x^2\right)^{3/2}\Bigg]_0^a = \frac{\pi a^2 b}{3}$$



37. $y_1 = \dfrac{\omega^2 x^2}{2g}$. The depth of the liquid below the x-axis is $y_2 = h - \dfrac{\omega^2 r^2}{2g}$. So the volume is

$$\begin{aligned} V &= 2\pi \int_0^r x \left(\frac{\omega^2 x^2}{2g} + h - \frac{\omega^2 r^2}{2g} \right) dx \\ &= 2\pi \int_0^r \left(\frac{\omega^2}{2g} x^3 + \frac{2hg - \omega^2 r^2}{2g} x \right) dx \\ &= 2\pi \left(\frac{\omega^2}{8g} x^4 + \frac{2hg - \omega^2 r^2}{4g} x^2 \right) \Bigg]_0^r \\ &= \frac{\pi\omega^2 r^4}{4g} + \frac{4\pi h g r^2 - 2\pi\omega^2 r^4}{4g} = \pi r^2 h - \frac{\pi\omega^2 r^4}{4g}. \end{aligned}$$

38. The liquid will touch the bottom of the bucket when $y_2 = h - \dfrac{\omega^2 r^2}{2g} = 0$, or $\omega = \dfrac{\sqrt{2hg}}{r}$. The volume of the liquid is then

$$V = \pi r^2 h - \frac{\pi\omega^2 r^4}{4g} = \pi r^2 h - \frac{\pi(2hg/r^2) r^4}{4g} = \pi r^2 h - \frac{1}{2}\pi r^2 h = \frac{1}{2}\pi r^2 h$$

6.5 Length of a Graph

1. $y' = 1; \qquad s = \displaystyle\int_{-1}^1 \sqrt{1 + 1^2}\, dx = 2\sqrt{2}$

2. $y' = 2; \qquad s = \displaystyle\int_0^3 \sqrt{1 + 2^2}\, dx = 3\sqrt{5}$

3. $y' = \dfrac{3}{2} x^{1/2}$

$$s = \int_0^1 \sqrt{1 + \frac{9}{4} x}\, dx = \frac{8}{27} \left(1 + \frac{9}{4} x \right)^{3/2} \Bigg]_0^1 = \frac{8}{27} \left[\left(\frac{13}{4} \right)^{3/2} - 1 \right] = \frac{13^{3/2} - 8}{27} \approx 1.4397$$

4. $y' = 2x^{-1/3}$

$$s = \int_1^8 \sqrt{1 + 4x^{-2/3}}\, dx = \int_1^8 \sqrt{\frac{x^{2/3} + 4}{x^{2/3}}}\, dx = \int_1^8 x^{-1/3} \sqrt{x^{2/3} + 4}\, dx$$

$$\boxed{u = x^{2/3} + 4,\ du = \frac{2}{3} x^{-1/3}\, dx}$$

$$= \int_5^8 u^{1/2} \left(\frac{3}{2}\, du \right) = u^{3/2} \Big]_5^8 = 8^{3/2} - 5^{3/2} \approx 11.4471$$

5. $y' = 2x(x^2+1)^{1/2}$

$$s = \int_1^4 \sqrt{1+4x^2(x^2+1)}\,dx = \int_1^4 \sqrt{(2x^2+1)^2}\,dx = \int_1^4 (2x^2+1)\,dx$$
$$= \left(\frac{2}{3}x^3+x\right)\Big]_1^4 = \frac{140}{3} - \frac{5}{3} = 45$$

6. $y = 2(x+1)^{3/2} - 1; \qquad y' = 3(x+1)^{1/2}$

$$s = \int_{-1}^0 \sqrt{1+9(x+1)}\,dx = \int_{-1}^0 \sqrt{9x+10}\,dx = \frac{2}{27}(9x+10)^{3/2}\Big]_{-1}^0$$
$$= \frac{2}{27}(10^{3/2}-1) \approx 2.2684$$

7. $y' = \frac{1}{2}x^{1/2} - \frac{1}{2}x^{-1/2} = \frac{x-1}{2x^{1/2}}$

$$s = \int_1^4 \sqrt{1+\frac{(x-1)^2}{4x}}\,dx = \int_1^4 \sqrt{\frac{4x+x^2-2x+1}{4x}}\,dx = \int_1^4 \sqrt{\frac{(x+1)^2}{4x}}\,dx$$
$$= \frac{1}{2}\int_1^4 \frac{x+1}{x^{1/2}}\,dx = \frac{1}{2}\int_1^4 (x^{1/2}+x^{-1/2})\,dx = \frac{1}{2}\left(\frac{2}{3}x^{3/2}+2x^{1/2}\right)\Big]_1^4 = \frac{1}{2}\left(\frac{28}{3}-\frac{8}{3}\right) = \frac{10}{3}$$

8. $y' = \frac{1}{2}x^2 - \frac{1}{2x^2} = \frac{x^4-1}{2x^2}$

$$s = \int_2^4 \sqrt{1+\frac{(x^4-1)^2}{4x^4}}\,dx = \int_2^4 \sqrt{\frac{4x^4+x^8-2x^4+1}{4x^4}}\,dx$$
$$= \int_2^4 \sqrt{\frac{(x^4+1)^2}{4x^4}}\,dx = \frac{1}{2}\int_2^4 (x^2+x^{-2})\,dx = \frac{1}{2}\left(\frac{1}{3}x^3-\frac{1}{x}\right)\Big]_2^4 = \frac{1}{2}\left(\frac{253}{12}-\frac{13}{6}\right) = \frac{227}{24}$$

9. $y' = x^3 - \frac{1}{4x^3} = \frac{4x^6-1}{4x^3}$

$$s = \int_2^3 \sqrt{1+\frac{(4x^6-1)^2}{16x^6}}\,dx = \int_2^3 \sqrt{\frac{16x^6+16x^{12}-8x^6+1}{16x^6}}\,dx = \int_2^3 \sqrt{\frac{(4x^6+1)^2}{16x^6}}\,dx$$
$$= \int_2^3 \frac{4x^6+1}{4x^3}\,dx = \frac{1}{4}\int_2^3 (4x^3+x^{-3})\,dx = \frac{1}{4}\left(x^4-\frac{1}{2x^2}\right)\Big]_2^3$$
$$= \frac{1}{4}\left(\frac{1457}{18}-\frac{127}{8}\right) = \frac{4685}{288} \approx 16.2674$$

10. $y' = x^4 - \dfrac{1}{4x^4} = \dfrac{4x^8 - 1}{4x^4}$

$$s = \int_1^2 \sqrt{1 + \frac{(4x^8-1)^2}{16x^8}}\,dx = \int_1^2 \sqrt{\frac{16x^8 + 16x^{16} - 8x^8 + 1}{16x^8}}\,dx = \int_1^2 \sqrt{\frac{(4x^8+1)^2}{16x^8}}\,dx$$
$$= \frac{1}{4}\int_1^2 \frac{4x^8+1}{x^4}\,dx = \frac{1}{4}\int_1^2 (4x^4 + x^{-4})\,dx = \frac{1}{4}\left(\frac{4}{5}x^5 + \frac{1}{3x^3}\right)\Big]_1^2$$
$$= \frac{1}{4}\left(\frac{3067}{120} - \frac{56}{120}\right) = \frac{3011}{480} \approx 6.2729$$

11. $y' = -\dfrac{(4-x^{2/3})^{1/2}}{x^{1/3}}$; $\quad s = \displaystyle\int_1^8 \sqrt{1 + \frac{4 - x^{2/3}}{x^{2/3}}}\,dx = \int_1^8 \frac{2}{x^{1/3}}\,dx = 3x^{2/3}\Big]_1^8 = 9$

12. $y' = \begin{cases} 1, & 2 < x < 3 \\ \dfrac{2}{3}(x-2)^{-1/3}, & 3 < x < 10 \\ \dfrac{3}{4}(x-6)^{1/2}, & 10 < x < 15 \end{cases}$

$$s = \int_2^3 \sqrt{1+1}\,dx + \int_3^{10} \sqrt{1 + \frac{4}{9}(x-2)^{-2/3}}\,dx + \int_{10}^{15} \sqrt{1 + \frac{9}{16}(x-6)}\,dx$$
$$= \sqrt{2} + \int_3^{10} \sqrt{(x-2)^{2/3} + \frac{4}{9}}\,(x-2)^{-1/3}\,dx + \frac{1}{4}\int_{10}^{15} \sqrt{9x-38}\,dx$$
$$= \sqrt{2} + \left[(x-2)^{2/3} + \frac{4}{9}\right]^{3/2}\Bigg|_3^{10} + \frac{1}{4}\left(\frac{1}{9}\right)\left(\frac{2}{3}\right)(9x-38)^{3/2}\Big]_{10}^{15}$$
$$= \sqrt{2} + \left(\frac{40}{9}\right)^{3/2} - \left(\frac{13}{9}\right)^{3/2} + \frac{1}{54}(97^{3/2} - 52^{3/2}) \approx 19.7954$$

13. $y' = 2x$; $\quad s = \displaystyle\int_{-1}^3 \sqrt{1+4x^2}\,dx$

14. $y' = (x+1)^{-1/2}$; $\quad s = \displaystyle\int_{-1}^3 \sqrt{1 + (x+1)^{-1}}\,dx = \int_{-1}^3 \sqrt{\frac{x+2}{x+1}}\,dx$

15. $y' = \cos x$; $\quad s = \displaystyle\int_0^\pi \sqrt{1 + \cos^2 x}\,dx$

16. $y' = \sec^2 x$; $\quad s = \displaystyle\int_{-\pi/4}^{\pi/4} \sqrt{1 + \sec^4 x}\,dx$

17. $\dfrac{dx}{dy} = -\dfrac{2}{3}y^{-1/3}$

$$s = \int_0^8 \sqrt{1 + \frac{4}{9}y^{-2/3}}\, dy = \int_0^8 \sqrt{\frac{9y^{2/3}+4}{9y^{2/3}}}\, dy = \frac{1}{3}\int_0^8 y^{-1/3}\sqrt{9y^{2/3}+4}\, dy$$

$$\boxed{u = 9y^{2/3}+4,\ du = 6y^{-1/3}\, dy}$$

$$= \frac{1}{18}\int_4^{40} u^{1/2}\, du = \frac{1}{27}u^{3/2}\Big]_4^{40} = \frac{1}{27}(40^{3/2} - 8) \approx 9.0734$$

18. $\dfrac{dx}{dy} = \dfrac{1}{2}y^{3/2} - \dfrac{1}{2}y^{-3/2} = \dfrac{y^3 - 1}{2y^{3/2}}$

$$s = \int_4^9 \sqrt{1 + \frac{(y^3-1)^2}{4y^3}}\, dy = \int_4^9 \sqrt{\frac{4y^3 + y^6 - 2y^3 + 1}{4y^3}}\, dy = \int_4^9 \sqrt{\frac{(y^3+1)^2}{4y^3}}\, dy$$

$$= \frac{1}{2}\int_4^9 \frac{y^3+1}{y^{3/2}}\, dy = \frac{1}{2}\int_4^9 (y^{3/2} + y^{-3/2})\, dy = \frac{1}{2}\left(\frac{2}{5}y^{5/2} - 2y^{-1/2}\right)\Big]_4^9$$

$$= \frac{1}{2}\left(\frac{1448}{15} - \frac{59}{5}\right) = \frac{1271}{30} \approx 42.3667$$

19. (a) $y = (1 - x^{2/3})^{3/2}$; $\qquad y' = \dfrac{3}{2}(1 - x^{2/3})^{1/2}\left(-\dfrac{2}{3}x^{-1/3}\right) = -\dfrac{(1-x^{2/3})^{1/2}}{x^{1/3}}$

$$s = \int_0^1 \sqrt{1 + \frac{1 - x^{2/3}}{x^{2/3}}}\, dx = \int_0^1 \sqrt{\frac{1}{x^{2/3}}}\, dx = \int_0^1 \frac{1}{x^{1/3}}\, dx$$

At $x = 0$, $\dfrac{1}{x^{1/3}}$ is discontinuous.

(b) The graph is symmetric with respect to both coordinate axes, so

$$s = 4\int_0^1 \frac{1}{x^{1/3}}\, dx = 4\left(\frac{3}{2}x^{2/3}\right)\Big]_0^1 = 6.$$

20. $y = b\left(1 - \dfrac{x^2}{a^2}\right)^{1/2}$; $\qquad y' = -\dfrac{bx}{a^2}\left(1 - \dfrac{x^2}{a^2}\right)^{-1/2}$

$$s = 4\int_0^a \sqrt{1 + \frac{b^2x^2}{a^4}\left(1 - \frac{x^2}{a^2}\right)^{-1}}\, dx = 4\int_0^a \sqrt{1 + \frac{b^2x^2}{a^4}\left(\frac{a^2}{a^2 - x^2}\right)}\, dx$$

$$= 4\int_0^a \sqrt{1 + \frac{b^2x^2}{a^2(a^2 - x^2)}}\, dx = \frac{4}{a}\int_0^a \sqrt{\frac{a^4 - a^2x^2 + b^2x^2}{a^2 - x^2}}\, dx$$

21. $y = \sqrt{r^2 - x^2}$; $\qquad y' = -x(r^2 - x^2)^{-1/2}$

$$2\pi r = s = 4\int_0^r \sqrt{1 + x^2(r^2 - x^2)^{-1}}\, dr = 4r\int_0^r \left(\frac{1}{\sqrt{r^2 - x^2}}\, dx\right) dr$$

Letting $r = 1$ we have $2\pi = 4\displaystyle\int_0^1 \frac{1}{\sqrt{1 - x^2}}\, dx$ or $\displaystyle\int_0^1 \frac{1}{\sqrt{1 - x^2}}\, dx = \frac{\pi}{2}$.

22. $y' = x^3$. Let $s(x) = \displaystyle\int_2^x \sqrt{1+t^6}\,dt$. We want $s(2.1)$ or $s(2+0.1)$. Using approximation by differentials, we have

$$\int_2^{2.1} \sqrt{1+x^6}\,dx = s(2+0.1) \approx s(2) + s'(2)\,dx = 0 + \sqrt{1+2^6}(0.1) = 0.1\sqrt{65} \approx 0.8062.$$

6.6 Area of a Surface of Revolution

1. $y' = x^{-1/2}$

$$S = 2\pi\int_0^8 2\sqrt{x}\sqrt{1+x^{-1}}\,dx = 4\pi\int_0^8 \sqrt{x+1}\,dx = \frac{8\pi}{3}(x+1)^{3/2}\Big]_0^8 = \frac{8\pi}{3}(27-1) = \frac{208\pi}{3}$$

2. $y' = \dfrac{1}{2}(x+1)^{-1/2}$

$$S = 2\pi\int_1^5 \sqrt{x+1}\sqrt{1+\frac{1}{4}(x+1)^{-1}}\,dx = 2\pi\int_1^5 \sqrt{x+1+\frac{1}{4}}\,dx = \frac{4\pi}{3}\left(x+\frac{5}{4}\right)^{3/2}\Bigg]_1^5$$
$$= \frac{4\pi}{3}\left[\left(\frac{25}{4}\right)^{3/2} - \left(\frac{9}{4}\right)^{3/2}\right] = \frac{\pi}{6}(25^{3/2} - 9^{3/2}) \approx 51.3127$$

3. $y' = 3x^2$

$$S = 2\pi\int_0^1 x^3\sqrt{1+9x^4}\,dx \quad \boxed{u = 1+9x^4,\ du = 36x^3\,dx}$$
$$= 2\pi\int_1^{10} \sqrt{u}\left(\frac{1}{36}\,du\right) = \frac{\pi}{27}u^{3/2}\Big]_1^{10} = \frac{\pi}{27}(10^{3/2}-1) \approx 3.5631$$

4. $y' = \dfrac{1}{3}x^{-2/3}$

$$S = 2\pi\int_1^8 x\sqrt{1+\frac{x^{-4/3}}{9}}\,dx = 2\pi\int_1^8 \frac{x^{1/3}}{3}\sqrt{9x^{4/3}+1}\,dx \quad \boxed{u = 9x^{4/3}+1,\ du = 12x^{1/3}\,dx}$$
$$= \frac{2\pi}{3}\int_{10}^{145} \sqrt{u}\left(\frac{1}{12}du\right) = \frac{\pi}{27}u^{3/2}\Big]_{10}^{145} = \frac{\pi}{27}(145^{3/2} - 10^{3/2}) \approx 199.4805$$

5. $y' = 2x$; $\quad S = 2\pi\displaystyle\int_0^3 x\sqrt{1+4x^2}\,dx = \frac{\pi}{6}(1+4x^2)^{3/2}\Big]_0^3 = \frac{\pi}{6}(37^{3/2}-1) \approx 117.3187$

6. $y' = -2x$; $\quad S = 2\pi\displaystyle\int_0^2 x\sqrt{1+4x^2}\,dx = \frac{\pi}{6}(1+4x^2)^{3/2}\Big]_0^2 = \frac{\pi}{6}(17^{3/2}-1) \approx 36.1769$

7. $y' = 2$

$$S = 2\pi\int_2^7 (2x+1)\sqrt{1+4}\,dx = 2\sqrt{5}\pi\int_2^7 (2x+1)\,dx = 2\sqrt{5}\pi\left(x^2+x\right)\Big]_2^7$$
$$= 2\sqrt{5}\pi(56-6) = 100\sqrt{5}\pi$$

8. $y' = -x(16-x^2)^{-1/2}$

$$S = 2\pi \int_0^{\sqrt{7}} x\sqrt{1 + x^2(16-x^2)^{-1}}\,dx = 2\pi \int_0^{\sqrt{7}} x\sqrt{\frac{16}{16-x^2}}\,dx = 8\pi \int_0^{\sqrt{7}} x(16-x^2)^{-1/2}\,dx$$

$$\boxed{u = 16 - x^2,\ du = -2x\,dx}$$

$$= 8\pi \int_{16}^{9} u^{-1/2}\left(-\frac{1}{2}\,du\right) = 4\pi \int_9^{16} u^{-1/2}\,du = 4\pi(2\sqrt{u})\Big]_9^{16} = 4\pi(8-6) = 8\pi$$

9. $y' = x^3 - \frac{1}{4}x^{-3} = \frac{4x^6-1}{4x^3}$

$$S = 2\pi \int_1^2 x\sqrt{1 + \frac{16x^{12} - 8x^6 + 1}{16x^6}}\,dx = 2\pi \int_1^2 x\sqrt{\frac{16x^6 + 16x^{12} - 8x^6 + 1}{16x^6}}\,dx$$

$$= \frac{\pi}{2}\int_1^2 \frac{\sqrt{(4x^6+1)^2}}{x^2}\,dx = \frac{\pi}{2}\int_1^2 (4x^4 + x^{-2})\,dx = \frac{\pi}{2}\left(\frac{4}{5}x^5 - \frac{1}{x}\right)\Big]_1^2$$

$$= \frac{\pi}{2}\left[\frac{251}{10} - \left(-\frac{1}{5}\right)\right] = \frac{253\pi}{20}$$

10. $y' = x^2 - \frac{1}{4}x^{-2} = \frac{4x^4-1}{4x^2}$

$$S = 2\pi \int_1^2 \left(\frac{1}{3}x^3 + \frac{1}{4x}\right)\sqrt{1 + \frac{16x^8 - 8x^4 + 1}{16x^4}}\,dx = \frac{\pi}{6}\int_1^2 \frac{4x^4+3}{x}\sqrt{\frac{(4x^4+1)^2}{16x^4}}\,dx$$

$$= \frac{\pi}{6}\int_1^2 \frac{(4x^4+3)(4x^4+1)}{4x^3}\,dx = \frac{\pi}{6}\int_1^2 \left(4x^5 + 4x + \frac{3}{4}x^{-3}\right)dx$$

$$= \frac{\pi}{6}\left(\frac{2}{3}x^6 + 2x^2 - \frac{3}{8x^2}\right)\Big]_1^2 = \frac{\pi}{6}\left(\frac{4855}{96} - \frac{55}{24}\right) = \frac{4635\pi}{576}$$

11. (a) $f'(x) = -\frac{r}{2h}\left(1 - \frac{x}{h}\right)^{-1/2}$; $\quad 1 + [f'(x)]^2 = \frac{4h^2 - 4hx + r^2}{4h^2(1 - x/h)}$

$$f(x)\sqrt{1 + [f'(x)]^2} = r\sqrt{1 - x/h}\,\frac{\sqrt{4h^2 - 2hx + r^2}}{2h\sqrt{1-x^2}} = \frac{r}{2h}\sqrt{4h^2 - 4hx + r^2}$$

$$S = 2\pi \int_0^h \frac{r}{2h}\sqrt{r^2 + 4h^2 - 4hx}\,dx = \frac{\pi r}{h}\left(\frac{2}{3}\right)\left(-\frac{1}{4h}\right)(r^2 + 4h^2 - 4hx)^{3/2}\Big]_0^h$$

$$= \frac{\pi r}{6h^2}[(r^2 + 4h^2)^{3/2} - r^3]$$

(b) With $h = 0.1r$ we have

$$S = \frac{\pi r}{6(0.01)r^2}[r^3(1.04)^{3/2} - r^3] \approx \pi r^2\left(\frac{1.04^{3/2} - 1}{0.06}\right) \approx \pi r^2 \frac{0.060596}{0.06} \approx \pi r^2.$$

The approximate percentage error is $\frac{(0.060596/0.06)\pi r^2 - \pi r^2}{\pi r^2} = \frac{0.060596}{0.06} - 1 \approx 0.0099$, or approximately 1%.

12. $y = \sqrt{r^2 - x^2}$; $\qquad y' = -x(r^2 - x^2)^{-1/2}$

$$S = 2\pi \int_a^b \sqrt{r^2 - x^2}\sqrt{1 + x^2(r^2 - x^2)^{-1}}\, dx = 2\pi \int_a^b \sqrt{r^2 - x^2}\sqrt{\frac{r^2}{r^2 - x^2}}\, dx$$
$$= 2\pi r \int_a^b dx = 2\pi r(b - a)$$

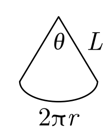
13. For $x < -2$, $y = -x - 2$ and $y' = -1$. For $x > 2$, $y = x + 2$ and $y' = 1$.

$$S = 2\pi \int_{-4}^{-2} (-x - 2)\sqrt{1 + 1}\, dx + 2\pi \int_{-2}^{2} (x + 2)\sqrt{1 + 1}\, dx$$
$$= 2\sqrt{2}\pi \int_{-4}^{-2} (-x - 2)\, dx + 2\sqrt{2}\pi \int_{-2}^{2} (x + 2)\, dx$$
$$= 2\sqrt{2}\pi \left(-\frac{1}{2}x^2 - 2x\right)\Bigg]_{-4}^{-2} + 2\sqrt{2}\pi \left(\frac{1}{2}x^2 + 2x\right)\Bigg]_{-2}^{2}$$
$$= 2\sqrt{2}\pi[(-2 + 4) - (-8 + 8)] + 2\sqrt{2}\pi[(2 + 4) - (2 - 4)] = 20\sqrt{2}\pi$$

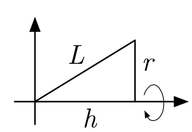
14. Since the graph is symmetric with respect to the y-axis, we will find the area on $[0, a]$ and multiply by 2.

$$y = (a^{2/3} - x^{2/3})^{3/2}; \qquad y' = -\frac{(a^{2/3} - x^{2/3})^{1/2}}{x^{1/3}}$$
$$S = 4\pi \int_0^a (a^{2/3} - x^{2/3})^{3/2}\sqrt{1 + \frac{(a^{2/3} - x^{2/3})}{x^{2/3}}}\, dx = 4\pi \int_0^a (a^{2/3} - x^{2/3})^{3/2}\sqrt{\frac{a^{2/3}}{x^{2/3}}}\, dx$$
$$= 4\pi a^{1/3} \int_0^a x^{-1/3}(a^{2/3} - x^{2/3})^{3/2}\, dx \qquad \boxed{u = a^{2/3} - x^{2/3},\ du = -\frac{2}{3}x^{-1/3}\, dx}$$
$$= 4\pi a^{1/3} \int_{a^{2/3}}^{0} u^{3/2}\left(-\frac{3}{2}\, du\right) = 4\pi a^{1/3}\left(-\frac{3}{5}u^{5/2}\right)\Bigg]_{a^{2/3}}^{0} = -\frac{12\pi a^{1/3}}{5}(0 - a^{5/3}) = \frac{12\pi a^2}{5}$$

15. Let θ be the angle formed when the cone is cut and flattened out. The length of the arc of the sector is $2\pi r$, the circumference of the base of the cone. Then $\theta/2\pi r = 2\pi/2\pi L$ (the angle subtended by the sector is to the length of the sector as 2π radians is to the circumference of the circle of radius L), and $\theta = 2\pi r/L$. Using the hint in the text, the lateral surface area is $\dfrac{1}{2}L^2\left(\dfrac{2\pi r}{L}\right) = \pi r L$.

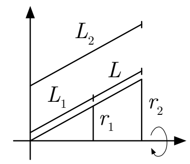


16. If L is the slant height, then $L^2 = r^2 + h^2$ and the surface area is $\pi r L = \pi r\sqrt{r^2 + h^2}$. The surface can also be obtained by revolving the line $y = \dfrac{r}{h}x$ about the x-axis:



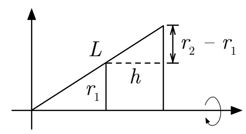
$$S = 2\pi \int_0^h \frac{r}{h}x\sqrt{1 + \left(\frac{r}{h}\right)^2}\, dx = \frac{2\pi r}{h}\int_0^h x\sqrt{\frac{r^2 + h^2}{h^2}}\, dx$$
$$= \frac{2\pi r\sqrt{r^2 + h^2}}{h^2}\left(\frac{1}{2}x^2\right)\Bigg]_0^h = \pi r\sqrt{r^2 + h^2}$$

17. By similar triangles, $\dfrac{r_1}{L_1} = \dfrac{r_2}{L_2}$, $r_1L_2 = r_2L_1$, and $r_1L_2 - r_2L_1 = 0$.
From Problem 15, the lateral surface area of the frustum is



$$\begin{aligned}S &= \pi r_2L_2 - \pi r_1L_1 = \pi(r_2L_2 - r_1L_1) = \pi[r_2L_2 + \overbrace{(r_1L_2 - r_2L_1)}^{0} - r_1L_1] \\ &= \pi[(r_2 + r_1)L_2 - (r_2 + r_1)L_1] = \pi(r_1 + r_2)(L_2 - L_1) = \pi(r_1 + r_2)L.\end{aligned}$$

18. By the Pythagorean Theorem, $L^2 = h^2 + (r_2 - r_1)^2$ or $L = \sqrt{h^2 + (r_2 - r_1)^2}$. From (1) in Section 6.6,



$$S = \pi(r_1 + r_2)L = \pi(r_1 + r_2)\sqrt{h^2 + (r_2 - r_1)^2}.$$

19. We need to extend (3) in Section 6.6 to include functions which are not necessarily non-negative. In this case we have $S = 2\pi\displaystyle\int_a^b |f(x)|\sqrt{1 + [f'(x)]^2}\,dx$. Next, we require the fact that the surface area obtained by revolving f around $y = L$ is the same as that obtained by revolving $F(x) = f(x) - L$ around the x-axis. Then $F'(x) = f'(x)$ and

$$S = 2\pi\int_a^b |F(x)|\sqrt{1 + [F'(x)]^2}\,dx = 2\pi\int_a^b |f(x) - L|\sqrt{1 + [f'(x)]^2}\,dx.$$

20. $y' = \dfrac{2}{3}x^{-1/3}$; $\qquad S = 2\pi\displaystyle\int_1^8 |x^{2/3} - 4|\sqrt{1 + \frac{4}{9}x^{-2/3}}\,dx = \frac{2\pi}{3}\int_1^8 \frac{4 - x^{2/3}}{x^{1/3}}\sqrt{9x^{2/3} + 4}\,dx$

21. (a) Since $\triangle BCT$ is similar to $\triangle TCS$ we have $\overline{CB}/\overline{TC} = \overline{CT}/\overline{CS}$ or $\dfrac{y_B}{R} = \dfrac{R}{R + h}$, which gives $y_B = \dfrac{R^2}{R + h}$. Now, revolving $x = \sqrt{R^2 - y^2}$ for $y_B \le y \le R$ around the y-axis, we obtain the surface area

$$\begin{aligned}A_S &= 2\pi\int_{y_B}^R \sqrt{R^2 - y^2}\sqrt{1 + \left(\frac{-y}{\sqrt{R^2 - y^2}}\right)^2}\,dy = 2\pi\int_{\frac{R^2}{R+h}}^R R\,dy \\ &= 2\pi R\left(R - \frac{R^2}{R + h}\right) = \frac{2\pi R^2 h}{R + h}.\end{aligned}$$

Since $A_E = 4\pi R^2$, we have $\dfrac{A_S}{A_E} = \dfrac{h}{2(R + h)}$.

(b) With $h = 2000$ and $R = 6380$, $\dfrac{A_S}{A_E} = \dfrac{2000}{2(6380 + 2000)} \approx 0.119332 \approx 11.9\%$.

(c) Setting $\dfrac{h}{2(R + h)} = \dfrac{1}{4}$ we obtain $2h = R + h$ or $h = R = 6380$ km.

(d) $\displaystyle\lim_{h\to\infty}\frac{A_S}{A_E}=\lim_{h\to\infty}\frac{h}{2R+2h}=\frac{1}{2}$
From a large distance we would expect to see half of the earth's surface.

(e) $\displaystyle\frac{A_S}{A_E}=\frac{3.76\times 10^5}{2(6380+3.76\times 10^5)}\approx 0.4917=49.17\%$

6.7 Average Value of a Function

1. $\displaystyle f_{\text{ave}}=\frac{1}{1-(-3)}\int_{-3}^{1}4x\,dx=\frac{1}{4}(2x^2)\Big]_{-3}^{1}=\frac{1}{4}(2-18)=-4$

2. $\displaystyle f_{\text{ave}}=\frac{1}{5-(-2)}\int_{-2}^{5}(2x+3)\,dx=\frac{1}{7}(x^2+3x)\Big]_{-2}^{5}=\frac{1}{7}[40-(-2)]=\frac{42}{7}$

3. $\displaystyle f_{\text{ave}}=\frac{1}{2-0}\int_{0}^{2}(x^2+10)\,dx=\frac{1}{2}\left(\frac{1}{3}x^3+10x\right)\Big]_{0}^{2}=\frac{1}{2}\left(\frac{68}{3}-0\right)=\frac{34}{3}$

4. $\displaystyle f_{\text{ave}}=\frac{1}{1-(-1)}\int_{-1}^{1}(2x^3-3x^2+4x-1)\,dx=\frac{1}{2}\left(\frac{1}{2}x^4-x^3+2x^2-x\right)\Big]_{-1}^{1}$
$\displaystyle =\frac{1}{2}\left(\frac{1}{2}-\frac{9}{2}\right)=-2$

5. $\displaystyle f_{\text{ave}}=\frac{1}{3-(-1)}\int_{-1}^{3}(3x^2-4x)\,dx=\frac{1}{4}(x^3-2x^2)\Big]_{-1}^{3}=\frac{1}{4}[9-(-3)]=3$

6. $\displaystyle f_{\text{ave}}=\frac{1}{2-0}\int_{0}^{2}(x+1)^2\,dx=\frac{1}{2}\cdot\frac{1}{3}(x+1)^3\Big]_{0}^{2}=\frac{1}{2}\left(9-\frac{1}{3}\right)=\frac{13}{3}$

7. $\displaystyle f_{\text{ave}}=\frac{1}{2-(-2)}\int_{-2}^{2}x^3\,dx=\frac{1}{4}\left(\frac{1}{4}x^4\right)\Big]_{-2}^{2}=\frac{1}{4}(4-4)=0$

8. $\displaystyle f_{\text{ave}}=\frac{1}{1-0}\int_{0}^{1}x(3x-1)^2\,dx=\int_{0}^{1}(9x^3-6x^2+x)\,dx=\left(\frac{9}{4}x^4-2x^3+\frac{1}{2}x^2\right)\Big]_{0}^{1}=\frac{3}{4}$

9. $\displaystyle f_{\text{ave}}=\frac{1}{9-0}\int_{0}^{9}x^{1/2}\,dx=\frac{1}{9}\left(\frac{2}{3}x^{3/2}\right)\Big]_{0}^{9}=2$

10. $\displaystyle f_{\text{ave}}=\frac{1}{3-0}\int_{0}^{3}(5x+1)^{1/2}\,dx=\frac{1}{3}\cdot\frac{2}{15}(5x+1)^{3/2}\Big]_{0}^{3}=\frac{2}{45}(64-1)=\frac{14}{5}$

11. $\displaystyle f_{\text{ave}}=\frac{1}{3-0}\int_{0}^{3}x\sqrt{x^2+16}\,dx=\frac{1}{3}\cdot\frac{1}{3}(x^2+16)^{3/2}\Big]_{0}^{3}=\frac{1}{9}(125-64)=\frac{61}{9}$

12. $f_{\text{ave}} = \dfrac{1}{1-1/2}\displaystyle\int_{1/2}^{1}\left(1+\frac{1}{x}\right)^{1/3}\frac{1}{x^2}\,dx$ $\quad\boxed{u = 1+\frac{1}{x},\ du = -\frac{1}{x^2}\,dx}$

$$= 2\int_3^2 -u^{1/3}\,du = 2\cdot\frac{3}{4}(-u^{4/3})\Big]_3^2 = -\frac{3}{2}(2^{4/3}-3^{4/3}) \approx 2.7104$$

13. $f_{\text{ave}} = \dfrac{1}{1/2-1/4}\displaystyle\int_{1/4}^{1/2} x^{-3}\,dx = 4\left(-\frac{1}{2}x^{-2}\right)\Big]_{1/4}^{1/2} = -2\left(\frac{1}{x^2}\right)\Big]_{1/4}^{1/2} = -2(4-16) = 24$

14. $f_{\text{ave}} = \dfrac{1}{4-1}\displaystyle\int_1^4 (x^{2/3}-x^{-2/3})\,dx = \frac{1}{3}\left(\frac{3}{5}x^{5/3}-3x^{1/3}\right)\Big]_1^4 = \frac{1}{3}\left[\frac{3}{5}4^{5/3}-3\cdot 4^{1/3}-\left(-\frac{12}{5}\right)\right]$

$$= \frac{1}{5}4^{5/3}-4^{1/3}+\frac{4}{5} \approx 1.2285$$

15. $f_{\text{ave}} = \dfrac{1}{5-3}\displaystyle\int_3^5 2(x+1)^{-2}\,dx = \frac{1}{2}[-2(x+1)^{-1}]\Big]_3^5 = -\frac{1}{x+1}\Big]_3^5 = -\left(\frac{1}{6}-\frac{1}{4}\right) = \frac{1}{12}$

16. $f_{\text{ave}} = \dfrac{1}{9-4}\displaystyle\int_4^9 \frac{(\sqrt{x}-1)^3}{\sqrt{x}}\,dx$ $\quad\boxed{u = \sqrt{x}-1,\ du = \frac{1}{2\sqrt{x}}\,dx}$

$$= \frac{1}{5}\int_1^2 2u^3\,du = \frac{1}{5}\cdot\frac{1}{2}u^4\Big]_1^2 = \frac{1}{10}(16-1) = \frac{3}{2}$$

17. $f_{\text{ave}} = \dfrac{1}{\pi-(-\pi)}\displaystyle\int_{-\pi}^{\pi}\sin x\,dx = \frac{1}{2\pi}(-\cos x)\Big]_{-\pi}^{\pi} = -\frac{1}{2\pi}[-1-(-1)] = 0$

18. $f_{\text{ave}} = \dfrac{1}{\pi/4-0}\displaystyle\int_0^{\pi/4}\cos 2x\,dx = \frac{4}{\pi}\cdot\frac{1}{2}\sin 2x\Big]_0^{\pi/4} = \frac{2}{\pi}(1-0) = \frac{2}{\pi}$

19. $f_{\text{ave}} = \dfrac{1}{\pi/2-\pi/6}\displaystyle\int_{\pi/6}^{\pi/2}\csc^2 x\,dx = \frac{3}{\pi}(-\cot x)\Big]_{\pi/6}^{\pi/2} = -\frac{3}{\pi}(0-\sqrt{3}) = \frac{3\sqrt{3}}{\pi}$

20. $f_{\text{ave}} = \dfrac{1}{1/3-(-1/3)}\displaystyle\int_{-1/3}^{1/3}\frac{\sin\pi x}{\cos^2\pi x}\,dx$ $\quad\boxed{u = \cos\pi x,\ du = -\pi\sin\pi x\,dx}$

$$= \frac{3}{2}\int_{1/2}^{1/2} -\frac{1}{\pi u^2}\,du = 0$$

21. $f_{\text{ave}} = \dfrac{1}{1-(-1)}\displaystyle\int_{-1}^{1}(x^2+2x)\,dx = \frac{1}{2}\left(\frac{1}{3}x^3+x^2\right)\Big]_{-1}^{1} = \frac{1}{2}\left(\frac{4}{3}-\frac{2}{3}\right) = \frac{1}{3}$

Setting $f(c) = c^2+c = \dfrac{1}{3}$, we obtain $3c^2+6c-1=0$. Then $c = \dfrac{-6\pm\sqrt{36+12}}{6} = -1\pm\dfrac{2}{3}\sqrt{3}$. The only solution on $[-1,1]$ is $-1+\dfrac{2}{3}\sqrt{3}$.

22. $f_{\text{ave}} = \dfrac{1}{6-1}\displaystyle\int_1^6 (x+3)^{1/2}\,dx = \frac{1}{5}\cdot\frac{2}{3}(x+3)^{3/2}\Big]_1^6 = \frac{2}{15}(27-8) = \frac{38}{15}$

Setting $f(c) = \sqrt{c+3} = 38/15$, we obtain $c+3 = 1444/225$. Thus, $c = 769/225$.

23. We are given $\dfrac{1}{5-1}\displaystyle\int_1^5 f(x)\,dx = 3$. The area under the graph is $\displaystyle\int_1^5 f(x)\,dx = 12$.

24. Solving $\dfrac{1}{b}\displaystyle\int_0^b (1-\sqrt{x})\,dx = \dfrac{1}{b}\left(x - \dfrac{2}{3}x^{3/2}\right)\Big]_0^b = 1 - \dfrac{2}{3}\sqrt{b} = 0$ gives $b = 9/4$. At $b = 9/4$ the area bounded by the graph above the x-axis is the same as the area below the x-axis.

25. $T_{\text{ave}} = \dfrac{1}{6-0}\displaystyle\int_0^6 \left(100 + 3t - \dfrac{1}{2}t^2\right)\,dt = \dfrac{1}{6}\left(100t + \dfrac{3}{2}t^2 - \dfrac{1}{6}t^3\right)\Big]_0^6 = 103°$

26. $R_{\text{ave}} = \dfrac{1}{5-1}\displaystyle\int_1^5 (50 + 4x + 3x^2)\,dx = \dfrac{1}{4}(50x + 2x^2 + x^3)\Big]_1^5 = \dfrac{1}{4}(425 - 53) = 93$

$\dfrac{1}{5}\displaystyle\sum_{k=1}^{5} R(k) = \dfrac{1}{5}(57 + 70 + 89 + 114 + 145) = 95$

27. Using $s'(t) = v(t)$ we have

$$v_{\text{ave}} = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} v(t)\,dt = \frac{1}{t_2 - t_1}s(t)\Big]_{t_1}^{t_2} = \frac{s(t_2) - s(t_1)}{t_2 - t_1} = \overline{v}.$$

28.
$$\begin{aligned}
U_{\text{ave}} &= \frac{1}{2\pi/\omega - 0}\int_0^{2\pi/\omega} \frac{1}{2}kx^2\,dt = \frac{\omega k}{4\pi}\int_0^{2\pi/\omega} A^2\cos^2(\omega t + \phi)\,dt \\
&= \frac{\omega k A^2}{4\pi}\int_0^{2\pi/\omega} \frac{1}{2}[1 + \cos 2(\omega t + \phi)]\,dt = \frac{\omega k A^2}{8\pi}\left[t + \frac{1}{2\omega}\sin(2\omega t + 2\pi)\right]\Bigg|_0^{2\pi/\omega} \\
&= \frac{\omega k A^2}{8\pi}\left[\frac{2\pi}{\omega} + \frac{1}{2\omega}\sin(4\pi + 2\phi) - \frac{1}{2\omega}\sin 2\phi\right] = \frac{kA^2}{4} \\
K_{\text{ave}} &= \frac{1}{2\pi/\omega - 0}\int_0^{2\pi/\omega} \frac{1}{2}mv^2\,dt = \frac{\omega m}{4\pi}\int_0^{2\pi/\omega} [x'(t)]^2\,dt = \frac{\omega m}{4\pi}\int_0^{2\pi/\omega} \omega^2 A^2\sin^2(\omega t + \phi)\,dt \\
&= \frac{A^2\omega(m\omega^2)}{4\pi}\int_0^{2\pi/\omega} \frac{1}{2}[1 - \cos 2(\omega t + \phi)]\,dt = \frac{A^2\omega k}{8\pi}\left[t - \frac{1}{2\omega}\sin(2\omega t + 2\phi)\right]\Bigg|_0^{2\pi/\omega} \\
&= \frac{\omega k A^2}{8\pi}\left[\frac{2\pi}{\omega} - \frac{1}{2\omega}\sin(4\pi + 2\phi) - \left(-\frac{1}{2\omega}\sin 2\phi\right)\right] = \frac{kA^2}{4}
\end{aligned}$$

29.
$$\begin{aligned}
mv_1 - mv_0 = (t_1 - 0)\overline{F} &= \frac{t_1}{t_1 - 0}\int_0^{t_1} k\left[1 - \left(\frac{2t}{t_1} - 1\right)^2\right]\,dt = k\int_0^{t_1}\left(1 - \frac{4}{t_1^2}t^2 + \frac{4}{t_1}t - 1\right)\,dt \\
&= k\left(-\frac{4}{3t_1^2}t^3 + \frac{2}{t_1}t^2\right)\Big]_0^{t_1} = k\left(-\frac{4}{3}t_1 + 2t_1\right) = \frac{2kt_1}{3}
\end{aligned}$$

30. $v_{\text{ave}} = \dfrac{1}{R-0}\displaystyle\int_0^R \dfrac{P}{4vl}(R^2 - r^2)\,dr = \dfrac{P}{4vlR}\left(R^2 r - \dfrac{1}{3}r^3\right)\Big]_0^R = \dfrac{P}{4vlR}\left(\dfrac{2}{3}R^3\right) = \dfrac{PR^2}{6vl}$

31. 0, since $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$.

32. Intuitively, the average value of the linear function $f(x) = ax + b$ on $[x_1, x_2]$ should be the value of f at the midpoint of that interval, or $X = \dfrac{x_1 + x_2}{2}$. This can be proven as follows:

$$\begin{aligned} f_{\text{ave}} &= \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} (ax + b)\,dx = \frac{1}{x_2 - x_1} \left(\frac{a}{2}x^2 + bx \right) \Bigg]_{x_1}^{x_2} \\ &= \frac{1}{x_2 - x_1} \left(\frac{ax_2^2}{2} + bx_2 - \frac{ax_1^2}{2} - bx_1 \right) = \frac{1}{x_2 - x_1} \left[\frac{a(x_2^2 - x_1^2)}{2} + b(x_2 - x_1) \right] \\ &= a\left(\frac{x_2 + x_1}{2} \right) + b = aX + b \end{aligned}$$

33. $f'_{\text{ave}} = \dfrac{1}{h} \displaystyle\int_x^{x+h} f'(x)\,dx = \dfrac{1}{h} f(x) \Bigg]_x^{x+h} = \dfrac{f(x+h) - f(x)}{h}$

34. $f_{\text{ave}} = \dfrac{1}{a-1} \displaystyle\int_1^a (n+1)x^n\,dx = \dfrac{1}{a-1} x^{n+1} \Bigg]_1^a = \dfrac{a^{n+1} - 1}{a - 1} = a^n + a^{n-1} + \cdots + a + 1$

35. If $\displaystyle\int_a^b [f(x) - f_{\text{ave}}]\,dx = 0$, then $\displaystyle\int_a^b f(x)\,dx = \int_a^b f_{\text{ave}}\,dx$. Thus, $\displaystyle\int_a^b f(x)\,dx = f_{\text{ave}}(b-a)$ and therefore $f_{\text{ave}} = \dfrac{1}{b-a} \displaystyle\int_a^b f(x)\,dx$.

36. The average of f on $[0,1]$ is 0. On $[0,2]$, it is $1/2$. On $[0,3]$, it is 1, and on $[0,4]$, it is $3/2$. The average value of f on the interval $[0,n]$ appears to be $\dfrac{1}{2}(n-1)$, which can be proven as follows:

$$\begin{aligned} f_{\text{ave}} &= \frac{1}{n-0} \int_0^n f(x)\,dx = \frac{1}{n} \left[\int_0^1 0\,dx + \int_1^2 1\,dx + \cdots + \int_{n-1}^n (n-1)\,dx \right] \\ &= \frac{1}{n}[0 + 1 + \cdots + (n-1)] = \frac{1}{n} \sum_{k=1}^{n-1} k = \frac{1}{n} \left[\frac{(n-1)n}{2} \right] = \frac{1}{2}(n-1) \end{aligned}$$

37. There is no unique answer to this question; among several possible approaches, here is probably the simplest. Suppose the circle is centered at the origin and that one of the points on the circle is $(-1,0)$. If (x,y) is any other point on the circle, then the length of the chords between $(-1,0)$ and (x,y) is the distance between the points:

$$\sqrt{(x+1)^2 + y^2} = \sqrt{x^2 + y^2 + 2x + 1} = \sqrt{2x + 2}.$$

The average chord length L_{ave} is then

$$L_{\text{ave}} = \frac{1}{1 - (-1)} \int_{-1}^1 \sqrt{2x+2}\,dx = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{(2x+2)^{3/2}}{3/2} \Bigg]_{-1}^1 = \frac{1}{6} \cdot 4^{3/2} = \frac{8}{6} = \frac{4}{3}.$$

38. (a) The surface area is $S = 2\pi \int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx$. If $f'(x) = 0$, then $S = 2\pi \int_a^b f(x)\,dx$. If $0 \le |f'(x)| < \epsilon$ for $a \le x \le b$, then

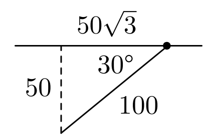
$$2\pi \int_a^b f(x)\,dx \le S \le 2\pi \int_a^b f(x)\sqrt{1+\epsilon^2}\,dx = 2\pi\sqrt{1+\epsilon^2}\int_a^b f(x)\,dx.$$

(b) At any x in $[a, b]$ the circumference of a circular cross-section is $2\pi f(x)$. The average circumference is then $\overline{C} = \dfrac{1}{b-a}\int_a^b 2\pi f(x)\,dx$. Letting $L = b - a$ be the length of the limb, we have $\overline{C}L = 2\pi \int_a^b f(x)\,dx$. Thus, from part (a), $\overline{C}L \le S \le \sqrt{1+\epsilon^2}\,\overline{C}L$.

6.8 Work

1. $W = 55 \cdot 20 = 1100$ yd-lb $= 3300$ ft-lb

2. The horizontal component of force is $50\sqrt{3}$ N.
$W = 50\sqrt{3} \cdot 8 = 400\sqrt{3}$ joules.



3. Since $10 = k\left(\frac{1}{2}\right)$, $k = 20$ and $F = 20x$. Solving $8 = 20x$ we obtain $x = \frac{2}{5}$ ft.

4. (a) Since $50 = k(0.1)$, $k = 500$ and $F = 500x$.
(b) When $x = 0.5$, $F = 250$ N.
(c) Solving $200 = 500x$, we obtain $x = 0.4$ m. The length of the spring is $0.5 + 0.4 = 0.9$ m.

5. (a) $W = \int_0^{0.2} 500x\,dx = 250x^2\Big]_0^{0.2} = 10$ joules
(b) $W = \int_{0.5}^{0.6} 500x\,dx = 250x^2\Big]_{0.5}^{0.6} = 27.5$ joules

6. (a) $W = \int_0^6 \frac{3}{2}x\,dx = \frac{3}{4}x^2\Big]_0^6 = 27$ in-lb $= \frac{27}{12}$ ft-lb $= \frac{9}{4}$ ft-lb
(b) $W = \int_0^{16} \frac{3}{2}x\,dx = \frac{3}{4}x^2\Big]_0^{16} = 192$ in-lb $= 16$ ft-lb

7. Since $10 = k\left(\frac{2}{3}\right)$, $k = 15$ and $F = 15x$.

(a) $W = \int_0^1 15x\,dx = \frac{15}{2}x^2\Big]_0^1 = \frac{15}{2}$ ft-lb
(b) $W = \int_2^3 15x\,dx = \frac{15}{2}x^2\Big]_2^3 = \frac{75}{2}$ ft-lb

8. Since $50 = k \cdot 3$, $k = \dfrac{50}{3}$ and $F = \dfrac{50}{3}x$. $W = \displaystyle\int_0^{10} \frac{50}{3}x\,dx = \frac{25}{3}x^2\Big]_0^{10} = \frac{2500}{3}$ in-lb $= \dfrac{625}{9}$ ft-lb

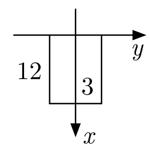
9. We use 500 km $= 0.5 \times 10^6$ m.

$$W = (6.67 \times 10^{-11})(6.0 \times 10^{24})(10^4)\left(\frac{1}{6.4 \times 10^6} - \frac{1}{6.9 \times 10^6}\right) \approx 4.531 \times 10^{10}$$
$$= 453.1 \times 10^8 \text{ joules}$$

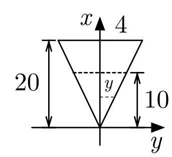
10. We use 200 km $= 0.2 \times 10^6$ m.

$$W = (6.67 \times 10^{-11})(7.3 \times 10^{22})(5 \times 10^4)\left(\frac{1}{1.7 \times 10^6} - \frac{1}{1.9 \times 10^6}\right) \approx 1.507 \times 10^{10}$$
$$= 150.7 \times 10^8 \text{ joules}$$

11. $W = \displaystyle\int_0^{12} 62.4\pi(3)^2 x\,dx = 9(62.4\pi)\frac{1}{2}x^2\Big]_0^{12} = 9(62.4\pi)(72) \approx 127{,}030.9$ ft-lb



12. $y = \dfrac{1}{5}x$

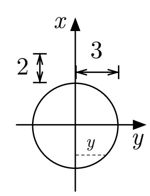


$$W = \int_0^{10} 62.4\pi y^2(20 - x)\,dx = 62.4\pi \int_0^{10} \frac{1}{25}x^2(20 - x)\,dx$$
$$= 62.4\pi \int_0^{10} \left(\frac{4}{5}x^2 - \frac{1}{25}x^3\right)dx = 62.4\pi\left(\frac{4}{15}x^3 - \frac{1}{100}x^4\right)\Big]_0^{10}$$
$$= 62.4\pi\left(\frac{500}{3}\right) \approx 32{,}672.6 \text{ ft-lb}$$

13. $W = \displaystyle\int_0^{10} 62.4\pi\left(\frac{1}{25}x^2\right)(25 - x)\,dx = 62.4\pi \int_0^{10}\left(x^2 - \frac{1}{25}x^3\right)dx$

$$= 62.4\pi\left(\frac{1}{3}x^3 - \frac{1}{100}x^4\right)\Big]_0^{10} = 62.4\pi\left(\frac{700}{3}\right) \approx 45{,}741.6 \text{ ft-lb}$$

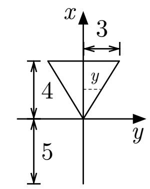
14. $x^2 + y^2 = 9; \quad y = \sqrt{9 - x^2}$



$$W = \int_{-3}^{3} 62.4(2y \cdot 12)(5 - x)\,dx = 1497.6 \int_{-3}^{3} \sqrt{9 - x^2}(5 - x)\,dx$$
$$= 1497.6\left(5\int_{-3}^{3} \sqrt{9 - x^2}\,dx - \int_{-3}^{3} x\sqrt{9 - x^2}\,dx\right)$$

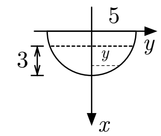
The first integral represents the area of the semicircle and is thus $\dfrac{1}{2}\pi(3)^2$. The second integral has an odd integrand and is thus 0. Therefore $W = 1497.6\left[5 \cdot \dfrac{1}{2}\pi(3)^2\right] = 33{,}696\pi$ ft-lb.

15. $y = \dfrac{3}{4}x$



$$W = \int_0^4 62.4(2y \cdot 10)(x+5)\,dx = 62.4(20)\int_0^4 \frac{3}{4}x(x+5)\,dx$$
$$= 936\int_0^4 (x^2+5x)\,dx = 936\left(\frac{1}{3}x^3 + \frac{5}{2}x^2\right)\Bigg]_0^4 = 936\left(\frac{184}{3}\right) = 57,408 \text{ ft-lb}$$

16. $x^2 + y^2 = 25; \quad y = \sqrt{25-x^2}$



$$W = \int_2^5 80(2y \cdot 25)x\,dx = 4000\int_2^5 x\sqrt{25-x^2}\,dx$$
$$= 4000\left[-\frac{1}{3}(25-x^2)^{3/2}\right]\Bigg|_2^5 = -\frac{4000}{3}(25-x^2)^{3/2}\Bigg]_2^5 = -\frac{4000}{3}(0-21^{3/2})$$
$$\approx 128,312.1 \text{ ft-lb}$$

17. The weight of the chain is $F(x) = 20(100-x)$ lb when x feet of chain have been pulled up.
$$W = \int_0^{40} 20(100-x)\,dx = 20\left(100x - \frac{1}{2}x^2\right)\Bigg]_0^{40} = 64,000 \text{ ft-lb}$$

18. The weight of the system is $F(x) = 3000 + 40(200-x)$ lb when x feet of chain have been pulled up.
$$W = \int_0^{100} [3000 + 40(200-x)]\,dx = \left[3000x + 40\left(200x - \frac{1}{2}x^2\right)\right]\Bigg|_0^{100} = 900,000 \text{ ft-lb}$$

19. (a) $W = 80 \cdot 65 = 5200$ ft-lb

(b) The weight of the system is $F(x) = 80 + \dfrac{1}{2}(65-x)$ when the bucket has been lifted x feet.
$$W = \int_0^{65}\left[80 + \frac{1}{2}(65-x)\right]dx = \left[80x + \frac{1}{2}\left(65x - \frac{1}{2}x^2\right)\right]\Bigg|_0^{65} = 6256.25 \text{ ft-lb}$$

20. The weight of the bucket after it has been lifted x feet is $62.4(20 - x/2)$ lb. The bucket will become empty when it has been lifted 40 feet.
$$W = \int_0^{40} 62.4\left(20 - \frac{x}{2}\right)dx = 62.4\left(20x - \frac{1}{4}x^2\right)\Bigg]_0^{40} \approx 24,960 \text{ ft-lb}$$

21. If x is the distance separating the electron and the nucleus, then the force is $F(x) = k/x^2$.
$$W = \int_1^4 \frac{k}{x^2}\,dx = -\frac{k}{x}\Bigg]_1^4 = -k\left(\frac{1}{4} - 1\right) = \frac{3k}{4}$$

22. (a) If x is the distance above the earth, then the weight of the system is $F(x) = 2,700,000 - 100x$.

(b) $W = \int_0^{1000} (2,700,000 - 100x)\, dx = 2,700,000x - 50x^2\Big]_0^{1000} = 2,650,000,000$ ft-lb

23. Since $p = kv^{-\gamma}$,

$$W = \int_{v_1}^{v_2} p\, dv = \int_{v_1}^{v_2} kv^{-\gamma}\, dv = \frac{k}{1-\gamma} v^{1-\gamma}\Big]_{v_1}^{v_2} = \frac{k}{1-\gamma}(v_2^{1-\gamma} - v_1^{1-\gamma})$$
$$= \frac{1}{1-\gamma}(kv_2^{-\gamma}v_2 - kv_1^{-\gamma}v_1) = \frac{1}{1-\gamma}(p_2v_2 - p_1v_1),$$

where p_1 and p_2 are the pressures corresponding to volumes v_1 and v_2, respectively.

24. Using Newton's second law $F = ma = mg$, we have

$$W = \int_{y_1}^{y_2} F(y)\, dy = \int_{y_1}^{y_2} mg\, dy = mgy]_{y_1}^{y_2} = mgy_2 - mgy_1.$$

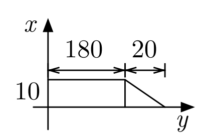
25. Since the distance moved is 0, no work is done.

26. The force is $F(x) = \begin{cases} 2x, & 0 \le x \le 1 \\ 2, & 1 \le x \le 2 \\ -x+4, & 2 \le x \le 4 \\ 0, & 4 \le x \le 5 \\ x-5, & 5 \le x \le 6 \end{cases}$.

The work is $$W = \int_0^1 2x\, dx + \int_1^2 2\, dx + \int_2^4 (-x+4)\, dx + 0 + \int_5^6 (x-5)\, dx$$
$$= x^2]_0^1 + 2x]_1^2 + \left(-\frac{1}{2}x^2 + 4x\right)\Big]_2^4 + \left(\frac{1}{2}x^2 - 5x\right)\Big]_5^6$$
$$= (1-0) + (4-2) + (8-6) + (-12+12.5) = 5.5 \text{ N-m.}$$

27. $W = 165 \cdot 1350 = 222,750$ ft-lb

28. Since the water leaks out of the bucket at a constant rate and the weight of the rope is negligible, it is reasonable to approximate that the overall lifted weight is the midpoint or average of the bucket's starting and ending weights of 200 and 180, respectively, or $(200 + 180)/2 = 190$ lb. Moving this average weight by 10 ft yields 1900 ft-lb of work done.



Without integration, it can be seen from the figure that the overall work done is the sum of the areas of a 180×10 rectangle and a right triangle of width 20 and height 10, or $(180 \times 10) + (20 \times 10)/2 = 1800 + 100 = 1900$ ft-lb.

29. Using $F = ma = mv'$ and $dx = x'(t)\, dt = v\, dt$, we have

$$W = \int_{x_1}^{x_2} F(x)\, dx = \int_{t_1}^{t_2} mv'v\, dt \qquad \boxed{u = v,\ du = v'\, dt}$$
$$= \int_{v_1}^{v_2} mu\, du = \frac{1}{2}mu^2\Big]_{v_1}^{v_2} = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2.$$

30. From the figure in the text, we see that $\sin\theta = x/30$ so that $x = 30\sin\theta$ and $dx = 30\cos\theta\,d\theta$. Also, when $x = 3$, $\sin\theta = 0.1$ and $\theta \approx 0.1$. Then

$$W = \int_0^3 mg\tan\theta\,dx = 550(9.8)\int_0^{0.1}\frac{\sin\theta}{\cos\theta}(30\cos\theta)\,d\theta = 16500(9.8)\int_0^{0.1}\sin\theta\,d\theta$$
$$= 161700(-\cos\theta)\Big]_0^{0.1} = 161700(1-\cos 0.1) \approx 807.8 \text{ N-m}.$$

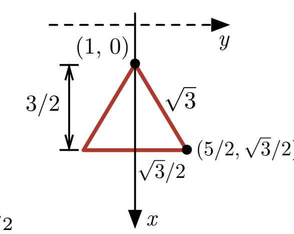
6.9 Fluid Pressure and Force

1. (a) pressure $= 62.4(20) = 1248$ lb/ft^2; $F = (1248)(25\pi) = 31200\pi$ lb
   (b) pressure $= 62.4(20) = 1248$ lb/ft^2; $F = (1248)(4\pi) = 4992\pi$ lb
   (c) pressure $= 62.4(20) = 1248$ lb/ft^2; $F = (1248)(100\pi) = 124800\pi$ lb

2. (a) pressure$_{\text{oil}} = 55$ lb/ft$^3 \cdot 96$ ft $= 5280$ lb/ft^2
   (b) pressure$_{\text{water}} = 62.4$ lb/ft$^3 \cdot 85$ ft $= 5304$ lb/ft^2
   (c) force$_{\text{oil}} = 5280 \cdot 125 \cdot 350 = 231,000,000$ lb
   (d) force$_{\text{water}} = 5304 \cdot 125 \cdot 350 = 232,050,000$ lb

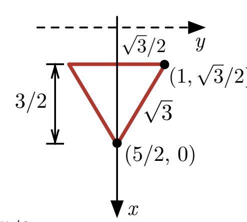
3. (a) pressure $= (62.4)(8) = 499.2$ lb/ft^2; $F = (499.2)(30)(15) = 224,640$ lb
   (b) sidewall force $= \displaystyle\int_0^8 62.4x(30)\,dx = 62.4(15)x^2\Big]_0^8 = 59,904$ lb
   end force $= \displaystyle\int_0^8 62.4x(15)\,dx = 62.4\left(\frac{15}{2}\right)x^2\Big]_0^8 = 29,952$ lb

4. The equation of the line through $(1,0)$ and $(5/2, \sqrt{3}/2)$ is $y = \dfrac{\sqrt{3}}{3}x - \dfrac{\sqrt{3}}{3}$. Using symmetry,



$$F = \int_1^{5/2} 62.4x(2y)\,dx = 124.8\int_1^{5/2} x\left(\frac{\sqrt{3}}{3}x - \frac{\sqrt{3}}{3}\right)dx$$
$$= 124.8\int_1^{5/2}\left(\frac{\sqrt{3}}{3}x^2 - \frac{\sqrt{3}}{3}x\right)dx = 124.8\left(\frac{\sqrt{3}}{9}x^3 - \frac{\sqrt{3}}{6}x^2\right)\Bigg]_1^{5/2}$$
$$\approx 124.8[1.20 - (-0.10)] = 162.12 \text{ lb}.$$

5. The equation of the line through $(5/2, 0)$ and $(1, \sqrt{3}/2)$ is $y = -\dfrac{\sqrt{3}}{3}x + \dfrac{5\sqrt{3}}{6}$. Using symmetry,

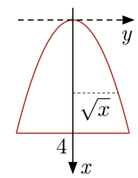


$$F = \int_1^{5/2} 62.4x(2y)\,dx = 124.8\int_1^{5/2} x\left(-\frac{\sqrt{3}}{3}x + \frac{5\sqrt{3}}{6}\right)dx$$
$$= 124.8\int_1^{5/2}\left(-\frac{\sqrt{3}}{3}x^2 + \frac{5\sqrt{3}}{6}x\right)dx = 124.8\left(-\frac{\sqrt{3}}{9}x^3 + \frac{5\sqrt{3}}{12}x^2\right)\Bigg]_1^{5/2}$$
$$\approx 124.8(1.50 - 0.53) = 121.59 \text{ lb}.$$

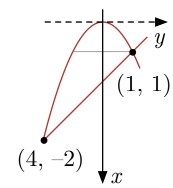
6. The equations of the lines are $y_1 = \dfrac{3}{2}x - 6$ and $y_2 = -\dfrac{1}{2}x + 2$.

$$F = \int_2^4 62.4x\left[\left(-\frac{1}{2}x + 2\right) - \left(\frac{3}{2}x - 6\right)\right] dx = 62.4\int_2^4 x(-2x+8)\,dx$$
$$= 62.4\int_2^4 (-2x^2 + 8x)\,dx = 62.4\left(-\frac{2}{3}x^3 + 4x^2\right)\Big]_2^4 = 62.4\left(\frac{64}{3} - \frac{32}{3}\right) = 665.60 \text{ lb}$$

7. $F = \displaystyle\int_0^4 50x(2\sqrt{x})\,dx = 100\int_0^4 x^{3/2}\,dx = 40x^{5/2}\Big]_0^4 = 1,280$ lb

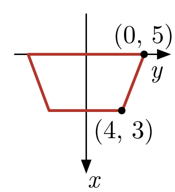


8. Solving $x = y^2$ and $y = -x + 2$ simultaneously, we find that the graphs intersect at $(4, -2)$ and $(1, 1)$. To compute the force we divide the plate into two parts with the line $x = 1$. For the upper part we use symmetry.



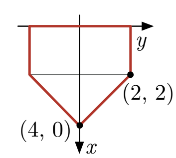
$$F = \int_0^1 62.4x(2\sqrt{x})\,dx + \int_1^4 62.4x[(-x+2) - (-\sqrt{x})]\,dx$$
$$= 124.8\int_0^1 x^{3/2}\,dx + 62.4\int_1^4 (x^{3/2} - x^2 + 2x)\,dx$$
$$= 124.8\left(\frac{2}{5}\right)x^{5/2}\Big]_0^1 + 62.4\left(\frac{2}{5}x^{5/2} - \frac{1}{3}x^3 + x^2\right)\Big]_1^4$$
$$= 49.92 + 62.4\left(\frac{112}{15} - \frac{16}{15}\right) = 49.92 + 399.36 = 499.28 \text{ lb}$$

9. The equation of the line through $(0, 5)$ and $(4, 3)$ is $y = -\dfrac{1}{2}x + 5$. Using symmetry,



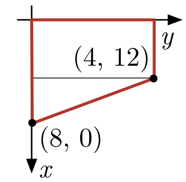
$$F = \int_0^4 62.4x\left[2\left(-\frac{1}{2}x + 5\right)\right] dx = 124.8\int_0^4 \left(-\frac{1}{2}x^2 + 5x\right) dx$$
$$= 124.8\left(-\frac{1}{6}x^3 + \frac{5}{2}x^2\right)\Big]_0^4 = 3660.8 \text{ lb}$$

10. The equation of the line through $(4, 0)$ and $(2, 2)$ is $y = -x + 4$. Using symmetry,



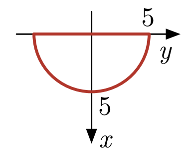
$$F = \int_0^2 62.4x(4)\,dx + \int_2^4 62.4x[2(-x+4)]\,dx$$
$$= 124.8\int_0^2 2x\,dx + 124.8\int_2^4 (-x^2 + 4x)\,dx$$
$$= 124.8x^2\Big]_0^2 + 124.8\left(-\frac{1}{3}x^3 2x^2\right)\Big]_2^4 = 499.2 + 665.6 = 1,164.8 \text{ lb}$$

11. The equation of the line through $(8, 0)$ and $(4, 12)$ is $y = -3x + 24$.



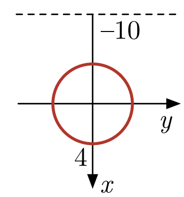
$$\begin{aligned} F &= \int_0^4 62.4x(12)\,dx + \int_4^8 62.4x(-3x+24)\,dx \\ &= 374.4\int_0^4 2x\,dx + 62.4\int_4^8 (-3x^2+24x)\,dx \\ &= 374.4x^2\Big]_0^4 + 62.4(-x^3+12x^2)\Big]_4^8 = 5,990.4 + 7,987.2 = 13,977.6 \text{ lb} \end{aligned}$$

12. $y = \sqrt{25 - x^2}$. Using symmetry,



$$\begin{aligned} F &= \int_0^5 60x\left(2\sqrt{25-x^2}\right)\,dx = 120\int_0^5 x(25-x^2)^{1/2}\,dx \\ &= -40(25-x^2)^{3/2}\Big]_0^5 = -40(0-125) = 5,000 \text{ lb} \end{aligned}$$

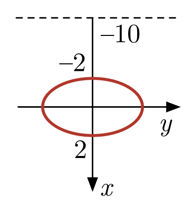
13. $y = \sqrt{16 - x^2}$. Using symmetry,



$$\begin{aligned} F &= \int_{-4}^4 62.4(x+10)\left(2\sqrt{16-x^2}\right)\,dx \\ &= 124.8\int_{-4}^4 x\sqrt{16-x^2}\,dx + 624\int_{-4}^4 2\sqrt{16-x^2}\,dx. \end{aligned}$$

The first integral has an odd integrand and is thus 0. The second integral represents the area of the circle and is thus $\pi(4)^2$. Therefore $F = 624(16\pi) = 9984\pi$ lb.

14. $y = \sqrt{9 - \frac{9}{4}x^2}$. Using symmetry,

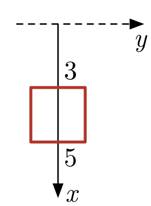


$$\begin{aligned} F &= \int_{-2}^2 \rho(x+10)\left(2\sqrt{9-\frac{9}{4}x^2}\right)\,dx \\ &= 2\rho\int_{-2}^2 x\sqrt{9-\frac{9}{4}x^2}\,dx + 10\rho\int_{-2}^2 2\sqrt{9-\frac{9}{4}x^2}\,dx. \end{aligned}$$

The first integral has an odd integrand and is thus 0. The second integral represents the area of the ellipse and is thus $\pi(2)(3)$. Therefor $F = 10\rho(6\pi) = 60\pi\rho$ lb.

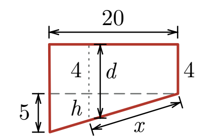
15. $F = F_{\text{top}} + F_{\text{bottom}} + 4F_{\text{side}} = 62.4(3)(4) + 62.4(5)(4) + 4\displaystyle\int_3^5 62.4x(2)\,dx$

$= 748.8 + 1248 + 249.6x^2\Big]_3^5 = 1996.8 + 3993.6 = 5990.4$ lb



16. $F_{\text{bottom}} - F_{\text{top}} = 62.4(5)(4) - 62.4(3)(4) = 499.2$ lb. Since the volume of the block is 8 ft^3, the weight of the water displaced is $62.4(8) = 499.2$ lb (illustrating Archimedes' principle).

17. The length of the bottom of the pool is $\sqrt{20^2+5^2} = 5\sqrt{17}$. Using similar triangles, $h/x = 5/5\sqrt{17}$ and $h = x/\sqrt{17}$. Then $d = 4 + x/\sqrt{17}$. Now

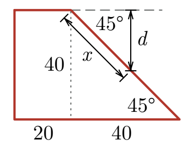


$$F = \lim_{\|P\|\to 0} \sum_{k=1}^{n} 62.4\left(4 + \frac{x_k^*}{\sqrt{17}}\right)(15)\Delta x_k = \int_0^{5\sqrt{17}} 62.4\left(4 + \frac{x}{\sqrt{17}}\right) 15\, dx$$

$$= 936 \int_0^{5\sqrt{17}} \left(4 + \frac{1}{\sqrt{17}}x\right) dx = 936\left(4x + \frac{1}{2\sqrt{17}}x^2\right)\Bigg]_0^{5\sqrt{17}}$$

$$= 936\left(\frac{65\sqrt{17}}{2}\right) = 30{,}420\sqrt{17} \text{ lb.}$$

18. The slant height of the dam is $40\sqrt{2}$ ft. Since $x = d\sqrt{2}$, we have $d = x/\sqrt{2}$. Then, using the fact that the area of the rectangular element is $100\Delta x_k$,

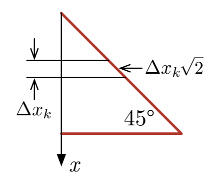


$$F = \lim_{\|P\|\to 0} \sum_{k=1}^{n} 62.4\left(\frac{x_k^*}{\sqrt{2}}\right) 100\Delta x_k = \int_0^{40\sqrt{2}} \frac{6240}{\sqrt{2}} x\, dx$$

$$= \frac{3120}{\sqrt{2}} x^2 \Bigg]_0^{40\sqrt{2}} = 4{,}992{,}000\sqrt{2} \text{ lb.}$$

19. $$F = \lim_{\|P\|\to 0} \sum_{k=1}^{n} 62.4x_k^*(100\sqrt{2}\Delta x_k) = \int_0^{40} 6240\sqrt{2}x\, dx = 3120\sqrt{2}x^2\Big]_0^{40}$$

$$= 4{,}992{,}000\sqrt{2} \text{ lb}$$



6.10 Centers of Mass and Centroids

1. $\overline{x} = \dfrac{2(4)+5(-2)}{2+5} = -\dfrac{2}{7}$

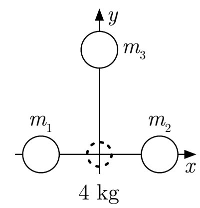
2. $\overline{x} = \dfrac{6(-1/2)+1(-3)+3(8)}{6+1+3} = \dfrac{18}{10} = \dfrac{9}{5}$

3. $\overline{x} = \dfrac{10(-5)+5(2)+8(6)+7(-3)}{10+5+8+7} = -\dfrac{13}{30}$

4. $\overline{x} = \dfrac{2(9)+(3/2)(-4)+(7/2)(-6)+(1/2)(-10)}{2+3/2+7/2+1/2} = \dfrac{-14}{15/2} = -\dfrac{28}{15}$

5. $\overline{x} = \dfrac{10(-5)+15(5)}{10+15} = \dfrac{25}{25} = 1$ m (to the right of the center)

6. The center of mass of m_1 and m_2 is $\overline{x}_{12} = \dfrac{2(-1)+2(1)}{2+2} = 0$. Thus, we consider masses m_1 and m_2 to be concentrated at the midpoint of the base of the triangle. Taking this point to be the origin, with the positive direction up through m_3, we have $\overline{y} = \dfrac{4(0)+1(2\sqrt{3})}{4+1} = \dfrac{2\sqrt{3}}{5}$.



7. $m = \displaystyle\int_0^5 (2x+1)\,dx = (x^2+x)\Big]_0^5 = 30;\quad M_0 = \int_0^5 x(2x+1)\,dx = \left(\frac{2}{3}x^3 + \frac{1}{2}x^2\right)\Big]_0^5 = \frac{575}{6}$

$\overline{x} = \dfrac{575/6}{30} = \dfrac{115}{36}$

8. $m = \displaystyle\int_0^2 (-x^2+2x)\,dx = \left(-\frac{1}{3}x^3 + x^2\right)\Big]_0^2 = \frac{4}{3}$

$M_0 = \displaystyle\int_0^2 x(-x^2+2x)\,dx = \int_0^2 (-x^3+2x^2)\,dx = \left(-\frac{1}{4}x^4 + \frac{2}{3}x^3\right)\Big]_0^2 = \frac{4}{3};\quad \overline{x} = \frac{4/3}{4/3} = 1$

9. $m = \displaystyle\int_0^1 x^{1/3}\,dx = \frac{3}{4}x^{4/3}\Big]_0^1 = \frac{3}{4}$

$M_0 = \displaystyle\int_0^1 xx^{1/3}\,dx = \int_0^1 x^{4/3}\,dx = \frac{3}{7}x^{7/3}\Big]_0^1 = \frac{3}{7};\quad \overline{x} = \frac{3/7}{3/4} = \frac{4}{7}$

10. $m = \displaystyle\int_0^1 (-x^2+1)\,dx = \left(-\frac{1}{3}x^3 + x\right)\Big]_0^1 = \frac{2}{3}$

$M_0 = \displaystyle\int_0^1 x(-x^2+1)\,dx = \int_0^1 (-x^3+x)\,dx = \left(-\frac{1}{4}x^4 + \frac{1}{2}x^2\right)\Big]_0^1 = \frac{1}{4};\quad \overline{x} = \frac{1/4}{2/3} = \frac{3}{8}$

11. $m = \displaystyle\int_0^4 |x-3|\,dx = \int_0^3 -(x-3)\,dx + \int_3^4 (x-3)\,dx = \left(-\frac{1}{2}x^2 + 3x\right)\Big]_0^3 + \left(\frac{1}{2}x^2 - 3x\right)\Big]_3^4$

$= \dfrac{9}{2} + \left(-4 + \dfrac{9}{2}\right) = 5$

$M_0 = \displaystyle\int_0^4 x|x-3|\,dx = \int_0^3 -(x^2-3x)\,dx + \int_3^4 (x^2-3x)\,dx$

$= \displaystyle\left(-\frac{1}{3}x^3 + \frac{3}{2}x^2\right)\Big]_0^3 + \left(\frac{1}{3}x^3 - \frac{3}{2}x^2\right)\Big]_3^4 = \left(-9 + \frac{27}{2}\right) + \left(\frac{64}{3} - 24\right) - \left(9 - \frac{27}{2}\right) = \frac{19}{3}$

$\overline{x} = \dfrac{19/3}{5} = \dfrac{19}{15}$

12. $$m = \int_0^3 (1 + |x-1|)\,dx = \int_0^1 [1-(x-1)]\,dx + \int_1^3 [1+(x-1)]\,dx$$
$$= \int_0^1 (2-x)\,dx + \int_1^3 x\,dx = \left(2x - \frac{1}{2}x^2\right)\Big]_0^1 + \frac{1}{2}x^2\Big]_1^3 = \frac{3}{2} + \left(\frac{9}{2} - \frac{1}{2}\right) = \frac{11}{2}$$
$$M_0 = \int_0^3 x(1+|x-1|)\,dx = \int_0^1 x[1-(x-1)]\,dx + \int_1^3 x[1+(x-1)]\,dx$$
$$= \int_0^1 (2x - x^2)\,dx + \int_1^3 x^2\,dx = \left(x^2 - \frac{1}{3}x^3\right)\Big]_0^1 + \frac{1}{3}x^3\Big]_1^3 = \frac{2}{3} + \left(9 - \frac{1}{3}\right) = \frac{28}{3}$$
$$\overline{x} = \frac{28/3}{11/2} = \frac{56}{33}$$

13. $$m = \int_0^2 \rho(x)\,dx = \int_0^1 x^2\,dx + \int_1^2 (2-x)\,dx = \frac{1}{3}x^3\Big]_0^1 + \left(2x - \frac{1}{2}x^2\right)\Big]_1^2$$
$$= \frac{1}{3} + \left(2 - \frac{3}{2}\right) = \frac{5}{6}$$
$$M_0 = \int_0^2 x\rho(x)\,dx = \int_0^1 x^3\,dx + \int_1^2 (2x - x^2)\,dx = \frac{1}{4}x^4\Big]_0^1 + \left(x^2 - \frac{1}{3}x^3\right)\Big]_1^2$$
$$= \frac{1}{4} + \left(\frac{4}{3} - \frac{2}{3}\right) = \frac{11}{12}$$
$$\overline{x} = \frac{11/12}{5/6} = \frac{11}{10}$$

14. $$m = \int_0^3 \rho(x)\,dx = \int_0^2 x\,dx + \int_2^3 2\,dx = \frac{1}{2}x^2\Big]_0^2 + 2x\Big]_2^3 = 2 + 2 = 4$$
$$M_0 = \int_0^3 x\rho(x)\,dx = \int_0^2 x^2\,dx + \int_2^3 2x\,dx = \frac{1}{3}x^3\Big]_0^2 + x^2\Big]_2^3 = \frac{8}{3} + 5 = \frac{23}{3}; \quad \overline{x} = \frac{23/3}{4} = \frac{23}{12}$$

15. Since $\rho(x) = kx^2$ and $12.5 = \rho(5) = 25k$, $k = \frac{1}{2}$ and $\rho(x) = \frac{1}{2}x^2$.
$$m = \int_0^{10} \frac{1}{2}x^2\,dx = \frac{1}{6}x^3\Big]_0^{10} = \frac{500}{3}; \quad M_0 = \int_0^{10} \frac{1}{2}x^3\,dx = \frac{1}{8}x^4\Big]_0^{10} = 1250$$
$$\overline{x} = \frac{1250}{500/3} = 7.5 \text{ ft (from the left end)}$$

16. Place the origin at the right end of the rod with the positive direction to the left. Then $\rho(x) = kx$ and $6 = \int_0^3 \rho(x)\,dx = \int_0^3 kx\,dx = \frac{1}{2}kx^2\Big]_0^3 = \frac{9}{2}k$. Solving for k we obtain $k = \frac{4}{3}$. Then $\rho(x) = \frac{4}{3}x$ and $\rho(3/2) = 2$ kg/m.

17. $m = 3 + 4 = 7$; $\quad \overline{x} = \dfrac{3(-2) + 4(1)}{7} = -\dfrac{2}{7}$; $\quad \overline{y} = \dfrac{3(3) + 4(2)}{7} = \dfrac{17}{7}$

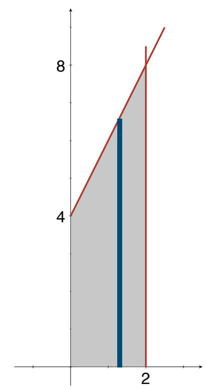
18. $m = 1 + 3 + 2 = 6; \quad \overline{x} = \dfrac{1(-4) + 3(2) + 2(5)}{6} = 2; \quad \overline{y} = \dfrac{1(1) + 3(2) + 2(-2)}{6} = \dfrac{1}{2}$

19. $m = 4 + 8 + 10 = 22; \quad \overline{x} = \dfrac{4(1) + 8(-5) + 10(7)}{22} = \dfrac{17}{11}; \quad \overline{y} = \dfrac{4(1) + 8(2) + 10(-6)}{22} = -\dfrac{20}{11}$

20. $m = 1 + 1/2 + 4 + 5/2 = 8; \quad \overline{x} = \dfrac{1(9) + (1/2)(-4) + 4(3/2) + (5/2)(-2)}{8} = 1$

$$\overline{y} = \frac{1(3) + (1/2)(-6) + 4(-1) + (5/2)(10)}{8} = \frac{21}{8}$$

21. $A = \displaystyle\int_0^2 (2x+4)\,dx = \left(x^2 + 4x\right)\Big]_0^2 = 12$



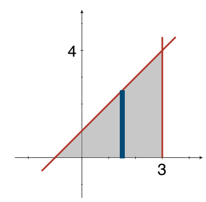
$$M_y = \int_0^2 x(2x+4)\,dx = \int_0^2 (2x^2 + 4x)\,dx = \left(\frac{2}{3}x^3 + 2x^2\right)\Big]_0^2 = \frac{40}{3}$$

$$M_x = \frac{1}{2}\int_0^2 (2x+4)^2\,dx = \frac{1}{2}\int_0^2 (4x^2 + 16x + 16)\,dx$$

$$= 2\left(\frac{1}{3}x^3 + 2x^2 + 4x\right)\Big]_0^2 = \frac{112}{3}$$

$$\overline{x} = \frac{40/3}{12} = \frac{10}{9}; \quad \overline{y} = \frac{112/3}{12} = \frac{28}{9}$$

22. $A = \displaystyle\int_{-1}^3 (x+1)\,dx = \left(\frac{1}{2}x^2 + x\right)\Big]_{-1}^3 = \frac{15}{2} - \left(-\frac{1}{2}\right) = 8$



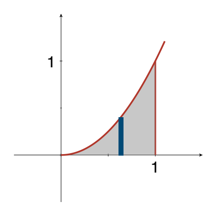
$$M_y = \int_{-1}^3 x(x+1)\,dx = \int_{-1}^3 (x^2 + x)\,dx = \left(\frac{1}{3}x^3 + \frac{1}{2}x^2\right)\Big]_{-1}^3$$

$$= \frac{27}{2} - \frac{1}{6} = \frac{40}{3}$$

$$M_x = \frac{1}{2}\int_{-1}^3 (x+1)^2\,dx = \frac{1}{6}(x+1)^3\Big]_{-1}^3 = \frac{32}{3} - 0 = \frac{32}{3}$$

$$\overline{x} = \frac{40/3}{8} = \frac{5}{3}; \quad \overline{y} = \frac{32/3}{8} = \frac{4}{3}$$

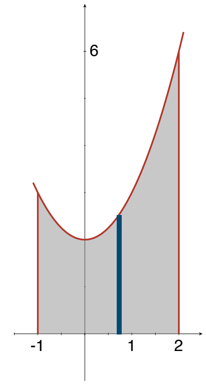
23. $A = \displaystyle\int_0^1 x^2\,dx = \frac{1}{3}x^3\Big]_0^1 = \frac{1}{3}$



$$M_y = \int_0^1 x^3\,dx = \int_0^2 (2x^2 + 4x)\,dx = \frac{1}{4}x^4\Big]_0^1 = \frac{1}{4}$$

$$M_x = \frac{1}{2}\int_0^1 x^4\,dx = \frac{1}{10}x^5\Big]_0^1 = \frac{1}{10}; \quad \overline{x} = \frac{1/4}{1/3} = \frac{3}{4}; \quad \overline{y} = \frac{1/10}{1/3} = \frac{3}{10}$$

24. $A = \int_{-1}^{2} (x^2+2)\,dx = \left(\frac{1}{3}x^3 + 2x\right)\Big]_{-1}^{2} = \frac{20}{3} - \left(-\frac{7}{3}\right) = 9$



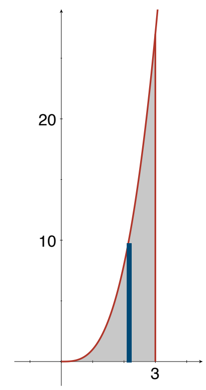
$$M_y = \int_{-1}^{2} x(x^2+2)\,dx = \frac{1}{4}(x^2+2)^2\Big]_{-1}^{2} = 9 - \frac{9}{4} = \frac{27}{4}$$

$$M_x = \frac{1}{2}\int_{-1}^{2} (x^2+2)^2\,dx = \frac{1}{2}\int_{-1}^{2} (x^4+4x^2+4)\,dx$$

$$= \frac{1}{2}\left(\frac{1}{5}x^5 + \frac{4}{3}x^3 + 4x\right)\Big]_{-1}^{2} = \frac{1}{2}\left[\frac{376}{15} - \left(-\frac{83}{15}\right)\right] = \frac{153}{10}$$

$$\overline{x} = \frac{27/4}{9} = \frac{3}{4};\quad \overline{y} = \frac{153/10}{9} = \frac{17}{10}$$

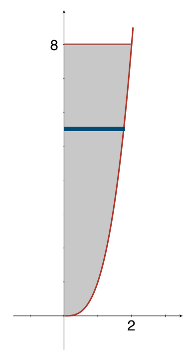
25. $A = \int_{0}^{3} x^3\,dx = \frac{1}{4}x^4\Big]_{0}^{3} = \frac{81}{4}$



$$M_y = \int_{0}^{3} x^4\,dx = \frac{1}{5}x^5\Big]_{0}^{3} = \frac{243}{5};\quad M_x = \frac{1}{2}\int_{0}^{3} x^6\,dx = \frac{1}{14}x^7\Big]_{0}^{3} = \frac{2187}{14}$$

$$\overline{x} = \frac{243/5}{81/4} = \frac{12}{5};\quad \overline{y} = \frac{2187/14}{81/4} = \frac{54}{7}$$

26. $x = y^{1/3};\quad A = \int_{0}^{8} y^{1/3}\,dy = \frac{3}{4}y^{4/3}\Big]_{0}^{8} = 12$

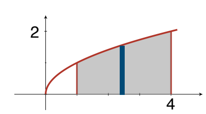


$$M_x = \int_{0}^{8} y^{4/3}\,dy = \frac{3}{7}y^{7/3}\Big]_{0}^{8} = \frac{384}{7}$$

$$M_y = \frac{1}{2}\int_{0}^{8} y^{2/3}\,dy = \frac{3}{10}y^{5/3}\Big]_{0}^{8} = \frac{48}{5}$$

$$\overline{x} = \frac{48/5}{12} = \frac{4}{5};\quad \overline{y} = \frac{384/7}{12} = \frac{32}{7}$$

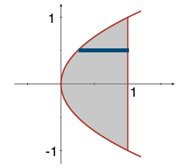
27. $A = \int_{1}^{4} x^{1/2}\,dx = \frac{2}{3}x^{3/2}\Big]_{1}^{4} = \frac{16}{3} - \frac{2}{3} = \frac{14}{3}$



$$M_y = \int_{1}^{4} x^{3/2}\,dx = \frac{2}{5}x^{5/2}\Big]_{1}^{4} = \frac{64}{5} - \frac{2}{5} = \frac{62}{5}$$

$$M_x = \frac{1}{2}\int_{1}^{4} x\,dx = \frac{1}{4}x^2\Big]_{1}^{4} = 4 - \frac{1}{4} = \frac{15}{4};\quad \overline{x} = \frac{62/5}{14/3} = \frac{93}{35};\quad \overline{y} = \frac{15/4}{14/3} = \frac{45}{56}$$

28. $A = \int_{-1}^{1} (1 - y^2)\, dy = \left(y - \frac{1}{3}y^3\right)\Big]_{-1}^{1} = \frac{2}{3} - \left(-\frac{2}{3}\right) = \frac{4}{3}$

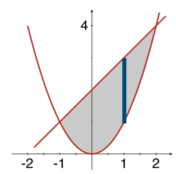


$$M_x = \int_{-1}^{1} y(1 - y^2)\, dy = \int_{-1}^{1} (y - y^3)\, dy = \left(\frac{1}{2}y^2 - \frac{1}{4}y^4\right)\Big]_{-1}^{1} = \frac{1}{4} - \frac{1}{4} = 0$$

$$M_y = \frac{1}{2}\int_{-1}^{1} [1^2 - (y^2)^2]\, dy = \frac{1}{2}\int_{-1}^{1} (1 - y^4)\, dy = \frac{1}{2}\left(y - \frac{1}{5}y^5\right)\Big]_{-1}^{1}$$

$$= \frac{1}{2}\left[\frac{4}{5} - \left(-\frac{4}{5}\right)\right] = \frac{4}{5}; \qquad \overline{x} = \frac{4/5}{4/3} = \frac{3}{5}; \quad \overline{y} = \frac{0}{4/3} = 0$$

29. $A = \int_{-1}^{2} [(x + 2) - x^2]\, dx = \left(\frac{1}{2}x^2 + 2x - \frac{1}{3}x^3\right)\Big]_{-1}^{2} = \frac{10}{3} - \left(-\frac{7}{6}\right) = \frac{9}{2}$



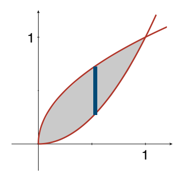
$$M_y = \int_{-1}^{2} x(x + 2 - x^2)\, dx = \int_{-1}^{2} (x^2 + 2x - x^3)\, dx$$

$$= \left(\frac{1}{3}x^3 + x^2 - \frac{1}{4}x^4\right)\Big]_{-1}^{2} = \frac{8}{3} - \frac{5}{12} = \frac{9}{4}$$

$$M_x = \frac{1}{2}\int_{-1}^{2} [(x + 2)^2 - x^4]\, dx = \frac{1}{2}\left[\frac{1}{3}(x + 2)^3 - \frac{1}{5}x^5\right]\Big|_{-1}^{2} = \frac{1}{2}\left(\frac{224}{15} - \frac{8}{15}\right) = \frac{36}{5}$$

$$\overline{x} = \frac{9/4}{9/2} = \frac{1}{2}; \quad \overline{y} = \frac{36/5}{9/2} = \frac{8}{5}$$

30. $A = \int_{0}^{1} (x^{1/2} - x^2)\, dx = \left(\frac{2}{3}x^{3/2} - \frac{1}{3}x^3\right)\Big]_{0}^{1} = \frac{1}{3}$

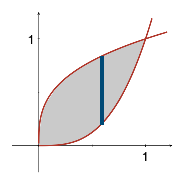


$$M_y = \int_{0}^{1} x(x^{1/2} - x^2)\, dx = \int_{0}^{1} (x^{3/2} - x^3)\, dx = \left(\frac{2}{5}x^{5/2} - \frac{1}{4}x^4\right)\Big]_{0}^{1} = \frac{3}{20}$$

$$M_x = \frac{1}{2}\int_{0}^{1} [(x^{1/2})^2 - (x^2)^2]\, dx = \frac{1}{2}\int_{0}^{1} (x - x^4)\, dx = \frac{1}{2}\left(\frac{1}{2}x^2 - \frac{1}{5}x^5\right)\Big]_{0}^{1} = \frac{3}{20}$$

$$\overline{x} = \frac{3/20}{1/3} = \frac{9}{20}; \quad \overline{y} = \frac{3/20}{1/3} = \frac{9}{20}$$

31. $A = \int_{0}^{1} (x^{1/3} - x^3)\, dx = \left(\frac{3}{4}x^{4/3} - \frac{1}{4}x^4\right)\Big]_{0}^{1} = \frac{1}{2}$

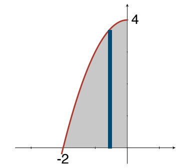


$$M_y = \int_{0}^{1} x(x^{1/3} - x^3)\, dx = \int_{0}^{1} (x^{4/3} - x^4)\, dx = \left(\frac{3}{7}x^{7/3} - \frac{1}{5}x^5\right)\Big]_{0}^{1} = \frac{8}{35}$$

$$M_x = \frac{1}{2}\int_{0}^{1} [(x^{1/3})^2 - (x^3)^2]\, dx = \frac{1}{2}\int_{0}^{1} (x^{2/3} - x^6)\, dx = \frac{1}{2}\left(\frac{3}{5}x^{5/3} - \frac{1}{7}x^7\right)\Big]_{0}^{1} = \frac{8}{35}$$

$$\overline{x} = \frac{8/35}{1/2} = \frac{16}{35}; \quad \overline{y} = \frac{8/35}{1/2} = \frac{16}{35}$$

32. $A = \int_{-2}^{0} (4 - x^2)\, dx = \left(4x - \frac{1}{3}x^3\right)\Big]_{-2}^{0} = \frac{16}{3}$

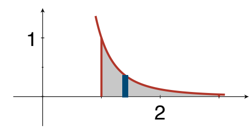


$M_y = \int_{-2}^{0} x(4 - x^2)\, dx = \int_{-2}^{0} (4x - x^3)\, dx = \left(2x^2 - \frac{1}{4}x^4\right)\Big]_{-2}^{0} = -4$

$M_x = \frac{1}{2}\int_{-2}^{0} (4 - x^2)^2\, dx = \frac{1}{2}\int_{-2}^{0} (16 - 8x^2 + x^4)\, dx = \frac{1}{2}\left(16x - \frac{8}{3}x^3 + \frac{1}{5}x^5\right)\Big]_{-2}^{0} = \frac{128}{15}$

$\overline{x} = \frac{-4}{16/3} = -\frac{3}{4}; \quad \overline{y} = \frac{128/15}{16/3} = \frac{8}{5}$

33. $y = x^{-3}; \quad A = \int_{1}^{3} x^{-3}\, dx = -\frac{1}{2x^2}\Big]_{1}^{3} = -\left(\frac{1}{18} - \frac{1}{2}\right) = \frac{4}{9}$

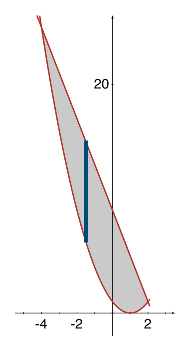


$M_y = \int_{1}^{3} x^{-2}\, dx = -\frac{1}{x}\Big]_{1}^{3} = -\left(\frac{1}{3} - 1\right) = \frac{2}{3}$

$M_x = \frac{1}{2}\int_{1}^{3} x^{-6}\, dx = -\frac{1}{10x^5}\Big]_{1}^{3} = -\left(\frac{1}{2430} - \frac{1}{10}\right) = \frac{121}{1215}$

$\overline{x} = \frac{2/3}{4/9} = \frac{3}{2}; \quad \overline{y} = \frac{121/1215}{4/9} = \frac{121}{540}$

34. $A = \int_{-4}^{2} [(-4x + 9) - (x^2 - 2x + 1)]\, dx = \int_{-4}^{2} (8 - 2x - x^2)\, dx$



$= \left(8x - x^2 - \frac{1}{3}x^3\right)\Big]_{-4}^{2} = \frac{28}{3} - \left(-\frac{80}{3}\right) = 36$

$M_y = \int_{-4}^{2} x(8 - 2x - x^2)\, dx = \int_{-4}^{2} (8x - 2x^2 - x^3)\, dx$

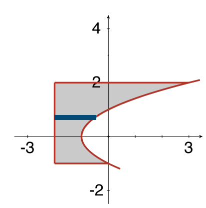
$= \left(4x^2 - \frac{2}{3}x^3 - \frac{1}{4}x^4\right)\Big]_{-4}^{2} = \frac{20}{3} - \frac{128}{3} = -36$

$M_x = \frac{1}{2}\int_{-4}^{2} [(-4x + 9)^2 - (x^2 - 2x + 1)^2]\, dx = \frac{1}{2}\int_{-4}^{2} [16x^2 - 72x + 81 - (x - 1)^4]\, dx$

$= \frac{1}{2}\left[\frac{16}{3}x^3 - 36x^2 + 81x - \frac{1}{5}(x - 1)^5\right]\Big|_{-4}^{2} = \frac{1}{2}\left[\frac{907}{15} - \left(-\frac{1849}{3}\right)\right] = \frac{1692}{5}$

$\overline{x} = \frac{-36}{36} = -1; \quad \overline{y} = \frac{1692/5}{36} = \frac{47}{5}$

35. $A = \int_{-1}^{2} (y^2 - 1 + 2)\, dy = \left(\frac{1}{3}y^3 + y\right)\Big]_{-1}^{2} = \frac{14}{3} - \left(-\frac{4}{3}\right) = 6$

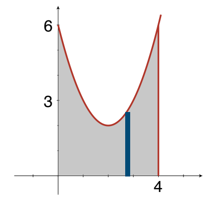


$M_x = \int_{-1}^{2} y(y^2 + 1)\, dy = \int_{-1}^{2} (y^3 + y)\, dy = \left(\frac{1}{4}y^4 + \frac{1}{2}y^2\right)\Big]_{-1}^{2}$

$= 6 - \frac{3}{4} = \frac{21}{4}$

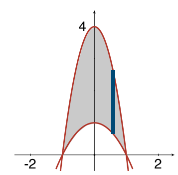
$$M_y = \frac{1}{2}\int_{-1}^{2}[(y^2-1)^2-(-2)^2]\,dy = \frac{1}{2}\int_{-1}^{2}(y^4-2y^2-3)\,dy$$
$$= \frac{1}{2}\left(\frac{1}{5}y^5-\frac{2}{3}y^3-3y\right)\bigg]_{-1}^{2} = \frac{1}{2}\left(-\frac{74}{15}-\frac{52}{15}\right) = -\frac{21}{5}$$
$$\overline{x} = \frac{-21/5}{6} = -\frac{7}{10};\quad \overline{y} = \frac{21/4}{6} = \frac{7}{8}$$

36. $A = \int_0^4 (x^2-4x+6)\,dx = \left(\frac{1}{3}x^3-2x^2+6x\right)\bigg]_0^4 = \frac{40}{3}$



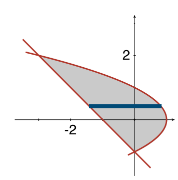
$$M_y = \int_0^4 x(x^2-4x+6)\,dx = \int_0^4 (x^3-4x^2+6x)\,dx$$
$$= \left(\frac{1}{4}x^4-\frac{4}{3}x^3+3x^2\right)\bigg]_0^4 = \frac{80}{3}$$
$$M_x = \frac{1}{2}\int_0^4 (x^2-4x+6)^2\,dx = \frac{1}{2}\int_0^4 [(x-2)^2+2]^2\,dx$$
$$= \frac{1}{2}\int_0^4 [(x-2)^4+4(x-2)^2+4]\,dx = \frac{1}{2}\left[\frac{1}{5}(x-2)^5+\frac{4}{3}(x-2)^3+4x\right]\bigg|_0^4$$
$$= \frac{1}{2}\left[\frac{496}{15}-\left(-\frac{256}{15}\right)\right] = \frac{376}{15}$$
$$\overline{x} = \frac{80/3}{40/3} = 2;\quad \overline{y} = \frac{376/15}{40/3} = \frac{47}{25}$$

37. $A = \int_{-1}^{1}[(4-4x^2)-(1-x^2)]\,dx = \int_{-1}^{1}(3-3x^2)\,dx = (3x-x^3)\big]_{-1}^{1} = 4$



$$M_y = \int_{-1}^{1} x(3-3x^2)\,dx = \int_{-1}^{1}(3x-3x^3)\,dx$$
$$= \left(\frac{3}{2}x^2-\frac{3}{4}x^4\right)\bigg]_{-1}^{1} = \frac{3}{4}-\frac{3}{4} = 0$$
$$M_x = \frac{1}{2}\int_{-1}^{1}[(4-4x^2)^2-(1-x^2)^2]\,dx = \frac{1}{2}\int_{-1}^{1}[16(1-x^2)^2-(1-x^2)^2]\,dx$$
$$= \frac{15}{2}\int_{-1}^{1}(1-2x^2+x^4)\,dx = \frac{15}{2}\left(x-\frac{2}{3}x^3+\frac{1}{5}x^5\right)\bigg]_{-1}^{1} = \frac{15}{2}\left[\frac{8}{15}-\left(-\frac{8}{15}\right)\right] = 8$$
$$\overline{x} = \frac{0}{4} = 0;\quad \overline{y} = \frac{8}{4} = 2$$

38. $x = 1-y^2;\quad x = -(1+y)$



$$A = \int_{-1}^{2}[(1-y^2)+(1+y)]\,dy = \int_{-1}^{2}(2+y-y^2)\,dy$$
$$= \left(2y+\frac{1}{2}y^2-\frac{1}{3}y^3\right)\bigg]_{-1}^{2} = \frac{10}{3}-\left(-\frac{7}{6}\right) = \frac{9}{2}$$

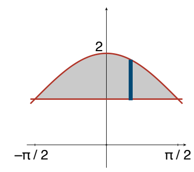
$$M_x = \int_{-1}^{2} y(2+y-y^2)\,dy = \int_{-1}^{2} (2y+y^2-y^3)\,dy = \left(y^2+\frac{1}{3}y^3-\frac{1}{4}y^4\right)\Big]_{-1}^{2} = \frac{8}{3}-\frac{5}{12} = \frac{9}{4}$$

$$M_y = \frac{1}{2}\int_{-1}^{2} [(1-y^2)^2-(1+y)^2]\,dy = \frac{1}{2}\int_{-1}^{2} (y^4-3y^2-2y)\,dy = \frac{1}{2}\left(\frac{1}{5}y^5-y^3-y^2\right)\Big]_{-1}^{2}$$

$$= \frac{1}{2}\left[-\frac{28}{5}-\left(-\frac{1}{5}\right)\right] = -\frac{27}{10}$$

$$\overline{x} = \frac{-27/10}{9/2} = -\frac{3}{5}; \quad \overline{y} = \frac{9/4}{9/2} = \frac{1}{2}$$

39. By symmetry, $\overline{x} = 0$.



$$A = 2\int_{0}^{\pi/2} (1+\cos x - 1)\,dx = 2\int_{0}^{\pi/2} \cos x\,dx = 2\sin x]_0^{\pi/2} = 2$$

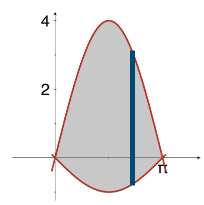
$$M_x = \frac{1}{2}\int_{-\pi/2}^{\pi/2} [(1+\cos x)^2-1^2]\,dx = \frac{1}{2}\int_{-\pi/2}^{\pi/2} (\cos^2 x + 2\cos x)\,dx$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left(\frac{1}{2}\cos 2x + \frac{1}{2} + 2\cos x\right) dx = \frac{1}{2}\left(\frac{1}{4}\sin 2x + \frac{1}{2}x + 2\sin x\right)\Big]_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{4}+2\right)-\left(-\frac{\pi}{4}-2\right)\right] = \frac{\pi+8}{4}$$

$$\overline{y} = \frac{(\pi+8)/4}{2} = \frac{\pi+8}{8}$$

40. By symmetry, $\overline{x} = \pi/2$.



$$A = \int_{0}^{\pi} (4\sin x + \sin x)\,dx = \int_{0}^{\pi} 5\sin x\,dx = -5\cos x]_0^{\pi}$$

$$= -5(-1-1) = 10$$

$$M_x = \frac{1}{2}\int_{0}^{\pi} (16\sin^2 x - \sin^2 x)\,dx = \frac{15}{2}\int_{0}^{\pi} \sin^2 x\,dx$$

$$= \frac{15}{2}\int_{0}^{\pi} \left(\frac{1}{2}-\frac{1}{2}\cos 2x\right) dx = \frac{15}{2}\left(\frac{1}{2}x - \frac{1}{4}\sin 2x\right)\Big]_0^{\pi} = \frac{15\pi}{4}$$

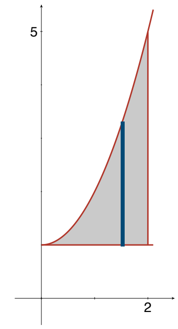
$$\overline{y} = \frac{15\pi/4}{10} = \frac{3\pi}{8}$$

41. (a) The circumference of a circle with radius $\overline{y}$ is $2\pi\overline{y}$. Thus, $V = 2\pi\overline{y}A$.

(b) By the same reasoning, $V = 2\pi\overline{x}A$ when the region R is revolved around the y-axis.

42. $A = \int_0^2 (x^2+1-1)\,dx = \frac{1}{3}x^3\Big]_0^2 = \frac{8}{3}$

$$M_x = \frac{1}{2}\int_0^2 [(x^2+1)^2 - 1^2]\,dx = \frac{1}{2}\int_0^2 (x^4+2x^2)\,dx$$
$$= \frac{1}{2}\left(\frac{1}{5}x^5 + \frac{2}{3}x^3\right)\Big]_0^2 = \frac{88}{15}; \qquad \overline{y} = \frac{88/15}{8/3} = \frac{11}{5}$$

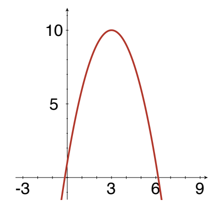


By the theorem of Pappus, $V = 2\pi\left(\frac{11}{5}\right)\left(\frac{8}{3}\right) = \frac{176\pi}{15}$. To verify this result we compute the volume using the washer method.

$$V = \pi\int_0^2 [(x^2+1)^2 - 1^2]\,dx = \pi\int_0^2 (x^4+2x)\,dx = \pi\left(\frac{1}{5}x^5 + \frac{2}{3}x^3\right)\Big]_0^2 = \frac{176\pi}{15}.$$

43. We identify $A = \pi a^2$. The centroid of the region R is b units from L, so $V = 2\pi b(\pi a^2) = 2\pi^2 a^2 b$.

44. Since the graph is symmetric around $x = 3$, we would expect the center of mass to occur at the center of the rod.



$$m = \int_0^6 (6x - x^2 + 1)\,dx = \left(3x^2 - \frac{1}{3}x^3 + x\right)\Big]_0^6 = 42$$
$$M_0 = \int_0^6 (6x^2 - x^3 + x)\,dx = \left(2x^3 - \frac{1}{4}x^4 + \frac{1}{2}x^2\right)\Big]_0^6 = 126$$
$$\overline{x} = \frac{126}{42} = 3$$

45. Thinking geometrically, the centroid of a triangle would appear to be the intersection of its three medians (a median is a line segment from one of the triangle's vertices to the midpoint of the opposing side). Doing some research on the centroid of a triangle shows this to be true, and in fact this intersection is the mean of the coordinates of the triangle's vertices.

46. Decomposing the region R into three 1×1 squares whose centers are $(1/2, 3/2)$, $(3/2, 3/2)$, and $(3/2, 1/2)$, we get the centroid of $R = ([1/2 + 3/2 + 3/2]/3, [3/2 + 3/2 + 1/2]/3) = ([7/2]/3, [7/2]/3) = (7/6, 7/6)$.

Chapter 6 in Review

A. True/False

1. False. $\int_a^b f(x)\,dx$ may be positive even though a portion of the graph of f lies below the x-axis.

2. False. On $[0, 3]$ a portion of the graph of $f(x) = x - 1$ lies below the x-axis.

3. True

4. True
5. True
6. True
7. True
8. True
9. True
10. False. Liquid pressure depends on the density and depth of the liquid, not the area covered.
11. False. The distance moved is given by $\int_{t_1}^{t_2} |v(t)|\, dt$.
12. False. The formula for $s(t)$ depends only on acceleration, initial velocity, and initial position. The mass and shape of the object are immaterial.

B. Fill in the Blanks

1. Newton-meter of joule
2. 400 mi-lb or $2,112,000$ ft-lb
3. $(100\cos 60^\circ)(50) = 2,500$ ft-lb
4. $80 = k(1/2)$, $k = 160$, $F = 160x$. When $F = 100$, $x = 5/8 = 0.625$ and the spring will measure 1.625 m.
5. 6
6. 62.4
7. smooth
8. $v_{\text{impact}} = -gT$ and $v_{\text{ave}} = \dfrac{1}{T-0}\displaystyle\int_0^T (-gt)\, dt = \dfrac{1}{T}\left(-\dfrac{1}{2}gT^2\right) = \dfrac{-gT}{2}$. Thus, $v_{\text{ave}} = \dfrac{v_{\text{impact}}}{2}$.

C. Exercises

1. $-\int_0^a f(x)\, dx$
2. $\int_a^b f(x)\, dx - \int_b^c f(x)\, dx + \int_c^d f(x)\, dx$
3. The line in Figure 6.R.3 is $y = \dfrac{f(a)}{a}x$. Thus, the integral is $\displaystyle\int_0^a \left[f(x) - \frac{f(a)}{a}x\right] dx$.
4. $\int_a^b [c - f(x)]\, dx$

5. $-\int_a^b 2f(x)\,dx + \int_b^c 2f(x)\,dx$

6. $\int_a^b g(x)\,dx + \int_b^c f(x)\,dx$

7. $\int_b^c [a - f(y)]\,dy + \int_c^d [f(y) - a]\,dy$

8. $\int_a^b [f(x) - g(x)]\,dx + \int_b^c [g(x) - f(x)]\,dx + \int_c^d [f(x) - g(x)]\,dx$

9. $A = -\int_a^0 \frac{x}{2}\,dx + \int_0^{2b} \left(b - \frac{x}{2}\right)\,dx = -\frac{1}{4}x^2\Big]_a^0 + \left(bx - \frac{1}{4}x^2\right)\Big]_0^{2b} = \frac{a^2}{4} + 2b^2 - b^2 = \frac{a^2}{4} + b^2$

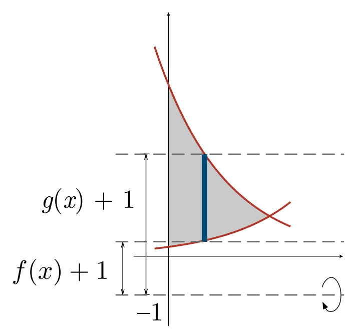
10. $A = \int_{a^2}^{b^2} \sqrt{y}\,dy = \frac{2}{3}y^{3/2}\Big]_{a^2}^{b^2} = \frac{2}{3}(b^3 - a^3)$

11. $\overline{x} = \dfrac{\int_0^2 x[g(x) - f(x)]\,dx}{\int_0^2 [g(x) - f(x)]\,dx}$; $\quad \overline{y} = \dfrac{\frac{1}{2}\int_0^2 \left\{[g(x)]^2 - [f(x)]^2\right\}\,dx}{\int_0^2 [g(x) - f(x)]\,dx}$

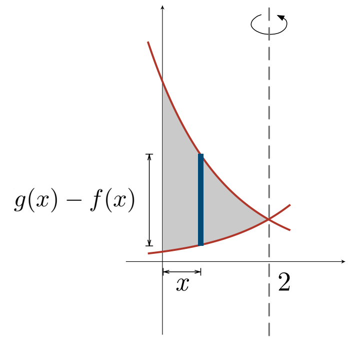
12. $V = \pi\int_0^2 \left\{[g(x)]^2 - [f(x)]^2\right\}\,dx$

13. $V = 2\pi\int_0^2 x[g(x) - f(x)]\,dx$

14. $V = \pi\int_0^2 \left\{[g(x) + 1]^2 - [f(x) + 1]^2\right\}\,dx$

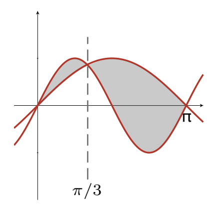


15. $V = 2\pi \int_0^2 (2-x)[g(x) - f(x)]\,dx$



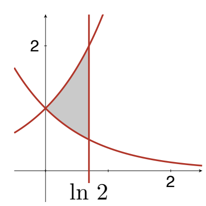
16. $V = \int_0^2 [g(x) - f(x)]^2\,dx$

17. Solving $\sin x = \sin 2x$, we get $\sin x = 2\sin x \cos x$, $\cos x = \dfrac{1}{2}$, and $x = \dfrac{\pi}{3}$ on $(0, \pi)$. Thus,



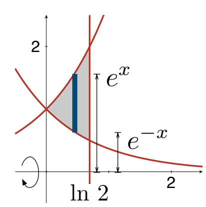
$$\begin{aligned} A &= \int_0^{\pi/3} (\sin 2x - \sin x)\,dx + \int_{\pi/3}^{\pi} (\sin x - \sin 2x)\,dx \\ &= \left(-\frac{1}{2}\cos 2x + \cos x\right)\Big]_0^{\pi/3} + \left(-\cos x + \frac{1}{2}\cos 2x\right)\Big]_{\pi/3}^{\pi} \\ &= \frac{1}{4} + \frac{1}{2} - \left(-\frac{1}{2}\right) - 1 + 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} = \frac{5}{2} \end{aligned}$$

18. (a) $A = \int_0^{\ln 2} (e^x - e^{-x})\,dx = (e^x + e^{-x})\Big]_0^{\ln 2} = 2 + \dfrac{1}{2} - 1 - 1 = \dfrac{1}{2}$



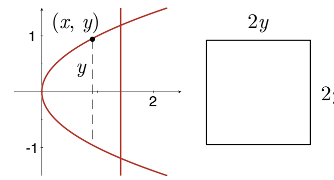
(b) $V = \pi \int_0^{\ln 2} [(e^x)^2 - (e^{-x})^2]\,dx = \pi \int_0^{\ln 2} (e^{2x} - e^{-2x})\,dx$

$$= \frac{\pi}{2}(e^{2x} + e^{-2x})\Big]_0^{\ln 2} = \frac{\pi}{2}\left[\left(4 + \frac{1}{4}\right) - (1+1)\right] = \frac{9\pi}{8}$$



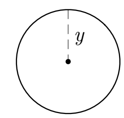
19. (a) $x = y^2;\quad A(x) = (2y)^2 = (2\sqrt{x})^2 = 4x$

$$V = 4\int_0^{\sqrt{2}} x\,dx = 4\left(\frac{1}{2}x^2\right)\Bigg]_0^{\sqrt{2}} = 4$$



(b) $x = y^2;\quad A(x) = \pi r^2 = \pi y^2 = \pi x$

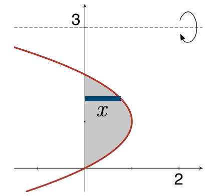
$$V = \pi\int_0^{\sqrt{2}} x\,dx = \pi\frac{1}{2}x^2\Bigg]_0^{\sqrt{2}} = \pi$$



20. $x = 2y - y^2$

$$V = 2\pi\int_0^2 (3-y)(2y-y^2)\,dy = 2\pi\int_0^2 (y^3 - 5y^2 + 6y)\,dy$$

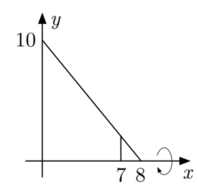
$$= 2\pi\left(\frac{1}{4}y^4 - \frac{5}{3}y^3 + 3y^2\right)\Bigg]_0^2 = \frac{16}{3}\pi$$



21. The equation of the line is $y = -\dfrac{5}{4}x + 10$. Then $y' = -5/4$ and $\sqrt{1 + (y')^2} = \sqrt{1 + 25/16} = \dfrac{1}{4}\sqrt{41}$. The surface area is

$$S = 2\pi\int_0^7 \left(-\frac{5}{4}x + 10\right)\frac{\sqrt{41}}{4}\,dx = \frac{\sqrt{41}\pi}{2}\left(-\frac{5}{8}x^2 + 10x\right)\Bigg]_0^7$$

$$= \frac{\sqrt{41}\pi}{2}\left(-\frac{5}{8}\cdot 49 + 70\right) = \frac{315\pi\sqrt{41}}{16} \approx 396.03\text{ ft}^2.$$



22. $f_{\text{ave}} = \dfrac{1}{4-(-3)}\displaystyle\int_{-3}^4 f(x)\,dx = \dfrac{1}{7}(21) = 3$

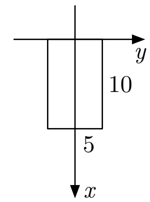
23. $f_{\text{ave}} = \dfrac{1}{4-1}\displaystyle\int_1^4 (x^{3/2} + x^{1/2})\,dx = \dfrac{1}{3}\left(\dfrac{2}{5}x^{5/2} + \dfrac{2}{3}x^{3/2}\right)\Bigg]_1^4 = \dfrac{1}{3}\left(\dfrac{272}{15} - \dfrac{16}{15}\right) = \dfrac{256}{45}$

24. $f_{\text{ave}} = \dfrac{1}{3-0}\displaystyle\int_0^3 (2x-1)\,dx = \dfrac{1}{3}(x^2 - x)\Bigg]_0^3 = 2$. Setting $f(c) = 2c - 1 = 2$ we obtain $c = 3/2$.

25. $50 = k(1/2),\ k = 100,\ F = 100x;\qquad W = \displaystyle\int_{1/2}^1 100x\,dx = 50x^2\Big]_{1/2}^1 = 50 - \dfrac{25}{2} = 37.5$ joules

26. Using 6 in $= \dfrac{1}{2}$ ft, we need to solve $10 = \displaystyle\int_0^{1/2} kx\,dx$ for x. We have $\dfrac{k}{2}x^2\Bigg]_0^{1/2} = 10$, $k/8 = 10$, and $k = 80$.

27. $W = \int_0^{10} 62.4(10)^2(x+5)\,dx = 6240\left(\frac{1}{2}x^2 + 5x\right)\Big]_0^{10} = 624,000$ ft-lb



28. The weight of the bucket after it has been lifted x feet is $32 - \frac{1}{4}x$ pounds.

$$W = \int_0^5 \left(32 - \frac{1}{4}x\right)\,dx = \left(32x - \frac{1}{8}x^2\right)\Big]_0^5 = 156.875 \text{ ft-lb}$$

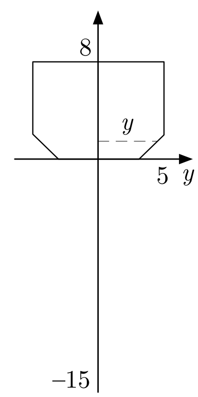
29. At a rate of loss of 1/4 pound per second, it will take 120 seconds to lose the entire 30 pounds. In 120 seconds, the bucket will be raised 120 feet.

$$W = \int_0^{120} \left(32 - \frac{1}{4}x\right)\,dx = \left(32x - \frac{1}{8}x^2\right)\Big]_0^{120} = 2040 \text{ ft-lb}$$

30. The weight of the rope after the bucket has been lifted x feet is $\frac{1}{8}(5-x)$. Thus, the weight of the system after the bucket has been lifted x feet is $\left(32 - \frac{1}{4}x\right) + \frac{1}{8}(5-x) = \frac{261}{8} - \frac{3}{8}x$.

$$W = \int_0^5 \left(\frac{261}{8} - \frac{3}{8}x\right)\,dx = \left(\frac{261}{8}x - \frac{3}{16}x^2\right)\Big]_0^5 = \frac{2535}{16} = 158.4375 \text{ ft-lb}$$

31. $y = \begin{cases} x+3 & 0 \le x \le 2 \\ 5 & 2 \le x \le 8 \end{cases}$



$$\begin{aligned}
W &= \int_0^8 62.4(\pi y^2)(x+15)\,dx \\
&= \int_0^2 62.4\pi(x+3)^2(x+15)\,dx + \int_2^8 62.4\pi(25)(x+15)\,dx \\
&= 62.4\pi \int_0^2 (x^3 + 21x^2 + 99x + 135)\,dx + 1560\pi \int_2^8 (x+15)\,dx \\
&= 62.4\pi\left(\frac{1}{4}x^4 + 7x^3 + \frac{99}{2}x^2 + 135x\right)\Big]_0^2 + 1560\pi\left(\frac{1}{2}x^2 + 15x\right)\Big]_2^8 \\
&= 62.4\pi(528) + 1560\pi(152 - 32) = 220,147.2\pi \approx 691,612.83 \text{ ft-lb}
\end{aligned}$$

32. (a) $a = -5.5$, $v(0) = 44$, $s(0) = 0$; $\quad v(t) = \int -5.5\,dt = -5.5t + c$; $\quad 44 = v(0) = c$

$s(t) = \int(-5.5t + 44)\,dt = -2.75t^2 + 44t + c$; $\quad 0 = s(0) = c$; $\quad s(t) = -2.75t^2 + 44t$

The maximum height is attained when $v(t) = -5.5t + 44 = 0$, or at $t = 8$ seconds. The maximum height is $s(8) = 176$ feet. On the earth, $a = -32$, $v(t) = -32t + 44$, and $s(t) = -16t^2 + 44t$. The maximum height is attained when $v(t) = -32t + 44 = 0$, or at $t = 1.375$ seconds. The maximum height is $s(1.375) = 30.25$ feet.

(b) When the rock hits the astronaut, $6 = s(t) = -2.75t^2 + 44t$ or $2.75t^2 - 44t + 6 = 0$. Solving for t, we obtain $t \approx 1.04$ (on the way up) and $t \approx 15.86$ (on the way down). The velocity is $v(15.86) \approx -43.24$ ft/s.

33. $y' = \frac{3}{2}(x-1)^{1/2}$

$$s = \int_1^5 \sqrt{1 + \frac{9}{4}(x-1)}\, dx = \frac{1}{2}\int_1^5 \sqrt{9x-5}\, dx = \frac{1}{27}(9x-5)^{3/2}\Big]_1^5 = \frac{1}{27}(40^{3/2} - 4^{3/2})$$
$$= \frac{40^{3/2} - 8}{27} \approx 9.07$$

34. Place the origin at the left end of the rod. Then the density of the rod is $\rho(x) = ax + b$. Since $\rho(3) = 11$ and $\rho(6) = 17$, we solve the system $3a + b = 11$, $6a + b = 17$ to obtain $\rho(x) = 2x + 5$.

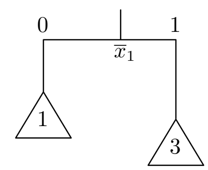
$$m = \int_0^6 (2x+5)\, dx = \left(x^2 + 5x\right)\Big]_0^6 = 66$$
$$M_0 = \int_0^6 x(2x+5)\, dx = \int_0^6 (2x^2 + 5x)\, dx = \left(\frac{2}{3}x^3 + \frac{5}{2}x^2\right)\Big]_0^6 = 234; \quad \overline{x} = \frac{234}{66} = \frac{39}{11}$$

35. $y = \sqrt{16 - x^2}$

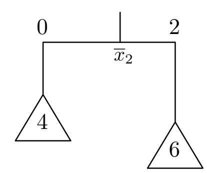
$$F = \int_0^4 800x\sqrt{16 - x^2}\, dx = -\frac{800}{3}(16 - x^2)^{3/2}\Big]_0^4 = -\frac{800}{3}(0 - 64) = \frac{51,200}{3} \approx 17,066.67 \text{ N}$$

36. Taking the origin at the center of the bar, $\overline{x} = \dfrac{3(-1) + 8(1)}{3+8} = \dfrac{5}{11}$ m.

37. Consider first the system of 1 kg and 3 kg weights. Taking the origin at the left end, we obtain $\overline{x}_1 = \dfrac{1(0) + 3(1)}{1+3} = \dfrac{3}{4}$.



Now consider the system with the left two weights concentrated at the left end of the upper bar. Taking the origin at the left, we obtain $\overline{x}_2 = \dfrac{4(0) + 6(2)}{4+6} = \dfrac{12}{10} = \dfrac{6}{5}$.



Chapter 7

Techniques of Integration

7.1 Integration — Three Resources

1. $$\int 5^{-5x}\,dx = -\frac{1}{5}\int 5^{-5x}(-5\,dx) \qquad \boxed{u = -5x,\ du = -5\,dx}$$
$$= -\frac{1}{5}\int 5^u\,du = -\frac{1}{5}\left(\frac{1}{\ln 5}5^u\right) + C = -\frac{5^{-5x}}{5\ln 5} + C = -\frac{1}{5^{5x+1}\ln 5} + C$$

2. $$\int \frac{1}{\sqrt{x}e^{\sqrt{x}}}\,dx = \int x^{-1/2}e^{-x^{1/2}}\,dx = -2\int e^{-x^{1/2}}\left(-\frac{1}{2}x^{-1/2}\,dx\right)$$
$$\boxed{u = -x^{1/2},\ du = -\frac{1}{2}x^{-1/2}\,dx}$$
$$= -2\int e^u\,du = -2e^u + C = -2e^{-x^{1/2}} + C = -\frac{2}{e^{\sqrt{x}}} + C$$

3. $$\int \frac{\sin\sqrt{1+x}}{\sqrt{1+x}} = 2\int [\sin(1+x)^{1/2}]\left[\frac{1}{2}(1+x)^{-1/2}\right]dx$$
$$\boxed{u = (1+x)^{1/2},\ du = \frac{1}{2}(1+x)^{-1/2}\,dx}$$
$$= 2\int \sin u\,du = -2\cos u + C = -2\cos\sqrt{1+x} + C$$

4. $$\int \frac{\cos e^{-x}}{e^x}\,dx = -\int(\cos e^{-x})(-e^{-x}\,dx) \qquad \boxed{u = e^{-x},\ du = -e^{-x}\,dx}$$
$$= -\int \cos u\,du = -\sin u + C = -\sin e^{-x} + C$$

5. $$\int \frac{x}{\sqrt{25-4x^2}}\,dx = -\frac{1}{8}\int(25-4x^2)^{-1/2}(-8x\,dx) \qquad \boxed{u = 25-4x^2,\ du = -8x\,dx}$$
$$= -\frac{1}{8}\int u^{-1/2}\,du = -\frac{1}{8}(2u^{1/2}) + C = -\frac{1}{4}\sqrt{25-4x^2} + C$$

6. $$\int \frac{1}{\sqrt{25-4x^2}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{5^2-(2x)^2}}(2\,dx) \qquad \boxed{u=2x,\ du=2\,dx}$$
$$= \frac{1}{2}\int \frac{1}{\sqrt{5^2-u^2}}\,du = \frac{1}{2}\left(\sin^{-1}\frac{u}{5}\right)+C = \frac{1}{2}\sin^{-1}\frac{2x}{5}+C$$

7. $$\int \frac{1}{x\sqrt{4x^2-25}}\,dx = \int \frac{1}{2x\sqrt{(2x)^2-5^2}}(2\,dx) \qquad \boxed{u=2x,\ du=2\,dx}$$
$$= \int \frac{1}{u\sqrt{u^2-5^2}}\,du = \frac{1}{5}\sec^{-1}\left|\frac{u}{5}\right|+C = \frac{1}{5}\sec^{-1}\left|\frac{2x}{5}\right|+C$$

8. $$\int \frac{1}{\sqrt{25+4x^2}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{5^2+(2x)^2}}(2\,dx) \qquad \boxed{u=2x,\ du=2\,dx}$$
$$= \frac{1}{2}\int \frac{1}{\sqrt{5^2+u^2}}\,du = \frac{1}{2}\left(\ln\left|u+\sqrt{u^2+5^2}\right|\right)+C$$
$$= \frac{1}{2}\ln\left|2x+\sqrt{4x^2+25}\right|+C$$

9. $$\int \frac{1}{25+4x^2}\,dx = \frac{1}{2}\int \frac{1}{5^2+(2x)^2}(2\,dx) \qquad \boxed{u=2x,\ du=2\,dx}$$
$$= \frac{1}{2}\int \frac{1}{5^2+u^2}\,du = \frac{1}{2}\left(\frac{1}{5}\tan^{-1}\frac{u}{5}\right)+C = \frac{1}{10}\tan^{-1}\frac{2x}{5}+C$$

10. $$\int \frac{x}{25+4x^2}\,dx = \frac{1}{8}\int \frac{1}{25+4x^2}(8x\,dx) \qquad \boxed{u=25+4x^2,\ du=8x\,dx}$$
$$= \frac{1}{8}\int \frac{1}{u}\,du = \frac{1}{8}\left(\ln|u|\right)+C = \frac{1}{8}\ln|25+4x^2|+C$$

11. $$\int \frac{1}{4x^2-25}\,dx = \frac{1}{2}\int \frac{1}{(2x)^2-5^2}(2\,dx) \qquad \boxed{u=2x,\ du=2\,dx}$$
$$= \frac{1}{2}\int \frac{1}{u^2-5^2}\,du = \frac{1}{2}\left[\frac{1}{2(5)}\ln\left|\frac{u-5}{u+5}\right|\right]+C$$
$$= \frac{1}{20}\ln\left|\frac{2x-5}{2x+5}\right|+C$$

12. $$\int \frac{1}{x\sqrt{4x^2+25}}\,dx = \int \frac{1}{2x\sqrt{5^2+(2x)^2}}(2\,dx) \qquad \boxed{u=2x,\ du=2\,dx}$$
$$= \int \frac{1}{u\sqrt{5^2+u^2}}\,du = -\frac{1}{5}\ln\left|\frac{5+\sqrt{25+u^2}}{u}\right|+C$$
$$= -\frac{1}{5}\ln\left|\frac{5+\sqrt{25+4x^2}}{2x}\right|+C$$

13. $$\int \cot 10x\,dx = \frac{1}{10}\int \cot 10x(10\,dx) \qquad \boxed{u=10x,\ du=10\,dx}$$
$$= \frac{1}{10}\int \cot u\,du = \frac{1}{10}\left(\ln|\sin u|\right)+C = \frac{1}{10}\ln|\sin 10x|+C$$

14. $$\int x\csc^2 x^2\,dx = \frac{1}{2}\int (\csc^2 x^2)(2x\,dx) \qquad \boxed{u = x^2,\ du = 2x\,dx}$$
$$= \frac{1}{2}\int \csc^2 u\,du = \frac{1}{2}(-\cot u) + C = -\frac{1}{2}\cot x^2 + C$$

15. $$\int \frac{6}{(3-5t)^{2.2}}\,dx = -\frac{6}{5}\int (3-5t)^{-2.2}(-5\,dt) \qquad \boxed{u = 3-5t,\ du = -5\,dt}$$
$$= -\frac{6}{5}\int u^{-2.2}\,du = -\frac{6}{5}\left(-\frac{1}{1.2}u^{-1.2}\right) + C = \frac{1}{(3-5t)^{1.2}} + C$$

16. $$\int x^2\sqrt{(1-x^3)^5}\,dx = -\frac{1}{3}\int (1-x^3)^{5/2}(-3x^2\,dx) \qquad \boxed{u = 1-x^3,\ du = -3x^2\,dx}$$
$$= -\frac{1}{3}\int u^{5/2}\,du = -\frac{1}{3}\left(\frac{2}{7}u^{7/2}\right) + C = -\frac{2}{21}\sqrt{(1-x^3)^7} + C$$

17. $$\int \sec 3x\,dx = \frac{1}{3}\int (\sec 3x)(3\,dx) \qquad \boxed{u = 3x,\ du = 3\,dx}$$
$$= \frac{1}{3}\int \sec u\,du = \frac{1}{3}(\ln|\sec u + \tan u|) + C = \frac{1}{3}\ln|\sec 3x + \tan 3x| + C$$

18. $$\int 2\csc 2x\,dx = \int (\csc 2x)(2\,dx) \qquad \boxed{u = 2x,\ du = 2\,dx}$$
$$= \int \csc u\,du = \ln|\csc u - \cot u| + C = \ln|\csc 2x - \cot 2x| + C$$

19. $$\int \frac{\sin^{-1} x}{\sqrt{1-x^2}}\,dx = \int (\sin^{-1} x)\left(\frac{1}{\sqrt{1-x^2}}\,dx\right) \qquad \boxed{u = \sin^{-1} x,\ du = \frac{1}{\sqrt{1-x^2}}\,dx}$$
$$= \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}(\sin^{-1} x)^2 + C$$

20. $$\int \frac{1}{(1+x^2)\tan^{-1} x}\,dx = \int \left(\frac{1}{\tan^{-1} x}\right)\left(\frac{1}{1+x^2}\,dx\right) \qquad \boxed{u = \tan^{-1} x,\ du = \frac{1}{1+x^2}\,dx}$$
$$= \int \frac{1}{u}\,du = \ln|u| + C = \ln|\tan^{-1} x| + C$$

21. $$\int \frac{\sin x}{1+\cos^2 x}\,dx = -\int \frac{1}{1^2+\cos^2 x}(-\sin x\,dx) \qquad \boxed{u = \cos x,\ du = -\sin x\,dx}$$
$$= -\int \frac{1}{1^2+u^2}\,du = -\tan^{-1} u + C = -\tan^{-1}(\cos x) + C$$

22. $$\int \frac{\cos(\ln 9x)}{x}\,dx = \int [\cos(\ln 9x)]\left(\frac{1}{x}\,dx\right) \qquad \boxed{u = \ln 9x,\ du = 9\left(\frac{1}{9x}\right)dx = \frac{1}{x}\,dx}$$
$$= \int \cos u\,du = \sin u + C = \sin(\ln 9x) + C$$

23. $$\int \frac{x^3}{\cosh^2 x^4}\,dx = \frac{1}{4}\int (\operatorname{sech}^2 x^4)(4x^3\,dx) \qquad \boxed{u = x^4,\ du = 4x^3\,dx}$$
$$= \frac{1}{4}\int \operatorname{sech}^2 u\,du = \frac{1}{4}\tanh u + C = \frac{1}{4}\tanh x^4 + C$$

24. $$\int \tanh x\,dx = \int \left(\frac{1}{\cosh x}\right)(\sinh x\,dx) \qquad \boxed{u = \cosh x,\ du = \sinh x\,dx}$$
$$= \int \frac{1}{u}\,du = \ln|u| + C = \ln|\cosh x| + C$$

25. $$\int \tan 2x \sec 2x\,dx = \frac{1}{2}\int (\sec 2x \tan 2x)(2\,dx) \qquad \boxed{u = 2x,\ du = 2\,dx}$$
$$= \frac{1}{2}\int \sec u \tan u\,du = \frac{1}{2}\sec u + C = \frac{1}{2}\sec 2x + C$$

26. $$\int \sin x \sin(\cos x)\,dx = -\int [\sin(\cos x)](-\sin x\,dx) \qquad \boxed{u = \cos x,\ du = -\sin x\,dx}$$
$$= -\int \sin u\,du = -(-\cos u) + C = \cos(\cos x) + C$$

27. $$\int \sin x \csc(\cos x)\cot(\cos x)\,dx = -\int [\csc(\cos x)\cot(\cos x)](-\sin x\,dx)$$
$$\boxed{u = \cos x,\ du = -\sin x}$$
$$= -\int \csc u \cot u\,du = -(-\csc u) + C = \csc(\cos x) + C$$

28. $$\int \cos x \csc^2(\sin x)\,dx = \int [\csc^2(\sin x)](\cos x\,dx) \qquad \boxed{u = \sin x,\ du = \cos x\,dx}$$
$$= \int \csc^2 u\,du = -\cot u + C = -\cot(\sin x) + C$$

29. $$\int (1+\tan x)^2 \sec^2 x\,dx \qquad \boxed{u = 1+\tan x,\ du = \sec^2 x\,dx}$$
$$= \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}(1+\tan x)^3 + C$$

30. $$\int \frac{1}{x(\ln x)^2}\,dx = \int (\ln x)^{-2}\left(\frac{1}{x}\,dx\right) \qquad \boxed{u = \ln x,\ du = \frac{1}{x}\,dx}$$
$$= \int u^{-2}\,du = -u^{-1} + C = -\frac{1}{\ln x} + C$$

31. $$\int \frac{e^{2x}}{1+e^{2x}}\,dx = \frac{1}{2}\int (1+e^{2x})^{-1}(2e^{2x}\,dx) \qquad \boxed{u = 1+e^{2x},\ du = 2e^{2x}\,dx}$$
$$= \frac{1}{2}\int u^{-1}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(1+e^{2x}) + C$$

32. $$\int \frac{e^x}{1+e^{2x}}\,dx = \int \frac{1}{1^2+(e^x)^2}(e^x\,dx) \qquad \boxed{u = e^x,\ du = e^x\,dx}$$
$$= \int \frac{1}{1^2+u^2}\,du = \tan^{-1} u + C = \tan^{-1} e^x + C$$

7.2 Integration by Substitution

1. $$\int x(x+1)^3\,dx \qquad \boxed{u = x+1,\ x = u-1,\ dx = du}$$
$$= \int (u-1)u^3\,du = \int (u^4 - u^3)\,du$$
$$= \frac{1}{5}u^5 - \frac{1}{4}u^4 + C = \frac{1}{5}(x+1)^5 - \frac{1}{4}(x+1)^4 + C$$

2. $$\int \frac{x^2-3}{(x+1)^3}\,dx \qquad \boxed{u = x+1,\ x = u-1,\ dx = du}$$
$$= \int \frac{(u-1)^2-3}{u^3}\,du = \int \frac{u^2-2u-2}{u^3}\,du = \int (u^{-1} - 2u^{-2} - 2u^{-3})\,du$$
$$= \ln|u| + 2u^{-1} + u^{-2} + C = \ln|x+1| + \frac{2}{x+1} + \frac{1}{(x+1)^2} + C$$

3. $$\int (2x+1)\sqrt{x-5}\,dx \qquad \boxed{u = x-5,\ x = u+5,\ dx = du}$$
$$= \int (2u+11)u^{1/2}\,du = \int (2u^{3/2} + 11u^{1/2})\,du = \frac{4}{5}u^{5/2} + \frac{22}{3}u^{3/2} + C$$
$$= \frac{4}{5}(x-5)^{5/2} + \frac{22}{3}(x-5)^{3/2} + C$$

4. $$\int (x^2-1)\sqrt{2x+1}\,dx \qquad \boxed{u = 2x+1,\ x = \frac{1}{2}(u-1),\ dx = \frac{1}{2}\,du}$$
$$= \int \left[\frac{(u-1)^2}{4} - 1\right] u^{1/2}\left(\frac{1}{2}\,du\right) = \int \left(\frac{u^2-2u+1}{4} - 1\right)\frac{u^{1/2}}{2}\,du$$
$$= \int \left(\frac{u^{5/2}}{8} - \frac{u^{3/2}}{4} - \frac{3u^{1/2}}{8}\right)du = \frac{1}{28}u^{7/2} - \frac{1}{10}u^{5/2} - \frac{1}{4}u^{3/2} + C$$
$$= \frac{1}{28}(2x+1)^{7/2} - \frac{1}{10}(2x+1)^{5/2} - \frac{1}{4}(2x+1)^{3/2} + C$$

5. $$\int \frac{x}{\sqrt{x-1}}\,dx \qquad \boxed{u = x-1,\ x = u+1,\ dx = du}$$
$$= \int \frac{u+1}{u^{1/2}}\,du = \int (u^{1/2} + u^{-1/2})\,du = \frac{2}{3}u^{3/2} + 2u^{1/2} + C$$
$$= \frac{2}{3}(x-1)^{3/2} + 2(x-1)^{1/2} + C$$

6. $\displaystyle\int \frac{x^2}{\sqrt{x+2}}\,dx$ $\boxed{u = x+2,\ x = u-2,\ dx = du}$

$$= \int \frac{(u-2)^2}{u^{1/2}}\,du = \int (u^{3/2} - 4u^{1/2} + 4u^{-1/2})\,du$$
$$= \frac{2}{5}u^{5/2} - \frac{8}{3}u^{3/2} + 8u^{1/2} + C = \frac{2}{5}(x+2)^{5/2} - \frac{8}{3}(x+2)^{3/2} + 8(x+2)^{1/2} + C$$

7. $\displaystyle\int \frac{x+3}{(3x-4)^{3/2}}\,dx$ $\boxed{u = 3x-4,\ x = \frac{1}{3}(u+4),\ dx = \frac{1}{3}\,du}$

$$= \int \frac{u/3 + 13/3}{u^{3/2}}\left(\frac{1}{3}\,du\right) = \int \frac{1}{9}u^{-1/2} + \frac{13}{9}u^{-3/2} = \frac{2}{9}u^{1/2} - \frac{26}{9}u^{-1/2} + C$$
$$= \frac{2}{9}(3x-4)^{1/2} - \frac{26}{9}(3x-4)^{-1/2} + C$$

8. $\displaystyle\int (x^2+x)\sqrt[3]{x+7}\,dx$ $\boxed{u = x+7,\ x = u-7,\ dx = du}$

$$= \int [(u-7)^2 + (u-7)]u^{1/3}\,du = \int (u^2 - 13u + 42)u^{1/3}\,du$$
$$= \int (u^{7/3} - 13u^{4/3} + 42u^{1/3})\,du = \frac{3}{10}u^{10/3} - \frac{39}{7}u^{7/3} + \frac{63}{2}u^{4/3} + C$$
$$= \frac{3}{10}(x+7)^{10/3} - \frac{39}{7}(x+7)^{7/3} + \frac{63}{2}(x+7)^{4/3} + C$$

9. $\displaystyle\int \frac{\sqrt{x}}{x+1}\,dx$ $\boxed{u = \sqrt{x},\ x = u^2,\ dx = 2u\,du}$

$$= \int \frac{u}{u^2+1}(2u\,du) = \int \frac{2u^2}{u^2+1}\,du = \int \left(2 - \frac{2}{u^2+1}\right)du$$
$$= 2u - 2\tan^{-1} u + C = 2\sqrt{x} - 2\tan^{-1}\sqrt{x} + C$$

10. $\displaystyle\int \frac{t}{\sqrt{t}+1}\,dt$ $\boxed{u = \sqrt{t}+1,\ t = (u-1)^2,\ dt = 2(u-1)\,du}$

$$= \int \frac{(u-1)^2}{u}[2(u-1)\,du] = \int \frac{2(u-1)^3}{u}\,du$$
$$= \int \left(2u^2 - 6u + 6 - \frac{2}{u}\right)du = \frac{2}{3}u^3 - 3u^2 + 6u - 2\ln|u| + C$$
$$= \frac{2}{3}(\sqrt{t}+1)^3 - 3(\sqrt{t}+1)^2 + 6(\sqrt{t}+1) - 2\ln(\sqrt{t}+1) + C$$

11. $\displaystyle\int \frac{\sqrt{t}-3}{\sqrt{t}+1}\,dt$ $\boxed{u = \sqrt{t}+1,\ t = (u-1)^2,\ dt = 2(u-1)\,du}$

$$= \int \frac{u-4}{u}[2(u-1)\,du] = \int \left(2u - 10 + \frac{8}{u}\right)du = u^2 - 10u + 8\ln|u| + C$$
$$= (\sqrt{t}+1)^2 - 10(\sqrt{t}+1) + 8\ln(\sqrt{t}+1) + C$$

12. $\displaystyle\int \frac{\sqrt{r}+3}{r+3}\,dr \qquad \boxed{u=\sqrt{r},\ r=u^2,\ dr=2u\,du}$

$$= \int \frac{u+3}{u^2+3}(2u\,du) = \int \frac{2u^2+6u}{u^2+3}\,du = \int \left(2+\frac{6u}{u^2+3}-\frac{6}{u^2+3}\right)du$$
$$= 2u + 3\ln(u^2+3) - \frac{6}{\sqrt{3}}\tan^{-1}\frac{u}{\sqrt{3}} + C$$
$$= 2\sqrt{r} + 3\ln(r+3) - 2\sqrt{3}\tan^{-1}\sqrt{\frac{r}{3}} + C$$

13. $\displaystyle\int \frac{x^3}{\sqrt[3]{x^2+1}}\,dx = \int \frac{x^2}{(x^2+1)^{1/3}}(x\,dx) \qquad \boxed{u=x^2+1,\ du=2x\,dx}$

$$= \int \frac{u-1}{u^{1/3}}\left(\frac{1}{2}\,du\right) = \frac{1}{2}\int (u^{2/3}-u^{-1/3})\,du = \frac{3}{10}u^{5/3} - \frac{3}{4}u^{2/3} + C$$
$$= \frac{3}{10}(x^2+1)^{5/3} - \frac{3}{4}(x^2+1)^{2/3} + C$$

14. $\displaystyle\int \frac{x^5}{\sqrt[5]{x^2+4}}\,dx = \int \frac{x^4}{(x^2+4)^{1/5}}(x\,dx) \qquad \boxed{u=x^2+4,\ du=2x\,dx}$

$$= \int \frac{(u-4)^2}{u^{1/5}}\left(\frac{1}{2}\,du\right) = \frac{1}{2}\int (u^{9/5}-8u^{4/5}+16u^{-1/5})\,du$$
$$= \frac{5}{28}u^{14/5} - \frac{20}{9}u^{9/5} + 10u^{4/5} + C$$
$$= \frac{5}{28}(x^2+4)^{14/5} - \frac{20}{9}(x^2+4)^{9/5} + 10(x^2+4)^{4/5} + C$$

15. $\displaystyle\int \frac{x^2}{(x-1)^4}\,dx \qquad \boxed{u=x-1,\ du=dx}$

$$= \int \frac{(u+1)^2}{u^4}\,du = \int (u^{-2}+2u^{-3}+u^{-4})\,du = -u^{-1} - u^{-2} - \frac{1}{3}u^{-3} + C$$
$$= -\frac{1}{x-1} - \frac{1}{(x-1)^2} - \frac{1}{3(x-1)^3} + C$$

16. $\displaystyle\int \frac{2x+1}{(x+7)^2}\,dx \qquad \boxed{u=x+7,\ du=dx}$

$$= \int \frac{2(u-7)+1}{u^2}\,du = \int (2u^{-1}-13u^{-2})\,du = 2\ln|u| + 13u^{-1} + C$$
$$= 2\ln|x+7| + \frac{13}{x+7} + C$$

17. $\displaystyle\int \sqrt{e^x-1}\,dx \qquad \boxed{u=\sqrt{e^x-1},\ x=\ln(u^2+1),\ dx=\frac{2u}{u^2+1}\,du}$

$$= \int u\left(\frac{2u}{u^2+1}\right)du = \int \left(2-\frac{2u}{u^2+1}\right)du = 2u - 2\tan^{-1}u + C$$
$$= 2\sqrt{e^x-1} - 2\tan^{-1}\sqrt{e^x-1} + C$$

18. $\displaystyle\int \frac{1}{\sqrt{e^x-1}}\,dx$ $\quad \boxed{u=\sqrt{e^x-1},\ x=\ln(u^2+1),\ dx=\frac{2u}{u^2+1}\,du}$

$$= \int \frac{1}{u}\left(\frac{2u}{u^2+1}\right)du = \int \frac{2}{u^2+1}\,du = 2\tan^{-1}u + C = 2\tan^{-1}\sqrt{e^x-1} + C$$

19. $\displaystyle\int \sqrt{1-\sqrt{v}}\,dv$ $\quad \boxed{u=1-\sqrt{v},\ v=(1-u)^2,\ dv=-2(1-u)\,du}$

$$= \int u^{1/2}[-2(1-u)\,du] = \int (2u^{3/2}-2u^{1/2})\,du = \frac{4}{5}u^{5/2} - \frac{4}{3}u^{3/2} + C$$
$$= \frac{4}{5}(1-\sqrt{v})^{5/2} - \frac{4}{3}(1-\sqrt{v})^{3/2} + C$$

20. $\displaystyle\int \frac{\sqrt{w}}{\sqrt{1-\sqrt{w}}}\,dw$ $\quad \boxed{u=1-\sqrt{w},\ w=(1-u)^2,\ dw=-2(1-u)\,du}$

$$= \int \frac{1-u}{u^{1/2}}[-2(1-u)\,du] = -2\int (u^{-1/2}-2u^{1/2}+u^{3/2})\,du$$
$$= -2\left(2u^{1/2} - \frac{4}{3}u^{3/2} + \frac{2}{5}u^{5/2} + C\right)$$
$$= -4(1-\sqrt{w})^{1/2} + \frac{8}{3}(1-\sqrt{w})^{3/2} - \frac{4}{5}(1-\sqrt{w})^{5/2} + C_1$$

21. $\displaystyle\int \frac{\sqrt{1+\sqrt{t}}}{\sqrt{t}}\,dt$ $\quad \boxed{u=1+\sqrt{t},\ du=\frac{1}{2\sqrt{t}}\,dt}$

$$= \int \sqrt{u}(2\,du) = \frac{4}{3}u^{3/2} + C = \frac{4}{3}(1+\sqrt{t})^{3/2} + C$$

22. $\displaystyle\int \sqrt{t}\sqrt{1+t\sqrt{t}}\,dt$ $\quad \boxed{u=1+t\sqrt{t}=1+t^{3/2},\ du=\frac{3}{2}t^{1/2}\,dt}$

$$= \int \frac{2}{3}\sqrt{u}\,du = \frac{4}{9}u^{3/2} + C = \frac{4}{9}(1+t\sqrt{t})^{3/2} + C$$

23. $\displaystyle\int \frac{2x+7}{x^2+2x+5}\,dx = \int \frac{2(x+1)+5}{(x+1)^2+4}\,dx = \int \frac{2(x+1)}{(x+1)^2+4}\,dx + \int \frac{5}{(x+1)^2+4}\,dx$

$$= \ln[(x+1)^2+4] + \frac{5}{2}\tan^{-1}\frac{x+1}{2} + C$$
$$= \ln(x^2+2x+5) + \frac{5}{2}\tan^{-1}\frac{x+1}{2} + C$$

24. $$\int \frac{6x-1}{4x^2+4x+10}\,dx = \frac{3}{4}\int \frac{2(x+1/2)-4/3}{(x+1/2)^2+9/4}\,dx$$
$$= \frac{3}{4}\int \frac{2(x+1/2)}{(x+1/2)^2+9/4}\,dx - \int \frac{1}{(x+1/2)^2+9/4}\,dx$$
$$= \frac{3}{4}\ln[(x+1/2)^2+9/4] - \frac{2}{3}\tan^{-1}\frac{x+1/2}{3/2} + C$$
$$= \frac{3}{4}\ln\frac{4x^2+4x+10}{4} - \frac{2}{3}\tan^{-1}\frac{2x+1}{3} + C$$
$$= \frac{3}{4}\ln(4x^2+4x+10) - \frac{2}{3}\tan^{-1}\frac{2x+1}{3} + C_1$$

25. $$\int \frac{2x+5}{\sqrt{16-16x-x^2}}\,dx = \int \frac{2(x+3)-1}{\sqrt{25-(x+3)^2}}\,dx$$
$$= \int \frac{2(x+3)}{\sqrt{25-(x+3)^2}}\,dx - \int \frac{1}{\sqrt{25-(x+3)^2}}\,dx$$
$$\boxed{u = 25-(x+3)^2,\ du = -2(x+3)\,dx}$$
$$= \int \frac{-1}{\sqrt{u}}\,du - \sin^{-1}\frac{x+3}{5} + C = -2\sqrt{u} - \sin^{-1}\frac{x+3}{5} + C$$
$$= -2\sqrt{16-6x-x^2} - \sin^{-1}\frac{x+3}{5} + C$$

26. $$\int \frac{4x-3}{\sqrt{11+10x-x^2}}\,dx = \int \frac{4(x-5)+17}{\sqrt{36-(x-5)^2}}\,dx$$
$$= \int \frac{4(x-5)}{\sqrt{36-(x-5)^2}}\,dx + \int \frac{17}{\sqrt{36-(x-5)^2}}\,dx$$
$$\boxed{u = 36-(x-5)^2,\ du = -2(x-5)\,dx}$$
$$= \int \frac{-2}{\sqrt{u}}\,du + 17\sin^{-1}\frac{x-5}{6} + C = -4\sqrt{u} + 17\sin^{-1}\frac{x-5}{6} + C$$
$$= -4\sqrt{11+10x-x^2} + 17\sin^{-1}\frac{x-5}{6} + C$$

27. $$\int \frac{1}{\sqrt{x}-\sqrt[3]{x}}\,dx \qquad \boxed{u = x^{1/6},\ x = u^6,\ dx = 6u^5\,du}$$
$$= \int \frac{6u^5}{u^3-u^2}\,du = 6\int \left(u^2+u+1+\frac{1}{u-1}\right)du$$
$$= 2u^3+3u^2+6u+6\ln|u-1| + C$$
$$= 2\sqrt{x} + 3\sqrt[3]{x} + 6\sqrt[6]{x} + 6\ln|\sqrt[6]{x}-1| + C$$

28. $\displaystyle\int \frac{\sqrt[6]{x}}{\sqrt[3]{x}+1}\,dx \qquad \boxed{u = x^{1/6},\ x = u^6,\ dx = 6u^5\,du}$

$$\begin{aligned}
&= \int \frac{u}{u^2+1}(6u^5\,du) = \int \frac{6u^6}{u^2+1}\,du = \int \left(6u^4 - 6u^2 + 6 - \frac{6}{u^2+1}\right) du \\
&= \frac{6}{5}u^5 - 2u^3 + 6u - 6\tan^{-1} u + C \\
&= \frac{6}{5}x^{5/6} - 2x^{1/2} + 6x^{1/6} - 6\tan^{-1} x^{1/6} + C
\end{aligned}$$

29. $\displaystyle\int_0^1 x\sqrt{5x+4}\,dx \qquad \boxed{u = 5x+4,\ x = \frac{1}{5}(u-4),\ dx = \frac{1}{5}\,du}$

$$\begin{aligned}
&= \int_4^9 \frac{1}{5}(u-4)u^{1/2}\left(\frac{1}{5}\,du\right) = \frac{1}{25}\int_4^9 (u^{3/2} - 4u^{1/2})\,du \\
&= \frac{1}{25}\left(\frac{2}{5}u^{5/2} - \frac{8}{3}u^{3/2}\right)\Big]_4^9 = \frac{1}{25}\left[\left(\frac{486}{5} - 72\right) - \left(\frac{64}{5} - \frac{64}{3}\right)\right] = \frac{506}{375}
\end{aligned}$$

30. $\displaystyle\int_{-1}^0 x\sqrt[3]{x+1}\,dx \qquad \boxed{u = x+1,\ du = dx}$

$$\begin{aligned}
&= \int_0^1 (u-1)u^{1/3}\,du = \int_0^1 (u^{4/3} - u^{1/3})\,du \\
&= \left(\frac{3}{7}u^{7/3} - \frac{3}{4}u^{4/3}\right)\Big]_0^1 = \frac{3}{7} - \frac{3}{4} = -\frac{9}{28}
\end{aligned}$$

31. $\displaystyle\int_1^{16} \frac{1}{10+\sqrt{x}}\,dx \qquad \boxed{u = 10+\sqrt{x},\ x = (u-10)^2,\ dx = 2(u-10)\,du}$

$$\begin{aligned}
&= \int_{11}^{14} \frac{2(u-10)}{u}\,du = \int_{11}^{14}\left(2 - \frac{20}{u}\right) du = \left(2u - 20\ln|u|\right)\Big]_{11}^{14} \\
&= (28 - 20\ln 14) - (22 - 20\ln 11) = 6 - 20\ln\frac{14}{11}
\end{aligned}$$

32. $\displaystyle\int_4^9 \frac{\sqrt{x}-1}{\sqrt{x}+1}\,dx \qquad \boxed{u = \sqrt{x}+1,\ x = (u-1)^2,\ dx = 2(u-1)\,du}$

$$\begin{aligned}
&= \int_3^4 \frac{u-2}{u}[2(u-1)\,du] = \int_3^4 \left(2u - 6 + \frac{4}{u}\right) du = \left(u^2 - 6u + 4\ln|u|\right)\Big]_3^4 \\
&= (-8 + 4\ln 4) - (-9 + 4\ln 3) = 1 + 4\ln\frac{4}{3}
\end{aligned}$$

33. $\displaystyle\int_2^9 \frac{5x-6}{\sqrt[3]{x-1}}\,dx \qquad \boxed{u = \sqrt[3]{x-1},\ x = u^3+1,\ dx = 3u^2\,du}$

$$\begin{aligned}
&= \int_1^2 \frac{5(u^3+1)-6}{u}(3u^2\,du) = \int_1^2 (15u^4 - 3u)\,du = \left(3u^5 - \frac{3}{2}u^2\right)\Big]_1^2 \\
&= 90 - \frac{3}{2} = \frac{177}{2}
\end{aligned}$$

34. $$\int_{-\sqrt{3}}^{0} \frac{2x^3}{\sqrt{x^2+1}}\,dx = \int_{-\sqrt{3}}^{0} \frac{x^2}{\sqrt{x^2+1}}(2x\,dx) \qquad \boxed{u = x^2+1,\ du = 2x\,dx}$$
$$= \int_4^1 \frac{u-1}{u^{1/2}}\,du = \int_4^1 (u^{1/2} - u^{-1/2})\,du = \left(\frac{2}{3}u^{3/2} - 2u^{1/2}\right)\Big]_4^1$$
$$= -\frac{4}{3} - \frac{4}{3} = -\frac{8}{3}$$

35. $$\int_0^1 (1-\sqrt{x})^{50}\,dx \qquad \boxed{u = 1-\sqrt{x},\ x = (1-u)^2,\ dx = -2(1-u)\,du}$$
$$= \int_1^0 u^{50}[-2(1-u)\,du] = \int_1^0 (2u^{51} - 2u^{50})\,du = \left(\frac{1}{26}u^{52} - \frac{2}{51}u^{51}\right)\Big]_1^0$$
$$= 0 - \left(-\frac{1}{1326}\right) = \frac{1}{1326}$$

36. $$\int_0^4 \frac{1}{(1+\sqrt{x})^3}\,dx \qquad \boxed{u = 1+\sqrt{x},\ x = (u-1^2),\ dx = 2(1-u)\,du}$$
$$= \int_1^3 \frac{2(u-1)}{u^3}\,du = \int_1^3 (2u^{-2} - 2u^{-3})\,du = (-2u^{-1} + u^{-2})]_1^3$$
$$= \left(\frac{1}{u^2} - \frac{2}{u}\right)\Big]_1^3 = -\frac{5}{9} - (-1) = \frac{4}{9}$$

37. $$\int_1^8 \frac{1}{x^{1/3} + x^{2/3}}\,dx \qquad \boxed{u = x^{1/3},\ x = u^3,\ dx = 3u^2\,du}$$
$$= \int_1^2 \frac{3u^2}{u+u^2}\,du = \int_1^2 \left(3 - \frac{3}{1+u}\right)du = (3u - 3\ln|1+u|)]_1^2$$
$$= (6 - 3\ln 3) - (3 - 3\ln 2) = 3 + 3\ln\frac{2}{3}$$

38. $$\int_1^{64} \frac{x^{1/3}}{x^{2/3}+2}\,dx \qquad \boxed{u = x^{1/3},\ x = u^3,\ dx = 3u^2\,du}$$
$$= \int_1^4 \frac{u}{u^2+2}(3u^2\,du) = \int_1^4 \left(3u - \frac{6u}{u^2+2}\right)du = \left[\frac{3}{2}u^2 - 3\ln(u^2+2)\right]\Big|_1^4$$
$$= (24 - 3\ln 18) - \left(\frac{3}{2} - 3\ln 3\right) = \frac{45}{2} - 3\ln 6$$

39. $$\int_0^1 x^2(1-x)^5\,dx \qquad \boxed{u = 1-x,\ du = -dx}$$
$$= \int_1^0 (1-u)^2 u^5(-du) = \int_0^1 (u^5 - 2u^6 + u^7)\,du$$
$$= \left(\frac{1}{6}u^6 - \frac{2}{7}u^7 + \frac{1}{8}u^8\right)\Big]_0^1 = \frac{1}{6} - \frac{2}{7} + \frac{1}{8} = \frac{1}{168}$$

40. $\displaystyle\int_0^6 \frac{2x+5}{\sqrt{2x+4}}\,dx$ $\boxed{u = 2x+4,\ du = 2\,dx}$

$$= \int_4^{16} \frac{u+1}{\sqrt{u}}\left(\frac{1}{2}\,du\right) = \frac{1}{2}\int_4^{16} (u^{1/2} + u^{-1/2})\,du = \frac{1}{2}\left(\frac{2}{3}u^{3/2} + 2u^{1/2}\right)\Big]_4^{16}$$
$$= \frac{1}{2}\left[\left(\frac{128}{3} + 8\right) - \left(\frac{16}{3} + 4\right)\right] = \frac{62}{3}$$

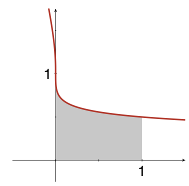
41. $\displaystyle\int_1^{x^2} \frac{1}{t}\,dt$ $\boxed{u = \sqrt{t},\ t = u^2,\ dt = 2u\,du}$

$$= \int_1^{x} \frac{1}{u^2}(2u\,du) = 2\int_1^x \frac{1}{u}\,du = 2\int_1^x \frac{1}{t}\,dt$$

42. $\displaystyle\int_1^{\sqrt{x}} \frac{1}{t}\,dt$ $\boxed{u = t^2,\ t = \sqrt{u},\ dt = \frac{1}{2\sqrt{u}}\,du}$

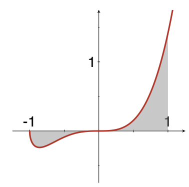
$$= \int_1^{x} \frac{1}{\sqrt{u}}\left(\frac{1}{2\sqrt{u}}\,du\right) = \frac{1}{2}\int_1^x \frac{1}{u}\,du = \frac{1}{2}\int_1^x \frac{1}{t}\,dt$$

43. $\displaystyle A = \int_0^1 \frac{1}{x^{1/3}+1}\,dx$ $\boxed{u = x^{1/3},\ x = u^3,\ dx = 3u^2\,du}$



$$= \int_0^1 \frac{3u^2}{u+1}\,du = \int_0^1 \left(3u - 3 + \frac{3}{u+1}\right) du$$
$$= \left(\frac{3}{2}u^2 - 3u + 3\ln|u+1|\right)\Big]_0^1 = \left(-\frac{3}{2} + 3\ln 2\right) - 0 \approx 0.5794$$

44. $\displaystyle A = -\int_{-1}^0 x^3\sqrt{x+1}\,dx + \int_0^1 x^3\sqrt{x+1}\,dx$ $\boxed{u = x+1,\ du = dx}$

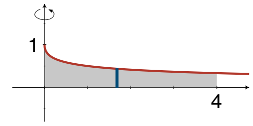


$$= -\int_0^1 (u-1)^3 u^{1/2}\,du + \int_1^2 (u-1)^3 u^{1/2}\,du$$
$$= -\int_0^1 (u^{7/2} - 3u^{5/2} + 3u^{3/2} - u^{1/2})\,du + \int_1^2 (u^{7/2} - 3u^{5/2} + 3u^{3/2} - u^{1/2})\,du$$
$$= -\left(\frac{2}{9}u^{9/2} - \frac{6}{7}u^{7/2} + \frac{6}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right)\Big]_0^1 + \left(\frac{2}{9}u^{9/2} - \frac{6}{7}u^{7/2} + \frac{6}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right)\Big]_1^2$$
$$= -\left(-\frac{32}{315} - 0\right) + \left[\frac{52}{315}\sqrt{2} - \left(-\frac{32}{315}\right)\right] = \frac{64 + 52\sqrt{2}}{315} \approx 0.4366$$

45. $\displaystyle V = 2\pi\int_0^4 x\frac{1}{\sqrt{x}+1}\,dx$ $\boxed{u = \sqrt{x}+1,\ x = (u-1)^2,\ dx = 2(u-1)\,du}$

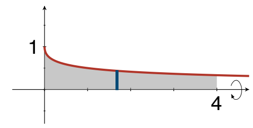
$$= 2\pi\int_1^3 \frac{(u-1)^2}{u}[2(u-1)\,du] = 2\pi\int_1^3 \left(2u^2 - 6u + 6 - \frac{2}{u}\right) du$$

$$= 2\pi\left(\frac{2}{3}u^3 - 3u^2 + 6u - 2\ln|u|\right)\Bigg]_1^3 = 2\pi\left[(9 - 2\ln 3) - \frac{11}{3}\right]$$
$$= 2\pi\left(\frac{16}{3} - 2\ln 3\right) \approx 19.7048$$



46. $V = \pi\displaystyle\int_0^4 \frac{1}{(\sqrt{x}+1)^2}\,dx$ $\quad\boxed{u = \sqrt{x}+1,\ x = (u-1)^2,\ dx = 2(u-1)\,du}$

$$= \pi\int_1^3 \frac{2(u-1)}{u^2}\,du = \pi\int_1^3\left(\frac{2}{u} - \frac{2}{u^2}\right)du = \pi\left(2\ln|u| + \frac{2}{u}\right)\Bigg]_1^3$$
$$= \pi\left[\left(2\ln 3 + \frac{2}{3}\right) - 2\right] = \pi\left(2\ln 3 - \frac{4}{3}\right) \approx 2.7140$$



47. $y = x^{1/4}$

$$L = \int_0^9 \sqrt{1 + (x^{1/4})^2}\,dx = \int_0^9 \sqrt{1+\sqrt{x}}\,dx$$

$\boxed{u = 1+\sqrt{x},\ x = (u-1)^2,\ dx = 2(u-1)\,du}$

$$= \int_1^4 \sqrt{u}[2(u-1)\,du] = 2\int_1^4 (u^{3/2} - u^{1/2})\,du = 2\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right)\Bigg]_1^4$$
$$= 2\left[\left(\frac{64}{5} - \frac{16}{3}\right) - \left(\frac{2}{5} - \frac{2}{3}\right)\right] = \frac{232}{15}$$

48. $T = \displaystyle\int_{w_1}^{w_2} \frac{1}{Aw^{2/3} - Bw}\,dw$ $\quad\boxed{u = w^{1/3},\ w = u^3,\ dw = 3u^2\,du}$

$$= \int_{w_1^{1/3}}^{w_2^{1/3}} \frac{3u^2}{Au^2 - Bu^3}\,du = \int_{w_1^{1/3}}^{w_2^{1/3}} \frac{3}{A - Bu}\,du$$
$$= -\frac{3}{B}\ln|A - Bu|\Bigg]_{w_1^{1/3}}^{w_2^{1/3}} = \frac{3}{B}\ln\left|\frac{A - Bw_1^{1/3}}{A - Bw_2^{1/3}}\right|.$$

Assuming that $A - Bw_2^{1/3} > 0$ we see that $w_2 < \left(\dfrac{A}{B}\right)^3$.

7.3 Integration by Parts

1. $\displaystyle\int x\sqrt{x+3}\,dx$ $\quad\boxed{u = x,\ du = dx;\quad dv = (x+3)^{1/2}\,dx,\ v = \frac{2}{3}(x+3)^{3/2}}$

$$= \frac{2}{3}x(x+3)^{3/2} - \int \frac{2}{3}(x+3)^{3/2}\,dx = \frac{2}{3}x(x+3)^{3/2} - \frac{4}{15}(x+3)^{5/2} + C$$

2. $\displaystyle\int \frac{x}{\sqrt{2x-5}}\,dx$ $\quad\boxed{u = x,\ du = dx;\quad dv = (2x-5)^{-1/2}\,dx,\ v = (2x-5)^{1/2}}$

$$= x(2x-5)^{1/2} - \int (2x-5)^{1/2}\,dx = x(2x-5)^{1/2} - \frac{1}{3}(2x-5)^{3/2} + C$$

3. $\int \ln 4x\,dx$ $\quad \boxed{u = \ln 4x,\ du = \frac{1}{x}\,dx; \quad dv = dx,\ v = x}$

$$= x\ln 4x - \int dx = x\ln 4x - x + C$$

4. $\int \ln(x+1)\,dx$ $\quad \boxed{u = \ln(x+1),\ du = \frac{1}{x+1}\,dx; \quad dv = dx,\ v = x}$

$$= x\ln(x+1) - \int \frac{x}{x+1}\,dx = x\ln(x+1) - \int \left(1 - \frac{1}{x+1}\right)dx$$
$$= x\ln(x+1) - x + \ln(x+1) + C$$

5. $\int x\ln 2x\,dx$ $\quad \boxed{u = \ln 2x,\ du = \frac{1}{x}\,dx; \quad dv = x\,dx,\ v = \frac{1}{2}x^2}$

$$= \frac{1}{2}x^2\ln 2x - \int \frac{x^2}{2x}\,dx = \frac{1}{2}x^2\ln 2x - \frac{1}{2}\int x\,dx = \frac{1}{2}x^2\ln 2x - \frac{1}{4}x^2 + C$$

6. $\int x^{1/2}\ln x\,dx$ $\quad \boxed{u = \ln x,\ du = \frac{1}{x}\,dx; \quad dv = x^{1/2}\,dx,\ v = \frac{2}{3}x^{3/2}}$

$$= \frac{2}{3}x^{3/2}\ln x - \int \frac{2}{3}\left(\frac{x^{3/2}}{x}\right)dx = \frac{2}{3}x^{3/2}\ln x - \frac{2}{3}\int x^{1/2}\,dx$$
$$= \frac{2}{3}x^{3/2}\ln x - \frac{4}{9}x^{3/2} + C$$

7. $\int \frac{\ln x}{x^2}\,dx$ $\quad \boxed{u = \ln x,\ du = \frac{1}{x}\,dx; \quad dv = \frac{1}{x^2}\,dx,\ v = -\frac{1}{x}}$

$$= -\frac{1}{x}\ln x - \int \left(-\frac{1}{x^2}\right)dx = -\frac{1}{x}\ln x - \frac{1}{x} + C$$

8. $\int \frac{\ln x}{\sqrt{x^3}}\,dx$ $\quad \boxed{u = \ln x,\ du = \frac{1}{x}\,dx; \quad dv = x^{-3/2}\,dx,\ v = -2x^{-1/2}}$

$$= -2x^{-1/2}\ln x - \int \frac{-2x^{-1/2}}{x}\,dx$$
$$= -2x^{-1/2}\ln x + 2\int x^{-3/2}\,dx = -2x^{-1/2}\ln x - 4x^{-1/2} + C$$

9. $\int (\ln t)^2\,dt$ $\quad \boxed{u = (\ln t)^2,\ du = \frac{2\ln t}{t}\,dt; \quad dv = dt,\ v = t}$

$$= t(\ln t)^2 - \int 2\ln t\,dt \quad \boxed{u = 2\ln t,\ du = \frac{2}{t}\,dt; \quad dv = dt,\ v = t}$$
$$= t(\ln t)^2 - \left(2t\ln t - \int 2\,dt\right) = t(\ln t)^2 - 2t\ln t + 2t + C$$

10. $\displaystyle\int (t\ln t)^2\,dt$ $\quad\boxed{u=(\ln t)^2,\ du=\frac{2\ln t}{t}\,dt;\quad dv=t^2\,dt,\ v=\frac{1}{3}t^3}$

$$= \frac{1}{3}t^3(\ln t)^2 - \int \frac{2}{3}t^2\ln t\,dt \qquad \boxed{u=\ln t,\ du=\frac{1}{t}\,dt;\quad dv=\frac{2}{3}t^2\,dt,\ v=\frac{2}{9}t^3}$$

$$= \frac{1}{3}t^3(\ln t)^2 - \left(\frac{2}{9}t^3\ln t - \int \frac{2}{9}t^2\,dt\right) = \frac{1}{3}t^3(\ln t)^2 - \frac{2}{9}t^3\ln t + \frac{2}{27}t^3 + C$$

11. $\displaystyle\int \sin^{-1}x\,dx$ $\quad\boxed{u=\sin^{-1}x,\ du=\frac{1}{\sqrt{1-x^2}}\,dx;\quad dv=dx,\ v=x}$

$$= x\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}}\,dx \qquad \boxed{u=1-x^2,\ du=-2x\,dx}$$

$$= x\sin^{-1}x - \int \frac{1}{\sqrt{u}}\left(-\frac{1}{2}\,du\right) = x\sin^{-1}x + \sqrt{u} + C$$

$$= x\sin^{-1}x + \sqrt{1-x^2} + C$$

12. $\displaystyle\int x^2\tan^{-1}x\,dx$ $\quad\boxed{u=\tan^{-1}x,\ du=\frac{1}{1+x^2}\,dx;\quad dv=x^2\,dx,\ v=\frac{1}{3}x^3}$

$$= \frac{1}{3}x^3\tan^{-1}x - \int \frac{1}{3}\left(\frac{x^3}{1+x^2}\right)dx = \frac{1}{3}x^3\tan^{-1}x - \frac{1}{3}\int\left(x-\frac{x}{1+x^2}\right)dx$$

$$= \frac{1}{3}x^3\tan^{-1}x - \frac{1}{6}x^2 + \frac{1}{6}\ln(1+x^2) + C$$

13. $\displaystyle\int xe^{3x}\,dx$ $\quad\boxed{u=x,\ du=dx;\quad dv=e^{3x}\,dx,\ v=\frac{1}{3}e^{3x}}$

$$= \frac{1}{3}xe^{3x} - \int \frac{1}{3}e^{3x}\,dx = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C$$

14. $\displaystyle\int x^2e^{5x}\,dx$ $\quad\boxed{u=x^2,\ du=2x\,dx;\quad dv=e^{5x}\,dx,\ v=\frac{1}{5}e^{5x}}$

$$= \frac{1}{5}x^2e^{5x} - \int \frac{2}{5}xe^{5x}\,dx \qquad \boxed{u=\frac{2}{5}x,\ du=\frac{2}{5}\,dx;\quad dv=e^{5x}\,dx,\ v=\frac{1}{5}e^{5x}}$$

$$= \frac{1}{5}x^2e^{5x} - \left(\frac{2}{25}xe^{5x} - \int \frac{2}{25}e^{5x}\,dx\right) = \frac{1}{5}x^2e^{5x} - \frac{2}{25}xe^{5x} + \frac{2}{125}e^{5x} + C$$

15. $$\int x^3 e^{-4x}\,dx = -\frac{1}{4}x^3e^{-4x} - \frac{3}{16}x^2e^{-4x} - \frac{3}{32}xe^{-4x} - \frac{3}{128}e^{-4x} + C$$

x^3	$+$	e^{-4x}
$3x^2$	$-$	$-\frac{1}{4}e^{-4x}$
$6x$	$+$	$\frac{1}{16}e^{-4x}$
6	$-$	$-\frac{1}{64}e^{-4x}$
		$\frac{1}{256}e^{-4x}$

16. $$\int x^5e^x\,dx = x^5e^x - 5x^4e^x + 20x^3e^x - 60x^2e^x + 120xe^x - 120e^x + C$$

x^5	$+$	e^x
$5x^4$	$-$	e^x
$20x^3$	$+$	e^x
$60x^2$	$-$	e^x
$120x$	$+$	e^x
120	$-$	e^x
		e^x

17. $$\int x^3e^{x^2}\,dx \qquad \boxed{u = x^2,\ du = 2x\,dx; \quad dv = xe^{x^2}\,dx,\ v = \frac{1}{2}e^{x^2}}$$
$$= \frac{1}{2}x^2e^{x^2} - \int xe^{x^2}\,dx = \frac{1}{2}x^2e^{x^2} - \frac{1}{2}e^{x^2} + C$$

18. $$\int x^5e^{2x^3}\,dx \qquad \boxed{u = x^3,\ du = 3x^2\,dx; \quad dv = x^2e^{2x^3}\,dx,\ v = \frac{1}{6}e^{2x^3}}$$
$$= \frac{1}{6}x^3e^{2x^3} - \int \frac{1}{2}x^2e^{2x^3}\,dx = \frac{1}{6}x^3e^{2x^3} - \frac{1}{12}e^{2x^3} + C$$

19. $$\int t\cos 8t\,dt \qquad \boxed{u = t,\ du = dt; \quad dv = \cos 8t\,dt,\ v = \frac{1}{8}\sin 8t}$$
$$= \frac{1}{8}t\sin 8t - \int \frac{1}{8}\sin 8t\,dt = \frac{1}{8}t\sin 8t + \frac{1}{64}\cos 8t + C$$

20. $$\int x\sinh x\,dx \qquad \boxed{u = x,\ du = dx; \quad dv = \sinh x\,dx,\ v = \cosh x}$$
$$= x\cosh x - \int \cosh x\,dx = x\cosh x - \sinh x + C$$

21. $$\int x^2\sin x\,dx \qquad \boxed{u = x^2,\ du = 2x\,dx; \quad dv = \sin x\,dx,\ v = -\cos x}$$
$$= -x^2\cos x + \int 2x\cos x\,dx \qquad \boxed{u = 2x,\ du = 2\,dx; \quad dv = \cos x\,dx,\ v = \sin x}$$
$$= -x^2\cos x + 2x\sin x - \int 2\sin x\,dx = -x^2\cos x + 2x\sin x + 2\cos x + C$$

22. $\displaystyle\int x^2 \cos\frac{x}{2}\,dx$ $\quad\boxed{u = x^2,\ du = 2x\,dx;\quad dv = \cos\frac{x}{2}\,dx,\ v = 2\sin\frac{x}{2}}$

$$= 2x^2\sin\frac{x}{2} - 4\int x\sin\frac{x}{2}\,dx$$

$$\boxed{u = x,\ du = dx;\quad dv = \sin\frac{x}{2}\,dx,\ v = -2\cos\frac{x}{2}}$$

$$= 2x^2\sin\frac{x}{2} - 4\left(-2x\cos\frac{x}{2} + 2\int\cos\frac{x}{2}\,dx\right)$$

$$= 2x^2\sin\frac{x}{2} - 4\left(-2x\cos\frac{x}{2} + 4\sin\frac{x}{2}\right) + C$$

$$= 2x^2\sin\frac{x}{2} + 8x\cos\frac{x}{2} - 16\sin\frac{x}{2} + C$$

23. $\displaystyle\int x^3\cos 3x\,dx$ $\quad\boxed{u = x^3,\ du = 3x^2\,dx;\quad dv = \cos 3x\,dx,\ v = \frac{1}{3}\sin 3x}$

$$= \frac{1}{3}x^3\sin 3x - \int x^2\sin 3x\,dx$$

$$\boxed{u = x^2,\ du = 2x\,dx;\quad dv = \sin 3x\,dx,\ v = -\frac{1}{3}\cos 3x}$$

$$= \frac{1}{3}x^3\sin 3x - \left(-\frac{1}{3}x^2\cos 3x + \frac{2}{3}\int x\cos 3x\,dx\right)$$

$$\boxed{u = x,\ du = dx;\quad dv = \cos 3x\,dx,\ v = \frac{1}{3}\sin 3x}$$

$$= \frac{1}{3}x^3\sin 3x + \frac{1}{3}x^2\cos 3x - \frac{2}{3}\left(\frac{1}{3}x\sin 3x - \frac{1}{3}\int\sin 3x\,dx\right)$$

$$= \frac{1}{3}x^3\sin 3x + \frac{1}{3}x^2\cos 3x - \frac{2}{9}x\sin 3x - \frac{2}{27}\cos 3x + C$$

24. $\displaystyle\int x^4\sin 2x\,dx$ $\quad\boxed{u = x^4,\ du = 4x^3\,dx;\quad dv = \sin 2x\,dx,\ v = -\frac{1}{2}\cos 2x}$

$$= -\frac{1}{2}x^4\cos 2x + 2\int x^3\cos 2x\,dx$$

$$\boxed{u = x^3,\ du = 3x^2\,dx;\quad dv = \cos 2x\,dx,\ v = \frac{1}{2}\sin 2x}$$

$$= -\frac{1}{2}x^4\cos 2x + 2\left(\frac{1}{2}x^3\sin 2x - \frac{3}{2}\int x^2\sin 2x\,dx\right)$$

$$\boxed{u = x^2,\ du = 2x\,dx;\quad dv = \sin 2x\,dx,\ v = -\frac{1}{2}\cos 2x}$$

$$= -\frac{1}{2}x^4\cos 2x + x^3\sin 2x - 3\left(-\frac{1}{2}x^2\cos 2x + \int x\cos 2x\,dx\right)$$

$$\boxed{u = x,\ du = dx; \quad dv = \cos 2x\,dx,\ v = \frac{1}{2}\sin 2x}$$

$$= -\frac{1}{2}x^4\cos 2x + x^3\sin 2x + \frac{3}{2}x^2\cos 2x - 3\left(\frac{1}{2}x\sin 2x - \frac{1}{2}\int \sin 2x\,dx\right)$$

$$= -\frac{1}{2}x^4\cos 2x + x^3\sin 2x + \frac{3}{2}x^2\cos 2x - \frac{3}{2}x\sin 2x - \frac{3}{4}\cos 2x + C$$

25. $\displaystyle\int e^x \sin 4x\,dx$ $\quad\boxed{u = \sin 4x,\ du = 4\cos 4x\,dx; \quad dv = e^x\,dx,\ v = e^x}$

$$= e^x\sin 4x - \int 4e^x\cos 4x\,dx$$

$$\boxed{u = \cos 4x,\ du = -4\sin 4x\,dx; \quad dv = 4e^x\,dx,\ v = 4e^x}$$

$$= e^x\sin 4x - \left(4e^x\cos 4x + 16\int e^x\sin 4x\,dx\right)$$

Solving for the integral, we have $\displaystyle 17\int e^x\sin 4x\,dx = e^x\sin 4x - 4e^x\cos 4x + C$ or

$$\int e^x\sin 4x\,dx = \frac{e^x}{17}(\sin 4x - 4\cos 4x) + C.$$

26. $\displaystyle\int e^{-x}\cos 5x\,dx$ $\quad\boxed{u = e^{-x},\ du = -e^{-x}\,dx; \quad dv = \cos 5x\,dx,\ v = \frac{1}{5}\sin 5x}$

$$= \frac{1}{5}e^{-x}\sin 5x - \int -\frac{1}{5}e^{-x}\sin 5x\,dx = \frac{1}{5}e^{-x}\sin 5x + \frac{1}{5}\int e^{-x}\sin 5x\,dx$$

$$\boxed{u = e^{-x},\ du = -e^{-x}\,dx; \quad dv = \sin 5x\,dx,\ v = -\frac{1}{5}\cos 5x}$$

$$= \frac{1}{5}e^{-x}\sin 5x + \frac{1}{5}\left(-\frac{1}{5}e^{-x}\cos 5x - \frac{1}{5}\int e^{-x}\cos 5x\,dx\right)$$

$$= \frac{1}{5}e^{-x}\sin 5x - \frac{1}{25}e^{-x}\cos 5x - \frac{1}{25}\int e^{-x}\cos 5x\,dx$$

Solving for the integral, we have $\displaystyle \frac{26}{25}\int e^{-x}\cos 5x\,dx = \frac{1}{5}e^{-x}\sin 5x - \frac{1}{25}e^{-x}\cos 5x + C$ or

$$\int e^{-x}\cos 5x\,dx = \frac{5\sin 5x - \cos 5x}{26e^x} + C.$$

27. $\displaystyle\int e^{-2\theta}\cos\theta\,d\theta$ $\boxed{u = e^{-2\theta},\ du = -2e^{-2\theta}\,d\theta;\quad dv = \cos\theta\,d\theta,\ v = \sin\theta}$

$$= e^{-2\theta}\sin\theta - \int -2e^{-2\theta}\sin\theta\,d\theta$$

$$\boxed{u = e^{-2\theta},\ du = -2e^{-2\theta}\,d\theta;\quad dv = \sin\theta\,d\theta,\ v = -\cos\theta}$$

$$= e^{-2\theta}\sin\theta + 2\left(-e^{-2\theta}\cos\theta - \int 2e^{-2\theta}\cos\theta\,d\theta\right)$$

$$= e^{-2\theta}\sin\theta - 2e^{-2\theta}\cos\theta - 4\int e^{-2\theta}\cos\theta\,d\theta$$

Solving for the integral, we have $\displaystyle\int e^{-2\theta}\cos\theta\,d\theta = \frac{\sin\theta - 2\cos\theta}{5e^{2\theta}} + C.$

28. $\displaystyle\int e^{\alpha x}\sin\beta x\,dx$ $\boxed{u = e^{\alpha x},\ du = \alpha e^{\alpha x}\,dx;\quad dv = \sin\beta x\,dx,\ v = -\frac{1}{\beta}\cos\beta x}$

$$= -\frac{1}{\beta}e^{\alpha x}\cos\beta x - \int -\frac{\alpha}{\beta}e^{\alpha x}\cos\beta x\,dx$$

$$\boxed{u = e^{\alpha x},\ du = \alpha e^{\alpha x}\,dx;\quad dv = \cos\beta x\,dx,\ v = \frac{1}{\beta}\sin\beta x}$$

$$= -\frac{1}{\beta}e^{\alpha x}\cos\beta x + \frac{\alpha}{\beta}\left(\frac{1}{\beta}e^{\alpha x}\sin\beta x - \int \frac{\alpha}{\beta}e^{\alpha x}\sin\beta x\,dx\right)$$

$$= \frac{\alpha}{\beta^2}e^{\alpha x}\sin\beta x - \frac{1}{\beta}e^{\alpha x}\cos\beta x - \frac{\alpha^2}{\beta^2}\int e^{\alpha x}\sin\beta x\,dx$$

Solving for the integral, we have $\displaystyle\int e^{\alpha x}\sin\beta x\,dx = \frac{e^{\alpha x}(\alpha\sin\beta x - \beta\cos\beta x)}{\alpha^2+\beta^2} + C.$

29. $\displaystyle\int \theta\sec\theta\tan\theta\,d\theta$ $\boxed{u = \theta,\ du = d\theta;\quad dv = \sec\theta\tan\theta\,d\theta,\ v = \sec\theta}$

$$= \theta\sec\theta - \int \sec\theta\,d\theta = \theta\sec\theta - \ln|\sec\theta + \tan\theta| + C$$

30. $\displaystyle\int e^{2t}\cos e^t\,dt$ $\boxed{u = e^t,\ du = e^t\,dt;\quad dv = e^t\cos e^t\,dt,\ v = \sin e^t}$

$$= e^t\sin e^t - \int e^t\sin e^t\,dt = e^t\sin e^t + \cos e^t + C$$

31. $\displaystyle\int \sin x\cos 2x\,dx$ $\boxed{u = \cos 2x,\ du = -2\sin 2x\,dx;\quad dv = \sin x\,dx,\ v = -\cos x}$

$$= -\cos x\cos 2x - \int 2\cos x\sin 2x\,dx$$

$$\boxed{u = 2\sin 2x,\ du = 4\cos 2x\,dx;\quad dv = \cos x\,dx,\ v = \sin x}$$

$$= -\cos x\cos 2x - \left(2\sin x\sin 2x - 4\int \sin x\cos 2x\,dx\right)$$

Solving for the integral, we have $\int \sin x \cos 2x \, dx = \frac{1}{3} \cos x \cos 2x + \frac{2}{3} \sin x \sin 2x + C$.

32. $\int \cosh x \cosh 2x \, dx$ $\boxed{u = \cosh 2x,\ du = 2\sinh 2x \, dx; \quad dv = \cosh x \, dx,\ v = \sinh x}$

$$= \sinh x \cosh 2x - \int 2 \sinh x \sinh 2x \, dx$$

$$\boxed{u = 2\sinh 2x,\ du = 4\cosh 2x \, dx; \quad dv = \sinh x \, dx,\ v = \cosh x}$$

$$= \sinh x \cosh 2x - \left(2 \sinh 2x \cosh x - 4 \int \cosh x \cosh 2x \, dx\right)$$

Solving for the integral, we have $\int \cosh x \cosh 2x \, dx = \frac{2}{3} \sinh 2x \cosh x - \frac{1}{3} \sinh x \cosh 2x + C$.

33. $\int x^3 \sqrt{x^2+4} \, dx$ $\boxed{u = x^2,\ du = 2x \, dx; \quad dv = x\sqrt{x^2+4} \, dx,\ v = \frac{1}{3}(x^2+4)^{3/2}}$

$$= \frac{1}{3}x^2(x^2+4)^{3/2} - \frac{2}{3}\int x(x^2+4)^{3/2} \, dx$$

$$= \frac{1}{3}x^2(x^2+4)^{3/2} - \frac{2}{15}(x^2+4)^{5/2} + C$$

34. $\int \frac{t^5}{(t^3+1)^2} \, dx$ $\boxed{u = t^3,\ du = 3t^2 \, dt; \quad dv = t^2(t^3+1)^{-2} \, dt,\ v = -\frac{1}{3}(t^3+1)^{-1}}$

$$= -\frac{1}{3}t^3(t^3+1)^{-1} + \int t^2(t^3+1)^{-1} \, dt = -\frac{1}{3}t^3(t^3+1)^{-1} + \frac{1}{3}\ln|t^3+1| + C$$

35. $\int \sin(\ln x) \, dx$ $\boxed{u = \sin(\ln x),\ du = \frac{1}{x}\cos(\ln x) \, dx; \quad dv = dx,\ v = x}$

$$= x \sin(\ln x) - \int \cos(\ln x) \, dx$$

$$\boxed{u = \cos(\ln x),\ du = -\frac{1}{x}\sin(\ln x) \, dx; \quad dv = dx,\ v = x}$$

$$= x \sin(\ln x) - \left[x \cos(\ln x) + \int \sin(\ln x) \, dx\right]$$

Solving for the integral, we have $\int \sin(\ln x) \, dx = \frac{1}{2}x \sin(\ln x) - \frac{1}{2}x \cos(\ln x) + C$.

36. $\int \cos \ln(\sin x) \, dx$ $\boxed{u = \ln(\sin x),\ du = \frac{\cos x}{\sin x} \, dx; \quad dv = \cos x \, dx,\ v = \sin x}$

$$= \sin x \ln(\sin x) - \int \cos x \, dx = \sin x \ln(\sin x) - \sin x + C$$

37. $\displaystyle\int \csc^3 x\,dx$ $\quad\boxed{u = \csc x,\ du = -\csc x\cot x\,dx;\quad dv = \csc^2 x\,dx,\ v = -\cot x}$

$$\begin{aligned}
&= -\csc x\cot x - \int \csc x\cot^2 x\,dx = -\csc x\cot x - \int \csc x(\csc^2 x - 1)\,dx \\
&= -\csc x\cot x - \int \csc^3 x\,dx + \int \csc x\,dx \\
&= -\csc x\cot x - \int \csc^3 x\,dx + \ln|\csc x - \cot x|
\end{aligned}$$

Solving for the integral, we have $\displaystyle\int \csc^3 x\,dx = -\frac{1}{2}\csc x\cot x + \frac{1}{2}\ln|\csc x - \cot x| + C.$

38. $\displaystyle\int x\sec^{-1} x\,dx$ $\quad\boxed{u = \sec^{-1} x,\ du = \frac{1}{x\sqrt{x^2-1}}\,dx;\quad dv = x\,dx,\ v = \frac{1}{2}x^2}$

$$\begin{aligned}
&= \frac{1}{2}x^2\sec^{-1} x - \int \frac{1}{2}\left(\frac{x^2}{x\sqrt{x^2-1}}\right)dx = \frac{1}{2}x^2\sec^{-1} x - \frac{1}{2}\int x(x^2-1)^{-1/2}\,dx \\
&= \frac{1}{2}x^2\sec^{-1} x - \frac{1}{4}\int (x^2-1)^{-1/2}(2x\,dx) \qquad \boxed{t = x^2 - 1,\ dt = 2x\,dx} \\
&= \frac{1}{2}x^2\sec^{-1} x - \frac{1}{4}\int t^{-1/2}\,dt = \frac{1}{2}x^2\sec^{-1} x - \frac{1}{4}(2t^{1/2}) + C \\
&= \frac{1}{2}x^2\sec^{-1} x - \frac{1}{2}\sqrt{x^2-1} + C = \frac{1}{2}(x^2\sec^{-1} x - \sqrt{x^2-1}) + C
\end{aligned}$$

39. $\displaystyle\int x\sec^2 x\,dx$ $\quad\boxed{u = x,\ du = dx;\quad dv = \sec^2 x\,dx,\ v = \tan x}$

$$= x\tan x - \int \tan x\,dx = x\tan x - \ln|\sec x| + C$$

40. $\displaystyle\int x\tan^2 x\,dx = \int x(\sec^2 x - 1)\,dx = \int x\sec^2 x\,dx - \int x\,dx$

$$\boxed{u = x,\ du = dx;\quad dv = \sec^2 x\,dx,\ v = \tan x}$$

$$= x\tan x - \int \tan x\,dx - \frac{1}{2}x^2 = x\tan x - \ln|\sec x| - \frac{1}{2}x^2 + C$$

41. $\displaystyle\int_0^2 x\ln(x+1)\,dx$ $\quad\boxed{u = \ln(x+1),\ du = \frac{1}{x+1}\,dx;\quad dv = x\,dx,\ v = \frac{1}{2}x^2}$

$$\begin{aligned}
&= \frac{1}{2}x^2\ln(x+1)\Big]_0^2 - \int_0^2 \frac{1}{2}\left(\frac{x^2}{x+1}\right)dx \\
&= \frac{1}{2}(4\ln 3 - 0) - \frac{1}{2}\int_0^2\left(x - 1 + \frac{1}{x+1}\right)dx \\
&= 2\ln 3 - \frac{1}{2}\left[\frac{1}{2}x^2 - x + \ln(x+1)\right]\Bigg|_0^2 = 2\ln 3 - \frac{1}{2}(\ln 3 - 0) = \frac{3}{2}\ln 3
\end{aligned}$$

42. $\displaystyle\int_0^1 \ln(x^2+1)\,dx$ $\quad \boxed{u=\ln(x^2+1),\ du=\frac{2x}{x^2+1}\,dx;\quad dv=dx,\ v=x}$

$$= x\ln(x^2+1)\Big]_0^1 - \int_0^1 \frac{2x^2}{x^2+1}\,dx = (\ln 2 - 0) - 2\int_0^1 \left(1-\frac{1}{x^2+1}\right)dx$$

$$= \ln 2 - 2\left(x-\tan^{-1}x\right)\Big]_0^1 = \ln 2 - 2\left[\left(1-\frac{\pi}{4}\right)-(0-0)\right] = \frac{\pi}{2}+\ln 2 - 2$$

43. $\displaystyle\int_2^4 xe^{-x/2}\,dx$ $\quad \boxed{u=x,\ du=dx;\quad dv=e^{-x/2}\,dx,\ v=-2e^{-x/2}}$

$$= -2xe^{-x/2}\Big]_2^4 - \int_2^4 (-2e^{-x/2})\,dx = -2(4e^{-2}-2e^{-1}) - 4e^{-x/2}\Big]_2^4$$

$$= \frac{4}{e}-\frac{8}{e^2}-4(e^{-2}-e^{-1}) = \frac{8}{e}-\frac{12}{e^2} = \frac{8e-12}{e^2}$$

44. $\displaystyle\int_{-\pi}^{\pi} e^x\cos x\,dx$ $\quad \boxed{u=\cos x,\ du=-\sin x\,dx;\quad dv=e^x\,dx,\ v=e^x}$

$$= e^x\cos x\Big]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} e^x\sin x\,dx$$

$$\boxed{u=\sin x,\ du=\cos x\,dx;\quad dv=e^x\,dx,\ v=e^x}$$

$$= [-e^{\pi}-(-e^{-\pi})] + e^x\sin x\Big]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} e^x\cos x\,dx$$

$$= e^{-\pi}-e^{\pi}+(0-0) - \int_{-\pi}^{\pi} e^x\cos x\,dx$$

Solving for the integral, we have $\displaystyle\int_{-\pi}^{\pi} e^x\cos x\,dx = \frac{1}{2}(e^{-\pi}-e^{\pi})$.

45. $\displaystyle\int_0^1 \tan^{-1}x\,dx$ $\quad \boxed{u=\tan^{-1}x,\ du=\frac{1}{1+x^2}\,dx;\quad dv=dx,\ v=x}$

$$= x\tan^{-1}x\Big]_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx = \left(\frac{\pi}{4}-0\right) - \frac{1}{2}\ln(1+x^2)\Big]_0^1$$

$$= \frac{\pi}{4}-\frac{1}{2}(\ln 2 - 0) = \frac{\pi}{4}-\frac{1}{2}\ln 2$$

46. $\displaystyle\int_0^{\sqrt{2}/2} \cos^{-1}x\,dx$ $\quad \boxed{u=\cos^{-1}x,\ du=-\frac{1}{\sqrt{1-x^2}}\,dx;\quad dv=dx,\ v=x}$

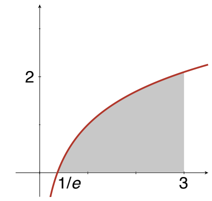
$$= x\cos^{-1}x\Big]_0^{\sqrt{2}/2} - \int_0^{\sqrt{2}/2} -\frac{x}{\sqrt{1-x^2}}\,dx$$

$$= \left(\frac{\sqrt{2}\pi}{8}-0\right) - \frac{1}{2}\int_0^{\sqrt{2}/2} (1-x^2)^{-1/2}(-2x\,dx)$$

$$\boxed{t=1-x^2,\ dt=-2x\,dx;\quad \int t^{-1/2}\,dt = 2t^{1/2} = 2\sqrt{t} = 2\sqrt{1-x^2}}$$

$$= \frac{\sqrt{2}\pi}{8} - \frac{1}{2}\left(2\sqrt{1-x^2}\right)\Big]_0^{\sqrt{2}/2} = \frac{\sqrt{2}\pi}{8} - \left(\sqrt{\frac{1}{2}} - \sqrt{1}\right) = \frac{\sqrt{2}\pi}{8} - \frac{\sqrt{2}}{2} + 1$$

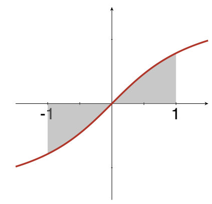
47. $A = \displaystyle\int_{e^{-1}}^{3} (1+\ln x)\,dx$ $\boxed{u = 1+\ln x,\ du = \frac{1}{x}\,dx; \quad dv = dx,\ v = x}$



$$= (x + x\ln x)]_{e^{-1}}^{3} - \int_{e^{-1}}^{3} dx = (3 + 3\ln 3) - (e^{-1} - e^{-1}) - x]_{e^{-1}}^{3}$$

$$= 3 + 3\ln 3 - \left(3 - \frac{1}{e}\right) = 3\ln 3 + \frac{1}{e}$$

48. $A = -\displaystyle\int_{-1}^{0} \tan^{-1} x\,dx + \int_0^1 \tan^{-1} x\,dx$



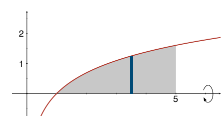
$$\boxed{u = \tan^{-1} x,\ du = \frac{1}{1+x^2}\,dx; \quad dv = dx,\ v = x}$$

$$= -\left(x\tan^{-1} x]_{-1}^{0} - \int_{-1}^{0} \frac{x}{1+x^2}\,dx\right) + x\tan^{-1} x]_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx$$

$$= -\left[\left(0 - \frac{\pi}{4}\right) - \frac{1}{2}\ln(1+x^2)\Big]_{-1}^{0}\right] + \left[\left(\frac{\pi}{4} - 0\right) - \frac{1}{2}\ln(1+x^2)\Big]_0^1\right]$$

$$= \frac{\pi}{4} + \frac{1}{2}(0 - \ln 2) + \frac{\pi}{4} - \frac{1}{2}(\ln 2 - 0) = \frac{\pi}{2} - \ln 2$$

49. $V = \pi \displaystyle\int_1^5 (\ln x)^2\,dx$ $\boxed{u = (\ln x)^2,\ du = \frac{2\ln x}{x}\,dx; \quad dv = dx,\ v = x}$



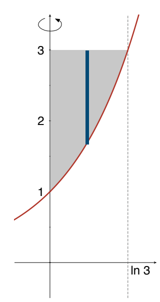
$$= \pi x(\ln x)^2]_1^5 - \pi \int_1^5 2\ln x\,dx$$

$$\boxed{u = 2\ln x,\ du = \frac{2}{x}\,dx; \quad dv = dx,\ v = x}$$

$$= \pi[5(\ln 5)^2 - 0] - \pi\left(2x\ln x]_1^5 - \int_1^5 2\,dx\right)$$

$$= 5\pi(\ln 5)^2 - \pi\left[(10\ln 5 - 0) - 2x]_1^5\right] = 5\pi(\ln 5)^2 - 10\pi\ln 5 + 8\pi$$

50. $V = 2\pi \displaystyle\int_0^{\ln 3} x(3 - e^x)\,dx = 6\pi\int_0^{\ln 3} x\,dx - 2\pi\int_0^{\ln 3} xe^x\,dx$

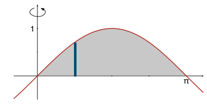


$$\boxed{u = x,\ du = dx; \quad dv = e^x\,dx,\ v = e^x}$$

$$= 3\pi x^2]_0^{\ln 3} - 2\pi\left(xe^x]_0^{\ln 3} - \int_0^{\ln 3} e^x\,dx\right)$$

$$= 3\pi(\ln 3)^2 - 2\pi\left[(3\ln 3 - 0) - e^x]_0^{\ln 3}\right] = 3\pi(\ln 3)^2 - 6\pi\ln 3 + 4\pi$$

51. $V = 2\pi \int_0^{\pi} x \sin x \, dx$ $\boxed{u = x,\ du = dx;\quad dv = \sin x \, dx,\ v = -\cos x}$



$$= 2\pi \left(-x \cos x \Big]_0^{\pi} + \int_0^{\pi} \cos x \, dx \right) = 2\pi \left[-(-\pi - 0) + \sin x]_0^{\pi} \right] = 2\pi^2$$

52. $y' = \dfrac{1}{\cos x}(-\sin x) = -\dfrac{\sin x}{\cos x} = -\tan x$

$$s = \int_0^{\pi/4} \sqrt{1 + \tan^2 x} \, dx = \int_0^{\pi/4} \sqrt{1 + (\sec^2 x - 1)} \, dx = \int_0^{\pi/4} \sqrt{\sec^2 x} \, dx = \int_0^{\pi/4} |\sec x| \, dx$$

On $[0, \pi/4]$, $\sec x > 0$, so we have

$$s = \int_0^{\pi/4} \sec x \, dx = \ln |\sec x + \tan x| \Big]_0^{\pi/4} = \ln |\sqrt{2} + 1| - \ln |1 + 0| = \ln(1 + \sqrt{2}).$$

53. $f_{\text{ave}} = \dfrac{1}{2-0} \displaystyle\int_0^2 \tan^{-1} \frac{x}{2} \, dx$ $\boxed{u = \tan^{-1} \frac{x}{2},\ du = \frac{2}{4+x^2} \, dx;\quad dv = dx,\ v = x}$

$$= \frac{1}{2} \left(x \tan^{-1} \frac{x}{2} \Big]_0^2 - \int_0^2 \frac{2x}{4+x^2} \, dx \right) = \frac{1}{2} \left[2 \left(\frac{\pi}{4} \right) - \ln(4 + x^2) \Big]_0^2 \right]$$

$$= \frac{1}{2} \left[\frac{\pi}{2} - (\ln 8 - \ln 4) \right] = \frac{\pi}{4} - \frac{1}{2} \ln 2$$

54. $s(t) = \displaystyle\int e^{-t} \sin t \, dt$ $\boxed{u = e^{-t},\ du = -e^{-t} \, dt;\quad dv = \sin t \, dt,\ v = -\cos t}$

$$= -e^{-t} \cos t - \int e^{-t} \cos t \, dt \qquad \boxed{u = e^{-t},\ du = -e^{-t} \, dt;\quad dv = \cos t \, dt,\ v = \sin t}$$

$$= -e^{-t} \cos t - \left(e^{-t} \sin t + \int e^{-t} \sin t \, dt \right)$$

Solving for the integral, we have $s(t) = \displaystyle\int e^{-t} \sin t \, dt = -\frac{1}{2} e^{-t} \cos t - \frac{1}{2} e^{-t} \sin t + C$. Now $0 = s(0) = -\dfrac{1}{2} + C$, so $C = \dfrac{1}{2}$ and $s(t) = \dfrac{1}{2}(1 - e^{-t} \cos t - e^{-t} \sin t)$.

55. $v(t) = \displaystyle\int t e^{-t} \, dt$ $\boxed{u = t,\ du = dt;\quad dv = e^{-t} \, dt,\ v = -e^{-t}}$

$$= -te^{-t} + \int e^{-t} \, dt = -te^{-t} - e^{-t} + C.$$

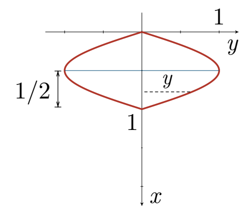
Now $1 = v(0) = -1 + C$, so $C = 2$ and $v(t) = -te^{-t} - e^{-t} + 2$.

$$s(t) = \int (-te^{-t} - e^{-t} + 2) \, dt = -\int te^{-t} \, dt + e^{-t} + 2t = -(-te^{-t} - e^{-t}) + e^{-t} + 2t + C$$

$$= te^{-t} + 2e^{-t} + 2t + C.$$

Now $-1 = s(0) = 2 + C$, so $C = -3$ and $s(t) = te^{-t} + 2e^{-t} + 2t - 3$.

56. $W = 62.4\pi \int_{1/2}^{1} x \sin^2 \pi x \, dx = 62.4\pi \int_{1/2}^{1} \frac{x}{2}(1 - \cos 2\pi x) \, dx$



$= 31.2\pi \int_{1/2}^{1} x \, dx - 31.2\pi \int_{1/2}^{1} x \cos 2\pi x \, dx$

$\boxed{u = x,\ du = dx; \quad dv = \cos 2\pi x \, dx,\ v = \frac{1}{2\pi} \sin 2\pi x}$

$= 15.6\pi x^2\Big]_{1/2}^{1} - 31.2\pi \left(\frac{1}{2\pi} x \sin 2\pi x \Big]_{1/2}^{1} - \int_{1/2}^{1} \frac{1}{2\pi} \sin 2\pi x \right)$

$= 15.6\pi \left(1 - \frac{1}{4}\right) - 31.2\pi \left(0 + \frac{1}{4\pi^2} \cos 2\pi x \Big]_{1/2}^{1} \right)$

$= 11.7\pi - \frac{7.8}{\pi}(1 + 1) = 11.7\pi - \frac{15.6}{\pi} \approx 31.7910$ ft-lb

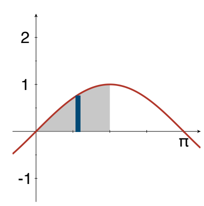
57. Using symmetry,

$F = 2\left(62.4 \int_0^2 x \cos \frac{\pi x}{4} \, dx\right)$ $\boxed{u = x,\ du = dx; \quad dv = \cos \frac{\pi x}{4} \, dx,\ v = \frac{4}{\pi} \sin \frac{\pi x}{4}}$

$= 124.8 \left(\frac{4x}{\pi} \sin \frac{\pi x}{4} \Big]_0^2 - \frac{4}{\pi} \int_0^2 \sin \frac{\pi x}{4} \, dx \right) = 124.8 \left(\frac{8}{\pi} + \frac{16}{\pi^2} \cos \frac{\pi x}{4} \Big]_0^2 \right)$

$= 124.8 \left(\frac{8}{\pi} - \frac{16}{\pi^2} \right) \approx 115.4825$ lb

58. $A = \int_0^{\pi/2} \sin x \, dx = -\cos x\Big]_0^{\pi/2} = -(0 - 1) = 1$



$M_y = \int_0^{\pi/2} x \sin x \, dx$ $\boxed{u = x,\ du = dx; \quad dv = \sin x \, dx,\ v = -\cos x}$

$= -x \cos x\Big]_0^{\pi/2} + \int_0^{\pi/2} \cos x \, dx = -(0 - 0) + \sin x\Big]_0^{\pi/2} = 1$

$M_x = \frac{1}{2} \int_0^{\pi/2} \sin^2 x \, dx = \frac{1}{4} \int_0^{\pi/2} (1 - \cos 2x) \, dx = \frac{1}{4} \left(x - \frac{1}{2} \sin 2x \right) \Big]_0^{\pi/2} = \frac{1}{4} \left(\frac{\pi}{2} \right) = \frac{\pi}{8}$

$\overline{x} = \frac{1}{1} = 1, \quad \overline{y} = \frac{\pi/8}{1} = \frac{\pi}{8}$

59. $\int_1^4 \frac{\tan^{-1} \sqrt{x}}{\sqrt{x}} \, dx$ $\boxed{t = \sqrt{x},\ dt = \frac{1}{2\sqrt{x}} \, dx}$

$= 2 \int_1^2 \tan^{-1} t \, dt$ $\boxed{u = \tan^{-1} t,\ du = \frac{1}{1 + t^2} \, dt; \quad dv = dt,\ v = t}$

$= 2 \left(t \tan^{-1} t\Big]_1^2 - \int_1^2 \frac{t}{1 + t^2} \, dt \right) = 2 \left[\left(2 \tan^{-1} 2 - \frac{\pi}{4} \right) - \frac{1}{2} \ln(1 + t^2) \Big]_1^2 \right]$

$= 4 \tan^{-1} 2 - \frac{\pi}{2} - (\ln 5 - \ln 2) = 4 \tan^{-1} 2 - \frac{\pi}{2} - \ln \frac{5}{2}$

60. $\displaystyle\int xe^{\sqrt{x}}\,dx \qquad \boxed{t=\sqrt{x},\ x=t^2,\ dx=2t\,dt}$

$$= \int t^2e^t(2t\,dt) = 2\int t^3e^t\,dt \qquad \boxed{u=t^3,\ du=3t^2\,dt;\quad dv=e^t\,dt,\ v=e^t}$$

$$= 2\left(t^3e^t - \int 3t^2e^t\,dt\right) \qquad \boxed{u=t^2,\ du=2t\,dt;\quad dv=e^t\,dt,\ v=e^t}$$

$$= 2t^3e^t - 6\left(t^2e^t - \int 2te^t\,dt\right) \qquad \boxed{u=t,\ du=dt;\quad dv=e^t\,dt,\ v=e^t}$$

$$= 2t^3e^t - 6t^2e^t + 12\left(te^t - \int e^t\,dt\right) = 2t^3e^t - 6t^2e^t + 12te^t - 12e^t + C$$

$$= 2x^{3/2}e^{\sqrt{x}} - 6xe^{\sqrt{x}} + 12\sqrt{x}e^{\sqrt{x}} - 12e^{\sqrt{x}} + C$$

61. $\displaystyle\int \sin\sqrt{x+2}\,dx \qquad \boxed{t=\sqrt{x+2},\ x=t^2-2,\ dx=2t\,dt}$

$$= \int (\sin t)2t\,dt \qquad \boxed{u=2t,\ du=2\,dt;\quad dv=\sin t\,dt,\ v=-\cos t}$$

$$= -2t\cos t + \int 2\cos t\,dt = -2t\cos t + 2\sin t + C$$

$$= -2\sqrt{x+2}\cos\sqrt{x+2} + 2\sin\sqrt{x+2} + C$$

62. $\displaystyle\int_0^{\pi^2} \cos\sqrt{t}\,dt \qquad \boxed{x=\sqrt{t},\ t=x^2,\ dt=2x\,dx}$

$$= \int_0^{\pi} \cos x(2x\,dx) = 2\int_0^{\pi} x\cos x\,dx$$

$$\boxed{u=x,\ du=dx;\quad dv=\cos x\,dx,\ v=\sin x}$$

$$= 2\left(x\sin x\Big]_0^{\pi} - \int_0^{\pi}\sin x\,dx\right) = 2\cos x\Big]_0^{\pi} = 2(-1-1) = -4$$

63. $\displaystyle\int (\ln x)^n\,dx \qquad \boxed{u=(\ln x)^n,\ du=\frac{n(\ln x)^{n-1}}{x}\,dx;\quad dv=dx,\ v=x}$

$$= x(\ln x)^n - n\int(\ln x)^{n-1}\,dx$$

64. $\displaystyle\int \sin^n x\,dx \qquad \boxed{u=\sin^{n-1}x,\ du=(n-1)\sin^{n-2}x\cos x\,dx;\quad dv=\sin x\,dx,\ v=-\cos x}$

$$= -\sin^{n-1}x\cos x + \int(n-1)\cos^2 x\sin^{n-2}x\,dx$$

$$= -\sin^{n-1}x\cos x + (n-1)\int(1-\sin^2 x)\sin^{n-2}x\,dx$$

$$= -\sin^{n-1}x\cos x + (n-1)\int\sin^{n-2}x\,dx - (n-1)\int\sin^n x\,dx$$

Solving for the integral $\int \sin^n x\,dx$, we have

$$\int \sin^n x\,dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx.$$

65. $\int \cos^n x\,dx$ $\boxed{u = \cos^{n-1} x,\ du = -(n-1)\cos^{n-2} x \sin x\,dx; \quad dv = \cos x\,dx,\ v = \sin x}$

$$\begin{aligned}
&= \cos^{n-1} x \sin x + \int (n-1)\sin^2 x \cos^{n-2} x\,dx \\
&= \cos^{n-1} x \sin x + (n-1)\int (1-\cos^2 x)\cos^{n-2} x\,dx \\
&= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx
\end{aligned}$$

Solving for the integral $\int \cos^n x\,dx$, we have

$$\int \cos^n x\,dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx.$$

66. $\int \sec^n x\,dx$ $\boxed{u = \sec^{n-2} x,\ du = (n-2)\sec^{n-2} x \tan x\,dx; \quad dv = \sec^2 x\,dx,\ v = \tan x}$

$$\begin{aligned}
&= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x \tan^2 x\,dx \\
&= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x(\sec^2 x - 1)\,dx \\
&= \sec^{n-2} x \tan x - (n-2)\int \sec^n x\,dx + (n-2)\int \sec^{n-2} x\,dx
\end{aligned}$$

Solving for $\int \sec^n x\,dx$, we have

$$\int \sec^n x\,dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx, \quad n \neq 1.$$

67. Using $\int \sin^n x\,dx = -\dfrac{\sin^{n-1} x \cos x}{n} + \dfrac{n-1}{n}\int \sin^{n-2} x\,dx$ with $n = 3$,

$$\int \sin^3 x\,dx = -\frac{\sin^2 x \cos x}{3} + \frac{2}{3}\int \sin x\,dx = -\frac{\sin^2 x \cos x}{3} - \frac{2}{3}\cos x + C.$$

68. Using $\int \sec^n x\,dx = \dfrac{\sec^{n-2} x \tan x}{n-1} + \dfrac{n-2}{n-1}\int \sec^{n-2} x\,dx$ with $n = 4$,

$$\int \sec^4 x\,dx = \frac{\sec^2 x \tan x}{3} + \frac{2}{3}\int \sec^2 x\,dx = \frac{\sec^2 x \tan x}{3} + \frac{2}{3}\tan x + C.$$

69. Using $\displaystyle\int \cos^n x\,dx = \frac{\cos^{n-1}x\sin x}{n} + \frac{n-1}{n}\int \cos^{n-2}x\,dx$ with $n=3$,

$$\begin{aligned}\int \cos^3 10x\,dx \qquad & \boxed{u = 10x,\ du = 10\,dx}\\ &= \frac{1}{10}\int \cos^3 u\,du = \frac{\cos^2 u\sin u}{10\cdot 3} + \frac{2}{10\cdot 3}\int \cos u\,du\\ &= \frac{\cos^2 u\sin u}{30} + \frac{1}{15}\sin u + C = \frac{\cos^2 10x\sin 10x}{30} + \frac{1}{15}\sin 10x + C.\end{aligned}$$

70. Using $\displaystyle\int \cos^n x\,dx = \frac{\cos^{n-1}x\sin x}{n} + \frac{n-1}{n}\int \cos^{n-2}x\,dx$ with $n=4$,

$$\begin{aligned}\int \cos^4 x\,dx &= \frac{\cos^3 x\sin x}{4} + \frac{3}{4}\int \cos^2 x\,dx = \frac{\cos^3 x\sin x}{4} + \frac{3}{4}\left(\frac{\cos x\sin x}{2} + \frac{1}{2}\int \cos^0 x\,dx\right)\\ &= \frac{\cos^3 x\sin x}{4} + \frac{3}{8}\left(\cos x\sin x + \int dx\right) = \frac{\cos^3 x\sin x}{4} + \frac{3}{8}(\cos x\sin x + x) + C.\end{aligned}$$

71. Using $\displaystyle\int \sin^n x\,dx = -\frac{\sin^{n-1}x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2}x\,dx$, we have

$$\begin{aligned}\int_0^{\pi/2} \sin^n x\,dx &= -\frac{\sin^{n-1}x\cos x}{n}\bigg]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2}\sin^{n-2}x\,dx\\ &= -\left[\frac{\sin^{n-1}(\pi/2)\cos(\pi/2)}{n} - \frac{\sin^{n-1}0\cos 0}{n}\right] + \frac{n-1}{n}\int_0^{\pi/2}\sin^{n-2}x\,dx\\ &= -(0-0) + \frac{n-1}{n}\int_0^{\pi/2}\sin^{n-2}x\,dx = \frac{n-1}{n}\int_0^{\pi/2}\sin^{n-2}x\,dx.\end{aligned}$$

72. Repeated use of $\displaystyle\int_0^{\pi/2}\sin^n x\,dx = \frac{n-1}{n}\int_0^{\pi/2}\sin^{n-2}x\,dx$ yields

$$\int_0^{\pi/2}\sin^n x\,dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{k+3}{k+4}\cdot\frac{k+1}{k+2}\int_0^{\pi/2}\sin^k x\,dx,$$

where $k=0$ when n is even and $n\ge 2$, and $k=1$ when n is odd and $n\ge 3$. Thus, we get, respectively:

(a)
$$\begin{aligned}\int_0^{\pi/2}\sin^n x\,dx &= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{3}{4}\cdot\frac{1}{2}\int_0^{\pi/2}\sin^0 x\,dx\\ &= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{3}{4}\cdot\frac{1}{2}\int_0^{\pi/2}dx\\ &= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{3}{4}\cdot\frac{1}{2}\left(\frac{\pi}{2}-0\right) = \frac{\pi}{2}\cdot\frac{1\cdot 3\cdot 5\cdots(n-1)}{2\cdot 4\cdot 6\cdots n}.\end{aligned}$$

(b) $$\int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{4}{5}\cdot\frac{2}{3}\int_0^{\pi/2}\sin^1 x\,dx$$
$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{4}{5}\cdot\frac{2}{3}\left(-\cos x\right)\Big]_0^{\pi/2}$$
$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{4}{5}\cdot\frac{2}{3}\left[0-(-1)\right] = \frac{2\cdot 4\cdot 6\cdots(n-1)}{3\cdot 5\cdot 7\cdots n}.$$

73. $$\int_0^{\pi/2}\sin^8 x\,dx = \frac{\pi}{2}\cdot\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8} = \frac{105\pi}{768} = \frac{35\pi}{256}$$

74. $$\int_0^{\pi/2}\sin^5 x\,dx = \cdot\frac{2\cdot 4}{3\cdot 5} = \frac{8}{15}$$

75. $$\int e^{2x}\tan^{-1}e^x\,dx \qquad \boxed{u=\tan^{-1}e^x,\ du=\frac{e^x}{1+e^{2x}}\,dx;\quad dv=e^{2x}\,dx,\ v=\frac{1}{2}e^{2x}}$$
$$= \frac{1}{2}e^{2x}\tan^{-1}e^x - \int\frac{1}{2}\left(\frac{e^{3x}}{1+e^{2x}}\right)dx \qquad \boxed{t=e^x,\ dt=e^x\,dx}$$
$$= \frac{1}{2}e^{2x}\tan^{-1}e^x - \frac{1}{2}\int\frac{t^2}{1+t^2}\,dt = \frac{1}{2}e^{2x}\tan^{-1}e^x - \frac{1}{2}\int\left(1-\frac{1}{1+t^2}\right)dt$$
$$= \frac{1}{2}e^{2x}\tan^{-1}e^x - \frac{1}{2}t + \frac{1}{2}\tan^{-1}t + C = \frac{1}{2}(e^{2x}+1)\tan^{-1}e^x - \frac{1}{2}e^x + C$$

76. $$\int(\sin^{-1}x)^2\,dx \qquad \boxed{u=(\sin^{-1}x)^2,\ du=\frac{2\sin^{-1}x}{\sqrt{1-x^2}}\,dx;\quad dv=dx,\ v=x}$$
$$= x(\sin^{-1}x)^2 - \int\frac{2x\sin^{-1}x}{\sqrt{1-x^2}}\,dx$$
$$\boxed{u=\sin^{-1}x,\ du=\frac{1}{\sqrt{1-x^2}}\,dx;\quad dv=\frac{2x}{\sqrt{1-x^2}}\,dx,\ v=-2\sqrt{1-x^2}}$$
$$= x(\sin^{-1}x)^2 - \left(-2\sqrt{1-x^2}\sin^{-1}x - \int -2\,dx\right)$$
$$= x(\sin^{-1}x)^2 + 2\sqrt{1-x^2}\sin^{-1}x - 2x + C$$

77. $$\int\frac{xe^x}{(x+1)^2}\,dx \qquad \boxed{u=xe^x,\ du=(x+1)e^x\,dx;\quad dv=\frac{1}{(x+1)^2}\,dx,\ v=-\frac{1}{x+1}}$$
$$= -\frac{x}{x+1}e^x + \int e^x\,dx = \left(1-\frac{x}{x+1}\right)e^x + C = \frac{1}{x+1}e^x + C = \frac{e^x}{x+1} + C$$

78. $$\int\frac{x^2e^x}{(x+2)^2}\,dx \qquad \boxed{u=x^2e^x,\ du=x(x+2)e^x\,dx;\quad dv=\frac{1}{(x+2)^2}\,dx,\ v=-\frac{1}{x+2}}$$
$$= -\frac{1}{x+2}x^2e^x + \int xe^x\,dx \qquad \boxed{u=x,\ du=dx;\quad dv=e^x\,dx,\ v=e^x}$$
$$= -\frac{x^2}{x+2}e^x + xe^x - \int e^x\,dx = \left(x-1-\frac{x^2}{x+2}\right)e^x + C = \frac{x-2}{x+2}e^x + C$$

79. We first compute $\int e^x \sin x\,dx$:

$$\begin{aligned}\int e^x \sin x\,dx \quad & \boxed{u = e^x,\ du = e^x\,dx; \quad dv = \sin x\,dx,\ v = -\cos x} \\ &= -e^x \cos x + \int e^x \cos x\,dx \\ & \boxed{u = e^x,\ du = e^x\,dx; \quad dv = \cos x\,dx,\ v = \sin x} \\ &= -e^x \cos x + e^x \sin x - \int e^x \sin x\,dx.\end{aligned}$$

Solving for the integral, we have $\int e^x \sin x\,dx = \frac{1}{2}e^x(\sin x - \cos x)$. Similarly, $\int e^x \cos x\,dx = \frac{1}{2}e^x(\sin x + \cos x)$. Then

$$\begin{aligned}\int xe^x \sin x\,dx \quad & \boxed{u = x,\ du = dx; \quad dv = e^x \sin x\,dx,\ v = \frac{1}{2}e^x(\sin x - \cos x)} \\ &= \frac{1}{2}xe^x(\sin x - \cos x) - \frac{1}{2}\int (e^x \sin x - e^x \cos x)\,dx \\ &= \frac{1}{2}xe^x(\sin x - \cos x) - \frac{1}{2}\left[\frac{1}{2}e^x(\sin x - \cos x) - \frac{1}{2}e^x(\sin x + \cos x)\right] + C \\ &= \frac{1}{2}xe^x \sin x - \frac{1}{2}xe^x \cos x + \frac{1}{2}e^x \cos x + C.\end{aligned}$$

80. We first compute $\int e^{-x} \cos 2x\,dx$:

$$\begin{aligned}\int e^{-x} \cos 2x\,dx \quad & \boxed{u = e^{-x},\ du = -e^{-x}\,dx; \quad dv = \cos 2x\,dx,\ v = \frac{1}{2}\sin 2x} \\ &= \frac{1}{2}e^{-x} \sin 2x + \frac{1}{2}\int e^{-x} \sin 2x\,dx \\ & \boxed{u = e^{-x},\ du = -e^{-x}\,dx; \quad dv = \sin 2x\,dx,\ v = -\frac{1}{2}\cos 2x} \\ &= \frac{1}{2}e^{-x} \sin 2x + \frac{1}{2}\left(-\frac{1}{2}e^{-x} \cos 2x - \frac{1}{2}\int e^{-x} \cos 2x\,dx\right).\end{aligned}$$

Solving for the integral, we have $\int e^{-x} \cos 2x\,dx = \frac{1}{5}e^{-x}(2\sin 2x - \cos 2x)$. Similarly,

$\int e^{-x}\sin 2x\,dx = -\frac{1}{5}e^{-x}(\sin 2x + 2\cos 2x)$. Then

$$\int xe^{-x}\cos 2x\,dx \qquad \boxed{u = x,\ du = dx;\quad dv = e^{-x}\cos 2x\,dx,\ v = \frac{1}{5}e^{x}(2\sin 2x - \cos 2x)}$$

$$\begin{aligned}
&= \frac{1}{5}xe^{-x}(2\sin 2x - \cos 2x) - \frac{1}{5}\int (2e^{-x}\sin 2x - e^{-x}\cos 2x)\,dx \\
&= \frac{1}{5}xe^{-x}(2\sin 2x - \cos 2x) - \frac{2}{5}\left[-\frac{1}{5}e^{-x}(\sin 2x + 2\cos 2x)\right] \\
&\qquad + \frac{1}{5}\left[\frac{1}{5}e^{-x}(2\sin 2x + \cos 2x)\right] + C \\
&= \frac{2}{5}xe^{-x}\sin 2x - \frac{1}{5}xe^{-x}\cos 2x + \frac{4}{25}e^{-x}\sin 2x + \frac{3}{25}e^{-x}\cos 2x + C.
\end{aligned}$$

81. $\int \ln\left(x + \sqrt{x^2+1}\right)\,dx$

$$\boxed{u = \ln\left(x + \sqrt{x^2+1}\right),\ du = \frac{1 + x/\sqrt{x^2+1}}{x + \sqrt{x^2+1}}\,dx;\quad dv = dx,\ v = x}$$

$$\begin{aligned}
&= x\ln\left(x + \sqrt{x^2+1}\right) - \int \frac{1}{x + \sqrt{x^2+1}}\left(1 + \frac{x}{\sqrt{x^2+1}}\right)x\,dx \\
&= x\ln\left(x + \sqrt{x^2+1}\right) - \int \frac{x - \sqrt{x^2+1}}{x^2 - (x^2+1)}\left(\frac{\sqrt{x^2+1} + x}{\sqrt{x^2+1}}\right)x\,dx \\
&= x\ln\left(x + \sqrt{x^2+1}\right) - \int \left(\sqrt{x^2+1} - x\right)\left(\sqrt{x^2+1} + x\right)\left(\frac{x}{\sqrt{x^2+1}}\right)dx \\
&= x\ln\left(x + \sqrt{x^2+1}\right) - \int \frac{x}{\sqrt{x^2+1}}\,dx \\
&= x\ln\left(x + \sqrt{x^2+1}\right) - \sqrt{x^2+1} + C
\end{aligned}$$

82. $\int e^{\sin^{-1}x}\,dx$ $\quad \boxed{u = e^{\sin^{-1}x},\ du = \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}\,dx;\quad dv = dx,\ v = x}$

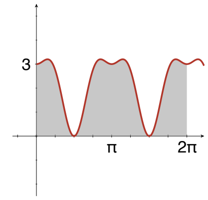
$$= xe^{\sin^{-1}x} - \int \frac{xe^{\sin^{-1}x}}{\sqrt{1-x^2}}\,dx$$

$$\boxed{u = e^{\sin^{-1}x},\ du = \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}\,dx;\quad dv = \frac{x}{\sqrt{1-x^2}}\,dx,\ v = -\sqrt{1-x^2}}$$

$$\begin{aligned}
&= xe^{\sin^{-1}x} - \left(-e^{\sin^{-1}x}\sqrt{1-x^2} - \int -e^{\sin^{-1}x}\,dx\right) \\
&= xe^{\sin^{-1}x} + e^{\sin^{-1}x}\sqrt{1-x^2} - \int e^{\sin^{-1}x}\,dx
\end{aligned}$$

Solving for the integral, we have $\int e^{\sin^{-1}x}\,dx = \frac{e^{\sin^{-1}x}}{2}(x + \sqrt{1-x^2}) + C$.

83. (a) Graph shown at right.



(b) We use the reduction formula

$$\int \sin^n x\,dx = -\frac{\sin^{n-1}x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2}x\,dx$$

with $n = 4$ and $n = 2$.

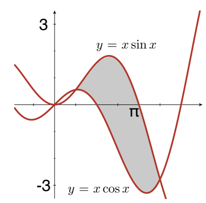
$$A = \int_0^{2\pi} (3 + \sin^2 x - 5\sin^4 x)\,dx$$

$$= 3x\Big]_0^{2\pi} + 2\int_0^{2\pi} \sin^2 x\,dx - 5\left(-\frac{\sin^3 x\cos x}{4}\Big]_0^{2\pi} + \frac{3}{4}\int_0^{2\pi}\sin^2 x\,dx\right)$$

$$= 6\pi + 5\left(-\frac{0}{4} + \frac{0}{4}\right) - \frac{7}{4}\int_0^{2\pi}\sin^2 x\,dx = 6\pi - \frac{7}{4}\left(-\frac{\sin x\cos x}{2}\Big]_0^{2\pi} + \frac{1}{2}\int_0^{2\pi} dx\right)$$

$$= 6\pi - \frac{7}{8}x\Big]_0^{2\pi} = \frac{17\pi}{4}.$$

84. (a) Graph shown at right.



(b) For $x > 0$, the first and second points of intersection of $y = x\sin x$ and $y = x\cos x$ are $x_1 = \pi/4$ and $x_2 = 5\pi/4$.

$$A = \int_{\pi/4}^{5\pi/4} (x\sin x - x\cos x)\,dx = \int_{\pi/4}^{5\pi/4} x(\sin x - \cos x)\,dx$$

$$\boxed{u = x,\ du = dx; \quad dv = (\sin x - \cos x)\,dx,\ v = -\cos x - \sin x}$$

$$= -x(\cos x + \sin x)\Big]_{\pi/4}^{5\pi/4} - \int_{\pi/4}^{5\pi/4} -(\cos x + \sin x)\,dx$$

$$= \left[-\frac{5\pi}{4}\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) + \frac{\pi}{4}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right)\right] + (\sin x - \cos x)\Big]_{\pi/4}^{5\pi/4}$$

$$= \frac{3\pi\sqrt{2}}{2} + \left[\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)\right] = \frac{3\pi\sqrt{2}}{2}$$

7.4 Powers of Trigonometric Functions

1. $\displaystyle\int \sin^{1/2} x\cos x\,dx$ $\quad\boxed{u = \sin x,\ du = \cos x\,dx}$

$$= \int u^{1/2}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(\sin x)^{3/2} + C$$

2. $\displaystyle\int \cos^4 5x\sin 5x\,dx$ $\quad\boxed{u = \cos 5x,\ du = -5\sin 5x\,dx}$

$$= \int u^4\left(-\frac{1}{5}\,du\right) = -\frac{1}{25}u^5 + C = -\frac{1}{25}\cos^5 5x + C$$

3. $$\int \cos^3 x\,dx = \int \cos^2 x \cos x\,dx = \int (1-\sin^2 x)\cos x\,dx = \int \cos x\,dx - \int \sin^2 x(\cos x\,dx)$$
$$= \sin x - \frac{1}{3}\sin^3 x + C$$

4. $$\int \sin^3 4x\,dx = \int \sin^2 4x \sin 4x\,dx = \int (1-\cos^2 4x)\sin 4x\,dx$$
$$= \int \sin 4x\,dx + \frac{1}{4}\int \cos^2 4x(-4\sin 4x\,dx) = \frac{1}{12}\cos^3 4x - \frac{1}{4}\cos 4x + C$$

5. $$\int \sin^5 t\,dt = \int \sin^4 t \sin t\,dt = \int (1-\cos^2 t)^2 \sin t\,dt = \int (1 - 2\cos^2 t + \cos^4 t)\sin t\,dt$$
$$= \int \sin t\,dt + 2\int \cos^2 t(-\sin t\,dt) - \int \cos^4 t(-\sin t\,dt)$$
$$= -\cos t + \frac{2}{3}\cos^3 t - \frac{1}{5}\cos^5 t + C$$

6. $$\int \cos^5 t\,dt = \int \cos^4 t \cos t\,dt = \int (1-\sin^2 t)^2 \cos t\,dt = \int (1 - 2\sin^2 t + \sin^4 t)\cos t\,dt$$
$$= \int \cos t\,dt - 2\int \sin^2 t(\cos t\,dt) + \int \sin^4 t(\cos t\,dt)$$
$$= \sin t - \frac{2}{3}\sin^3 t + \frac{1}{5}\sin^5 t + C$$

7. $$\int \sin^3 x \cos^3 x\,dx = \int \cos^2 x \sin^3 x \cos x\,dx = \int (1-\sin^2 x)\sin^3 x \cos x\,dx$$
$$= \int \sin^3 x(\cos x\,dx) - \int \sin^5 x(\cos x\,dx) = \frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x + C$$

8. $$\int \sin^5 2x \cos^2 2x\,dx = \int \sin^4 2x \cos^2 2x \sin 2x\,dx = \int (1-\cos^2 2x)^2 \cos^2 2x \sin 2x\,dx$$
$$= \int (1 - 2\cos^2 2x + \cos^4 2x)\cos^2 2x \sin 2x\,dx$$

9. $$\int \sin^4 t\,dt = \int (\sin^2 t)^2\,dt = \int \left(\frac{1-\cos 2t}{2}\right)^2 dt = \frac{1}{4}\int (1 - 2\cos 2t + \cos^2 2t)\,dt$$
$$= \frac{1}{4}\int \left(1 - 2\cos 2t + \frac{1+\cos 4t}{2}\right) dt = \frac{1}{4}\int \left(\frac{3}{2} - 2\cos 2t + \frac{1}{2}\cos 4t\right) dt$$
$$= \frac{3}{8}t - \frac{1}{4}\sin 2t + \frac{1}{32}\sin 4t + C$$

10. $$\begin{aligned}\int \cos^6\theta\,d\theta &= \int (\cos^2\theta)^3\,d\theta = \int \left(\frac{1+\cos 2\theta}{2}\right)^3 d\theta\\ &= \frac{1}{8}\int (1+3\cos 2\theta+3\cos^2 2\theta+\cos^3 2\theta)\,d\theta\\ &= \frac{1}{8}\int \left[1+3\cos 2\theta+3\left(\frac{1+\cos 4\theta}{2}\right)+(1-\sin^2 2\theta)\cos 2\theta\right] d\theta\\ &= \frac{1}{8}\int \left(\frac{5}{2}+4\cos 2\theta+\frac{3}{2}\cos 4\theta-\sin^2 2\theta\cos 2\theta\right) d\theta\\ &= \frac{5}{16}\theta+\frac{1}{4}\sin 2\theta+\frac{3}{64}\sin 4\theta-\frac{1}{6}\sin^3 2\theta+C\end{aligned}$$

11. $$\begin{aligned}\int \sin^2 x\cos^4 x\,dx &= \int (1-\cos^2 x)\cos^4 x\,dx = \int (\cos^4 x-\cos^6 x)\,dx\\ &= \int \left[\left(\frac{1+\cos 2x}{2}\right)^2-\left(\frac{1+\cos 2x}{2}\right)^3\right] dx\\ &= \frac{1}{8}\int [2(1+2\cos 2x+\cos^2 2x)-(1+3\cos 2x+3\cos^2 2x+\cos^3 2x)]\,dx\\ &= \frac{1}{8}\int (1+\cos 2x+\cos^2 2x-\cos^3 2x)\,dx\\ &= \frac{1}{8}\int \left[1+\cos 2x-\frac{1+\cos 4x}{2}-(1-\sin^2 2x)\cos 2x\right] dx\\ &= \frac{1}{8}\int \left(\frac{1}{2}-\frac{1}{2}\cos 4x+\sin^2 2x\cos 2x\right) dx\\ &= \frac{1}{16}x-\frac{1}{64}\sin 4x+\frac{1}{48}\sin^3 2x+C\end{aligned}$$

12. $$\begin{aligned}\int \frac{\cos^3 x}{\sin^2 x}\,dx &= \int \frac{\cos^2 x}{\sin^2 x}\cos x\,dx = \int \frac{1-\sin^2 x}{\sin^2 x}\cos x\,dx = \int (\sin x)^{-2}(\cos x\,dx)-\int \cos x\,dx\\ &= -\frac{1}{\sin x}-\sin x+C = -\csc x-\sin x+C\end{aligned}$$

13. $$\begin{aligned}\int \sin^4 x\cos^4 x\,dx &= \frac{1}{16}\int \sin^4 2x\,dx = \frac{1}{16}\int \left(\frac{1-\cos 4x}{2}\right)^2 dx\\ &= \frac{1}{64}\int (1-2\cos 4x+\cos^2 4x)\,dx = \frac{1}{64}\int \left(1-2\cos 4x+\frac{1+\cos 8x}{2}\right) dx\\ &= \frac{1}{64}\int \left(\frac{3}{2}-2\cos 4x+\frac{1}{2}\cos 8x\right) dx\\ &= \frac{3}{128}x-\frac{1}{128}\sin 4x+\frac{1}{1024}\sin 8x+C\end{aligned}$$

14. $$\begin{aligned}\int \sin^2 3x\cos^2 3x\,dx &= \int \left(\frac{1}{2}\sin 6x\right)^2 dx = \frac{1}{4}\int \sin^2 6x\,dx = \frac{1}{4}\int \frac{1-\cos 12x}{2}\,dx\\ &= \frac{1}{8}\int dx-\frac{1}{8}\int \cos 12x\,dx = \frac{1}{8}x-\frac{1}{96}\sin 12x+C\end{aligned}$$

15. $$\int \tan^3 2t \sec^4 2t\,dt = \int \tan^3 2t \sec^2 2t \sec^2 2t\,dt = \int \tan^3 2t(1+\tan^2 2t)\sec^2 2t\,dt$$
$$= \frac{1}{2}\int \tan^3 2t(2\sec^2 2t\,dt) + \frac{1}{2}\int \tan^5 2t(2\sec^2 2t\,dt)$$
$$= \frac{1}{8}\tan^4 2t + \frac{1}{12}\tan^6 2t + C$$

16. $$\int (2-\sqrt{\tan x})^2 \sec^2 x\,dx = \int 4\sec^2 x\,dx - \int 4(\tan x)^{1/2}\sec^2 x\,dx + \int \tan x \sec^2 x\,dx$$
$$= 4\tan x - \frac{8}{3}(\tan x)^{3/2} + \frac{1}{2}\tan^2 x + C$$

17. $$\int \tan^2 x \sec^3 x\,dx = \int (\sec^2 x - 1)\sec^3 x\,dx = \sec^5 x\,dx - \int \sec^3 x\,dx$$
$$\boxed{u = \sec^3 x,\ du = 3\sec^2 x \sec x \tan x\,dx; \quad dv = \sec^2 x\,dx,\ v = \tan x}$$
$$= \tan x \sec^3 x - 3\int \tan^2 x \sec^3 x\,dx - \int \sec^3 x\,dx$$

From Example 5 of Section 7.3, $\int \sec^3 x\,dx = \frac{1}{2}\sec x\tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$, so

$$\int \tan^2 x\sec^3 x\,dx = \tan x\sec^3 x - 3\int \tan^2 x \sec^3 x\,dx - \frac{1}{2}\sec x\tan x - \frac{1}{2}\ln|\sec x + \tan x|.$$

Solving for the integral, we have

$$\int \tan^2 x \sec^3 x\,dx = \frac{1}{4}\tan x\sec^3 x - \frac{1}{8}\sec x\tan x - \frac{1}{8}\ln|\sec x + \tan x| + C.$$

18. $$\int \tan^2 3x \sec^2 3x\,dx = \frac{1}{3}\int \tan^2 3x\sec^2 3x(3\,dx) \qquad \boxed{u = \tan 3x,\ du = \sec^2 3x(3\,dx)}$$
$$= \frac{1}{3}\int u^2\,du = \frac{1}{9}u^3 + C = \frac{1}{9}\tan^3 3x + C$$

19. $$\int \tan^3 x(\sec x)^{-1/2}\,dx = \int \tan^2 x(\sec x)^{-1/2}\tan x\,dx$$
$$= \int (\sec^2 x - 1)(\sec x)^{-3/2}\sec x\tan x\,dx$$
$$= \int (\sec x)^{1/2}(\sec x\tan x\,dx) - \int (\sec x)^{-3/2}(\sec x \tan x\,dx)$$
$$= \frac{2}{3}(\sec x)^{3/2} + 2(\sec x)^{-1/2} + C$$

20. $$\int \tan^3\frac{x}{2}\sec^3\frac{x}{2}\,dx = \int \tan^2\frac{x}{2}\sec^2\frac{x}{2}\sec\frac{x}{2}\tan\frac{x}{2}\,dx = \int \left(\sec^2\frac{x}{2} - 1\right)\sec^2\frac{x}{2}\sec\frac{x}{2}\tan\frac{x}{2}\,dx$$
$$= \int \sec^4\frac{x}{2}\left(\sec\frac{x}{2}\tan\frac{x}{2}\,dx\right) - \int \sec^2\frac{x}{2}\left(\sec\frac{x}{2}\tan\frac{x}{2}\,dx\right)$$
$$= \frac{2}{5}\sec^5\frac{x}{2} - \frac{2}{3}\sec^3\frac{x}{2} + C$$

21. $$\begin{aligned}\int \tan^3 x \sec^5 x\,dx &= \int \tan^2 x \sec^4 x \sec x \tan x\,dx = \int (\sec^2 x - 1)\sec^4 x \sec x \tan x\,dx\\ &= \int \sec^6 x(\sec x \tan x\,dx) - \int \sec^4 x(\sec x \tan x\,dx)\\ &= \frac{1}{7}\sec^7 x - \frac{1}{5}\sec^5 x + C\end{aligned}$$

22. $$\begin{aligned}\int \tan^5 x \sec x\,dx &= \int (\tan^2 x)^2(\sec x \tan x\,dx) = \int (\sec^2 x - 1)^2(\sec x \tan x\,dx)\\ &= \int (\sec^4 x - 2\sec^2 x + 1)(\sec x \tan x\,dx)\\ &= \int \sec^4 x(\sec x \tan x\,dx) - 2\int \sec^2 x(\sec x \tan x\,dx) + \int \sec x \tan x\,dx\\ &= \frac{1}{5}\sec^5 x - \frac{2}{3}\sec^3 x + \sec x + C\end{aligned}$$

23. $$\int \sec^5 x\,dx = \int \sec^3 x \sec^2 x\,dx = \int \sec^3 x(1+\tan^2 x)\,dx = \int \sec^3 x\,dx + \int \tan^2 x \sec^3 x\,dx$$

From Example 5 of Section 7.3, $\int \sec^3 x\,dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$. From Exercise 17, $\int \tan^2 x \sec^3 x\,dx = \frac{1}{4}\tan x \sec^3 x - \frac{1}{8}\sec x \tan x - \frac{1}{8}\ln|\sec x + \tan x| + C_1$.

Thus, $$\begin{aligned}\int \sec^5 x &= \left(\frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x|\right)\\ &\quad + \left(\frac{1}{4}\tan x \sec^3 x - \frac{1}{8}\sec x \tan x - \frac{1}{8}\ln|\sec x + \tan x|\right) + C_2\\ &= \frac{3}{8}\sec x \tan x + \frac{3}{8}\ln|\sec x + \tan x| + \frac{1}{4}\tan x \sec^3 x + C_2.\end{aligned}$$

24. $$\begin{aligned}\int \frac{1}{\cos^4 x}\,dx &= \int \sec^2 x \sec^2 x\,dx = \int (1+\tan^2 x)\sec^2 x\,dx\\ &= \int \sec^2 x\,dx + \int \tan^2 x \sec^2 x\,dx = \tan x + \frac{1}{3}\tan^3 x + C\end{aligned}$$

25. $$\begin{aligned}\int \cos^2 x \cot x\,dx &= \int \frac{\cos^3 x}{\sin x}\,dx = \int \frac{(1-\sin^2 x)}{\sin x}\cos x\,dx\\ &= \int (\sin x)^{-1}(\cos x\,dx) - \int \sin x(\cos x\,dx) = \ln|\sin x| - \frac{1}{2}\sin^2 x + C\end{aligned}$$

(Alternatively, $\int \sin x \cos x\,dx = \int \cos x(\sin x\,dx) = -\frac{1}{2}\cos^2 x + C$ or
$\int \sin x \cos x\,dx = \frac{1}{2}\int \sin 2x\,dx = -\frac{1}{4}\cos 2x + C$).

26. $$\int \sin x \sec^7 x\,dx = \int (\cos x)^{-7}(\sin x\,dx) = \frac{1}{6}(\cos x)^{-6} + C = \frac{1}{6}\sec^6 x + C$$

27. $$\int \cot^{10} x \csc^4 x\,dx = \int \cot^{10} x \csc^2 x \csc^2 x\,dx = \int \cot^{10} x(1+\cot^2 x)\csc^2 x\,dx$$
$$= -\int \cot^{10} x(-\csc^2 x)\,dx - \int \cot^{12} x(-\csc^2 x)\,dx$$
$$= -\frac{1}{11}\cot^{11} x - \frac{1}{13}\cot^{13} x + C$$

28. $$\int (1+\csc^2 t)^2\,dt = \int (1+2\csc^2 t + \csc^2 t\csc^2 t)\,dt = \int [1+2\csc^2 t + (1+\cot^2 t)\csc^2 t]\,dt$$
$$= \int (1+3\csc^2 t + \cot^2 t\csc^2 t)\,dt = t - 3\cot t - \frac{1}{3}\cot^3 t + C$$

29. $$\int \frac{\sec^4(1-t)}{\tan^8(1-t)}\,dt = \int \frac{\sec^2(1-t)}{\tan^8(1-t)}\sec^2(1-t)\,dt = \int \frac{1+\tan^2(1-t)}{\tan^8(1-t)}\sec^2(1-t)\,dt$$
$$= \int [\tan(1-t)]^{-8}\sec^2(1-t)\,dt + \int [\tan(1-t)]^{-6}\sec^2(1-t)\,dt$$
$$= \frac{1}{7}[\tan(1-t)]^{-7} + \frac{1}{5}[\tan(1-t)]^{-5} + C$$
$$= \frac{1}{7\tan^7(1-t)} + \frac{1}{5\tan^5(1-t)} + C$$

30. $$\int \frac{\sin^3\sqrt{t}\cos^2\sqrt{t}}{\sqrt{t}}\,dt \qquad \boxed{u=\sqrt{t},\ du=\frac{1}{2\sqrt{t}}}$$
$$= 2\int \sin^3 u\cos^2 u\,du = 2\int \sin^2 u\cos^2 u\sin u\,du$$
$$= 2\int (1-\cos^2 u)\cos^2 u\sin u\,du$$
$$= 2\int \cos^2 u(\sin u\,du) - 2\int \cos^4 u(\sin u\,du)$$
$$= -\frac{2}{3}\cos^3 u + \frac{2}{5}\cos^5 u + C = -\frac{2}{3}\cos^3\sqrt{t} + \frac{2}{5}\cos^5\sqrt{t} + C$$

31. $$\int (1+\tan x)^2\sec x\,dx = \int (1+2\tan x+\tan^2 x)\sec x\,dx$$
$$= \int \sec x\,dx + 2\int \tan x\sec x\,dx + \int \tan^2 x\sec x\,dx$$

The last integral is evaluated in Example 8 of Section 7.4. Thus,

$$\int (1+\tan x)^2\sec x\,dx = \ln|\sec x+\tan x| + 2\sec x + \frac{1}{2}\sec x\tan x - \frac{1}{2}\ln|\sec x+\tan x| + C$$
$$= \frac{1}{2}\ln|\sec x+\tan x| + 2\sec x + \frac{1}{2}\sec x\tan x + C.$$

32. $$\int (\tan x+\cot x)^2\,dx = \int (\tan^2 x + 2 + \cot^2 x)\,dx = \int (\sec^2 x - 1 + 2 + \csc^2 x - 1)\,dx$$
$$= \int (\sec^2 x + \csc^2 x)\,dx = \tan x - \cot x + C$$

33. $$\int \tan^4 x\,dx = \int \tan^2 x \tan^2 x\,dx = \int \tan^2 x(\sec^2 x - 1)\,dx$$
$$= \int \tan^2 x \sec^2 x\,dx - \int (\sec^2 x - 1)\,dx = \frac{1}{3}\tan^3 x - \tan x + x + C$$

34. $$\int \tan^5 x\,dx = \int (\tan^2 x)^2 \tan x\,dx = \int (\sec^2 x - 1)^2 \tan x\,dx$$
$$= \int (\sec^4 x - 2\sec^2 x + 1)\tan x\,dx$$
$$= \int \sec^3 x(\sec x \tan x\,dx) - 2\int \sec x(\sec x \tan x\,dx) + \int \tan x\,dx$$
$$= \frac{1}{4}\sec^4 x - \sec^2 x - \ln|\cos x| + C$$

35. $$\int \cot^3 t\,dt = \int \cot^2 t \cot t\,dt = \int (\csc^2 t - 1)\cot t\,dt = \int \csc t(\csc t \cot t\,dt) - \int \cot t\,dt$$
$$= -\frac{1}{2}\csc^2 t - \ln|\sin t| + C$$

36. $$\int \csc^5 t\,dt \qquad \boxed{u = \csc^3 t,\ du = -3\csc^2 t \cot t \csc t\,dt; \quad dv = \csc^2 t\,dt,\ v = -\cot t}$$
$$= -\cot t \csc^3 t - 3\int \cot^2 t \csc^3 t\,dt = -\cot t \csc^3 t - 3\int (\csc^2 t - 1)\csc^3 t\,dt$$
$$= -\cot t \csc^3 t - 3\int \csc^5 t\,dt + 3\int \csc^3 t\,dt \qquad \boxed{\text{See Exercises 7.3, Problem 37}}$$
$$= -\cot t \csc^3 t - 3\int \csc^5 t\,dt + 3\left(-\frac{1}{2}\cot t \csc t + \frac{1}{2}\ln|\csc t - \cot t|\right)$$
$$= -\cot t \csc^3 t - \frac{3}{2}\cot t \csc t \frac{3}{2}\ln|\csc t - \cot t| - 3\int \csc^5 t\,dt$$

Solving for the integral, we have

$$\int \csc^5 t\,dt = -\frac{1}{4}\cot t \csc^3 t - \frac{3}{8}\cot t \csc t + \frac{3}{8}\ln|\csc t - \cot t| + C.$$

37. $$\int (\tan^6 x - \tan^2 x)\,dx = \int (\tan^4 x \tan^2 x - \tan^2 x)\,dx = \int (\tan^4 x - 1)\tan^2 x\,dx$$
$$= \int (\tan^4 x - 1)(\sec^2 x - 1)\,dx$$
$$= \int (\tan^4 x \sec^2 x - \tan^4 x - \sec^2 x + 1)\,dx$$
$$= \int \tan^4 x(\sec^2 x\,dx) - \int \tan^4 x\,dx - \int \sec^2 x\,dx + \int dx$$
$$= \frac{1}{5}\tan^5 x - \int \tan^4 x\,dx - \tan x + x$$

From Exercise 33, $\displaystyle\int \tan^4 x = \frac{1}{3}\tan^3 x - \tan x + x + C$, so

$$\begin{aligned}\int (\tan^6 x - \tan^2 x)\,dx &= \frac{1}{5}\tan^5 x - \left(\frac{1}{3}\tan^3 x - \tan x + x\right) - \tan x + x + C_1 \\ &= \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + C_1.\end{aligned}$$

38. $$\begin{aligned}\int \cot 2x \csc^{5/2} 2x\,dx &= \int \csc^{3/2} 2x \csc 2x \cot 2x\,dx = -\frac{1}{2}\int (\csc^{3/2} 2x)(-2\csc 2x \cot 2x\,dx) \\ &\boxed{u = \csc 2x,\ du = -2\csc 2x \cot 2x\,dx} \\ &= -\frac{1}{2}\int u^{3/2}\,du = -\frac{1}{2}\left(\frac{2}{5}u^{5/2}\right) + C = -\frac{1}{5}\csc^{5/2} 2x + C\end{aligned}$$

39. $$\begin{aligned}\int x\sin^3 x^2\,dx &= \int x\sin^2 x^2 \sin x^2\,dx = \int x(1-\cos^2 x^2)\sin x^2\,dx \\ &= \int x\sin x^2\,dx - \int x\cos^2 x^2 \sin x^2\,dx \\ &= \frac{1}{2}\int \sin x^2(2x\,dx) + \frac{1}{2}\int \cos^2 x^2(-2x\sin x^2\,dx) \\ &\boxed{t = x^2,\ dt = 2x\,dx;\quad u = \cos x^2,\ du = -2x\sin x^2\,dx} \\ &= \frac{1}{2}\left(\int \sin t\,dt + \int u^2\,du\right) = \frac{1}{2}\left(-\cos t + \frac{1}{3}u^3\right) + C \\ &= \frac{1}{2}\left(-\cos x^2 + \frac{1}{3}\cos^3 x^2\right) + C = \frac{1}{6}\cos^3 x^2 - \frac{1}{2}\cos x^2 + C\end{aligned}$$

40. $$\begin{aligned}\int x\tan^8 x^2 \sec^2 x^2\,dx &= \frac{1}{2}\int \tan^8 x^2(2x\sec^2 x^2\,dx) \qquad \boxed{u = \tan x^2,\ du = 2x\sec^2 x^2\,dx} \\ &= \frac{1}{2}\int u^8\,du = \frac{1}{2}\left(\frac{1}{9}u^9\right) + C = \frac{1}{18}\tan^9 x^2 + C\end{aligned}$$

41. $$\begin{aligned}\int_{\pi/3}^{\pi/2} \sin^3\theta\sqrt{\cos\theta}\,d\theta &= \int_{\pi/3}^{\pi/2} \sin^2\theta(\cos\theta)^{1/2}\sin\theta\,d\theta = \int_{\pi/3}^{\pi/2} (1-\cos^2\theta)(\cos\theta)^{1/2}\sin\theta\,d\theta \\ &= -\int_{\pi/3}^{\pi/2} (\cos\theta)^{1/2}(-\sin\theta\,d\theta) + \int_{\pi/3}^{\pi/2} (\cos\theta)^{5/2}(-\sin\theta\,d\theta) \\ &= -\frac{2}{3}(\cos\theta)^{3/2}\Big]_{\pi/3}^{\pi/2} + \frac{2}{7}(\cos\theta)^{7/2}\Big]_{\pi/3}^{\pi/2} \\ &= -\frac{2}{3}\left[0 - \left(\frac{1}{2}\right)^{3/2}\right] + \frac{2}{7}\left[0 - \left(\frac{1}{2}\right)^{7/2}\right] = \frac{\sqrt{2}}{6} - \frac{\sqrt{2}}{56} = \frac{25\sqrt{2}}{168}\end{aligned}$$

42. $$\begin{aligned}\int_0^{\pi/2} \sin^5 x \cos^5 x\, dx &= \int_0^{\pi/2} \sin^5 x \cos^4 x \cos x\, dx = \int_0^{\pi/2} \sin^5 x(1-\sin^2 x)^2 \cos x\, dx \\ &= \int_0^{\pi/2} \sin^5 x(1 - 2\sin^2 x + \sin^4 x)\cos x\, dx \\ &= \int_0^{\pi/2} \sin^5 x(\cos x\, dx) - 2\int_0^{\pi/2} \sin^7 x(\cos x\, dx) + \int_0^{\pi/2} \sin^9 x(\cos x\, dx) \\ &= \frac{1}{6}\sin^6 x\Big]_0^{\pi/2} - \frac{1}{4}\sin^8 x\Big]_0^{\pi/2} + \frac{1}{10}\sin^{10} x\Big]_0^{\pi/2} = \frac{1}{6} - \frac{1}{4} + \frac{1}{10} = \frac{1}{60}\end{aligned}$$

43. $$\begin{aligned}\int_0^{\pi} \sin^3 2t\, dt &= \int_0^{\pi} \sin^2 2t \sin 2t\, dt = \int_0^{\pi} (1-\cos^2 2t)\sin 2t\, dt \\ &= \int_0^{\pi} \sin 2t\, dt + \frac{1}{2}\int_0^{\pi} \cos^2 2t(-2\sin 2t\, dt) = -\frac{1}{2}\cos 2t\Big]_0^{\pi} + \frac{1}{6}\cos^3 2t\Big]_0^{\pi} \\ &= -\frac{1}{2}(1-1) + \frac{1}{6}(1-1) = 0\end{aligned}$$

44. $$\begin{aligned}\int_{-\pi}^{\pi} \sin^4 x \cos^2 x\, dx &= \int_{-\pi}^{\pi} \sin^4 x(1-\sin^2 x)\, dx = \int_{-\pi}^{\pi} (\sin^4 x - \sin^6 x)\, dx \\ &= \int_{-\pi}^{\pi} \left[\left(\frac{1-\cos 2x}{2}\right)^2 - \left(\frac{1-\cos 2x}{2}\right)^3\right] dx \\ &= \frac{1}{8}\int_{-\pi}^{\pi} [2(1-2\cos 2x + \cos^2 2x) - (1 - 3\cos 2x + 3\cos^2 2x - \cos^3 2x)]\, dx \\ &= \frac{1}{8}\int_{-\pi}^{\pi} (1 - \cos 2x - \cos^2 2x + \cos^3 2x)\, dx \\ &= \frac{1}{8}\int_{-\pi}^{\pi} \left[1 - \cos 2x - \frac{1+\cos 4x}{2} + (1-\sin^2 2x)\cos 2x\right] dx \\ &= \frac{1}{8}\int_{-\pi}^{\pi} \left(\frac{1}{2} - \frac{1}{2}\cos 4x - \sin^2 2x \cos 2x\right) dx \\ &= \frac{1}{8}\left(\frac{1}{2}x - \frac{1}{8}\sin 4x - \frac{1}{6}\sin^3 2x\right)\Big]_{-\pi}^{\pi} = \frac{1}{16}(\pi+\pi) = \frac{\pi}{8}\end{aligned}$$

45. $$\begin{aligned}\int_0^{\pi/4} \tan y \sec^4 y\, dy &= \int_0^{\pi/4} \tan y \sec^2 y \sec^2 y\, dy = \int_0^{\pi/4} \tan y(1+\tan^2 y)\sec^2 y\, dy \\ &= \int_0^{\pi/4} \tan y \sec^2 y\, dy + \int_0^{\pi/4} \tan^3 y \sec^2 y\, dy \\ &= \frac{1}{2}\tan^2 y\Big]_0^{\pi/4} + \frac{1}{4}\tan^4 y\Big]_0^{\pi/4} = \frac{1}{2}(1-0) + \frac{1}{4}(1-0) = \frac{3}{4}\end{aligned}$$

46. $$\begin{aligned}\int_0^{\pi/3} \tan x \sec^{3/2} x\, dx &= \int_0^{\pi/3} \sec^{1/2} x(\sec x \tan x\, dx) = \frac{2}{3}\sec^{3/2} x\Big]_0^{\pi/3} \\ &= \frac{2}{3}(2\sqrt{2} - 1) = \frac{4\sqrt{2}-2}{3}\end{aligned}$$

47. $$\int \sin x \cos 2x\,dx = \frac{1}{2}\int [\sin 3x + \sin(-x)]\,dx = \frac{1}{2}\int (\sin 3x - \sin x)\,dx$$
$$= -\frac{1}{6}\cos 3x + \frac{1}{2}\cos x + C$$

48. $$\int \cos 3x \cos 5x\,dx = \frac{1}{2}\int (\cos 2x + \cos 8x)\,dx = \frac{1}{4}\sin 2x + \frac{1}{16}\sin 8x + C$$

49. $$\int \sin 2x \sin 4x\,dx = \frac{1}{2}\int (\cos 2x - \cos 6x)\,dx = \frac{1}{4}\sin 2x - \frac{1}{12}\sin 6x + C$$

50. $$\int \frac{5 - 3\sin 2x}{\sec 6x}\,dx = \int (5\cos 6x - 3\sin 2x \cos 6x)\,dx = \frac{5}{6}\sin 6x - \frac{3}{2}\int [\sin 8x + \sin(-4x)]\,dx$$
$$= \frac{5}{6}\sin 6x - \frac{3}{2}\int (\sin 8x - \sin 4x)\,dx = \frac{5}{6}\sin 6x + \frac{3}{16}\cos 8x - \frac{3}{8}\cos 4x + C$$

51. $$\int_0^{\pi/6} \cos 2x \cos x\,dx = \frac{1}{2}\int_0^{\pi/6} (\cos x + \cos 3x)\,dx = \frac{1}{2}\left(\sin x + \frac{1}{3}\sin 3x\right)\Big]_0^{\pi/6}$$
$$= \frac{1}{2}\left(\frac{1}{2} + \frac{1}{3}\right) = \frac{5}{12}$$

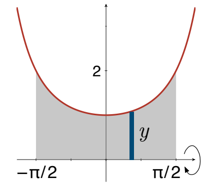
52. $$\int_0^{\pi/2} \sin \frac{3}{2}x \sin \frac{1}{2}x\,dx = \frac{1}{2}\int_0^{\pi/2} (\cos x - \cos 2x)\,dx = \frac{1}{2}\left(\sin x - \frac{1}{2}\sin 2x\right)\Big]_0^{\pi/2}$$
$$= \frac{1}{2}(1 - 0) = \frac{1}{2}$$

53. If $m \neq n$, then using the fact that $\sin mx \sin nx$ is an even function we have
$$\int_{-\pi}^{\pi} \sin mx \sin nx\,dx = \int_0^{\pi} [\cos(m-n)x - \cos(m+n)x]\,dx$$
$$= \frac{1}{m-n}\sin(m-n)x\Big]_0^{\pi} - \frac{1}{m+n}\sin(m+n)x\Big]_0^{\pi} = 0.$$

If $m = n$, then
$$\int_{-\pi}^{\pi} \sin mx \sin nx\,dx = \int_{-\pi}^{\pi} \sin^2 mx\,dx = 2\int_0^{\pi} \sin^2 mx\,dx = \int_0^{\pi} (1 + \cos 2mx)\,dx$$
$$= \left(x + \frac{1}{2m}\sin 2mx\right)\Big]_0^{\pi} = \pi.$$

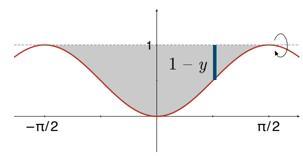
Thus, $\displaystyle\int_{-\pi}^{\pi} \sin mx \sin nx\,dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}.$

54. Since $\sin mx \cos nx$ is an odd function, $\displaystyle\int_{-\pi}^{\pi} \sin mx \sin nx\,dx = 0.$



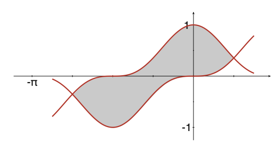
55. $y = \sec^2 \dfrac{x}{2}$

$$V = \pi \int_{-\pi/2}^{\pi/2} \sec^4 \frac{x}{2}\, dx = \pi \int_{-\pi/2}^{\pi/2} \sec^2 \frac{x}{2} \sec^2 \frac{x}{2}\, dx$$
$$= \pi \int_{-\pi/2}^{\pi/2} \left(1 + \tan^2 \frac{x}{2}\right) \sec^2 \frac{x}{2}\, dx$$
$$= 2\pi \left[\int_{-\pi/2}^{\pi/2} \left(\sec^2 \frac{x}{2}\right)\left(\frac{1}{2}\, dx\right) + \int_{-\pi/2}^{\pi/2} \left(\tan^2 \frac{x}{2}\right)\left(\frac{1}{2} \sec^2 \frac{x}{2}\, dx\right)\right]$$
$$= 2\pi \left(\tan \frac{x}{2}\Big]_{-\pi/2}^{\pi/2} + \frac{1}{3} \tan^3 \frac{x}{2}\Big]_{-\pi/2}^{\pi/2}\right) = 2\pi \left[1 - (-1) + \frac{1}{3} - \frac{1}{3}(-1)\right]$$
$$= 2\pi \left(2 + \frac{2}{3}\right) = \frac{16\pi}{3}$$



56. $y = \sin^2 x$

$$V = \pi \int_{-\pi/2}^{\pi/2} (1 - \sin^2 x)^2\, dx = \pi \int_{-\pi/2}^{\pi/2} (\cos^2 x)^2\, dx$$
$$= \pi \int_{-\pi/2}^{\pi/2} \left(\frac{1 + \cos 2x}{2}\right)^2 dx = \frac{\pi}{4} \int_{-\pi/2}^{\pi/2} (1 + 2\cos 2x + \cos^2 2x)\, dx$$
$$= \frac{\pi}{4} \int_{-\pi/2}^{\pi/2} \left(1 + 2\cos 2x + \frac{1 + \cos 4x}{2}\right) dx = \frac{\pi}{4} \int_{-\pi/2}^{\pi/2} \left(\frac{3}{2} + 2\cos 2x + \frac{1}{2}\cos 4x\right) dx$$
$$= \frac{\pi}{4} \left(\frac{3}{2}x + \sin 2x + \frac{1}{8}\sin 4x\right)\Big]_{-\pi/2}^{\pi/2} = \frac{\pi}{4}\left[\left(\frac{3\pi}{4} + 0 + 0\right) - \left(-\frac{3\pi}{4} + 0 + 0\right)\right] = \frac{3\pi^2}{8}$$



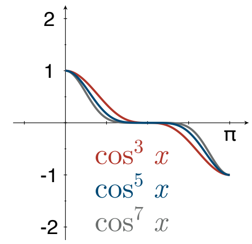
57. $A = \displaystyle\int_{-3\pi/4}^{\pi/4} (\cos^3 x - \sin^3 x)\, dx$

$$= \int_{-3\pi/4}^{\pi/4} (\cos^2 x \cos x - \sin^2 x \sin x)\, dx = \int_{-3\pi/4}^{\pi/4} [(1 - \sin^2 x)\cos x - (1 - \cos^2 x)\sin x]\, dx$$
$$= \int_{-3\pi/4}^{\pi/4} [\cos x - (\sin^2 x)\cos x - \sin x + (\cos^2 x)\sin x]\, dx$$
$$= \left(\sin x - \frac{1}{3}\sin^3 x + \cos x - \frac{1}{3}\cos^3 x\right)\Big]_{-3\pi/4}^{\pi/4}$$
$$= \left[\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{12} + \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{12}\right) - \left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{12} - \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{12}\right)\right] = 2\sqrt{2} - \frac{\sqrt{2}}{3} = \frac{5\sqrt{2}}{3}$$

58. $A = \frac{1}{2}\int_0^{2\pi} r^2\,d\theta = \frac{1}{2}\int_0^{2\pi} \sin^2 4\theta \sin^2 \frac{\theta}{2}\,d\theta = \frac{1}{8}\int_0^{2\pi}(1-\cos 8\theta)(1-\cos\theta)\,d\theta$

$$= \frac{1}{8}\int_0^{2\pi}(1-\cos\theta-\cos 8\theta+\cos 8\theta\cos\theta)\,d\theta$$
$$= \frac{1}{8}\left(\theta-\sin\theta-\frac{1}{8}\sin 8\theta\right)\Big]_0^{2\pi} + \frac{1}{16}\int(\cos 7\theta+\cos 9\theta)\,d\theta$$
$$= \frac{1}{8}(2\pi) + \frac{1}{16}\left(\frac{1}{7}\sin 7\theta+\frac{1}{9}\sin 9\theta\right)\Big]_0^{2\pi} = \frac{\pi}{4}$$

59. Based on the graphs, the values of $\int_0^{\pi}\cos^3 x\,dx$, $\int_0^{\pi}\cos^5 x\,dx$, and $\int_0^{\pi}\cos^7 x\,dx$ all appear to be 0. We note that, for every t such that $0 \le t \le \frac{\pi}{2}$, $cos\left(\frac{\pi}{2}-t\right) = -\cos\left(\frac{\pi}{2}+t\right)$, thus lending credence to this conjecture.



60. Based on Problem 59, we conjecture that the value of $\int_0^{\pi}\cos^n x\,dx$, where n is a positive odd integer, is 0. To prove this, we evaluate $\int_0^{\pi}\cos^n x\,dx$:

$$\int_0^{\pi}\cos^n x\,dx = \int_0^{\pi}\cos^{n-1} x\cos x\,dx$$

Since n is a positive odd integer, $n-1$ is guaranteed to be even, and thus $k = \frac{n-1}{2}$ is an integer. Further, we substitute $\cos^2 x = 1-\sin^2 x$:

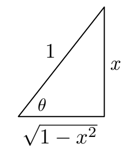
$$\int_0^{\pi}\cos^n x\,dx = \int_0^{\pi}(\cos^2 x)^k\cos x\,dx = \int_0^{\pi}(1-\sin^2 x)^k\cos x\,dx.$$

By the binomial theorem, $(1-\sin^2 x)^k$ expands into an expression of the form $1+c_1\sin^2 x + c_2(\sin^2 x)^2 + c_3(\sin^2 x)^3 + \cdots + c_k(\sin^2 x)^k$. For this particular proof, it is not necessary to specify the precise values of the binary coefficients c_i. Using the u substitution $u = \sin x$, $du = \cos x\,dx$, integration is accomplished as follows:

$$\int_0^{\pi}\cos^n x\,dx = \int_0^{\pi}(1+c_1\sin^2 x+c_2\sin^4 x+\cdots+c_k\sin^{2k} x)(\cos x\,dx)$$
$$= \left(\sin x+\frac{c_1}{3}\sin^3 x+\frac{c_2}{5}\sin^5 x+\cdots+\frac{c_k}{2k+1}\sin^{2k+1} x\right)\Big]_0^{\pi}$$

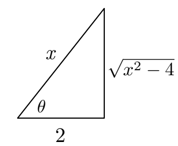
Evaluating this, we note that $\sin\pi = \sin 0 = 0$, and therefore $\int_0^{\pi}\cos^n x\,dx = 0$.

7.5 Trigonometric Substitutions



1. $\displaystyle\int \frac{\sqrt{1-x^2}}{x^2}\,dx \qquad \boxed{x=\sin\theta,\ dx=\cos\theta\,d\theta}$

$$= \int \frac{\sqrt{1-\sin^2\theta}}{\sin^2\theta}\cos\theta\,d\theta = \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta = \int \cot^2 d\theta$$
$$= \int (\csc^2\theta - 1)\,d\theta = -\cot\theta - \theta + C = -\frac{\sqrt{1-x^2}}{x} - \sin^{-1}x + C$$

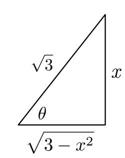


2. $\displaystyle\int \frac{x^3}{\sqrt{x^2-4}}\,dx \qquad \boxed{x=2\sec\theta,\ dx=2\sec\theta\tan\theta\,d\theta}$

$$= \int \frac{8\sec^3\theta}{\sqrt{4\sec^2\theta-4}}(2\sec\theta\tan\theta)\,d\theta = 8\int \sec^4\theta\,d\theta$$
$$= 8\int (1+\tan^2\theta)\sec^2\theta\,d\theta = 8\left(\tan\theta + \frac{1}{3}\tan^3\theta\right) + C$$
$$= 4\sqrt{x^2-4} + \frac{1}{3}(x^2-4)^{3/2} + C$$

3. $\displaystyle\int \frac{1}{\sqrt{x^2-36}}\,dx = \cosh^{-1}\frac{x}{6} + C = \ln\left(x + \sqrt{x^2-36^2}\right) + C, \quad x > 6$

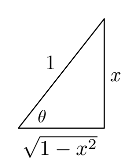
Alternatively, the substitution $x = 6\sec\theta$ could have been used.



4. $\displaystyle\int \sqrt{3-x^2}\,dx \qquad \boxed{x=\sqrt{3}\sin\theta,\ dx=\sqrt{3}\cos\theta\,d\theta}$

$$= \int \sqrt{3-3\sin^2\theta}\sqrt{3}\cos\theta\,d\theta = 3\int \cos^2\theta\,d\theta$$
$$= 3\int \frac{1+\cos 2\theta}{2}\,d\theta = \frac{3}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C = \frac{3}{2}\theta + \frac{3}{2}\sin\theta\cos\theta + C$$
$$= \frac{3}{2}\sin^{-1}\frac{x}{\sqrt{3}} + \frac{3}{2}\left(\frac{x}{\sqrt{3}}\right)\frac{\sqrt{3-x^2}}{\sqrt{3}} + C = \frac{3}{2}\sin^{-1}\frac{x}{\sqrt{3}} + \frac{x}{2}\sqrt{3-x^2} + C$$

5. $\displaystyle\int x\sqrt{x^2+7}\,dx = \frac{1}{2}\int (x^2+7)^{1/2}(2x\,dx) = \frac{1}{3}(x^2+7)^{3/2} + C$



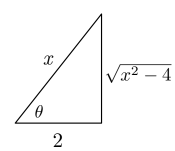
6. $\displaystyle\int (1-x^2)^{3/2}\,dx \qquad \boxed{x=\sin\theta,\ dx=\cos\theta\,d\theta}$

$$= \int (1-\sin^2\theta)^{3/2}\cos\theta\,d\theta = \int \cos^4\theta\,d\theta = \int \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$
$$= \frac{1}{4}\int (1+2\cos 2\theta+\cos^2 2\theta)\,d\theta = \frac{1}{4}\int \left(1+2\cos 2\theta+\frac{1+\cos 4\theta}{2}\right) d\theta$$
$$= \frac{1}{4}\int \left(\frac{3}{2}+2\cos 2\theta+\frac{1}{2}\cos 4\theta\right) d\theta = \frac{3}{8}\theta+\frac{1}{4}\sin 2\theta+\frac{1}{32}\sin 4\theta + C$$
$$= \frac{3}{8}\theta+\frac{1}{4}\sin 2\theta+\frac{1}{16}\sin 2\theta\cos 2\theta + C = \frac{3}{8}\theta+\frac{1}{16}\sin 2\theta(4+\cos 2\theta)+C$$
$$= \frac{3}{8}\theta+\frac{1}{8}\sin\theta\cos\theta(4+1-2\sin^2\theta)+C$$
$$= \frac{3}{8}\theta+\frac{1}{8}\sin\theta\cos\theta(5-2\sin^2\theta)+C$$
$$= \frac{3}{8}\sin^{-1}x+\frac{1}{8}x\sqrt{1-x^2}(5-2x^2)+C$$

7. $$\int x^3\sqrt{1-x^2}\,dx = -\frac{1}{2}\int x^2\sqrt{1-x^2}(-2x\,dx) \qquad \boxed{u=1-x^2,\ x^2=1-u,\ 2x\,dx=-du}$$
$$= -\frac{1}{2}\int (1-u)u^{1/2}\,du = -\frac{1}{2}\int (u^{1/2}-u^{3/2})\,du$$
$$= -\frac{1}{2}\left(\frac{2}{3}u^{3/2}-\frac{2}{5}u^{5/2}\right)+C = -\frac{1}{3}(1-x^2)^{3/2}+\frac{1}{5}(1-x^2)^{5/2}+C$$

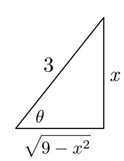
8. $$\int x^3\sqrt{x^2-1}\,dx = \frac{1}{2}\int x^2\sqrt{x^2-1}(2x\,dx) \qquad \boxed{u=x^2-1,\ x^2=u+1,\ 2x\,dx=du}$$
$$= \frac{1}{2}\int (u+1)u^{1/2}\,du = \frac{1}{2}\int (u^{3/2}+u^{1/2})\,du = \frac{1}{2}\left(\frac{2}{5}u^{5/2}+\frac{2}{3}u^{3/2}\right)+C$$
$$= \frac{1}{5}(x^2-1)^{5/2}+\frac{1}{3}(x^2-1)^{3/2}+C$$

9. $$\int \frac{1}{(x^2-4)^{3/2}}\,dx \qquad \boxed{x=2\sec\theta,\ dx=2\sec\theta\tan\theta\,d\theta}$$



$$= \int \frac{2\sec\theta\tan\theta}{(4\sec^2\theta-4)^{3/2}}\,d\theta$$
$$= \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta = \frac{1}{4}\int (\sin\theta)^{-2}\cos\theta\,d\theta$$
$$= -\frac{1}{4}(\sin\theta)^{-1}+C = -\frac{1}{4}\csc\theta + C = -\frac{1}{4}\frac{x}{\sqrt{x^2-4}}+C$$

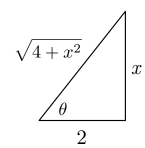
10. $$\int (9-x^2)^{3/2}\,dx \qquad \boxed{x=3\sin\theta,\ dx=3\cos\theta\,d\theta}$$



$$= \int (9-9\sin^2\theta)^{3/2}(3\cos\theta\,d\theta)$$
$$= -81\int (1-\sin^2\theta)^{3/2}(-\cos\theta\,d\theta) = 81\int \cos^4\theta\,d\theta$$
$$= 81\int (\cos^2\theta)^2\,d\theta = 81\int \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

$$= \frac{81}{4}\int (1 + 2\cos 2\theta + \cos^2 2\theta)\,d\theta = \frac{81}{4}\int \left(1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right) d\theta$$
$$= \frac{81}{4}\int \left(\frac{3}{2} + 2\cos 2\theta + \frac{1}{2}\cos 4\theta\right) d\theta = \frac{81}{4}\left(\frac{3}{2}\theta + \sin 2\theta + \frac{1}{8}\sin 4\theta\right) + C$$
$$= \frac{243}{8}\theta + \frac{81}{4}\sin 2\theta + \frac{81}{32}\sin 4\theta + C$$
$$= \frac{243}{8}\theta + \frac{81}{2}\sin\theta\cos\theta + \frac{81}{16}\sin 2\theta\cos 2\theta + C$$
$$= \frac{243}{8}\theta + \frac{81}{2}\sin\theta\cos\theta + \frac{81}{8}\sin\theta\cos\theta(1 - 2\sin^2\theta) + C$$
$$= \frac{243}{8}\theta + 81\sin\theta\cos\theta\left(\frac{1}{2} + \frac{1}{8} - \frac{1}{4}\sin^2\theta\right) + C$$
$$= \frac{243}{8}\theta + 81\sin\theta\cos\theta\left(\frac{5}{8} - \frac{1}{4}\sin^2\theta\right) + C$$
$$= \frac{243}{8}\sin^{-1}\frac{x}{3} + 9x\sqrt{9-x^2}\left(\frac{5}{8} - \frac{1}{36}x^2\right) + C$$
$$= \frac{243}{8}\sin^{-1}\frac{x}{3} + \frac{1}{8}x\sqrt{9-x^2}(45 - 2x^2) + C$$

11. $\displaystyle\int \sqrt{x^2+4}\,dx$ $\quad\boxed{x = 2\tan\theta,\ dx = 2\sec^2\theta\,d\theta}$



$$= \int \sqrt{4\tan^2\theta + 4}\;2\sec^2\theta\,d\theta = 4\int \sec^3\theta\,d\theta$$

$\boxed{\text{See Section 7.3, Example 5}}$

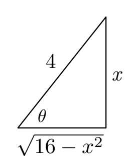
$$= 2\sec\theta\tan\theta + 2\ln|\sec\theta + \tan\theta| + C$$
$$= 2\frac{\sqrt{x^2+4}}{2}\left(\frac{x}{2}\right) + 2\ln\left|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right| + C$$
$$= \frac{x}{2}\sqrt{x^2+4} + 2\ln\left|\sqrt{x^2+4} + x\right| + C_1$$

12. $\displaystyle\int \frac{x}{25+x^2}\,dx = \frac{1}{2}\int \frac{1}{25+x^2}(2x\,dx) = \frac{1}{2}\ln(25+x^2) + C$

13. $\displaystyle\int \frac{1}{\sqrt{25-x^2}}\,dx = \sin^{-1}\frac{x}{5} + C$

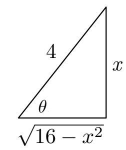
14. $\displaystyle\int \frac{1}{x\sqrt{x^2-25}}\,dx = \frac{1}{5}\sec^{-1}\frac{|x|}{5} + C$

15. $\displaystyle\int \frac{1}{x\sqrt{16-x^2}}\,dx$ $\quad\boxed{x = 4\sin\theta,\ dx = 4\cos\theta\,d\theta}$



$$= \int \frac{4\cos\theta}{4\sin\theta\sqrt{16 - 16\sin^2\theta}}\,d\theta = \frac{1}{4}\int \csc\theta\,d\theta$$
$$= \frac{1}{4}\ln|\csc\theta - \cot\theta| + C = \frac{1}{4}\ln\left|\frac{4}{x} - \frac{\sqrt{16-x^2}}{x}\right| + C$$

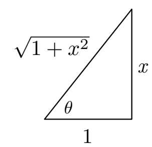
16. $\displaystyle\int \frac{1}{x^2\sqrt{16-x^2}}\,dx \qquad \boxed{x = 4\sin\theta,\ dx = 4\cos\theta\,d\theta}$



$$= \int \frac{4\cos\theta}{16\sin^2\theta\sqrt{16-16\sin^2\theta}} = \frac{1}{16}\int \csc^2\theta\,d\theta$$

$$= -\frac{1}{16}\cot\theta + C = -\frac{1}{16}\left(\frac{\sqrt{16-x^2}}{x}\right) + C$$

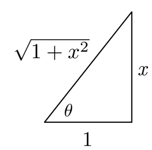
17. $\displaystyle\int \frac{1}{x\sqrt{1+x^2}}\,dx \qquad \boxed{x = \tan\theta,\ dx = \sec^2\theta\,d\theta}$



$$= \int \frac{\sec^2\theta}{\tan\theta\sqrt{1+\tan^2\theta}}\,d\theta = \int \frac{\sec\theta}{\tan\theta}\,d\theta = \int \csc\theta\,d\theta$$

$$= \ln|\csc\theta - \cot\theta| + C = \ln\left|\frac{\sqrt{1+x^2}}{x} - \frac{1}{x}\right| + C$$

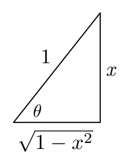
18. $\displaystyle\int \frac{1}{x^2\sqrt{1+x^2}}\,dx \qquad \boxed{x = \tan\theta,\ dx = \sec^2\theta\,d\theta}$



$$= \int \frac{\sec^2\theta}{\tan^2\theta\sqrt{1+\tan^2\theta}}\,d\theta = \int \frac{\sec\theta}{\tan^2\theta}\,d\theta$$

$$= \int (\sin\theta)^{-2}\cos\theta\,d\theta = -(\sin\theta)^{-1} + C = -\csc\theta + C = -\frac{\sqrt{1+x^2}}{x} + C$$

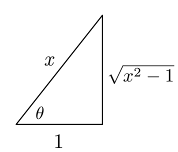
19. $\displaystyle\int \frac{\sqrt{1-x^2}}{x^4}\,dx \qquad \boxed{x = \sin\theta,\ dx = \cos\theta\,d\theta}$



$$= \int \frac{\sqrt{1-\sin^2\theta}}{\sin^4\theta}\cos\theta\,d\theta = \int \frac{\cos^2\theta}{\sin^4\theta}\,d\theta = \int \cot^2\theta\csc^2\theta\,d\theta$$

$$= -\frac{1}{3}\cot^3\theta + C = -\frac{1}{3}\left(\frac{\sqrt{1-x^2}}{x}\right)^3 + C = -\frac{1}{3x^3}(1-x^2)^{3/2} + C$$

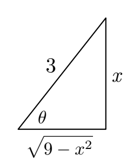
20. $\displaystyle\int \frac{\sqrt{x^2-1}}{x^4}\,dx \qquad \boxed{x = \sec\theta,\ dx = \sec\theta\tan\theta\,d\theta}$



$$= \int \frac{\tan\theta}{\sec^4\theta}\sec\theta\tan\theta\,d\theta = \int \frac{\tan^2\theta}{\sec^3\theta}\,d\theta = \int \sin^2\theta\cos\theta\,d\theta$$

$$= \frac{1}{3}\sin^3\theta + C = \frac{1}{3}\left(\frac{\sqrt{x^2-1}}{x}\right)^3 + C = \frac{1}{3x^3}(x^2-1)^{3/2} + C$$

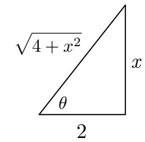
21. $\displaystyle\int \frac{x^2}{(9-x^2)^{3/2}}\,dx \qquad \boxed{x = 3\sin\theta,\ dx = 3\cos\theta\,d\theta}$



$$= \int \frac{9\sin^2\theta}{(9-9\sin^2\theta)^{3/2}}3\cos\theta\,d\theta = \int \frac{\sin^2\theta}{\cos^2\theta}\,d\theta = \int \tan^2\theta\,d\theta$$

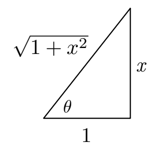
$$= \int (\sec^2\theta - 1)\,d\theta = \tan\theta - \theta + C = \frac{x}{\sqrt{9-x^2}} - \sin^{-1}\frac{x}{3} + C$$

22. $\displaystyle\int \frac{x^2}{(4+x^2)^{3/2}}\,dx \qquad \boxed{x = 2\tan\theta,\ dx = 2\sec^2\theta\,d\theta}$



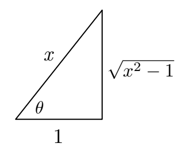
$$\begin{aligned}
&= \int \frac{4\tan^2\theta}{(4+4\tan^2\theta)^{3/2}} 2\sec^2\theta\,d\theta = \int \frac{\tan^2\theta}{\sec\theta}\,d\theta \\
&= \int \frac{\sec^2\theta - 1}{\sec\theta}\,d\theta = \int \sec\theta\,d\theta - \int \cos\theta\,d\theta = \ln|\sec\theta + \tan\theta| - \sin\theta + C \\
&= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| - \frac{x}{\sqrt{4+x^2}} + C = \ln\left|\sqrt{4+x^2} + x\right| - \frac{x}{\sqrt{4+x^2}} + C_1
\end{aligned}$$

23. $\displaystyle\int \frac{1}{(1+x^2)^2}\,dx \qquad \boxed{x = \tan\theta,\ dx = \sec^2\theta\,d\theta}$



$$\begin{aligned}
&= \int \frac{\sec^2\theta}{(1+\tan^2\theta)^2}\,d\theta = \int \frac{1}{\sec^2\theta}\,d\theta = \int \cos^2\theta\,d\theta \\
&= \frac{1}{2}\int (1+\cos 2\theta)\,d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C = \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + C \\
&= \frac{1}{2}\tan^{-1}x + \frac{1}{2}\left(\frac{x}{\sqrt{1+x^2}}\right)\frac{1}{\sqrt{1+x^2}} + C = \frac{1}{2}\tan^{-1}x + \frac{1}{2}\left(\frac{x}{1+x^2}\right) + C
\end{aligned}$$

24. $\displaystyle\int \frac{x^2}{(x^2-1)^2}\,dx \qquad \boxed{x = \sec\theta,\ dx = \sec\theta\tan\theta\,d\theta}$

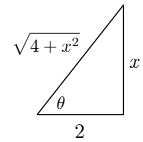


$$\begin{aligned}
&= \int \frac{\sec^2\theta}{(\sec^2\theta - 1)^2}\sec\theta\tan\theta\,d\theta = \int \frac{\sec^3\theta}{\tan^3\theta} = \int \csc^3\theta\,d\theta \\
&\qquad \boxed{u = \csc\theta,\ du = -\csc\theta\cot\theta\,d\theta; \quad dv = \csc^2\theta\,d\theta,\ v = -\cot\theta} \\
&= -\csc\theta\cot\theta - \int \csc\theta\cot^2\theta\,d\theta = -\csc\theta\cot\theta - \int \csc\theta(\csc^2\theta - 1)\,d\theta \\
&= -\csc\theta\cot\theta - \int \csc^3\theta\,d\theta + \int \csc\theta\,d\theta \\
&= -\csc\theta\cot\theta - \int \csc^3\theta\,d\theta + \ln|\csc\theta - \cot\theta|
\end{aligned}$$

Solving for the integral $\displaystyle\int \csc^3\theta\,d\theta$, we have

$$\begin{aligned}
\int \csc^3\theta\,d\theta &= -\frac{1}{2}\csc\theta\cot\theta + \frac{1}{2}\ln|\csc\theta - \cot\theta| + C \\
&= -\frac{1}{2}\left(\frac{x}{\sqrt{x^2-1}}\right)\frac{1}{\sqrt{x^2-1}} + \frac{1}{2}\ln\left|\frac{x}{\sqrt{x^2-1}} - \frac{1}{\sqrt{x^2-1}}\right| + C \\
&= -\frac{1}{2}\left(\frac{x}{\sqrt{x^2-1}}\right) + \frac{1}{2}\ln\left|\frac{x-1}{\sqrt{x^2-1}}\right| + C.
\end{aligned}$$

25. $\displaystyle\int \frac{1}{(4+x^2)^{5/2}}\,dx \qquad \boxed{x = 2\tan\theta,\ dx = 2\sec^2\theta\,d\theta}$

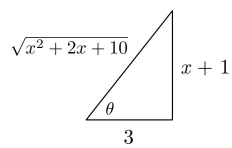


$$\begin{aligned}
&= \int \frac{2\sec^2\theta}{(4+4\tan^2\theta)^{5/2}}\,d\theta = \frac{1}{16}\int \frac{1}{\sec^3\theta}\,d\theta = \frac{1}{16}\int \cos^3\theta\,d\theta \\
&= \frac{1}{16}\int \cos^2\theta\cos\theta\,d\theta = \frac{1}{16}\int (1-\sin^2\theta)\cos\theta\,d\theta \\
&= \frac{1}{16}\int \cos\theta - \frac{1}{16}\int \sin^2\theta\cos\theta\,d\theta = \frac{1}{16}\sin\theta - \frac{1}{48}\sin^3\theta + C \\
&= \frac{1}{16}\left(\frac{x}{\sqrt{4+x^2}}\right) - \frac{1}{48}\left[\frac{x^3}{(4+x^2)^{3/2}}\right] + C
\end{aligned}$$

26. $\displaystyle\int \frac{x^3}{(1-x^2)^{5/2}}\,dx = \int \frac{x^2}{(1-x^2)^{5/2}}(x\,dx) \qquad \boxed{u = 1-x^2,\ du = -2x\,dx}$

$$\begin{aligned}
&= \int \frac{1-u}{u^{5/2}}\left(-\frac{1}{2}\,du\right) = -\frac{1}{2}\int (u^{-5/2} - u^{-3/2})\,du \\
&= -\frac{1}{2}\left(-\frac{2}{3}u^{-3/2} + 2u^{-1/2}\right) + C = \frac{1}{3}(1-x^2)^{-3/2} - (1-x^2)^{-1/2} + C
\end{aligned}$$

27. $\displaystyle\int \frac{1}{\sqrt{x^2+2x+10}}\,dx = \int \frac{1}{\sqrt{(x+1)^2+9}}\,dx$



$\boxed{x+1 = 3\tan\theta,\ dx = 3\sec^2\theta\,d\theta}$

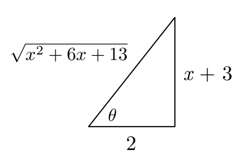
$$\begin{aligned}
&= \int \frac{3\sec^2\theta}{\sqrt{9\tan^2\theta+9}}\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta+\tan\theta| + C \\
&= \ln\left|\frac{\sqrt{x^2+2x+10}}{3} + \frac{x+1}{3}\right| + C = \ln\left|\sqrt{x^2+2x+10} + x + 1\right| + C_1
\end{aligned}$$

28. $\displaystyle\int \frac{x}{\sqrt{4x-x^2}}\,dx = \int \frac{x}{\sqrt{4-(x-2)^2}}\,dx = \int \frac{x-2}{\sqrt{4-(x-2)^2}}\,dx + \int \frac{2}{\sqrt{4-(x-2)^2}}\,dx$

$\boxed{u = 4-(x-2)^2,\ du = -2(x-2)\,dx}$

$$\begin{aligned}
&= -\frac{1}{2}\int \frac{1}{u^{1/2}}\,du + 2\sin^{-1}\frac{x-2}{2} = -u^{1/2} + 2\sin^{-1}\frac{x-2}{2} + C \\
&= -\sqrt{4-(x-2)^2} + 2\sin^{-1}\frac{x-2}{2} + C
\end{aligned}$$

29. $\displaystyle\int \frac{1}{(x^2+6x+13)^2}\,dx = \int \frac{1}{[(x+3)^2+4]^2}\,dx$

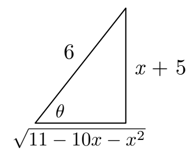


$\boxed{x+3 = 2\tan\theta,\ dx = 2\sec^2\theta\,d\theta}$

$$= \int \frac{2\sec^2\theta}{(4\tan^2\theta+4)^2}\,d\theta = \int \frac{\sec^2\theta}{8\sec^4\theta}\,d\theta = \frac{1}{8}\int \cos^2\theta\,d\theta$$

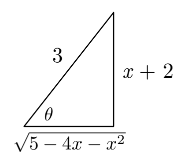
$$= \frac{1}{16}\int (1+\cos 2\theta)\,d\theta = \frac{1}{16}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$
$$= \frac{1}{16}\theta + \frac{1}{16}\sin\theta\cos\theta + C$$
$$= \frac{1}{16}\tan^{-1}\frac{x+3}{2} + \frac{1}{16}\left(\frac{x+3}{\sqrt{x^2+6x+13}}\right)\frac{2}{\sqrt{x^2+6x+13}} + C$$
$$= \frac{1}{16}\tan^{-1}\frac{x+3}{2} + \frac{x+3}{8(x^2+6x+13)} + C$$

30. $$\int \frac{1}{(11-10x-x^2)^2}\,dx = \int \frac{1}{[36-(x+5)^2]^2}\,dx$$



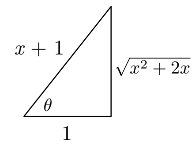
$$\boxed{x+5 = 6\sin\theta,\ \ dx = 6\cos\theta\,d\theta}$$
$$= \int \frac{6\cos\theta}{(36-36\sin^2\theta)^2}\,d\theta = \int \frac{\cos\theta}{216\cos^4\theta}\,d\theta$$
$$= \frac{1}{216}\int \sec^3\theta\,d\theta \qquad \boxed{\text{See Section 7.3, Example 5}}$$
$$= \frac{1}{216}\left(\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta|\right) + C$$
$$= \frac{1}{432}\left(\frac{6}{\sqrt{11-10x-x^2}}\right)\frac{x+5}{\sqrt{11-10x-x^2}}$$
$$+ \frac{1}{432}\ln\left|\frac{6}{\sqrt{11-10x-x^2}} + \frac{x+5}{\sqrt{11-10x-x^2}}\right| + C$$
$$= \frac{x+5}{72(11-10x-x^2)} + \frac{1}{432}\ln\left|\frac{x+11}{\sqrt{11-10x-x^2}}\right| + C$$

31. $$\int \frac{x-3}{(5-4x-x^2)^{3/2}}\,dx = \int \frac{x-3}{[9-(x+2)^2]^{3/2}}\,dx$$



$$\boxed{x+2 = 3\sin\theta,\ \ dx = 3\cos\theta\,d\theta}$$
$$= \int \frac{3\sin\theta - 5}{(9-9\sin^2\theta)^{3/2}}(3\cos\theta\,d\theta)$$
$$= \int \frac{9\sin\theta\cos\theta - 15\cos\theta}{27\cos^3\theta}\,d\theta$$
$$= \frac{1}{3}\int \frac{\sin\theta}{\cos^2\theta}\,d\theta - \frac{5}{9}\int \frac{1}{\cos^2\theta}\,d\theta = \frac{1}{3}\int \tan\theta\sec\theta\,d\theta - \frac{5}{9}\int \sec^2\theta\,d\theta$$
$$= \frac{1}{3}\sec\theta - \frac{5}{9}\tan\theta + C$$
$$= \frac{1}{3}\left(\frac{3}{\sqrt{5-4x-x^2}}\right) - \frac{5}{9}\left(\frac{x+2}{\sqrt{5-4x-x^2}}\right) + C$$
$$= \frac{-5x-1}{9\sqrt{5-4x-x^2}} + C$$

32. $$\int \frac{1}{(x^2+2x)^{3/2}}\,dx = \int \frac{1}{[(x+1)^2-1]^{3/2}}\,dx \quad \boxed{x+1=\sec\theta,\ dx=\sec\theta\tan\theta\,d\theta}$$



$$= \int \frac{\sec\theta\tan\theta}{(\sec^2\theta-1)^{3/2}}\,d\theta = \int \frac{\sec\theta}{\tan^2\theta}\,d\theta = \int (\sin\theta)^{-2}\cos\theta\,d\theta$$

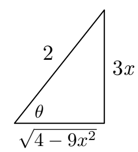
$$= -(\sin\theta)^{-1} + C = -\csc\theta + C = -\frac{x+1}{\sqrt{x^2+2x}} + C$$

33. $$\int \frac{2x+4}{x^2+4x+13}\,dx = \ln(x^2+4x+13) + C$$

34. $$\int \frac{1}{4+(x-3)^2}\,dx = \frac{1}{2}\tan^{-1}\frac{x-3}{2} + C$$

35. $$\int \frac{x^2}{x^2+16}\,dx = \int \left(1 - \frac{16}{x^2+16}\right)dx = x - 4\tan^{-1}\frac{x}{4} + C$$

36. $$\int \frac{\sqrt{4-9x^2}}{x}\,dx \quad \boxed{3x=2\sin\theta,\ 3\,dx=2\cos\theta\,d\theta}$$



$$= \int \frac{\sqrt{4-4\sin^2\theta}}{\frac{2}{3}\sin\theta}\left(\frac{2}{3}\cos\theta\,d\theta\right)$$

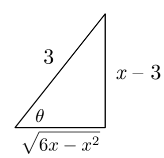
$$= 2\int \frac{\cos^2\theta}{\sin\theta}\,d\theta = 2\int \frac{1-\sin^2\theta}{\sin\theta}\,d\theta$$

$$= 2\int \csc\theta\,d\theta - 2\int \sin\theta\,d\theta = 2\ln|\csc\theta - \cot\theta| + 2\cos\theta + C$$

$$= 2\ln\left|\frac{2}{3x} - \frac{\sqrt{4-9x^2}}{3x}\right| + 2\frac{\sqrt{4-9x^2}}{2} + C$$

$$= 2\ln\left|\frac{2-\sqrt{4-9x^2}}{3x}\right| + \sqrt{4-9x^2} + C$$

37. $$\int \sqrt{6x-x^2}\,dx = \int \sqrt{9-(x-3)^2}\,dx \quad \boxed{x-3=3\sin\theta,\ dx=3\cos\theta\,d\theta}$$



$$= \int \sqrt{9-9\sin^2\theta}\,3\cos\theta\,d\theta = 9\int \cos^2\theta\,d\theta$$

$$= \frac{9}{2}\int (1+\cos 2\theta)\,d\theta = \frac{9}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{9}{2}\theta + \frac{9}{2}\sin\theta\cos\theta + C = \frac{9}{2}\sin^{-1}\frac{x-3}{3} + \frac{9}{2}\left(\frac{x-3}{3}\right)\frac{\sqrt{6x-x^2}}{3} + C$$

$$= \frac{9}{2}\sin^{-1}\frac{x-3}{3} + \frac{1}{2}(x-3)\sqrt{6x-x^2} + C$$

38. $$\int \frac{1}{\sqrt{6x-x^2}}\,dx = \int \frac{1}{\sqrt{9-(x-3)^2}}\,dx \quad \boxed{u=x-3,\ du=dx}$$

$$= \int \frac{1}{\sqrt{3^2-u^2}}\,du = \sin^{-1}\frac{u}{3} + C = \sin^{-1}\frac{x-3}{3} + C$$

39. $\displaystyle\int_{-1}^{1} \sqrt{4-x^2}\,dx \qquad \boxed{x = 2\sin\theta,\ dx = 2\cos\theta\,d\theta}$

$$= \int_{-\pi/6}^{\pi/6} \sqrt{4-4\sin^2\theta}\;2\cos\theta\,d\theta = 4\int_{-\pi/6}^{\pi/6} \cos^2\theta\,d\theta$$
$$= \int_{-\pi/6}^{\pi/6} (2+2\cos 2\theta)\,d\theta = (2\theta + \sin 2\theta)\Big]_{-\pi/6}^{\pi/6}$$
$$= \left[\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right) - \left(-\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)\right] = \frac{2\pi + 3\sqrt{3}}{3}$$

40. $\displaystyle\int_{-1}^{\sqrt{3}} \frac{x^2}{\sqrt{4-x^2}}\,dx \qquad \boxed{x = 2\sin\theta,\ dx = 2\cos\theta\,d\theta} \qquad = \int_{-\pi/6}^{\pi/3} \frac{4\sin^2\theta}{\sqrt{4-4\sin^2\theta}}\;2\cos\theta\,d\theta$

$$= 4\int_{-\pi/6}^{\pi/3} \sin^2\theta\,d\theta = \int_{-\pi/6}^{\pi/3} (2-2\cos 2\theta)\,d\theta = (2\theta - \sin 2\theta)\Big]_{-\pi/6}^{\pi/3}$$
$$= \left[\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) - \left(-\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right)\right] = \pi - \sqrt{3}$$

41. $\displaystyle\int_{0}^{5} \frac{1}{(x^2+25)^{3/2}}\,dx \qquad \boxed{x = 5\tan\theta,\ dx = 5\sec^2\theta\,d\theta}$

$$= \int_{0}^{\pi/4} \frac{5\sec^2\theta}{(25\tan^2\theta + 25)^{3/2}}\,d\theta = \frac{1}{25}\int_{0}^{\pi/4} \frac{1}{\sec\theta}\,d\theta = \frac{1}{25}\int_{0}^{\pi/4} \cos\theta\,d\theta$$
$$= \frac{1}{25}\sin\theta\Big]_{0}^{\pi/4} = \frac{\sqrt{2}}{50}$$

42. $\displaystyle\int_{\sqrt{2}}^{2} \frac{1}{x^3\sqrt{x^2-1}}\,dx \qquad \boxed{x = \sec\theta,\ dx = \sec\theta\tan\theta\,d\theta}$

$$= \int_{\pi/4}^{\pi/3} \frac{\sec\theta\tan\theta}{\sec^3\theta\sqrt{\sec^2\theta - 1}}\,d\theta = \int_{\pi/4}^{\pi/3} \frac{\tan\theta}{\sec^2\theta\tan\theta}\,d\theta = \int_{\pi/4}^{\pi/3} \cos^2\theta\,d\theta$$
$$= \frac{1}{2}\int_{\pi/4}^{\pi/3} (1+\cos 2\theta)\,d\theta = \frac{1}{2}\left(\theta + \frac{1}{2}\sin^2\theta\right)\Big]_{\pi/4}^{\pi/3}$$
$$= \frac{1}{2}\left[\left(\frac{\pi}{3} + \frac{\sqrt{3}}{4}\right) - \left(\frac{\pi}{4} + \frac{1}{2}\right)\right] = \frac{\pi + 3\sqrt{3} - 6}{24}$$

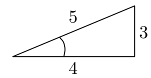
43. $\displaystyle\int_{1}^{6/5} \frac{16}{x^4\sqrt{4-x^2}}\,dx \qquad \boxed{x = 2\sin\theta,\ dx = 2\cos\theta\,d\theta}$

$$= \int_{\pi/6}^{\sin^{-1}(3/5)} \frac{32\cos\theta}{16\sin^4\theta\sqrt{4-4\sin^2\theta}}\,d\theta = \int_{\pi/6}^{\sin^{-1}(3/5)} \frac{2\cos\theta}{\sin^4(2\cos\theta)}\,d\theta$$
$$= \int_{\pi/6}^{\sin^{-1}(3/5)} \csc^2\theta\csc^2\theta\,d\theta = \int_{\pi/6}^{\sin^{-1}(3/5)} (1+\cot^2\theta)\csc^2\theta\,d\theta$$

$$= \int_{\pi/6}^{\sin^{-1}(3/5)} \csc^2\theta\,d\theta + \int_{\pi/6}^{\sin^{-1}(3/5)} \cot^2\theta\csc^2\theta\,d\theta$$

$$= -\cot\theta\Big]_{\pi/6}^{\sin^{-1}(3/5)} - \frac{1}{3}\cot^3\theta\Big]_{\pi/6}^{\sin^{-1}(3/5)}$$

$$= -[\cot(\sin^{-1}3/5) - \sqrt{3}] - \frac{1}{3}[\cot^3(\sin^{-1}3/5) - 3\sqrt{3}]$$



$$= -\left(\frac{4}{3} - \sqrt{3}\right) - \frac{1}{3}\left(\frac{64}{27} - 3\sqrt{3}\right) = 2\sqrt{3} - \frac{172}{81}$$

44. $\displaystyle\int_0^{1/2} x^3(1+x^2)^{-1/2}\,dx$ $\boxed{x = \tan\theta,\ dx = \sec^2\theta\,d\theta}$

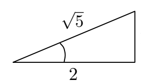
$$= \int_0^{\tan^{-1}(1/2)} \tan^3\theta(1+\tan^2\theta)^{-1/2}\sec^2\theta\,d\theta$$

$$= \int_0^{\tan^{-1}(1/2)} \frac{\tan^3\theta\sec^2\theta}{\sec\theta}\,d\theta = \int_0^{\tan^{-1}(1/2)} \tan^2\theta\tan\theta\sec\theta\,d\theta$$

$$= \int_0^{\tan^{-1}(1/2)} (\sec^2\theta - 1)\tan\theta\sec\theta\,d\theta$$

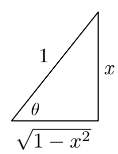
$$= \left(\frac{1}{3}\sec^3\theta - \sec\theta\right)\Bigg]_0^{\tan^{-1}(1/2)}$$

$$= \frac{1}{3}[\sec^3(\tan^{-1}1/2) - 1] - [\sec(\tan^{-1}1/2) - 1]$$



$$= \frac{1}{3}\left(\frac{5\sqrt{5}}{8} - 1\right) - \left(\frac{\sqrt{5}}{2} - 1\right) = \frac{16 - 7\sqrt{5}}{24}$$

45. $\displaystyle\int x^2\sin^{-1}x\,dx$ $\boxed{u = \sin^{-1}x,\ du = \frac{1}{\sqrt{1-x^2}}\,dx;\quad dv = x^2\,dx,\ v = \frac{1}{3}x^3}$



$$= \frac{1}{3}x^3\sin^{-1}x - \frac{1}{3}\int \frac{x^3}{\sqrt{1-x^2}}\,dx \qquad \boxed{x = \sin\theta,\ dx = \cos\theta\,d\theta}$$

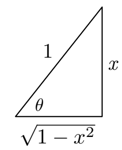
$$= \frac{1}{3}x^3\sin^{-1}x - \frac{1}{3}\int \frac{\sin^3\theta}{\sqrt{1-\sin^2\theta}}\cos\theta\,d\theta = \frac{1}{3}x^3\sin^{-1}x - \frac{1}{3}\int \sin^3\theta\,d\theta$$

$$= \frac{1}{3}x^3\sin^{-1}x - \frac{1}{3}\int (1-\cos^2\theta)\sin\theta\,d\theta$$

$$= \frac{1}{3}x^3\sin^{-1}x - \frac{1}{3}\left(-\cos\theta + \frac{1}{3}\cos^3\theta\right) + C$$

$$= \frac{1}{3}x^3\sin^{-1}x + \frac{1}{3}\sqrt{1-x^2} - \frac{1}{9}(1-x^2)^{3/2} + C$$

46. $\displaystyle\int x\cos^{-1}x\,dx$ $\quad \boxed{u=\cos^{-1}x,\ du=-\frac{1}{\sqrt{1-x^2}}\,dx;\quad dv=x\,dx,\ v=\frac{1}{2}x^2}$



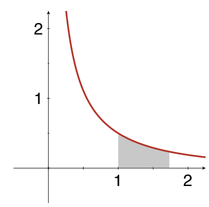
$$=\frac{1}{2}x^2\cos^{-1}x+\frac{1}{2}\int\frac{x^2}{\sqrt{1-x^2}}\,dx \qquad \boxed{x=\sin\theta,\ dx=\cos\theta\,d\theta}$$

$$=\frac{1}{2}x^2\cos^{-1}x+\frac{1}{2}\int\frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}}\cos\theta\,d\theta=\frac{1}{2}x^2\cos^{-1}x+\frac{1}{2}\int\sin^2\theta\,d\theta$$

$$=\frac{1}{2}x^2\cos^{-1}x+\frac{1}{4}\int(1-\cos 2\theta)\,d\theta=\frac{1}{2}x^2\cos^{-1}x+\frac{1}{4}\left(\theta-\frac{1}{2}\sin 2\theta\right)+C$$

$$=\frac{1}{2}x^2\cos^{-1}x+\frac{1}{4}\sin^{-1}x-\frac{1}{4}\sin\theta\cos\theta+C$$

47. $\displaystyle A=\int_1^{\sqrt{3}}\frac{1}{x\sqrt{3+x^2}}\,dx$ $\quad \boxed{x=\sqrt{3}\tan\theta,\ dx=\sqrt{3}\sec^2\theta\,d\theta}$

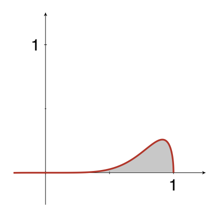


$$=\int_{\pi/6}^{\pi/4}\frac{\sqrt{3}\sec^2\theta}{\sqrt{3}\tan\theta\sqrt{3+3\tan^2\theta}}\,d\theta$$

$$=\frac{1}{\sqrt{3}}\int_{\pi/6}^{\pi/4}\frac{\sec\theta}{\tan\theta}\,d\theta=\frac{1}{\sqrt{3}}\int_{\pi/6}^{\pi/4}\csc\theta\,d\theta$$

$$=\frac{1}{\sqrt{3}}\ln|\csc\theta-\cot\theta|\Big]_{\pi/6}^{\pi/4}=\frac{1}{\sqrt{3}}(\ln|\sqrt{2}-1|-\ln|2-\sqrt{3}|)=\frac{1}{\sqrt{3}}\ln\frac{\sqrt{2}-1}{2-\sqrt{3}}\approx 0.2515$$

48. $\displaystyle A=\int_0^1 x^5\sqrt{1-x^2}\,dx=\int_0^1 x^4\sqrt{1-x^2}\,x\,dx$

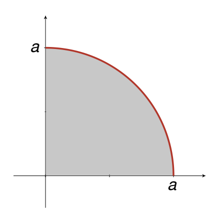


$$\boxed{u=1-x^2,\ x^2=1-u,\ 2x\,dx=-du}$$

$$=\int_1^0(1-u)^2u^{1/2}\left(-\frac{1}{2}\,du\right)=-\frac{1}{2}\int_1^0(u^{1/2}-2u^{3/2}+u^{5/2})\,du$$

$$=-\frac{1}{2}\left(\frac{2}{3}u^{3/2}-\frac{4}{5}u^{5/2}+\frac{2}{7}u^{7/2}\right)\Big]_1^0=-\frac{1}{2}\left(0-\frac{16}{105}\right)=\frac{8}{105}\approx 0.0762$$

49. We find the area in the first quadrant and use symmetry.

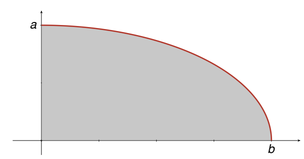


$$A=4\int_0^a\sqrt{a^2-x^2}\,dx \qquad \boxed{x=a\sin\theta,\ dx=a\cos\theta\,d\theta}$$

$$=4\int_0^{\pi/2}\sqrt{a^2-a^2\sin^2\theta}\;a\cos\theta\,d\theta=4a^2\int_0^{\pi/2}\cos^2\theta\,d\theta$$

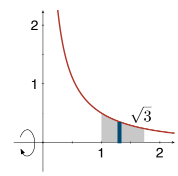
$$=2a^2\int_0^{\pi/2}(1+\cos 2\theta)\,d\theta=2a^2\left(\theta+\frac{1}{2}\sin 2\theta\right)\Big]_0^{\pi/2}=2a^2\left(\frac{\pi}{2}\right)=\pi a^2$$

50. We find the area in the first quadrant and use symmetry.



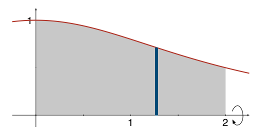
$$A = 4\int_0^b \frac{a}{b}\sqrt{b^2 - x^2}\,dx \qquad \boxed{x = b\sin\theta,\ dx = b\cos\theta\,d\theta}$$
$$= \frac{4a}{b}\int_0^{\pi/2} \sqrt{b^2 - b^2\sin^2\theta}\;b\cos\theta\,d\theta = 4ab\int_0^{\pi/2} \cos^2\theta\,d\theta$$
$$= 2ab\int_0^{\pi/2}(1+\cos 2\theta)\,d\theta = 2ab\left(\theta + \frac{1}{2}\sin 2\theta\right)\Bigg]_0^{\pi/2} = 2ab\left(\frac{\pi}{2}+0\right) = \pi ab$$

51. $$V = \pi\int_0^{\sqrt{3}} \frac{1}{x^2(3+x^2)}\,dx \qquad \boxed{x = \sqrt{3}\tan\theta,\ dx = \sqrt{3}\sec^2\theta\,d\theta}$$



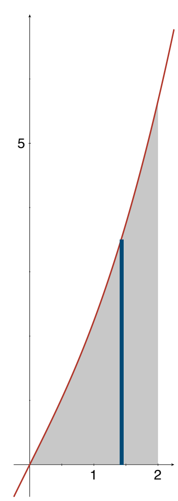
$$= \pi\int_{\pi/6}^{\pi/4} \frac{\sqrt{3}\sec^2\theta}{3\tan^2\theta(3+3\tan^2\theta)}\,d\theta = \frac{\pi\sqrt{3}}{9}\int_{\pi/6}^{\pi/4}\frac{1}{\tan^2\theta}\,d\theta$$
$$= \frac{\pi\sqrt{3}}{9}\int_{\pi/6}^{\pi/4}\cot^2\theta\,d\theta = \frac{\pi\sqrt{3}}{9}\int_{\pi/6}^{\pi/4}(\csc^2\theta - 1)\,d\theta = \frac{\pi\sqrt{3}}{9}(-\cot\theta - \theta)\Bigg]_{\pi/6}^{\pi/4}$$
$$= -\frac{\pi\sqrt{3}}{9}\left[\left(1+\frac{\pi}{4}\right) - \left(\sqrt{3}+\frac{\pi}{6}\right)\right] = \frac{\pi\sqrt{3}}{9}\left(\sqrt{3} - 1 - \frac{\pi}{12}\right) \approx 0.2843$$

52. Using the disk method,



$$V = \pi\int_0^2 \frac{16}{(4+x^2)^2}\,dx \qquad \boxed{x = 2\tan\theta,\ dx = 2\sec^2\theta\,d\theta}$$
$$= \pi\int_0^{\pi/4}\frac{16}{(4+4\tan^2\theta)^2}(2\sec^2\theta\,d\theta) = 2\pi\int_0^{\pi/4}\frac{\sec^2\theta}{\sec^4\theta}\,d\theta = 2\pi\int_0^{\pi/4}\cos^2\theta\,d\theta$$
$$= \pi\int_0^{\pi/4}(1+\cos 2\theta)\,d\theta = \pi\left(\theta + \frac{1}{2}\sin 2\theta\right)\Bigg]_0^{\pi/4} = \pi\left(\frac{\pi}{4}+\frac{1}{2}\right) = \frac{\pi^2+2\pi}{4}.$$

53. Using the shell method,



$$V = 2\pi\int_0^2 x^2\sqrt{4+x^2}\,dx \qquad \boxed{x = 2\tan\theta,\ dx = 2\sec^2\theta\,d\theta}$$
$$= 2\pi\int_0^{\pi/4} 4\tan^2\theta\sqrt{4+4\tan^2\theta}\;2\sec^2\theta\,d\theta = 32\pi\int_0^{\pi/4}\tan^2\theta\sec^3\theta\,d\theta$$
$$= 32\pi\int_0^{\pi/4}(\sec^2\theta - 1)\sec^3\theta\,d\theta = 32\pi\int_0^{\pi/4}\sec^5\theta\,d\theta - 32\pi\int_0^{\pi/4}\sec^3\theta\,d\theta.$$

From Section 7.3, Example 5 we obtain

$$\int_0^{\pi/4}\sec^3\theta\,d\theta = \left(\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta|\right)\Bigg]_0^{\pi/4}$$
$$= \frac{1}{2}(\sqrt{2})(1) + \frac{1}{2}\ln(\sqrt{2}+1)$$
$$= \frac{1}{2}[\sqrt{2} + \ln(\sqrt{2}+1)].$$

To find $\displaystyle\int_0^{\pi/4} \sec^5\theta\,d\theta$ we use integration by parts.

$$\int_0^{\pi/4} \sec^5\theta\,d\theta = \int_0^{\pi/4} \sec^3\theta\sec^2\theta\,d\theta$$

$$\boxed{u = \sec^3\theta,\ du = 3\sec^2\theta\sec\theta\tan\theta\,d\theta;\quad dv = \sec^2\theta\,d\theta,\ v = \tan\theta}$$

$$= \sec^3\theta\tan\theta\Big]_0^{\pi/4} - 3\int_0^{\pi/4} \sec^3\theta\tan^2\theta\,d\theta$$

$$= 2\sqrt{2} - 3\int_0^{\pi/4} \sec^3\theta(\sec^2\theta - 1)\,d\theta$$

$$= 2\sqrt{2} - 3\int_0^{\pi/4} \sec^5\theta\,d\theta + 3\int_0^{\pi/4} \sec^3\theta\,d\theta$$

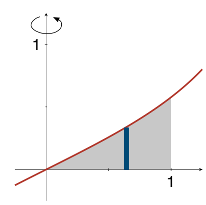
$$= 2\sqrt{2} - 3\int_0^{\pi/4} \sec^5\theta\,d\theta + \frac{3}{2}[\sqrt{2} + \ln(\sqrt{2}+1)]$$

Solving for $\displaystyle\int_0^{\pi/4} \sec^5\theta\,d\theta$ we obtain $\displaystyle\int_0^{\pi/4} \sec^5\theta\,d\theta = \frac{\sqrt{2}}{2} + \frac{3}{8}[\sqrt{2} + \ln(\sqrt{2}+1)]$. Then

$$V = 32\pi\left\{\frac{\sqrt{2}}{2} + \frac{3}{8}[\sqrt{2} + \ln(\sqrt{2}+1)]\right\} - 32\pi\left[\frac{\sqrt{2}}{2} + \frac{1}{2}\ln(\sqrt{2}+1)\right]$$

$$= 12\pi\sqrt{2} - 4\pi\ln(\sqrt{2}+1).$$

54. $\displaystyle V = 2\pi\int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,dx \qquad \boxed{x = 2\sin\theta,\ dx = 2\cos\theta\,d\theta}$



$$= 2\pi\int_0^{\pi/6} \frac{4\sin^2\theta}{\sqrt{4-4\sin^2\theta}}\,2\cos\theta\,d\theta = 8\pi\int_0^{\pi/6} \sin^2\theta\,d\theta$$

$$= 4\pi\int_0^{\pi/6} (1-\cos 2\theta)\,d\theta = 4\pi\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big]_0^{\pi/6}$$

$$= 4\pi\left[\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) - 0\right] = \frac{2\pi^2 - 3\pi\sqrt{3}}{3} \approx 1.1383$$

55. $y' = 1/x$

$$L = \int_1^{\sqrt{3}} \sqrt{1+(1/x)^2}\,dx = \int_1^{\sqrt{3}} \frac{\sqrt{x^2+1}}{x}\,dx \qquad \boxed{x = \tan\theta,\ dx = \sec^2\theta\,d\theta}$$

$$= \int_{\pi/4}^{\pi/3} \frac{\sqrt{\tan^2\theta+1}}{\tan\theta}\sec^2\theta\,d\theta = \int_{\pi/4}^{\pi/3} \frac{\sec^3\theta}{\tan\theta}\,d\theta = \int_{\pi/4}^{\pi/3} \frac{\sec\theta}{\tan\theta}\sec^2\theta\,d\theta$$

$$= \int_{\pi/4}^{\pi/3} \csc\theta(\tan^2\theta+1)\,d\theta = \int_{\pi/4}^{\pi/3} \left(\frac{\sin\theta}{\cos^2\theta} + \csc\theta\right)d\theta = \int_{\pi/4}^{\pi/3} (\sec\theta\tan\theta + \csc\theta)\,d\theta$$

$$= (\sec\theta + \ln|\csc\theta - \cot\theta|)\Big]_{\pi/4}^{\pi/3} = \left(2 + \ln\left|\frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right|\right) - (\sqrt{2} + \ln|\sqrt{2}-1|)$$

$= 2 - \sqrt{2} - \ln(\sqrt{6} - \sqrt{3}) \approx 0.9179$

56. $y' = -x + 2$

$$L = \int_1^2 \sqrt{1 + (2-x)^2}\,dx \qquad \boxed{2 - x = \tan\theta,\ -dx = \sec^2\theta\,d\theta}$$

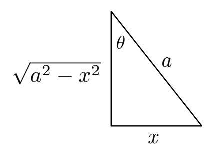
$$= \int_{\pi/4}^0 \sqrt{1+\tan^2\theta}(-\sec^2\theta\,d\theta) = \int_{\pi/4}^0 \sec^3\theta\,d\theta \qquad \boxed{\text{See Section 7.3, Example 5}}$$

$$= \left(\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta|\right)\Bigg]_0^{\pi/4} = \left[\frac{1}{2}\sqrt{2}(1) + \frac{1}{2}\ln|\sqrt{2}+1|\right] - \left(0 - \frac{1}{2}\ln 1\right)$$

$$= \frac{\sqrt{2}}{2} + \frac{1}{2}\ln(\sqrt{2}+1) \approx 1.1478$$

57. (a) The slope at (x, y) is $-\dfrac{\sqrt{a^2 - x^2}}{x}$, which is also $\dfrac{dy}{dx}$.

(b) Separating variables, $\displaystyle\int \frac{\sqrt{a^2-x^2}}{x}\,dx = -\int dy$. Now



$$\int \frac{\sqrt{a^2-x^2}}{x}\,dx \qquad \boxed{x = a\sin\theta,\ dx = a\cos\theta\,d\theta}$$

$$= \int \frac{\sqrt{a^2 - a^2\sin\theta}}{a\sin\theta}\,a\cos\theta\,d\theta = a\int \frac{\cos^2\theta}{\sin\theta}\,d\theta = a\int \frac{1-\sin^2\theta}{\sin\theta}\,d\theta$$

$$= a\int \csc\theta\,d\theta - a\int \sin\theta\,d\theta = a\ln|\csc\theta - \cot\theta| + a\cos\theta + C$$

$$= a\ln\left|\frac{a}{x} - \frac{\sqrt{a^2-x^2}}{x}\right| + a\left(\frac{\sqrt{a^2-x^2}}{a}\right) + C.$$

Then $a\ln\left|\dfrac{a - \sqrt{a^2-x^2}}{x}\right| + \sqrt{a^2-x^2} = -y + C_1$. Now $y(10) = 0$ and $a = 10$, so $10\ln\left|\dfrac{10 - \sqrt{100-100}}{10}\right| + \sqrt{100-100} = 0 + C_1$ and $C_1 = 0$. Thus

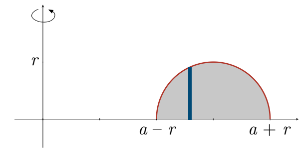
$$y = -10\ln\left|\frac{10 - \sqrt{100-x^2}}{x}\right| - \sqrt{100-x^2}.$$

Note: If the substitution $y = a\cos\theta$ is used, we obtain the equivalent solution

$$y = 10\ln\left|\frac{10 - \sqrt{100-x^2}}{x}\right| - \sqrt{100-x^2}.$$

58. Using symmetry with respect to the x-axis, we have

$$V = 4\pi \int_{a-r}^{a+r} x\sqrt{r^2-(x-a)^2}\,dx$$



$$\boxed{x-a = r\sin\theta,\ dx = r\cos\theta\,d\theta}$$

$$= 4\pi \int_{-\pi/2}^{\pi/2} (a+r\sin\theta)\sqrt{r^2-r^2\sin^2\theta}\; r\cos\theta\,d\theta$$

$$= 4\pi r^2 \int_{-\pi/2}^{\pi/2} (a+r\sin\theta)\cos^2\theta\,d\theta = 4\pi a r^2 \int_{-\pi/2}^{\pi/2} \frac{1+\cos 2\theta}{2}\,d\theta + 4\pi r^3 \int_{-\pi/2}^{\pi/2} \cos^2\theta\sin\theta\,d\theta$$

$$= 2\pi a r^2 \left(\theta + \frac{1}{2}\sin 2\theta\right)\Bigg]_{-\pi/2}^{\pi/2} + 4\pi r^3\left(-\frac{1}{3}\cos^3\theta\right)\Bigg]_{-\pi/2}^{\pi/2}$$

$$= 2\pi a r^2\left[\left(\frac{\pi}{2}+0\right) - \left(-\frac{\pi}{2}+0\right)\right] - \frac{4}{3}\pi r^3(0-0) = 2a\pi^2 r^2$$

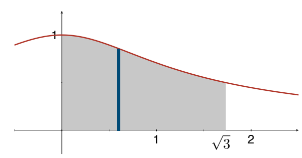
59. $F = 62.4 \int_1^2 x\sqrt{\dfrac{2-x}{x}}\,dx = 62.4\int_1^2 \sqrt{2x-x^2}\,dx = 62.4\int_1^2 \sqrt{1-(x-1)^2}\,dx$

$$\boxed{x-1 = \sin\theta,\ dx = \cos\theta\,d\theta}$$

$$= 62.4\int_0^{\pi/2} \sqrt{1-\sin^2\theta}\;\cos\theta\,d\theta = 62.4\int_0^{\pi/2}\cos^2\theta\,d\theta = 31.2\int_0^{\pi/2}(1+\cos 2\theta)\,d\theta$$

$$= 31.2\left(\theta + \frac{1}{2}\sin 2\theta\right)\Bigg]_0^{\pi/2} = 31.2\left(\frac{\pi}{2}\right) = 15.6\pi \approx 49.0088 \text{ lb}$$

60. $A = \displaystyle\int_0^{\sqrt{3}} \frac{1}{\sqrt{1+x^2}}\,dx \qquad \boxed{x = \tan\theta,\ dx = \sec^2\theta\,d\theta}$



$$= \int_0^{\pi/3} \frac{\sec^2\theta}{\sqrt{1+\tan^2\theta}}\,d\theta = \int_0^{\pi/3}\sec\theta\,d\theta$$

$$= \ln|\sec\theta + \tan\theta|\,\Bigg]_0^{\pi/3} = \ln(2+\sqrt{3})$$

$$M_y = \int_0^{\sqrt{3}} \frac{x}{\sqrt{1+x^2}}\,dx = \frac{1}{2}\int_0^{\sqrt{3}} 2x(1+x^2)^{-1/2}\,dx = \frac{1}{2}\cdot\frac{(1+x^2)^{1/2}}{1/2}\Bigg]_0^{\sqrt{3}}$$

$$= \sqrt{1+x^2}\,\Bigg]_0^{\sqrt{3}} = 2-1 = 1$$

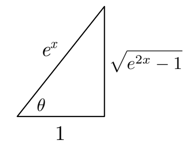
$$M_x = \frac{1}{2}\int_0^{\sqrt{3}} \frac{1}{1+x^2}\,dx = \frac{1}{2}\tan^{-1}x\,\Bigg]_0^{\sqrt{3}} = \frac{\pi}{6}$$

$$\overline{x} = \frac{1}{\ln(2+\sqrt{3})} \approx 0.76; \quad \overline{y} = \frac{\pi/6}{\ln(2+\sqrt{3})} = \frac{\pi}{6\ln(2+\sqrt{3})} \approx 0.40$$

61. (a) $\displaystyle\int \frac{1}{\sqrt{e^{2x}-1}}\,dx \qquad \boxed{e^x = \sec\theta,\ e^x\,dx = \tan\theta\sec\theta\,d\theta,\ dx = \tan\theta\,d\theta}$

$$= \int \frac{\tan\theta}{\sqrt{\sec^2\theta - 1}}\,d\theta = \int\frac{\tan\theta}{\tan\theta}\,d\theta = \int d\theta = \theta + C = \sec^{-1}e^x + C$$

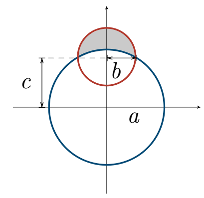
(b) $\int \sqrt{e^{2x}-1}\,dx$ $\quad \boxed{e^x = \sec\theta,\ e^x\,dx = \tan\theta\sec\theta\,d\theta,\ dx = \tan\theta\,d\theta}$



$$= \int \sqrt{\sec^2\theta - 1}\,\tan\theta\,d\theta = \int \tan^2\theta\,d\theta$$

$$= \int (\sec^2\theta - 1)\,d\theta = \tan\theta - \theta + C = \sqrt{e^{2x}-1} - \sec^{-1} e^x + C$$

62. The circle of radius a, which is centered on the origin, is defined by $x^2+y^2 = a^2$. Let c be the distance between the centers of the two circles. The circle of radius b is thus defined by $x^2 + (y-c)^2 = b^2$. The area is the integral of the difference between $y = \sqrt{b^2-x^2} + c$ and $y = \sqrt{a^2-x^2}$ from $-b$ to b. Using symmetry, we have



$$A = 2\int_0^b [(\sqrt{b^2-x^2} + c) - \sqrt{a^2-x^2}]\,dx = 2\int_0^b (\sqrt{b^2-x^2} - \sqrt{a^2-x^2} + c)\,dx$$

By using the substitution $\boxed{x = r\sin\theta,\ \ dx = r\cos\theta\,d\theta}$, we find

$$\int \sqrt{r^2-x^2}\,dx = \int \sqrt{r^2 - r^2\sin^2\theta}\,r\cos\theta\,d\theta = r^2\int \cos^2\theta\,d\theta$$

$$= r^2\int \frac{1+\cos 2\theta}{2}\,d\theta = \frac{r^2}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C = \frac{r^2}{2}\theta + \frac{r^2}{2}\sin\theta\cos\theta + C$$

$$= \frac{r^2}{2}\sin^{-1}\frac{x}{r} + \frac{r^2}{2}\left(\frac{x}{r}\right)\frac{\sqrt{r^2-x^2}}{r} = \frac{r^2}{2}\sin^{-1}\frac{x}{r} + \frac{1}{2}x\sqrt{r^2-x^2} + C.$$

Substituting b and a respectively, we get

$$A = 2\int_0^b \sqrt{b^2-x^2}\,dx - 2\int_0^b \sqrt{a^2-x^2}\,dx + 2\int_0^b c\,dx$$

$$= \left(b^2\sin^{-1}\frac{x}{b} + x\sqrt{b^2-x^2}\right)\Big]_0^b - \left(a^2\sin^{-1}\frac{x}{a} + x\sqrt{a^2-x^2}\right)\Big]_0^b + 2cx\Big]_0^b$$

$$= \left[\left(b^2\sin^{-1}1 + b\sqrt{b^2-b^2}\right) - (0+0)\right] - \left[\left(a^2\sin^{-1}\frac{b}{a} + b\sqrt{a^2-b^2}\right) - (0+0)\right] + 2bc$$

$$= \frac{\pi b^2}{2} - a^2\sin^{-1}\frac{b}{a} - b\sqrt{a^2-b^2} + 2bc.$$

From the figure, it can be seen that $a^2 = b^2 + c^2$ or $c = \sqrt{a^2-b^2}$. We substitute to simplify further:

$$A = \frac{\pi b^2}{2} - a^2\sin^{-1}\frac{b}{a} - b\sqrt{a^2-b^2} + 2bc = \frac{\pi b^2}{2} - a^2\sin^{-1}\frac{b}{a} - bc + 2bc$$

$$= \frac{\pi b^2}{2} - a^2\sin^{-1}\frac{b}{a} + b\sqrt{a^2-b^2}.$$

The special-case *lune of Hippocrates* specifies a lune where the triangle formed by the origin and the intersections of the two circles is a right isosceles triangle. For this lune, $b = \frac{\sqrt{2}}{2}a$ by the Pythagorean theorem and $\sin^{-1}\frac{b}{a}$ is $\frac{\pi}{4}$. Substituting these values above yields the

well-known result that the area of the lune of Hippocrates is the same as the area of the right isosceles triangle that defines it, or $\frac{1}{2}a^2$.

7.6 Partial Fractions

1. Write $\dfrac{x-1}{x^2+x} = \dfrac{x-1}{x(x+1)} = \dfrac{A}{x} + \dfrac{B}{x+1}$.

2. Write $\dfrac{9x-8}{(x-3)(2x-5)} = \dfrac{A}{x-3} + \dfrac{B}{2x-5}$.

3. Write $\dfrac{x^3}{(x-1)(x+2)^3} = \dfrac{A}{x-1} + \dfrac{B}{x+2} + \dfrac{C}{(x+2)^2} + \dfrac{D}{(x+2)^3}$.

4. Write $\dfrac{2x^2-3}{x^3+6x^2} = \dfrac{2x^2-3}{x^2(x+6)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+6}$.

5. Write $\dfrac{4}{x^3(x^2+3)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x^3} + \dfrac{Dx+E}{x^2+3}$.

6. Write $\dfrac{-x^2+3x+7}{(x+2)^2(x^2+x+1)} = \dfrac{A}{x+2} + \dfrac{B}{(x+2)^2} + \dfrac{Cx+D}{x^2+x+1}$.

7. Write $\dfrac{2x^3-x}{(x^2+9)^2} = \dfrac{Ax+B}{x^2+9} + \dfrac{Cx+D}{(x^2+9)^2}$.

8. $\dfrac{3x^2-x+4}{x^4+2x^3+x} = \dfrac{3x^2-x+4}{x(x^3+2x^2+1)}$.
   This expression does not fall under any of the four partial fraction decomposition cases covered in Section 7.6.

9. Write $\dfrac{1}{x(x-2)} = \dfrac{A}{x} + \dfrac{B}{x-2}$. Then $1 = A(x-2) + Bx$.
   Setting $x = 0$ and $x = 2$ gives $A = -1/2$ and $B = 1/2$. Thus
$$\begin{aligned}\int \frac{1}{x(x-2)}\,dx &= -\frac{1}{2}\int \frac{1}{x}\,dx + \frac{1}{2}\int \frac{1}{x-2}\,dx \\ &= -\frac{1}{2}\ln|x| + \frac{1}{2}\ln|x-2| + C = \frac{1}{2}\ln\left|\frac{x-2}{x}\right| + C.\end{aligned}$$

10. Write $\dfrac{1}{x(2x+3)} = \dfrac{A}{x} + \dfrac{B}{2x+3}$. Then $1 = A(2x+3) + Bx$.
    Setting $x = 0$ and $x = -3/2$ gives $A = 1/3$ and $B = -2/3$. Thus
$$\begin{aligned}\int \frac{1}{x(2x+3)}\,dx &= \frac{1}{3}\int \frac{1}{x}\,dx - \frac{2}{3}\int \frac{1}{2x+3}\,dx \\ &= \frac{1}{3}\ln|x| - \frac{1}{3}\ln|2x+3| + C = \frac{1}{3}\ln\left|\frac{x}{2x+3}\right| + C.\end{aligned}$$

11. Write $\dfrac{x+2}{2x^2-x} = \dfrac{x+2}{x(2x-1)} = \dfrac{A}{x} + \dfrac{B}{2x-1}$. Then $x+2 = A(2x-1) + Bx$.
Setting $x=0$ and $x=1/2$ gives $A=-2$ and $B=5$. Thus
$$\int \frac{x+2}{2x^2-x}\,dx = -2\int \frac{1}{x}\,dx + 5\int \frac{1}{2x-1}\,dx = -2\ln|x| + \frac{5}{2}\ln|2x-1| + C.$$

12. Write $\dfrac{3x+10}{x^2+2x} = \dfrac{A}{x} + \dfrac{B}{x+2}$. Then $3x+10 = A(x+2) + Bx$.
Setting $x=0$ and $x=-2$ gives $A=5$ and $B=-2$. Thus
$$\int \frac{3x+10}{x^2+2x}\,dx = 5\int \frac{1}{x}\,dx + -2\int \frac{1}{x+2}\,dx = 5\ln|x| - 2\ln|x+2| + C.$$

13. Write $\dfrac{x+1}{x^2-16} = \dfrac{A}{x+4} + \dfrac{B}{x-4}$. Then $x+1 = A(x-4) + B(x+4)$.
Setting $x=-4$ and $x=4$ gives $A=3/8$ and $B=5/8$. Thus
$$\int \frac{x+1}{x^2-16}\,dx = \frac{3}{8}\int \frac{1}{x+4}\,dx + \frac{5}{8}\int \frac{1}{x-4}\,dx = \frac{3}{8}\ln|x+4| + \frac{5}{8}\ln|x-4| + C.$$

14. Write $\dfrac{1}{4x^2-25} = \dfrac{A}{2x+5} + \dfrac{B}{2x-5}$. Then $1 = A(2x-5) + B(2x+5)$.
Setting $x=-5/2$ and $x=5/2$ gives $A=-1/10$ and $B=1/10$. Thus
$$\begin{aligned}\int \frac{1}{4x^2-25}\,dx &= -\frac{1}{10}\int \frac{1}{2x+5}\,dx + \frac{1}{10}\int \frac{1}{2x-5}\,dx \\ &= -\frac{1}{20}\ln|2x+5| + \frac{1}{20}\ln|2x-5| + C = \frac{1}{20}\ln\left|\frac{2x-5}{2x+5}\right| + C.\end{aligned}$$

15. Write $\dfrac{x}{2x^2+5x+2} = \dfrac{A}{2x+1} + \dfrac{B}{x+2}$.
Then $x = A(x+2) + B(2x+1)$.
Setting $x=-1/2$ and $x=-2$ gives $A=-1/3$ and $B=2/3$. Thus
$$\int \frac{x}{2x^2+5x+2}\,dx = -\frac{1}{3}\int \frac{1}{2x+1}\,dx + \frac{2}{3}\int \frac{1}{x+2}\,dx = -\frac{1}{6}\ln|2x+1| + \frac{2}{3}\ln|x+2| + C.$$

16. Write $\dfrac{x+5}{(x+4)(x^2-1)} = \dfrac{A}{x+4} + \dfrac{B}{x-1} + \dfrac{C}{x+1}$.
Then $x+5 = A(x^2-1) + B(x+4)(x+1) + C(x+4)(x-1)$.
Setting $x=-4$, $x=1$, and $x=-1$ gives $A=1/15$, $B=3/5$, and $C=-2/3$. Thus
$$\begin{aligned}\int \frac{x+5}{(x+4)(x^2-1)}\,dx &= \frac{1}{15}\int \frac{1}{x+4}\,dx + \frac{3}{5}\int \frac{1}{x-1}\,dx - \frac{2}{3}\int \frac{1}{x+1}\,dx \\ &= \frac{1}{15}\ln|x+4| + \frac{3}{5}\ln|x-1| - \frac{2}{3}\ln|x+1| + C.\end{aligned}$$

17. Write $\dfrac{x^2+2x+6}{x^3-x} = \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{C}{x+1}$.
Then $x^2+2x-6 = A(x^2-1) + B(x^2+x) + C(x^2-x)$.
Setting $x=0$, $x=1$, and $x=-1$ gives $A=6$, $B=-3/2$, and $C=-7/2$. Thus

$$\begin{aligned}\int \frac{x^2+2x-6}{x^3-x}\,dx &= 6\int \frac{1}{x}\,dx - \frac{3}{2}\int \frac{1}{x-1}\,dx - \frac{7}{2}\int \frac{1}{x+1}\,dx \\ &= 6\ln|x| - \frac{3}{2}\ln|x-1| - \frac{7}{2}\ln|x+1| + C.\end{aligned}$$

18. Write $\dfrac{5x^2-x+1}{x^3-4x} = \dfrac{A}{x} + \dfrac{B}{x-2} + \dfrac{C}{x+2}$.
Then $5x^2-x+1 = A(x^2-4) + B(x^2+2x) + C(x^2-2x)$.
Setting $x=0$, $x=2$, and $x=-2$ gives $A=-1/4$, $B=19/8$, and $C=23/8$. Thus

$$\begin{aligned}\int \frac{5x^2-x+1}{x^3-4x}\,dx &= -\frac{1}{4}\int \frac{1}{x}\,dx + \frac{19}{8}\int \frac{1}{x-2}\,dx + \frac{23}{8}\int \frac{1}{x+2}\,dx \\ &= -\frac{1}{4}\ln|x| + \frac{19}{8}\ln|x-2| - \frac{23}{8}\ln|x+2| + C.\end{aligned}$$

19. Write $\dfrac{1}{(x+1)(x+2)(x+3)} = \dfrac{A}{x+1} + \dfrac{B}{x+2} + \dfrac{C}{x+3}$.
Then $1 = A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)$.
Setting $x=-1$, $x=-2$, and $x=-3$ gives $A=1/2$, $B=-1$, and $C=1/2$. Thus

$$\begin{aligned}\int \frac{1}{(x+1)(x+2)(x+3)}\,dx &= \frac{1}{2}\int \frac{1}{x+1}\,dx - \int \frac{1}{x+2}\,dx + \frac{1}{2}\int \frac{1}{x+3}\,dx \\ &= \frac{1}{2}\ln|x+1| - \ln|x+2| + \frac{1}{2}\ln|x+3| + C.\end{aligned}$$

20. Write $\dfrac{1}{(4x^2-1)(x+7)} = \dfrac{A}{2x-1} + \dfrac{B}{2x+1} + \dfrac{C}{x+7}$.
Then $1 = A(2x+1)(x+7) + B(2x-1)(x+7) + C(4x^2-1)$.
Setting $x=1/2$, $x=-1/2$, and $x=-7$ gives $A=1/15$, $B=-1/13$, and $C=1/195$. Thus

$$\begin{aligned}\int \frac{1}{(4x^2-1)(x+7)}\,dx &= \frac{1}{15}\int \frac{1}{2x-1}\,dx - \frac{1}{13}\int \frac{1}{2x+1}\,dx + \frac{1}{195}\int \frac{1}{x+7}\,dx \\ &= \frac{1}{30}\ln|2x-1| - \frac{1}{26}\ln|2x+1| + \frac{1}{195}\ln|x+7| + C.\end{aligned}$$

21. Write $\dfrac{4t^2+3t-1}{t^3-t^2} = \dfrac{A}{t} + \dfrac{B}{t^2} + \dfrac{C}{t-1}$.
Then $4t^2+3t-1 = A(t^2-t) + B(t-1) + Ct^2 = (A+C)t^2 + (-A+B)t - B$.
Solving $\boxed{A+C=4 \qquad -A+B=3 \qquad -B=-1}$

gives $A = -2$, $B = 1$, and $C = 6$. Thus

$$\int \frac{4t^2+3t-1}{t^3-t^2}\,dt = -2\int \frac{1}{t}\,dt + \int \frac{1}{t^2}\,dt + 6\int \frac{1}{t-1}\,dt = -2\ln|t| - \frac{1}{t} + 6\ln|t-1| + C.$$

22. Write $\dfrac{2x-11}{x^3+2x^2} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+2}$.

Then $2x - 11 = A(x^2+2x) + B(x+2) + Cx^2 = (A+C)x^2 + (2A+B)x + 2B$.

Solving $\boxed{A + C = 0 \qquad 2A + B = 2 \qquad 2B = -11}$

gives $A = 15/4$, $B = -11/2$, and $C = -15/4$. Thus

$$\begin{aligned}\int \frac{2x-11}{x^3+2x^2}\,dx &= \frac{15}{4}\int \frac{1}{x}\,dx - \frac{11}{2}\int \frac{1}{x^2}\,dx - \frac{15}{4}\int \frac{1}{x+2}\,dx \\ &= \frac{15}{4}\ln|x| + \frac{11}{2}x^{-1} - \frac{15}{4}\ln|x+2| + C.\end{aligned}$$

23. Write $\dfrac{1}{x^3+2x^2+x} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$.

Then $1 = A(x+1)^2 + B(x^2+x) + Cx = (A+B)x^2 + (2A+B+C)x + A$.

Solving $\boxed{A + B = 0 \qquad 2A + B + C = 0 \qquad A = 1}$

gives $A = 1$, $B = -1$, and $C = -1$. Thus

$$\begin{aligned}\int \frac{1}{x^3+2x^2+x}\,dx &= \int \frac{1}{x}\,dx - \int \frac{1}{x+1}\,dx - \int \frac{1}{(x+1)^2}\,dx \\ &= \ln|x| - \ln|x+1| + \frac{1}{x+1} + C.\end{aligned}$$

24. Write $\dfrac{t-1}{t^4+6t^3+9t^2} = \dfrac{A}{t} + \dfrac{B}{t^2} + \dfrac{C}{t+3} + \dfrac{D}{(t+3)^2}$.

$$\begin{aligned}\text{Then } t-1 &= At(t+3)^2 + B(t+3)^2 + Ct^2(t+3) + Dt^2 \\ &= (A+C)t^3 + (6A+B+3C+D)t^2 + (9A+6B)t + 9B.\end{aligned}$$

Solving $\boxed{\begin{array}{ll} A + C = 0 & 6A + B + 3C + D = 0 \\ 9A + 6B = 1 & 9B = -1 \end{array}}$

gives $A = 5/27$, $B = -1/9$, $C = -5/27$, and $D = -4/9$. Thus

$$\begin{aligned}\int \frac{t-1}{t^4+6t^3+9t^2}\,dt &= \frac{5}{27}\int \frac{1}{t}\,dt - \frac{1}{9}\int \frac{1}{t^2}\,dt - \frac{5}{27}\int \frac{1}{t+3}\,dt - \frac{4}{9}\int \frac{1}{(t+3)^2}\,dt \\ &= \frac{5}{27}\ln|t| + \frac{1}{9}t^{-1} - \frac{5}{27}\ln|t+3| + \frac{4}{9}(t+3)^{-1} + C.\end{aligned}$$

25. $$\begin{aligned}\int \frac{2x-1}{(x+1)^3}\,dx &= \int \frac{2(x+1)-3}{(x+1)^3}\,dx = \int \frac{2}{(x+1)^2}\,dx - \int \frac{3}{(x+1)^3}\,dx \\ &= -\frac{2}{x+1} + \frac{3}{2}\left[\frac{1}{(x+1)^2}\right] + C = \frac{3}{2(x+1)^2} - \frac{2}{x+1} + C\end{aligned}$$

26. Write $\dfrac{1}{x^2(x^2-4)^2} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-2} + \dfrac{D}{(x-2)^2} + \dfrac{E}{x+2} + \dfrac{F}{(x+2)^2}$.

Then
$$\begin{aligned}1 &= Ax(x^2-4)^2 + B(x^2-4)^2 + Cx^2(x-2)(x+2)^2 + Dx^2(x+2)^2\\ &\quad + Ex^2(x-2)^2(x+2) + Fx^2(x-2)^2\\ &= (A+C+E)x^5 + (B+2C+D-2E+F)x^4 + (-8A-4C+4D-4E-4F)x^3\\ &\quad + (-8B-8C+4D+8E+4F)x^2 + 16Ax + 16B.\end{aligned}$$

Solving
$$\boxed{\begin{array}{rr} A+C+E=0 & B+2C+D-2E+F=0\\ -8A-4C+4D-4E-4F=0 & -8B-8C+4D+8E+4F=0\\ 16A=0 & 16B=1\end{array}}$$

gives $A=0$, $B=1/16$, $C=-3/128$, $D=1/64$, $E=3/128$, and $F=1/64$. Thus
$$\begin{aligned}\int \frac{1}{x^2(x^2-4)^2}\,dx &= \frac{1}{16}\int\frac{1}{x^2}\,dx - \frac{3}{128}\int\frac{1}{x-2}\,dx - \frac{1}{64}\int\frac{1}{(x-2)^2}\,dx\\ &\quad + \frac{3}{128}\int\frac{1}{x+2}\,dx + \frac{1}{64}\int\frac{1}{(x+2)^2}\,dx\\ &= -\frac{1}{16}x^{-1} - \frac{3}{128}\ln|x-2| - \frac{1}{64}(x-2)^{-1}\\ &\quad + \frac{3}{128}\ln|x+2| - \frac{1}{64}(x+2)^{-1} + C.\end{aligned}$$

27. Write $\dfrac{1}{(x^2+6x+5)^2} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{x+5} + \dfrac{D}{(x+5)^2}$.

Then
$$\begin{aligned}1 &= A(x+1)(x+5)^2 + B(x+5)^2 + C(x+5)(x+1)^2 + D(x+1)^2\\ &= (A+C)x^3 + (11A+B+7C+D)x^2 + (35A+10B+11C+2D)x\\ &\quad + 25A+25B+5C+D.\end{aligned}$$

Solving
$$\boxed{\begin{array}{rr} A+C=0 & 11A+B+7C+D=0\\ 35A+10B+11C+2D=0 & 25A+25B+5C+D=1\end{array}}$$

gives $A=-1/32$, $B=1/16$, $C=1/32$, and $D=1/16$. Thus
$$\begin{aligned}\int\frac{1}{(x^2+6x+5)^2}\,dx &= -\frac{1}{32}\int\frac{1}{x+1}\,dx + \frac{1}{16}\int\frac{1}{(x+1)^2}\,dx + \frac{1}{32}\int\frac{1}{x+5}\,dx\\ &\quad + \frac{1}{16}\int\frac{1}{(x+5)^2}\,dx\\ &= -\frac{1}{32}\ln|x+1| - \frac{1}{16}\left(\frac{1}{x+1}\right) + \frac{1}{32}\ln|x+5| - \frac{1}{16}\left(\frac{1}{x+5}\right) + C.\end{aligned}$$

28. Write $\dfrac{1}{(x^2-x-6)(x^2-2x-8)} = \dfrac{A}{x-4} + \dfrac{B}{x-3} + \dfrac{C}{x+2} + \dfrac{D}{(x+2)^2}$.

Then
$$\begin{aligned}1 &= A(x-3)(x+2)^2 + B(x-4)(x+2)^2 + C(x-4)(x-3)(x+2) + D(x-4)(x-3)\\ &= (A+B+C)x^3 + (A-5C+D)x^2 + (-8A-12B-2C-7D)x\\ &\quad + (-12A-16B+24C+12D).\end{aligned}$$

Solving

$$\boxed{\begin{array}{rr} A+B+C=0 & A-5C+D=0 \\ -8A-12B-2C-7D=0 & -12A-16B+24C+12D=1 \end{array}}$$

gives $A=1/36$, $B=-1/25$, $C=11/900$, and $D=1/30$. (Note that A and B can be easily obtained by substituting $x=4$ and $x=3$, respectively, in the initial equation.) Thus

$$\begin{aligned} \int \frac{1}{(x^2-x-6)(x^2-2x-8)}\,dx &= \frac{1}{36}\int \frac{1}{x-4}\,dx - \frac{1}{25}\int \frac{1}{x-3}\,dx + \frac{11}{900}\int \frac{1}{x+2}\,dx \\ &\quad + \frac{1}{30}\int \frac{1}{(x+2)^2}\,dx \\ &= \frac{1}{36}\ln|x-4| - \frac{1}{25}\ln|x-3| + \frac{11}{900}\ln|x+2| \\ &\quad - \frac{1}{30}(x+2)^{-1} + C. \end{aligned}$$

29. Write $\dfrac{x^4+2x^2-x+9}{x^5+2x^4} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x^3} + \dfrac{D}{x^4} + \dfrac{E}{x+2}$.

Then $x^4+2x^2-x+9 = Ax^3(x+2)+Bx^2(x+2)+Cx(x+2)+D(x+2)+Ex^4$
$= (A+E)x^4+(2A+B)x^3+(2B+C)x^2+(2C+D)x+2D.$

Solving

$$\boxed{\begin{array}{lll} A+E=1 & 2A+B=0 & 2B+C=2 \\ 2C+D=-1 & 2D=9 & \end{array}}$$

gives $A=-19/16$, $B=19/8$, $C=-11/4$, $D=9/2$, and $E=35/16$. Thus

$$\begin{aligned} \int \frac{x^4+2x^2-x+9}{x^5+2x^4}\,dx &= -\frac{19}{16}\int \frac{1}{x}\,dx + \frac{19}{8}\int \frac{1}{x^2}\,dx - \frac{11}{4}\int \frac{1}{x^3}\,dx \\ &\quad + \frac{9}{2}\int \frac{1}{x^4}\,dx + \frac{35}{16}\int \frac{1}{x+2}\,dx \\ &= -\frac{19}{16}\ln|x| - \frac{19}{8}\left(\frac{1}{x}\right) + \frac{11}{8}\left(\frac{1}{x^2}\right) - \frac{3}{2}\left(\frac{1}{x^3}\right) + \frac{35}{16}\ln|x+2| + C. \end{aligned}$$

30. Write $\dfrac{5x-1}{x(x-3)^2(x+2)^2} = \dfrac{A}{x} + \dfrac{B}{x-3} + \dfrac{C}{(x-3)^2} + \dfrac{D}{x+2} + \dfrac{E}{(x+2)^2}$.

Then $5x-1 = A(x-3)^2(x+2)^2+Bx(x-3)(x+2)^2+Cx(x+2)^2$

$$\begin{aligned} &\quad + Dx(x-3)^2(x+2)+Ex(x-3)^2 \\ &= (A+B+D)x^4+(-2A+B+C+4D+E)x^3 \\ &\quad + (11A-8B+4C-3D-6E)x^2+(12A-12B+4C+18D+9E)x \\ &\quad + 36A. \end{aligned}$$

Solving

$$\boxed{\begin{array}{rr} A+B+D=0 & -2A+B+C-4D+E=0 \\ 11A-8B+4C-3D-6E=0 & 12A-12B+4C+18D+9E=5 \\ 36A=-1 & \end{array}}$$

gives $A = -1/36$, $B = -79/1125$, $C = 14/75$, $D = 49/500$, and $E = 11/50$. Thus

$$\begin{aligned}\int \frac{5x-1}{x(x-3)^2(x+2)^2}\,dx &= -\frac{1}{36}\int \frac{1}{x}\,dx - \frac{79}{1125}\int \frac{1}{x-3}\,dx + \frac{14}{75}\int \frac{1}{(x-3)^2}\,dx \\ &\quad + \frac{49}{500}\int \frac{1}{x+2}\,dx + \frac{11}{50}\int \frac{1}{(x+2)^2}\,dx \\ &= -\frac{1}{36}\ln|x| - \frac{79}{1125}\ln|x-3| - \frac{14}{75}(x-3)^{-1} + \frac{49}{500}\ln|x+2| \\ &\quad - \frac{11}{50}(x+2)^{-1} + C.\end{aligned}$$

31. Write $\dfrac{x-1}{x(x^2+1)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1}$.

Then $x-1 = A(x^2+1) + (Bx+C)x = (A+B)x^2 + Cx + A$.

Solving $\boxed{A+B=0 \qquad C=1 \qquad A=-1}$

gives $A=-1$, $B=1$, and $C=1$. Thus

$$\begin{aligned}\int \frac{x-1}{x(x^2+1)}\,dx &= -\int \frac{1}{x}\,dx + \int \frac{x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx \\ &= -\ln|x| + \frac{1}{2}\ln(x^2+1) + \tan^{-1}x + C.\end{aligned}$$

32. Write $\dfrac{1}{(x-1)(x^2+3)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+3}$.

Then $1 = A(x^2+3) + (Bx+C)(x-1) = (A+B)x^2 + (-B+C)x + (3A-C)$.

Solving $\boxed{A+B=0 \qquad -B+C=0 \qquad 3A-C=1}$

gives $A = 1/4$, $B = -1/4$, and $C = -1/4$. Thus

$$\begin{aligned}\int \frac{1}{(x-1)(x^2+3)}\,dx &= \frac{1}{4}\int \frac{1}{x-1}\,dx - \frac{1}{4}\int \frac{x}{x^2+3}\,dx - \frac{1}{4}\int \frac{1}{x^2+3}\,dx \\ &= \frac{1}{4}\ln|x-1| - \frac{1}{8}\ln(x^2+3) - \frac{1}{4\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} + C.\end{aligned}$$

33. Write $\dfrac{x}{(x+1)^2(x^2+1)} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{Cx+D}{x^2+1}$.

Then $x = A(x+1)(x^2+1) + B(x^2+1) + (Cx+D)(x+1)^2$

$= (A+C)x^3 + (A+B+2C+D)x^2 + (A+C+2D)x + (A+B+D)$.

Solving $\boxed{\begin{array}{cc} A+C=0 & A+B+2C+D=0 \\ A+C+2D=1 & A+B+D=0 \end{array}}$

gives $A=0$, $B=-1/2$, $C=0$, and $D=1/2$. Thus

$$\int \frac{x}{(x+1)^2(x^2+1)}\,dx = -\frac{1}{2}\int \frac{1}{(x+1)^2}\,dx + \frac{1}{2}\int \frac{1}{x^2+1}\,dx = \frac{1}{2}\left(\frac{1}{x+1}\right) + \frac{1}{2}\tan^{-1}x + C.$$

34. Write $\dfrac{x^2}{(x-1)^3(x^2+4)} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{(x-1)^3} + \dfrac{Dx+E}{x^2+4}$.

Then $x^2 = A(x-1)^2(x^2+4) + B(x-1)(x^2+4) + C(x^2+4) + (Dx+E)(x-1)^3$

$$= (A+D)x^4 + (-2A+B-3D+E)x^3 + (5A-B+C+3D-3E)x^2 + (-8A+4B-D+3E)x + (4A-4B+4C-E).$$

Solving $\boxed{\begin{array}{rr} A+D=0 & -2A+B-3D+E=0 \\ 5A-B+C+3D-3E=1 & -8A+4B-D+3E=0 \\ 4A-4B+4C-E=0 & \end{array}}$

gives $A = 4/125$, $B = 8/25$, $C = 1/5$, $D = -4/125$, and $E = -44/125$. Thus

$$\begin{aligned} \int \frac{x^2}{(x-1)^3(x^2+4)}\,dx &= \frac{4}{125}\int \frac{1}{x-1}\,dx + \frac{8}{25}\int \frac{1}{(x-1)^2}\,dx + \frac{1}{5}\int \frac{1}{(x-1)^3}\,dx \\ &\quad - \frac{4}{125}\int \frac{x}{x^2+4}\,dx - \frac{44}{125}\int \frac{1}{x^2+4}\,dx \\ &= \frac{4}{125}\ln|x-1| - \frac{8}{25}(x-1)^{-1} - \frac{1}{10}(x-1)^{-2} \\ &\quad - \frac{2}{125}\ln(x^2+4) - \frac{22}{125}\tan^{-1}\frac{x}{2} + C. \end{aligned}$$

35. Write $\dfrac{1}{x^4+5x^2+4} = \dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{x^2+4}$.

Then $1 = (Ax+B)(x^2+4) + (Cx+D)(x^2+1)$

$= (A+C)x^3 + (B+D)x^2 + (4A+C)x + (4B+D)$.

Solving $\boxed{A+C=0 \qquad B+D=0 \qquad 4A+C=0 \qquad 4B+D=1}$

gives $A = 0$, $B = 1/3$, $C = 0$, and $D = -1/3$. Thus

$$\int \frac{1}{x^4+5x^2+4}\,dx = \frac{1}{3}\int \frac{1}{x^2+1}\,dx - \frac{1}{3}\int \frac{1}{x^2+4}\,dx = \frac{1}{3}\tan^{-1} x - \frac{1}{6}\tan^{-1}\frac{x}{2} + C.$$

36. Write $\dfrac{1}{x^4+13x^2+36} = \dfrac{Ax+B}{x^2+9} + \dfrac{Cx+D}{x^2+4}$.

Then $1 = (Ax+B)(x^2+4) + (Cx+D)(x^2+9)$

$= (A+C)x^3 + (B+D)x^2 + (4A+9C)x + (4B+9D)$.

Solving $\boxed{A+C=0 \qquad B+D=0 \qquad 4A+9C=0 \qquad 4B+9D=1}$

gives $A = 0$, $B = -1/5$, $C = 0$, and $D = 1/5$. Thus

$$\int \frac{1}{x^4+13x^2+36}\,dx = -\frac{1}{5}\int \frac{1}{x^2+9}\,dx + \frac{1}{5}\int \frac{1}{x^2+4}\,dx = -\frac{1}{15}\tan^{-1}\frac{x}{3} + \frac{1}{10}\tan^{-1}\frac{x}{2} + C.$$

37. Write $\dfrac{1}{x^3-1} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+x+1}$.

Then $1 = A(x^2+x+1) + (Bx+C)(x-1) = (A+B)x^2 + (A-B+C)x + (A-C)$.

Solving $\boxed{A+B=0 \qquad A-B+C=0 \qquad A-C=1}$

gives $A = 1/3$, $B = -1/3$, and $C = -2/3$. Thus

$$\begin{aligned}
\int \frac{1}{x^3-1}\,dx &= \frac{1}{3}\int \frac{1}{x-1}\,dx - \frac{1}{6}\int \frac{2x+4}{x^2+x+1}\,dx \\
&= \frac{1}{3}\ln|x-1| - \frac{1}{6}\int \frac{2x+1}{x^2+x+1}\,dx - \frac{1}{2}\int \frac{1}{x^2+x+1}\,dx \\
&= \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln|x^2+x+1| - \frac{1}{2}\int \frac{1}{(x+1/2)^2+3/4}\,dx \\
&= \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln|x^2+x+1| - \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}} + C.
\end{aligned}$$

38. Write $\dfrac{81}{x^4+27x} = \dfrac{A}{x} + \dfrac{B}{x+3} + \dfrac{Cx+D}{x^2-3x+9}$.

Then $81 = A(x^3+27) + B(x^3-3x^2+9x) + (Cx+D)(x^2+3x)$
$= (A+B+C)x^3 + (-3B+3C+D)x^2 + (9B+3D)x + 27A$.

Solving $\boxed{\begin{array}{ll} A+B+C=0 & -3B+3C+D=0 \\ 9B+3D=0 & 27A=81 \end{array}}$

gives $A = 3$, $B = -1$, $C = -2$, and $D = 3$. Thus

$$\begin{aligned}
\int \frac{81}{x^4+27x}\,dx &= 3\int \frac{1}{x}\,dx - \int \frac{1}{x+3}\,dx - \int \frac{2x-3}{x^2-3x+9}\,dx \\
&= 3\ln|x| - \ln|x+3| - \ln|x^2-3x+9| + C = \ln\left|\frac{x^3}{x^3+27}\right| + C.
\end{aligned}$$

39. Write $\dfrac{3x^2-x+1}{(x+1)(x^2+2x+2)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+2x+2}$.

Then $3x^2-x+1 = A(x^2+2x+2) + (Bx+C)(x+1)$
$= (A+B)x^2 + (2A+B+C)x + (2A+C)$.

Solving $\boxed{A+B=3 \qquad 2A+B+C=-1 \qquad 2A+C=1}$

gives $A = 5$, $B = -2$, and $C = -9$. Thus

$$\begin{aligned}
\int \frac{3x^2-x+1}{(x+1)(x^2+2x+2)}\,dx &= 5\int \frac{1}{x+1}\,dx - \int \frac{2x+9}{x^2+2x+2}\,dx \\
&= 5\ln|x+1| - \int \frac{2x+2}{x^2+2x+2}\,dx - \int \frac{7}{x^2+2x+2}\,dx \\
&= 5\ln|x+1| - \ln|x^2+2x+2| - \int \frac{7}{(x+1)^2+1}\,dx \\
&= 5\ln|x+1| - \ln|x^2+2x+2| - 7\tan^{-1}(x+1) + C.
\end{aligned}$$

40. Write $\dfrac{4x+12}{(x-2)(x^2+4x+8)} = \dfrac{A}{x-2} + \dfrac{Bx+C}{x^2+4x+8}$.

Then $4x+12 = A(x^2+4x+8) + (Bx+C)(x-2)$
$= (A+B)x^2 + (4A-2B+C)x + (8A-2C)$.

Solving $\boxed{A+B=0 \qquad 4A-2B+C=4 \qquad 8A-2C=12}$

gives $A=1$, $B=-1$, and $C=-2$. Thus

$$\begin{aligned}\int \frac{4x+12}{(x-2)(x^2+4x+8)}\,dx &= \int \frac{1}{x-2}\,dx - \int \frac{x+2}{x^2+4x+8}\,dx \\ &= \ln|x-2| - \frac{1}{2}\int \frac{2x+4}{x^2+4x+8}\,dx \\ &= \ln|x-2| - \frac{1}{2}\ln|x^2+4x+8| + C.\end{aligned}$$

41. Write $\dfrac{x^2-x+4}{(x^2+4)^2} = \dfrac{Ax+B}{x^2+4} + \dfrac{Cx+D}{(x^2+4)^2}$.

Then $x^2-x+4 = (Ax+B)(x^2+4) + Cx + D = Ax^3 + Bx^2 + (4A+C)x + (4B+D)$.

Solving $\boxed{A=0 \qquad B=1 \qquad 4A+C=-1 \qquad 4B+D=4}$

gives $A=0$, $B=1$, $C=-1$, and $D=0$. Thus

$$\int \frac{x^2-x+4}{(x^2+4)^2}\,dx = \int \frac{1}{x^2+4}\,dx - \int \frac{x}{(x^2+4)^2}\,dx = \frac{1}{2}\tan^{-1}\frac{x}{2} + \frac{1}{2}\left(\frac{1}{x^2+4}\right) + C$$

42. Write $\dfrac{1}{x^3(x^2+1)^2} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x^3} + \dfrac{Dx+E}{x^2+1} + \dfrac{Fx+G}{(x^2+1)^2}$.

Then $1 = Ax^2(x^2+1)^2 + Bx(x^2+1)^2 + C(x^2+1)^2 + (Dx+E)x^3(x^2+1) + (Fx+G)x^3$
$= (A+D)x^6 + (B+E)x^5 + (2A+C+D+F)x^4 + (2B+E+G)x^3$
$+ (A+2C)x^2 + Bx + C$.

Solving $\boxed{\begin{array}{llll} A+D=0 & B+E=0 & 2A+C+D+F=0 & \\ 2B+E+G=0 & A+2C=0 & B=0 & C=1 \end{array}}$

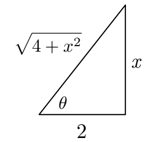
gives $A=-2$, $B=0$, $C=1$, $D=2$, $E=0$, $F=1$, and $G=0$. Thus

$$\begin{aligned}\int \frac{1}{x^3(x^2+1)^2}\,dx &= -2\int \frac{1}{x}\,dx + \int \frac{1}{x^3}\,dx + \int \frac{2x}{x^2+1}\,dx + \frac{1}{2}\int \frac{2x}{(x^2+1)^2}\,dx \\ &= -2\ln|x| - \frac{1}{2}x^{-2} + \ln(x^2+1) - \frac{1}{2}(x^2+1)^{-1} + C.\end{aligned}$$

43. For this and possibly later problems, we will encounter $\int \cos^2\theta\,d\theta$. Using Example 12 of Section 5.2 in the text, we have

$$\int \cos^2\theta\,d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C = \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + C.$$

Write $\dfrac{x^3-2x^2+x-3}{x^4+8x^2+16} = \dfrac{Ax+B}{x^2+4} + \dfrac{Cx+D}{(x^2+4)^2}$.



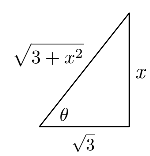
$$\begin{aligned}\text{Then } x^3-2x^2+x-3 &= (Ax+B)(x^2+4)+Cx+D\\ &= Ax^3+Bx^2+(4A+C)x+(4B+D).\end{aligned}$$

Solving $\boxed{A=1 \qquad B=-2 \qquad 4A+C=1 \qquad 4B+D=-3}$

gives $A=1$, $B=-2$, $C=-3$, and $D=5$. Thus

$$\begin{aligned}\int \frac{x^3-2x^2+x-3}{x^4+8x^2+16}\,dx &= \int \frac{x}{x^2+4}\,dx - 2\int \frac{1}{x^2+4}\,dx - 3\int \frac{x}{(x^2+4)^2}\,dx\\ &\quad + 5\int \frac{1}{(x^2+4)^2}\,dx \qquad \boxed{x=2\tan\theta,\ dx=2\sec^2\theta\,d\theta}\\ &= \frac{1}{2}\ln(x^2+4) - \tan^{-1}\frac{x}{2} + \frac{3}{2}\left(\frac{1}{x^2+4}\right) + 5\int \frac{2\sec^2\theta}{(4\tan^2\theta+4)^2}\,d\theta\\ &= \frac{1}{2}\ln(x^2+4) - \tan^{-1}\frac{x}{2} + \frac{3}{2(x^2+4)} + \frac{5}{8}\int \cos^2\theta\,d\theta\\ &= \frac{1}{2}\ln(x^2+4) - \tan^{-1}\frac{x}{2} + \frac{3}{2(x^2+4)} + \frac{5}{16}\theta + \frac{5}{16}\sin\theta\cos\theta + C\\ &= \frac{1}{2}\ln(x^2+4) - \tan^{-1}\frac{x}{2} + \frac{3}{2(x^2+4)} + \frac{5}{16}\tan^{-1}\frac{x}{2}\\ &\quad + \frac{5}{8}\left(\frac{x}{x^2+4}\right) + C\\ &= \frac{1}{2}\ln(x^2+4) - \frac{11}{6}\tan^{-1}\frac{x}{2} + \frac{5x+12}{8(x^2+4)} + C.\end{aligned}$$

44. Write $\dfrac{x^2}{(x^2+3)^2} = \dfrac{Ax+B}{x^2+3} + \dfrac{Cx+D}{(x^2+3)^2}$.



$$\begin{aligned}\text{Then } x^2 &= (Ax+B)(x^2+3)+Cx+D\\ &= Ax^3+Bx^2+(3A+C)x+(3B+D).\end{aligned}$$

Solving $\boxed{A=0 \qquad B=1 \qquad 3A+C=0 \qquad 3B+D=0}$

gives $A=0$, $B=1$, $C=0$, and $D=-3$. Thus

$$\begin{aligned}\int \frac{x^2}{(x^2+3)^2}\,dx &= \int \frac{1}{x^2+3}\,dx - 3\int \frac{1}{(x^2+3)^2}\,dx \qquad \boxed{x=\sqrt{3}\tan\theta,\ dx=\sqrt{3}\sec^2\theta\,d\theta}\\ &= \frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} - 3\int \frac{\sqrt{3}\sec^2\theta}{(3\tan^2\theta+3)^2}\,d\theta = \frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} - \frac{1}{\sqrt{3}}\int \cos^2\theta\,d\theta\\ &= \frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} - \frac{1}{2\sqrt{3}}\theta - \frac{1}{2\sqrt{3}}\sin\theta\cos\theta + C\\ &= \frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} - \frac{1}{2\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} - \frac{1}{2}\left(\frac{x}{x^2+3}\right) + C\\ &= \frac{1}{2\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} - \frac{x}{2x^2+6} + C\end{aligned}$$

45. Write $\dfrac{x^4+3x^2+4}{(x+1)^2} = x^2-2x+6-\dfrac{10x+2}{(x+1)^2} = x^2-2x+6-\dfrac{10(x+1)-8}{(x+1)^2}$.

Then $\displaystyle\int \frac{x^4+3x^2+4}{(x+1)^2}\,dx = \int (x^2-2x+6)\,dx - 10\int \frac{1}{x+1}\,dx + 8\int \frac{1}{(x+1)^2}\,dx$

$$= \frac{1}{3}x^3 - x^2 + 6x - 10\ln|x+1| - \frac{8}{x+1} + C.$$

46. Write $\dfrac{x^5-10x^3}{x^4-10x^2+9} = x - \dfrac{9x}{x^4-10x^2+9} = x - \left(\dfrac{A}{x-1}+\dfrac{B}{x+1}+\dfrac{C}{x-3}+\dfrac{D}{x+3}\right)$.

Then $9x = A(x+1)(x^2-9)+B(x-1)(x^2-9)+C(x+3)(x^2-1)+D(x-3)(x^2-1)$.

Setting $x=1$, $x=-1$, $x=3$, and $x=-3$ gives $A=-9/16$, $B=-9/16$, $C=9/16$, and $D=9/16$. Thus

$$\int \frac{x^5-10x^3}{x^4-10x^2+9}\,dx = \int x\,dx - \left(-\frac{9}{16}\int \frac{1}{x-1}\,dx - \frac{9}{16}\int \frac{1}{x+1}\,dx + \frac{9}{16}\int \frac{1}{x-3}\,dx + \frac{9}{16}\int \frac{1}{x+3}\,dx\right)$$

$$= \frac{1}{x^2} + \frac{9}{16}\ln|x-1| + \frac{9}{16}\ln|x+1| - \frac{9}{16}\ln|x-3| - \frac{9}{16}\ln|x+3| + C$$

$$= \frac{1}{2}x^2 + \frac{9}{16}\ln\left|\frac{x^2-1}{x^2-9}\right| + C.$$

47. Write $\dfrac{1}{x^2-6x+5} = \dfrac{A}{x-1}+\dfrac{B}{x-5}$.

Then $1 = A(x-5)+B(x-1)$. Setting $x=1$ and $x=5$ gives $A=-1/4$ and $B=1/4$. Thus

$$\int_2^4 \frac{1}{x^2-6x+5}\,dx = -\frac{1}{4}\int_2^4 \frac{1}{x-1}\,dx + \frac{1}{4}\int_2^4 \frac{1}{x-5}\,dx = -\frac{1}{4}\ln|x-1|\Big]_2^4 + \frac{1}{4}\ln|x-5|\Big]_2^4$$

$$= \frac{1}{4}\ln\left|\frac{x-5}{x-1}\right|\Big]_2^4 = \frac{1}{4}\left(\ln\frac{1}{3} - \ln 3\right) = -\frac{1}{2}\ln 3.$$

48. Write $\dfrac{1}{x^2-4} = \dfrac{A}{x-2}+\dfrac{B}{x+2}$.

Then $1 = A(x+2)+B(x-2)$. Setting $x=2$ and $x=-2$ gives $A=1/4$ and $B=-1/4$.

Thus $\displaystyle\int_0^1 \frac{1}{x^2-4}\,dx = \frac{1}{4}\int_0^1 \frac{1}{x-2}\,dx - \frac{1}{4}\int_0^1 \frac{1}{x+2}\,dx = \frac{1}{4}\ln|x-2|\Big]_0^1 - \frac{1}{4}\ln|x+2|\Big]_0^1$

$$= \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right|\Big]_0^1 = \frac{1}{4}\left(\ln\frac{1}{3} - \ln 1\right) = -\frac{1}{4}\ln 3.$$

49. $\displaystyle\int_0^2 \frac{2x-1}{(x+3)^2}\,dx = \int_0^2 \frac{2(x+3)-7}{(x+3)^2}\,dx = \int_0^2 \frac{2}{x+3}\,dx - 7\int_0^2 \frac{1}{(x+3)^2}\,dx$

$$= 2\ln|x+3|\Big]_0^2 + \frac{7}{x+3}\Big]_0^2 = 2(\ln 5 - \ln 3) + \left(\frac{7}{5} - \frac{7}{3}\right) = 2\ln\frac{5}{3} - \frac{14}{15}$$

50. Write $\dfrac{2x+6}{x(x+1)^2} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$.

Then $2x + 6 = A(x+1)^2 + B(x^2+x) + Cx = (A+B)x^2 + (2A+B+C)x + A$.

Solving $\boxed{A + B = 0 \qquad 2A + B + C = 2 \qquad A = 6}$

gives $A = 6$, $B = -6$, and $C = -4$. Thus

$$\begin{aligned}
\int_1^5 \frac{2x+6}{x(x+1)^2} &= 6\int_1^5 \frac{1}{x}\,dx - 6\int_1^5 \frac{1}{x+1}\,dx - 4\int_1^5 \frac{1}{(x+1)^2}\,dx \\
&= 6\ln|x|\Big]_1^5 - 6\ln|x+1|\Big]_1^5 + \frac{4}{x+1}\Big]_1^5 = 6\ln\left|\frac{x}{x+1}\right|\Big]_1^5 + \frac{4}{x+1}\Big]_1^5 \\
&= 6\left(\frac{5}{6} - \ln\frac{1}{2}\right) + \left(\frac{4}{6} - \frac{4}{2}\right) = 6\ln\frac{5}{3} - \frac{4}{3}.
\end{aligned}$$

51. Write $\dfrac{1}{x^3+x^2+2x+2} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+2}$.

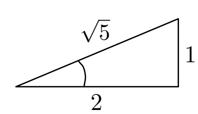
Then $1 = A(x^2+2) + (Bx+C)(x+1) = (A+B)x^2 + (B+C)x + (2A+C)$.

Solving $\boxed{A + B = 0 \qquad B + C = 0 \qquad 2A + C = 1}$

gives $A = 1/3$, $B = -1/3$, and $C = 1/3$. Thus

$$\begin{aligned}
\int_0^1 \frac{1}{x^3+x^2+2x+2}\,dx &= \frac{1}{3}\int_0^1 \frac{1}{x+1}\,dx - \frac{1}{6}\int_0^1 \frac{2x}{x^2+2}\,dx + \frac{1}{3}\int_0^1 \frac{1}{x^2+2}\,dx \\
&= \frac{1}{3}\ln|x+1|\Big]_0^1 + \frac{1}{6}\ln(x^2+2)\Big]_0^1 + \frac{1}{3\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}\Big]_0^1 \\
&= \frac{1}{3}\ln 2 - \frac{1}{6}(\ln 3 - \ln 2) + \frac{1}{3\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}} \\
&= \frac{1}{6}\ln\frac{8}{3} + \frac{1}{3\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}}.
\end{aligned}$$

52. $$\begin{aligned}
\int_0^1 \frac{x^2}{x^4+8x^2+16}\,dx &= \int_0^1 \frac{x^2+4-4}{(x^2+4)^2}\,dx = \int_0^1 \frac{1}{x^2+4}\,dx - 4\int_0^1 \frac{1}{(x^2+4)^2}\,dx \\
&\qquad \boxed{x = 2\tan\theta,\ dx = 2\sec^2\theta\,d\theta} \\
&= \frac{1}{2}\tan^{-1}\frac{x}{2}\Big]_0^1 - 4\int_0^{\tan^{-1}1/2} \frac{2\sec^2\theta}{(4\tan^2\theta+4)^2}\,d\theta \\
&= \frac{1}{2}\tan^{-1}\frac{1}{2} - \frac{1}{2}\int_0^{\tan^{-1}1/2} \cos^2\theta\,d\theta \\
&= \frac{1}{2}\tan^{-1}\frac{1}{2} - \frac{1}{2}\left(\frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta\right)\Big]_0^{\tan^{-1}1/2} \\
&= \frac{1}{2}\tan^{-1}\frac{1}{2} - \frac{1}{4}\tan^{-1}\frac{1}{2} - \frac{1}{4}\left(\frac{1}{\sqrt{5}}\right)\left(\frac{2}{\sqrt{5}}\right) = \frac{1}{4}\tan^{-1}\frac{1}{2} - \frac{1}{10}
\end{aligned}$$



53. $$\int_{-1}^{1} \frac{2x^3+5x}{x^4+5x^2+6}\,dx = \frac{1}{2}\int_{-1}^{1} \frac{4x^3+10x}{x^4+5x^2+6}\,dx = \frac{1}{2}\ln|x^4+5x^2+6|\Big]_{-1}^{1}$$
$$= \frac{1}{2}(\ln 12 - \ln 12) = 0$$

54. Write $\dfrac{1}{x^5+4x^4+5x^3} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x^3} + \dfrac{Dx+E}{x^2+4x+5}$.

Then $1 = Ax^2(x^2+4x+5) + Bx(x^2+4x+5) + C(x^2+4x+5) + (Dx+E)x^3$
$= (A+D)x^4 + (4A+B+E)x^3 + (5A+4B+C)x^2 + (5B+4C)x + 5C.$

Solving $\boxed{\begin{array}{lll} A+D=0 & 4A+B+E=0 & 5A+4B+C=0 \\ 5B+4C=0 & 5C=1 & \end{array}}$

gives $A = 11/125$, $B = -4/25$, $C = 1/5$, $D = -11/125$, and $E = -24/125$. Thus

$$\begin{aligned}\int_1^2 \frac{1}{x^5+4x^4+5x^3}\,dx &= \frac{11}{125}\int_1^2 \frac{1}{x}\,dx - \frac{4}{25}\int_1^2 \frac{1}{x^2}\,dx + \frac{1}{5}\int_1^2 \frac{1}{x^3}\,dx \\ &\quad - \frac{11}{250}\int_1^2 \frac{2x+4}{x^2+4x+5}\,dx - \frac{2}{125}\int_1^2 \frac{1}{(x+2)^2+1}\,dx \\ &= \frac{11}{125}\ln|x|\Big]_1^2 + \frac{4}{25}\left(\frac{1}{x}\right)\Big]_1^2 - \frac{1}{10}\left(\frac{1}{x^2}\right)\Big]_1^2 - \frac{11}{250}\ln|x^2+4x+5|\Big]_1^2 \\ &\quad - \frac{2}{125}\tan^{-1}(x+1)\Big]_1^2 \\ &= \frac{11}{125}\ln 2 + \frac{4}{25}\left(\frac{1}{2}-1\right) - \frac{1}{10}\left(\frac{1}{4}-1\right) - \frac{11}{250}(\ln 17 + \ln 10) \\ &\quad - \frac{2}{125}(\tan^{-1}4 - \tan^{-1}3) \\ &= \frac{11}{250}\ln\frac{40}{17} - \frac{1}{200} - \frac{2}{125}(\tan^{-1}4 - \tan^{-1}3).\end{aligned}$$

55. $$\int \frac{\sqrt{1-x^2}}{x^3}\,dx = \int \frac{\sqrt{1-x^2}}{x^4}x\,dx \qquad \boxed{u^2 = 1-x^2,\ 2u\,du = -2x\,dx}$$
$$= \int \frac{u}{(1-u^2)^2}(-u\,du) = -\int \frac{u^2}{(1-u^2)^2}\,du$$

Write $\dfrac{u^2}{(1-u^2)^2} = \dfrac{A}{1-u} + \dfrac{B}{(1-u)^2} + \dfrac{C}{1+u} + \dfrac{D}{(1+u)^2}$.

Then $u^2 = A(1-u)(1+u)^2 + B(1+u^2) + C(1+u)(1-u)^2 + D(1-u)^2$
$= (A+B+C+D) + (A+2B-C-2D)u + (-A+B-C+D)u^2 + (-A+C)u^3.$

Solving $\boxed{\begin{array}{ll} A+B+C+D=0 & A+2B-C-2D=0 \\ -A+B-C+D=1 & -A+C=0 \end{array}}$

gives $A = -1/4$, $B = 1/4$, $C = -1/4$, and $D = 1/4$. Thus

$$\begin{aligned}
\int \frac{\sqrt{1-x^2}}{x^3}\,dx &= -\int \frac{u^2}{(1-u^2)^2}\,du \\
&= \frac{1}{4}\int \frac{1}{1-u}\,du - \frac{1}{4}\int \frac{1}{(1-u)^2}\,du + \frac{1}{4}\int \frac{1}{1+u}\,du - \frac{1}{4}\int \frac{1}{(1+u)^2}\,du \\
&= -\frac{1}{4}\ln|1-u| - \frac{1}{4}\left(\frac{1}{1-u}\right) + \frac{1}{4}\ln|1+u| + \frac{1}{4}\left(\frac{1}{1+u}\right) + C \\
&= \frac{1}{4}\ln\left|\frac{1+u}{1-u}\right| - \frac{1}{2}\left(\frac{u}{1-u^2}\right) + C = \frac{1}{4}\ln\left|\frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}}\right| - \frac{1}{2}\left(\frac{\sqrt{1-x^2}}{x^2}\right) + C.
\end{aligned}$$

56. $\displaystyle\int \sqrt{\frac{x-1}{x+1}}\,dx$ $\quad \boxed{u^2 = \frac{x-1}{x+1},\ x = \frac{1+u^2}{1-u^2},\ dx = \frac{4u}{(u^2-1)^2}\,du}$

$$= \int |u| \frac{4u}{(u^2-1)^2}\,du = \int \frac{4u^2}{(u^2-1)^2}\,du$$

Write $\displaystyle\frac{4u^2}{(u^2-1)^2} = \frac{A}{u-1} + \frac{B}{(u-1)^2} + \frac{C}{u+1} + \frac{D}{(u+1)^2}$. Then

$$\begin{aligned}
4u^2 &= A(u-1)(u+1)^2 + B(u+1)^2 + C(u-1)^2(u+1) + D(u-1)^2 \\
&= (A+C)u^3 + (A+B-C+D)u^2 + (-A+2B-C-2D)u + (-A+B+C+D).
\end{aligned}$$

Solving $\boxed{\begin{array}{rr} A+C=0 & A+B-C+D=4 \\ -A+2B-C-2D=0 & -A+B+C+D=0 \end{array}}$

gives $A = 1$, $B = 1$, $C = -1$, and $D = 1$. Thus

$$\begin{aligned}
\int \sqrt{\frac{x-1}{x+1}}\,dx &= \int \frac{1}{u-1}\,du + \int \frac{1}{(u-1)^2}\,du - \int \frac{1}{u+1}\,du + \int \frac{1}{(u+1)^2}\,du \\
&= \ln|u-1| - \frac{1}{u-1} - \ln|u+1| - \frac{1}{u+1} + C \\
&= \ln\left|\frac{\sqrt{\frac{x-1}{x+1}} - 1}{\sqrt{\frac{x-1}{x+1}} + 1}\right| - \frac{2\sqrt{\frac{x-1}{x+1}}}{\frac{x-1}{x+1} - 1} + C = \ln\left|\sqrt{x^2-1} - x\right| + \sqrt{x^2-1} + C.
\end{aligned}$$

57. $\displaystyle\int \frac{\sqrt[3]{x+1}}{x}\,dx$ $\boxed{u^3 = x+1,\ 3u^2\,du = dx}$ $\displaystyle = \int \frac{u}{u^3-1}(3u^2\,du)$

$$= 3\int \frac{u^3-1+1}{u^3-1}\,du = 3\int du + 3\int \frac{1}{u^3-1}\,du = 3u + \int \frac{1}{u^3-1}\,du$$

$$= 3u + \ln|u-1| - \frac{1}{2}\ln|u^2+u+1| - \sqrt{3}\tan^{-1}\frac{2u+1}{\sqrt{3}} + C$$

$$= 3\sqrt[3]{x+1} + \ln|\sqrt[3]{x+1} - 1| - \frac{1}{2}\ln\left|\sqrt[3]{(x+1)^2} + \sqrt[3]{x+1} + 1\right|$$

$$- \sqrt{3}\tan^{-1}\frac{2\sqrt[3]{x+1}+1}{\sqrt{3}} + C.$$

58. $\displaystyle\int \frac{1}{\sqrt{x}(1+\sqrt[3]{x})^2}$ $\boxed{x = u^6,\ dx = 6u^5\,du}$

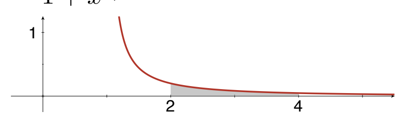
$$= \int \frac{6u^5}{u^3(1+u^2)^2}\,du = \int \frac{6u^2}{(1+u^2)^2}\,du \qquad \boxed{u = \tan\theta,\ du = \sec^2\theta\,d\theta}$$

$$= 6\int \frac{\tan^2\theta}{(1+\tan^2\theta)^2}\sec^2\theta\,d\theta = 6\int \frac{\tan^2\theta}{\sec^2\theta}\,d\theta = 6\int \sin^2\theta\,d\theta$$

$$= 3\int (1-\cos 2\theta)\,d\theta = 3\theta - \frac{3}{2}\sin 2\theta + C = 3\theta - 3\sin\theta\cos\theta + C$$

$$= 3\tan^{-1}u - 3\frac{u}{1+u^2} + C = 3\tan^{-1}x^{1/6} - \frac{3x^{1/6}}{1+x^{1/3}} + C$$

59. Write $\displaystyle\frac{1}{x^2+2x-3} = \frac{A}{x+3} + \frac{B}{x-1}$.

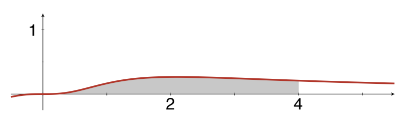


Then $1 = A(x-1) + B(x+3)$. Setting $x = -3$ and $x = 1$ gives $A = -1/4$ and $B = 1/4$. Thus

$$\text{Area} = -\frac{1}{4}\int_2^4 \frac{1}{x+3}\,dx + \frac{1}{4}\int_2^4 \frac{1}{x-1}\,dx = -\frac{1}{4}\ln|x+3|\Big]_2^4 + \frac{1}{4}\ln|x-1|\Big]_2^4$$

$$= \frac{1}{4}\ln\left|\frac{x-1}{x+3}\right|\Big]_2^4 = \frac{1}{4}\left(\ln\frac{3}{7} - \ln\frac{1}{5}\right) = \frac{1}{4}\ln\frac{15}{7} \approx 0.1905.$$

60. Write $\displaystyle\frac{x^3}{(x^2+1)(x^2+2)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+2}$.



Then $x^3 = (Ax+B)(x^2+2) + (Cx+D)(x^2+1)$

$$= (A+C)x^3 + (B+D)x^2 + (2A+C)x + (2B+D).$$

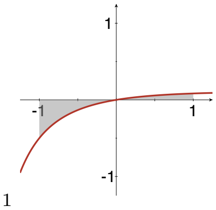
Solving $\boxed{A+C=1 \qquad B+D=0 \qquad 2A+C=0 \qquad 2B+D=0}$

gives $A=-1$, $B=0$, $C=2$, and $D=0$. Thus

$$\text{Area} = -\int_0^4 \frac{x}{x^2+1}\,dx + \int_0^4 \frac{2x}{x^2+2}\,dx = -\frac{1}{2}\ln(x^2+1)\Big]_0^4 + \ln(x^2+2)\Big]_0^4$$

$$= \ln\frac{x^2+2}{\sqrt{x^2+1}}\Big]_0^4 = \ln\frac{18}{\sqrt{17}} - \ln 2 = \ln\frac{9}{\sqrt{17}} \approx 0.7806$$

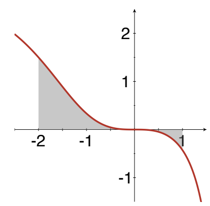
61. Write $\dfrac{x}{(x+2)(x+3)} = \dfrac{A}{x+2} + \dfrac{B}{x+3}$. Then $x = A(x+3) + B(x+2)$.



Setting $x = -2$ and $x = -3$ gives $A = -2$ and $B = 3$. Thus

$$\begin{aligned}
\text{Area} &= -\int_{-1}^{0} \left(\frac{-2}{x+2} + \frac{3}{x+3} \right) dx + \int_{0}^{1} \left(\frac{-2}{x+2} + \frac{3}{x+3} \right) dx \\
&= -(-2\ln|x+2| + 3\ln|x+3|)\Big]_{-1}^{0} + (-2\ln|x+2| + 3\ln|x+3|)\Big]_{0}^{1} \\
&= -[(-2\ln 2 + 3\ln 3) - (-2\ln 1 + 3\ln 2)] + [(-2\ln 3 + 3\ln 4) - (-2\ln 2 + 3\ln 3)] \\
&= 7\ln 2 - 8\ln 3 + 3\ln 4 = \ln\frac{8192}{6561} \approx 0.2220.
\end{aligned}$$

62. Write $\dfrac{3x^3}{x^3-8} = 3 + \dfrac{24}{x^3-8} = 3 + 24\left(\dfrac{A}{x-2} + \dfrac{Bx+C}{x^2+2x+4} \right)$.



Then $1 = A(x^2+2x+4) + (Bx+C)(x-2)$

$= (A+B)x^2 + (2A-2B+C)x + (4A-2C)$.

Solving $\boxed{A+B=0 \qquad 2A-2B+C=0 \qquad 4A-2C=1}$

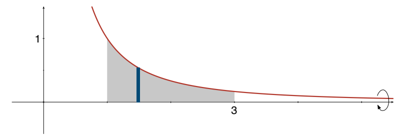
gives $A = 1/12$, $B = -1/12$, and $C = -1/3$. Thus

$$\begin{aligned}
\int \frac{3x^3}{x^3-8}\,dx &= \int \left[3 + 24\left(\frac{1/12}{x-2} - \frac{x/12}{x^2+2x+4} - \frac{1/3}{x^2+2x+4} \right) \right] dx \\
&= \int \left(3 + \frac{2}{x-2} - \frac{2x+8}{x^2+2x+4} \right) dx \\
&= 3\int dx + 2\int \frac{1}{x-2}\,dx - \int \frac{1}{x^2+2x+4}(2x+2)\,dx - 6\int \frac{1}{x^2+2x+4}\,dx \\
&= 3x + 2\ln|x-2| - \ln|x^2+2x+4| - 6\int \frac{1}{(x+1)^2+3}\,dx \\
&= 3x + \ln\left| \frac{x^2-4x+4}{x^2+2x+4} \right| - 2\sqrt{3}\tan^{-1}\frac{x+1}{\sqrt{3}} + C
\end{aligned}$$

Let $g(x) = \displaystyle\int \frac{3x^3}{x^3-8}\,dx = 3x + \ln\left| \frac{x^2-4x+4}{x^2+2x+4} \right| - 2\sqrt{3}\tan^{-1}\frac{x+1}{\sqrt{3}}$. Then

$$\begin{aligned}
\text{Area} &= \int_{-2}^{0} \frac{3x^3}{x^3-8}\,dx - \int_{0}^{1} \frac{3x^3}{x^3-8}\,dx = g(x)\Big]_{-2}^{0} - g(x)\Big]_{0}^{1} = [g(0) - g(-2)] - [g(1) - g(0)] \\
&= \left[\left(0 + \ln 1 - 2\sqrt{3}\tan^{-1}\frac{1}{\sqrt{3}} \right) - \left(-6 + \ln 4 - 2\sqrt{3}\tan^{-1}\frac{-1}{\sqrt{3}} \right) \right] \\
&\qquad - \left[\left(3 + \ln\frac{1}{7} - 2\sqrt{3}\tan^{-1}\frac{2}{\sqrt{3}} \right) - \left(0 + \ln 1 - 2\sqrt{3}\tan^{-1}\frac{1}{\sqrt{3}} \right) \right] \\
&= \left(-\frac{\sqrt{3}\pi}{3} + 6 - \ln 4 - \frac{\sqrt{3}\pi}{3} \right) - \left(3 - \ln 7 - 2\sqrt{3}\tan^{-1}\frac{2}{\sqrt{3}} + \frac{\sqrt{3}\pi}{3} \right) \\
&= 3 + \ln\frac{7}{4} - \sqrt{3}\pi + 2\sqrt{3}\tan^{-1}\frac{2}{\sqrt{3}} \approx 1.0872.
\end{aligned}$$

63. $V = \pi \int_1^3 \frac{4}{x^2(x+1)^2}\,dx$



Write $\frac{4}{x^2(x+1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2}$.

Then $4 = Ax(x+1)^2 + B(x+1)^2 + Cx^2(x+1) + Dx^2$

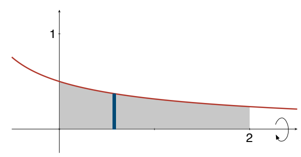
$= (A+C)x^3 + (2A+B+C+D)x^2 + (A+2B)x + B.$

Solving $\boxed{A+C=0 \qquad 2A+B+C+D=0 \qquad A+2B=0 \qquad B=4}$

gives $A=-8$, $B=4$, $C=8$, and $D=4$. Thus

$$\begin{aligned} V &= \pi\left[-8\int_1^3 \frac{1}{x}\,dx + 4\int_1^3 \frac{1}{x^2}\,dx + 8\int_1^3 \frac{1}{x+1}\,dx + 4\int_1^3 \frac{1}{(x+1)^2}\,dx\right] \\ &= \pi\left(-8\ln|x| - \frac{4}{x} + 8\ln|x+1| - \frac{4}{x+1}\right)\Big]_1^3 = \pi\left[8\ln\left|\frac{x+1}{x}\right| - \frac{8x+4}{x(x+1)}\right]\Big|_1^3 \\ &= \pi\left[\left(8\ln\frac{4}{3} - \frac{7}{3}\right) - (8\ln 2 - 6)\right] = 8\pi\ln\frac{2}{3} + \frac{11\pi}{3} \approx 1.3287. \end{aligned}$$

64. $V = \pi \int_0^2 \frac{1}{(x+1)(x+4)}\,dx$



Write $\frac{1}{(x+1)(x+4)} = \frac{A}{x+1} + \frac{B}{x+4}$.

Then $1 = A(x+4) + B(x+1)$. Setting $x=-1$ and $x=-4$ gives $A = 1/3$ and $B = -1/3$. Thus

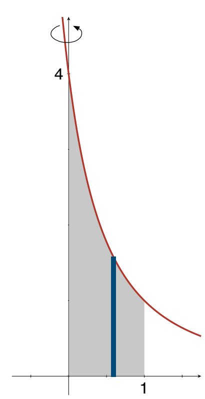
$$\begin{aligned} V &= \pi\left(\frac{1}{3}\int_0^2 \frac{1}{x+1}\,dx - \frac{1}{3}\int_0^2 \frac{1}{x+4}\,dx\right) = \pi\left(\frac{1}{3}\ln|x+1| - \frac{1}{3}\ln|x+4|\right)\Big]_0^2 \\ &= \frac{\pi}{3}\ln\left|\frac{x+1}{x+4}\right|\Big]_0^2 = \frac{\pi}{3}\left(\ln\frac{1}{2} - \ln\frac{1}{4}\right) = \frac{\pi}{3}\ln 2 \approx 0.7259. \end{aligned}$$

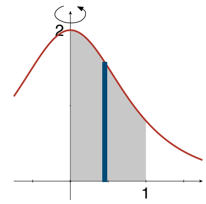
65. $V = 2\pi \int_0^1 \frac{4x}{(x+1)^2}\,dx = 8\pi\int_0^1 \frac{x+1-1}{(x+1)^2}\,dx$

$= 8\pi\left[\int_0^1 \frac{1}{x+1}\,dx - \int_0^1 \frac{1}{(x+1)^2}\,dx\right]$

$= 8\pi\left(\ln|x+1| + \frac{1}{x+1}\right)\Big]_0^1 = 8\pi\left[\left(\ln 2 + \frac{1}{2}\right) - (\ln 1 + 1)\right]$

$= 8\pi\ln 2 - 4\pi \approx 4.8543$





66. $V = 2\pi \int_0^1 \frac{8x}{(x^2+1)(x^2+4)}\,dx$

Write $\frac{8x}{(x^2+1)(x^2+4)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4}$.

Then $8x = (Ax+B)(x^2+4) + (Cx+D)(x^2+1)$
$= (A+C)x^3 + (B+D)x^2 + (4A+C)x + (4B+D)$.

Solving $\boxed{A+C=0 \qquad B+D=0 \qquad 4A+C=8 \qquad 4B+D=0}$

gives $A = 8/3$, $B = 0$, $C = -8/3$, and $D = 0$. Thus

$$V = 2\pi\left(\frac{8}{3}\int_0^1 \frac{x}{x^2+1}\,dx - \frac{8}{3}\int_0^1 \frac{x}{x^2+4}\,dx\right) = 2\pi\left[\frac{4}{3}\ln(x^2+1) - \frac{4}{3}\ln(x^2+4)\right]\Bigg|_0^1$$
$$= \frac{8\pi}{3}\ln\frac{x^2+1}{x^2+4}\Bigg]_0^1 = \frac{8\pi}{3}\left(\ln\frac{2}{5} - \ln\frac{1}{4}\right) = \frac{8\pi}{3}\ln\frac{8}{5} \approx 3.9375.$$

67. $\int \frac{\cos x}{\sin^2 x + 3\sin x + 2}\,dx$ $\boxed{u = \sin x,\ du = \cos x\,dx}$ $= \int \frac{1}{u^2+3u+2}\,du$

Write $\frac{1}{(u+1)(u+2)} = \frac{A}{u+1} + \frac{B}{u+2}$.

Then $1 = A(u+2) + B(u+1)$. Setting $u = -1$ and $u = -2$ gives $A = 1$ and $B = -1$. Thus

$$\int \frac{\cos x}{\sin^2 x + 3\sin x + 2}\,dx = \int \frac{1}{u+1}\,du - \int \frac{1}{u+2}\,du = \ln|u+1| - \ln|u+2| + C$$
$$= \ln\left|\frac{u+1}{u+2}\right| + C = \ln\left|\frac{\sin x + 1}{\sin x + 2}\right| + C.$$

68. $\int \frac{\sin x}{\cos^2 x - \cos^3 x}\,dx$ $\boxed{u = \cos x,\ du = -\sin x\,dx}$ $= \int \frac{-1}{u^2-u^3}\,du = \int \frac{1}{u^2(u-1)}\,du$

Write $\frac{1}{u^2(u-1)} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u-1}$.

Then $1 = A(u^2-u) + B(u-1) + Cu^2 = (A+C)u^2 + (-A+B)u - B$.

Solving $\boxed{A+C=0 \qquad -A+B=0 \qquad -B=1}$

gives $A = -1$, $B = -1$, and $C = 1$. Thus

$$\int \frac{\sin x}{\cos^2 x - \cos^3 x}\,dx = -\int \frac{1}{u}\,du - \int \frac{1}{u^2}\,du + \int \frac{1}{u-1}\,du = -\ln|u| + \frac{1}{u} + \ln|u-1| + C$$
$$= \ln\left|\frac{u-1}{u}\right| + \frac{1}{u} + C = \ln\left|\frac{\cos x - 1}{\cos x}\right| + \frac{1}{\cos x} + C$$
$$= \ln|1 - \sec x| + \sec x + C.$$

69. $\displaystyle\int \frac{e^t}{(e^t+1)^2(e^t-2)}\,dt \quad \boxed{u=e^t,\ du=e^t\,dt} \quad = \int \frac{1}{(u+1)^2(u-2)}\,du$

Write $\displaystyle\frac{1}{(u+1)^2(u-2)} = \frac{A}{u+1} + \frac{B}{(u+1)^2} + \frac{C}{u-2}$.

Then $1 = A(u+1)(u-2) + B(u-2) + C(u+1)^2$
$= (A+C)u^2 + (-A+B+2C)u + (-2A-2B+C)$.

Solving $\boxed{A+C=0 \qquad -A+B+2C=0 \qquad -2A-2B+C=1}$

gives $A=-1/9$, $B=-1/3$, and $C=1/9$. Thus

$$\begin{aligned}\int \frac{e^t}{(e^t+1)^2(e^t-2)}\,dt &= -\frac{1}{9}\int\frac{1}{u+1}\,du - \frac{1}{3}\int\frac{1}{(u+1)^2}\,du + \frac{1}{9}\int\frac{1}{u-2}\,du \\ &= -\frac{1}{9}\ln|u+1| + \frac{1}{3}\left(\frac{1}{u+1}\right) + \frac{1}{9}\ln|u-2| + C \\ &= \frac{1}{9}\ln\left|\frac{e^t-2}{e^t+1}\right| + \frac{1}{3(e^t+1)} + C.\end{aligned}$$

70. $\displaystyle\int \frac{e^{2t}}{(e^t+1)^3}\,dt = \int \frac{e^t}{(e^t+1)^3}\,e^t\,dt \quad \boxed{u=e^t+1,\ du=e^t\,dt} \quad = \int\frac{u-1}{u^3}\,du$

$$= \int (u^{-2}-u^{-3})\,du = -u^{-1} + \frac{1}{2}u^{-2} + C = \frac{1}{2(e^t+1)^2} - \frac{1}{e^t+1} + C$$

71. $y' = e^x$

$\displaystyle L = \int_0^{\ln 2}\sqrt{1+e^{2x}}\,dx \quad \boxed{u^2 = 1+e^{2x},\ 2u\,du = 2e^{2x}\,dx,\ dx = \frac{u}{e^{2x}}\,du = \frac{u}{u^2-1}\,du}$

$$= \int_{\sqrt{2}}^{\sqrt{5}} \frac{u^2}{u^2-1}\,du = \int_{\sqrt{2}}^{\sqrt{5}}\left(1+\frac{1}{u^2-1}\right)du$$

Write $\displaystyle\frac{1}{u^2-1} = \frac{A}{u-1} + \frac{B}{u+1}$.

Then $1 = A(u+1) + B(u-1)$. Setting $u=1$ and $u=-1$ gives $A=1/2$ and $B=-1/2$.

Thus $\displaystyle L = \int_{\sqrt{2}}^{\sqrt{5}} du + \frac{1}{u}\int_{\sqrt{2}}^{\sqrt{5}}\frac{1}{u-1}\,du - \frac{1}{2}\int_{\sqrt{2}}^{\sqrt{5}}\frac{1}{u+1}\,du$

$$\begin{aligned}&= \left(u + \frac{1}{2}\ln|u-1| - \frac{1}{2}\ln|u+1|\right)\Bigg]_{\sqrt{2}}^{\sqrt{5}} = \left(u + \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right|\right)\Bigg]_{\sqrt{2}}^{\sqrt{5}} \\ &= \left[\left(\sqrt{5} + \frac{1}{2}\ln\frac{\sqrt{5}-1}{\sqrt{5}+1}\right) - \left(\sqrt{2} + \frac{1}{2}\ln\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right] \approx 1.2220.\end{aligned}$$

72. (a) $\displaystyle\int\frac{x^3}{(x^2-1)(x^2+1)}\,dx = \int\frac{x^3}{x^4-1}\,dx = \frac{1}{4}\int\frac{1}{x^4-1}(4x^3\,dx)$. Partial fraction decomposition is unnecessary because the substitution $u = x^4-1$, $du = 4x^3\,dx$ will suffice.

(b) $\displaystyle\int \frac{3x+4}{x^2+4}\,dx = \frac{3}{2}\int \frac{1}{x^2+4}(2x\,dx) + 4\int \frac{1}{x^2+4}\,dx$. Partial fraction decomposition is unnecessary because the substitution $u = x^2+4$, $du = 2x\,dx$ will suffice for the first term, while the second term corresponds to formula 24 in Table 7.1.1.

(c) $\displaystyle\int \frac{x}{(x^2+5)^2}\,dx = \frac{1}{2}\int \frac{1}{(x^2+5)^2}(2x\,dx)$. Partial fraction decomposition is unnecessary because the substitution $u = x^2+5$, $du = 2x\,dx$ will suffice.

(d) $\displaystyle\int \frac{2x^3+5x}{x^4+5x^2+6}\,dx = \frac{1}{2}\int \frac{1}{x^4+5x^2+6}(4x^3+10x\,dx)$. Partial fraction decomposition is unnecessary because the substitution $u = x^4+5x^2+6$, $du = 4x^3+10x\,dx$ will suffice.

73. Rewrite the integral as

$$\int \frac{x^5}{(x-1)^{10}(x+1)^{10}}\,dx = \int \frac{x^5}{(x^2-1)^{10}}\,dx = \int \frac{x^4}{(x^2-1)^{10}}(x\,dx)$$

then integrate using $\boxed{u = x^2-1,\ x^2 = u+1,\ \frac{1}{2}u = x\,dx}$:

$$\begin{aligned}\int \frac{x^4}{(x^2-1)^{10}}(x\,dx) &= \frac{1}{2}\int \frac{(u+1)^2}{u^{10}}\,du = \frac{1}{2}\int \frac{u^2+2u+1}{u^{10}}\,du \\ &= \frac{1}{2}\int (u^{-8}+2u^{-9}+u^{-10})\,du = -\frac{1}{14}u^{-7} - \frac{1}{8}u^{-8} - \frac{1}{18}u^{-9} + C \\ &= -\frac{1}{14}(x^2-1)^{-7} - \frac{1}{8}(x^2-1)^{-8} - \frac{1}{18}(x^2-1)^{-9} + C\end{aligned}$$

74. The integrand in Problem 53 is an odd function and its definite integral is symmetric about the y-axis. Thus, the definite integral's value is known to be 0.

7.7 Improper Integrals

In this exercise set, the symbol "$\stackrel{h}{=}$" is used to denote the fact that L'Hôpital's Rule was applied to obtain the equality.

1. $\displaystyle\int_3^\infty \frac{1}{x^4}\,dx = \lim_{t\to\infty}\int_3^t x^{-4}\,dx = \lim_{t\to\infty}\left(-\frac{1}{3}x^{-3}\right)\Big]_3^t = \lim_{t\to\infty}\left(\frac{1}{81} - \frac{1}{3t^3}\right) = \frac{1}{81}$

2. $\displaystyle\int_{-\infty}^{-1} \frac{1}{\sqrt[3]{x}}\,dx = \lim_{s\to-\infty}\int_s^{-1} x^{-1/3}\,dx = \lim_{s\to-\infty}\frac{3}{2}x^{2/3}\Big]_s^{-1} = \lim_{s\to-\infty}\left(\frac{3}{2} - \frac{3}{2}s^{2/3}\right)$

   The integral diverges.

3. $\displaystyle\int_1^\infty \frac{1}{x^{0.99}}\,dx = \lim_{t\to\infty}\int_1^t x^{-0.99}\,dx = \lim_{t\to\infty}\frac{x^{0.01}}{0.01}\Big]_1^t = \lim_{t\to\infty}\left(\frac{t^{0.01}}{0.01} - \frac{1}{0.01}\right)$

   The integral diverges.

4. $\displaystyle\int_1^\infty \frac{1}{x^{1.01}}\,dx = \lim_{t\to\infty}\int_1^t x^{-1.01}\,dx = \lim_{t\to\infty} \frac{x^{-0.01}}{-0.01}\Big]_1^t = \lim_{t\to\infty}\left(\frac{1}{0.01} - \frac{1}{0.01t^{0.01}}\right) = 100$

5. $\displaystyle\int_{-\infty}^3 e^{2x}\,dx = \lim_{s\to-\infty}\int_s^3 e^{2x}\,dx = \lim_{s\to-\infty} \frac{1}{2}e^{2x}\Big]_s^3 = \lim_{s\to-\infty}\left(\frac{1}{2}e^6 - \frac{1}{2}e^{2x}\right) = \frac{1}{2}e^6$

6. $\displaystyle\int_{-\infty}^\infty e^{-x}\,dx = \int_{-\infty}^0 e^{-x}\,dx + \int_0^\infty e^{-x}\,dx$

   Since $\displaystyle\int_{-\infty}^0 e^{-x}\,dx = \lim_{s\to-\infty}\int_s^0 e^{-x}\,dx = \lim_{s\to-\infty}(-e^{-x})\Big]_s^0 = \lim_{s\to-\infty}(e^{-s} - 1)$,
   the integral diverges.

7. $\displaystyle\int_1^\infty \frac{\ln x}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{\ln x}{x}\,dx = \lim_{t\to\infty}\frac{1}{2}(\ln x)^2\Big]_1^t = \lim_{t\to\infty}\left[\frac{1}{2}(\ln t)^2 - 0\right]$

   The integral diverges.

8. $\displaystyle\int_1^\infty \frac{\ln 2}{t^2}\,dt = \lim_{k\to\infty}\int_1^k \frac{\ln t}{t^2}\,dt \qquad \boxed{u = \ln t,\ du = \frac{1}{t}\,dt;\quad dv = \frac{1}{t^2}\,dt,\ v = -\frac{1}{t}}$

$$= \lim_{k\to\infty}\left(-\frac{1}{t}\ln t\Big]_1^k + \int_1^k \frac{1}{t^2}\,dt\right) = \lim_{k\to\infty}\left(0 - \frac{1}{k}\ln k - \frac{1}{t}\Big]_1^k\right)$$

$$= \lim_{k\to\infty}\left(1 - \frac{1}{k} - \frac{1}{k}\ln k\right) = 1 - \lim_{k\to\infty}\frac{\ln k}{k} \stackrel{h}{=} 1 - \lim_{k\to\infty}\frac{1/k}{1} = 1$$

9. $\displaystyle\int_e^\infty \frac{1}{x(\ln x)^3}\,dx = \lim_{t\to\infty}\int_e^t (\ln x)^{-3}\frac{1}{x}\,dx = \lim_{t\to\infty}\left[-\frac{1}{2}(\ln x)^{-2}\right]\Big|_e^t = \lim_{t\to\infty}\left[\frac{1}{2} - \frac{1}{2(\ln t)^2}\right] = \frac{1}{2}$

10. $\displaystyle\int_e^\infty \ln x\,dx = \lim_{t\to\infty}\int_e^t \ln x\,dx = \lim_{t\to\infty}(x\ln x - x)\Big]_e^t = \lim_{t\to\infty}(t\ln t - t - e + e) = \lim_{t\to\infty}\frac{\ln t - 1}{1/t}$

    The limit has the form $\infty/0$, so the integral diverges.

11. $\displaystyle\int_{-\infty}^\infty \frac{x}{(x^2+1)^{3/2}}\,dx = \int_{-\infty}^0 \frac{x}{(x^2+1)^{3/2}}\,dx + \int_0^\infty \frac{x}{(x^2+1)^{3/2}}\,dx$

$$= \lim_{s\to-\infty}\int_s^0 \frac{x}{(x^2+1)^{3/2}}\,dx + \lim_{t\to\infty}\int_0^t \frac{x}{(x^2+1)^{3/2}}\,dx$$

$$= \lim_{s\to-\infty}[-(x^2+1)^{-1/2}]\Big]_s^0 + \lim_{t\to\infty}[-(x^2+1)^{-1/2}]\Big]_0^t$$

$$= \lim_{s\to-\infty}\left(-1 + \frac{1}{\sqrt{s^2+1}}\right) + \lim_{t\to\infty}\left(-\frac{1}{\sqrt{t^2+1}} + 1\right) = -1 + 1 = 0$$

12. $\displaystyle\int_{-\infty}^\infty \frac{x}{1+x^2}\,dx = \int_{-\infty}^0 \frac{x}{1+x^2}\,dx + \int_0^\infty \frac{x}{1+x^2}\,dx$

    Since $\displaystyle\int_{-\infty}^0 \frac{x}{1+x^2}\,dx = \lim_{s\to-\infty}\int_s^0 \frac{x}{1+x^2}\,dx = \lim_{s\to-\infty}\frac{1}{2}\ln(1+x^2)\Big]_s^0$

$$= \lim_{s\to-\infty}\frac{1}{2}[\ln 1 - \ln(1+s^2)] = -\frac{1}{2}\lim_{s\to-\infty}\ln(1+s^2),$$

the integral diverges.

13. $$\int_{-\infty}^{0} \frac{x}{(x^2+9)^2}\,dx = \lim_{s\to-\infty}\int_s^0 \frac{x}{(x^2+9)^2}\,dx = \lim_{s\to-\infty}\left[-\frac{1}{2}(x^2+9)^{-1}\right]\Bigg|_s^0$$
$$= \lim_{s\to-\infty}\left[\frac{1}{2}\left(\frac{1}{s^2+9}\right)-\frac{1}{18}\right] = -\frac{1}{18}$$

14. $$\int_5^{\infty} \frac{1}{\sqrt[4]{3x+1}}\,dx = \lim_{t\to\infty}\int_5^t (3x+1)^{-1/4}\,dx = \lim_{t\to\infty}\frac{4}{9}(3x+1)^{3/4}\Big]_5^t$$
$$= \lim_{t\to\infty}\left[\frac{4}{9}(3t+1)^{3/4}-\frac{4}{9}(8)\right]$$

The integral diverges.

15. $$\int_2^{\infty} ue^{-u}\,du = \lim_{t\to\infty}\int_2^t ue^{-u}\,du = \lim_{t\to\infty}(-ue^{-u}-e^{-u})\Big]_2^t = \lim_{t\to\infty}(2e^{-2}+e^{-2}-te^{-t}-e^{-t})$$
$$= 2e^{-2}+e^{-2}-\lim_{t\to\infty}\frac{t+1}{e^t} \stackrel{h}{=} 2e^{-2}+e^{-2}-\lim_{t\to\infty}\frac{1}{e^t} = 2e^{-2}+e^{-2} = 3e^{-2}$$

16. $$\int_{-\infty}^{3} \frac{x^3}{x^4+1}\,dx = \lim_{s\to-\infty}\int_s^3 \frac{x^3}{x^4+1}\,dx = \lim_{s\to-\infty}\frac{1}{4}\ln(x^4+1)\Big]_s^3$$
$$= \lim_{s\to-\infty}\left[\frac{1}{4}\ln 82-\frac{1}{4}\ln(s^4+1)\right]$$

The integral diverges.

17. $$\int_{2/\pi}^{\infty} \frac{\sin(1/x)}{x^2}\,dx = \lim_{t\to\infty}\int_{2/\pi}^t \frac{\sin(1/x)}{x^2}\,dx = \lim_{t\to\infty}\cos\frac{1}{x}\Big]_{2/\pi}^t = \lim_{t\to\infty}\left(\cos\frac{1}{t}-\cos\frac{\pi}{2}\right) = 1$$

18. $$\int_{-\infty}^{\infty} te^{-t^2}\,dt = \int_{-\infty}^{0} te^{-t^2}\,dt + \int_0^{\infty} te^{-t^2}\,dt = \lim_{s\to-\infty}\int_s^0 te^{-t^2}\,dt + \lim_{r\to\infty}\int_0^r te^{-t^2}\,dt$$
$$= \lim_{s\to-\infty}\left(-\frac{1}{2}e^{-t^2}\right)\Big]_s^0 + \lim_{r\to\infty}\left(-\frac{1}{2}e^{-t^2}\right)\Big]_0^r$$
$$= \lim_{s\to-\infty}\left(\frac{1}{2}e^{-s^2}-\frac{1}{2}\right) + \lim_{r\to\infty}\left(\frac{1}{2}-\frac{1}{2}e^{-r^2}\right) = -\frac{1}{2}+\frac{1}{2} = 0$$

19. $$\int_{-1}^{\infty} \frac{1}{x^2+2x+2}\,dx = \lim_{t\to\infty}\int_{-1}^t \frac{1}{(x+1)^2+1}\,dx = \lim_{t\to\infty}\tan^{-1}(x+1)\Big]_{-1}^t$$
$$= \lim_{t\to\infty}[\tan^{-1}(t+1)-0] = \frac{\pi}{2}$$

20. $$\int_{-\infty}^{0} \frac{1}{x^2+2x+3}\,dx = \lim_{s\to-\infty}\int_s^0 \frac{1}{(x+1)^2+2}\,dx = \lim_{s\to-\infty}\frac{1}{\sqrt{2}}\tan^{-1}\frac{x+1}{\sqrt{2}}\Big]_s^0$$
$$= \lim_{s\to-\infty}\left(\frac{1}{\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\tan^{-1}\frac{s+1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}}+\frac{\pi}{2\sqrt{2}}$$

21. $\displaystyle\int_0^\infty e^{-x}\sin x\,dx = \lim_{t\to\infty}\int_0^t e^{-x}\sin x\,dx$

$$\boxed{u = e^{-x},\ du = -e^{-x}\,dx;\quad dv = \sin x\,dx,\ v = -\cos x}$$

$$= \lim_{t\to\infty}\left(-e^{-x}\cos x\Big]_0^t - \int_0^t e^{-x}\cos x\,dx\right)$$

$$\boxed{u = e^{-x},\ du = -e^{-x}\,dx;\quad dv = \cos x\,dx,\ v = \sin x}$$

$$= \lim_{t\to\infty}\left(1 - e^{-t}\cos t - e^{-x}\sin x\Big]_0^t - \int_0^t e^{-x}\sin x\,dx\right)$$

$$= \lim_{t\to\infty}\left(1 - e^{-t}\cos t - e^{-t}\sin t - \int_0^t e^{-x}\sin x\,dx\right) = 1 - \int_0^\infty e^{-x}\sin x\,dx$$

Solving for the integral, $\displaystyle\int_0^\infty e^{-x}\sin x\,dx = \frac{1}{2}$.

22. $\displaystyle\int_{-\infty}^0 e^x\cos 2x\,dx = \lim_{s\to-\infty}\int_s^0 e^x\cos 2x\,dx$

$$\boxed{u = e^x,\ du = e^x\,dx;\quad dv = \cos 2x\,dx,\ v = \frac{1}{2}\sin 2x}$$

$$= \lim_{s\to-\infty}\left(\frac{1}{2}e^x\sin 2x\Big]_s^0 - \frac{1}{2}\int_s^0 e^x\sin 2x\,dx\right)$$

$$\boxed{u = e^x,\ du = e^x\,dx;\quad dv = \sin 2x\,dx,\ v = -\frac{1}{2}\cos 2x}$$

$$= -\frac{1}{2}\lim_{s\to-\infty}\left(-\frac{1}{2}e^x\cos 2x\Big]_s^0 + \frac{1}{2}\int_s^0 e^x\cos 2x\,dx\right)$$

$$= -\frac{1}{2}\left(-\frac{1}{2} - 0\right) - \frac{1}{4}\int_{-\infty}^0 e^x\cos 2x\,dx$$

Solving for the integral, $\displaystyle\int_{-\infty}^0 e^x\cos 2x\,dx = \frac{4}{5}\left(\frac{1}{4}\right) = \frac{1}{5}$.

23. $\displaystyle\int_{1/2}^\infty \frac{x+1}{x^3}\,dx = \lim_{t\to\infty}\int_{1/2}^t (x^{-2}+x^{-3})\,dx = \lim_{t\to\infty}\left(-x^{-1} - \frac{1}{2}x^{-2}\right)\Big]_{1/2}^t$

$$= \lim_{t\to\infty}\left(2 + 2 - \frac{1}{t} - \frac{1}{2t^2}\right) = 4$$

24. $\displaystyle\int_0^\infty (e^{-x} - e^{-2x})^2\,dx = \lim_{t\to\infty}\int_0^t (e^{-2x} - 2e^{-3x} + e^{-4x})\,dx$

$$= \lim_{t\to\infty}\left(-\frac{1}{2}e^{-2x} + \frac{2}{3}e^{-3x} - \frac{1}{4}e^{-4x}\right)\Big]_0^t$$

$$= \lim_{t\to\infty}\left(-\frac{1}{2}e^{-2t} + \frac{2}{3}e^{-3t} - \frac{1}{4}e^{-4t} + \frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = \frac{1}{12}$$

25. $$\int_1^\infty \left(\frac{1}{x} - \frac{1}{x+1}\right) dx = \lim_{t\to\infty} \int_1^t \left(\frac{1}{x} - \frac{1}{x+1}\right) dx = \lim_{t\to\infty} (\ln|x| - \ln|x+1|)\Big]_1^t$$
$$= \lim_{t\to\infty} [\ln t - \ln(t+1) + \ln 2] = \ln 2 + \lim_{t\to\infty} \ln \frac{t}{t+1}$$
$$= \ln 2 + \ln\left(\lim_{t\to\infty} \frac{t}{t+1}\right) \overset{h}{=} \ln 2 + \ln\left(\lim_{t\to\infty} \frac{1}{1}\right) = \ln 2 + \ln 1 = \ln 2$$

26. $$\int_3^\infty \left(\frac{1}{x} + \frac{1}{x^2+9}\right) dx = \lim_{t\to\infty} \int_3^t \left(\frac{1}{x} + \frac{1}{x^2+9}\right) dx = \lim_{t\to\infty} \left(\ln|x| + \frac{1}{3}\tan^{-1}\frac{x}{3}\right)\Big]_3^t$$
$$= \lim_{t\to\infty} \left(\ln t + \frac{1}{3}\tan^{-1}\frac{t}{3} - \ln 3 - \frac{\pi}{12}\right)$$

The integral diverges.

27. $$\int_2^\infty \frac{1}{x^2+6x+5}\, dx = \lim_{t\to\infty} \int_2^t \frac{1}{(x+1)(x+5)}\, dx = \lim_{t\to\infty} \int_2^t \left(\frac{1/4}{x+1} - \frac{1/4}{x+5}\right) dx$$
$$= \lim_{t\to\infty} \left(\frac{1}{4}\ln|x+1| - \frac{1}{4}\ln|x+5|\right)\Big]_2^t = \lim_{t\to\infty} \frac{1}{4}\ln\left|\frac{x+1}{x+5}\right|\Big]_2^t$$
$$= \lim_{t\to\infty} \left(\frac{1}{4}\ln\frac{t+1}{t+5} - \frac{1}{4}\ln\frac{3}{7}\right) = \frac{1}{4}\ln\left(\lim_{t\to\infty}\frac{t+1}{t+5}\right) - \frac{1}{4}\ln\frac{3}{7}$$
$$\overset{h}{=} \frac{1}{4}\ln\left(\lim_{t\to\infty}\frac{1}{1}\right) - \frac{1}{4}\ln\frac{3}{7} = -\frac{1}{4}\ln\frac{3}{7} = \frac{1}{4}\ln\frac{7}{3}$$

28. $$\int_{-\infty}^0 \frac{1}{x^2-3x+2}\, dx = \int_{-\infty}^0 \left(\frac{1}{x-2} - \frac{1}{x-1}\right) dx = \lim_{s\to-\infty} (\ln|x-2| - \ln|x-1|)\Big]_s^0$$
$$= \lim_{s\to-\infty} \ln\left|\frac{x-2}{x-1}\right|\Big]_s^0 = \ln 2 - \ln 1 = \ln 2$$

29. $$\int_{-\infty}^{-2} \frac{x^2}{(x^3+1)^2}\, dx = \lim_{s\to-\infty} \int_s^{-2} \frac{1}{3}\left[\frac{3x^2}{(x^3+1)^2}\right] dx = \lim_{s\to-\infty} \left[-\frac{1}{3}\left(\frac{1}{x^3+1}\right)\right]\Big|_s^{-2}$$
$$= -\frac{1}{3}\left(-\frac{1}{7}\right) = \frac{1}{21}$$

30. $$\int_0^\infty \frac{1}{e^x+e^{-x}}\, dx = \lim_{t\to\infty} \int_0^t \frac{e^x}{e^{2x}+1}\, dx \qquad \boxed{u = e^x,\ du = e^x\, dx} \qquad = \lim_{t\to\infty} \int_1^{e^t} \frac{1}{u^2+1}\, du$$
$$= \lim_{t\to\infty} \tan^{-1} u\Big]_1^{e^t} = \lim_{t\to\infty} \left(\tan^{-1} e^t - \frac{\pi}{4}\right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

31. $\displaystyle\int_0^5 \frac{1}{x}\, dx = \lim_{s\to0^+} \int_s^5 \frac{1}{x}\, dx = \lim_{s\to0^+} \ln|x|\Big]_s^5 = \lim_{s\to0^+} (\ln 5 - \ln s)$. The integral diverges.

32. $$\int_0^8 \frac{1}{x^{2/3}}\, dx = \lim_{s\to0^+} \int_s^8 x^{-2/3}\, dx = \lim_{s\to0^+} 3x^{1/3}\Big]_s^8 = \lim_{s\to0^+} (6 - 3s^{1/3}) = 6$$

33. $$\int_0^1 \frac{1}{x^{0.99}}\, dx = \lim_{s\to0^+} \int_s^1 x^{-0.99}\, dx = \lim_{s\to0^+} 100x^{0.01}\Big]_s^1 = \lim_{s\to0^+} (100 - 100s^{0.01}) = 100$$

34. $\displaystyle\int_0^1 \frac{1}{x^{1.01}}\,dx = \lim_{s\to 0^+}\int_s^1 x^{-1.01}\,dx = \lim_{s\to 0^+}\left(-100x^{-0.01}\right)\Big]_s^1 = \lim_{s\to 0^+}\left(\frac{100}{s^{0.01}} - 100\right)$

The integral diverges.

35. $\displaystyle\int_0^2 \frac{1}{\sqrt{2-x}}\,dx = \lim_{t\to 2^-}\int_0^t (2-x)^{-1/2}\,dx = \lim_{t\to 2^-}\left[-2(2-x)^{1/2}\right]\Big]_0^t$

$\displaystyle = \lim_{t\to 2^-}(2\sqrt{2} - 2\sqrt{2-t}) = 2\sqrt{2}$

36. $\displaystyle\int_1^3 \frac{1}{(x-1)^2}\,dx = \lim_{s\to 1^+}\int_s^3 (x-1)^{-2}\,dx = \lim_{s\to 1^+}\left[-(x-1)^{-1}\right]\Big]_s^3 = \lim_{s\to 1^+}\left(\frac{1}{s-1} - \frac{1}{2}\right)$

The integral diverges.

37. $\displaystyle\int_{-1}^1 \frac{1}{x^{5/3}}\,dx = \lim_{t\to 0^-}\int_{-1}^t \frac{1}{x^{5/3}}\,dx + \lim_{s\to 0^+}\int_s^1 \frac{1}{x^{5/3}}\,dx$

Since $\displaystyle\lim_{t\to 0^-}\int_{-1}^t \frac{1}{x^{5/3}}\,dx = \lim_{t\to 0^-}\left(-\frac{3}{2}x^{-2/3}\right)\Big]_{-1}^t = \lim_{t\to 0^-}\left(\frac{3}{2} - \frac{3}{2t^{2/3}}\right)$, the integral diverges.

38. $\displaystyle\int_0^2 \frac{1}{\sqrt[3]{x-1}}\,dx = \lim_{t\to 1^-}\int_0^t (x-1)^{-1/3}\,dx + \lim_{s\to 1^+}\int_s^2 (x-1)^{1/3}\,dx$

$\displaystyle = \lim_{t\to 1^-}\frac{3}{2}(x-1)^{2/3}\Big]_0^t + \lim_{s\to 1^+}\frac{3}{2}(x-1)^{2/3}\Big]_s^2$

$\displaystyle = \lim_{t\to 1^-}\left[\frac{3}{2}(t-1)^{2/3} - \frac{3}{2}\right] + \lim_{s\to 1^+}\left[\frac{3}{2} - \frac{3}{2}(s-1)^{2/3}\right] = -\frac{3}{2} + \frac{3}{2} = 0$

39. $\displaystyle\int_0^2 (x-1)^{-2/3}\,dx = \lim_{t\to 1^-}\int_0^t (x-1)^{-2/3}\,dx + \lim_{s\to 1^+}\int_s^2 (x-1)^{-2/3}\,dx$

$\displaystyle = \lim_{t\to 1^-} 3(x-1)^{1/3}\Big]_0^t + \lim_{s\to 1^+} 3(x-1)^{1/3}\Big]_s^2 = 3 + 3 = 6$

40. $\displaystyle\int_0^{27} \frac{e^{x^{1/3}}}{x^{2/3}}\,dx = \lim_{s\to 0^+}\int_s^{27} x^{-2/3}e^{x^{1/3}}dx$ $\boxed{u = x^{1/3},\ du = \frac{1}{3}x^{-2/3}\,dx}$

$\displaystyle = \lim_{s\to 0^+}\int_{s^{1/3}}^3 3e^u\,du = \lim_{s\to 0^+} 3e^u\Big]_{s^{1/3}}^3 = \lim_{s\to 0^+}\left(3e^3 - 3e^{s^{1/3}}\right) = 3e^3 - 3$

41. $\displaystyle\int_0^1 x\ln x\,dx = \lim_{s\to 0^+}\int_s^1 x\ln x\,dx$ $\boxed{u = \ln x,\ du = \frac{1}{x}\,dx;\quad dv = x\,dx,\ v = \frac{1}{2}x^2}$

$\displaystyle = \lim_{s\to 0^+}\left(\frac{1}{2}x^2\ln x\Big]_s^1 - \int_s^1 \frac{1}{2}x\,dx\right) = \lim_{s\to 0^+}\left(-\frac{1}{2}s^2\ln x - \frac{1}{4}x^2\Big]_s^1\right)$

$\displaystyle = \lim_{s\to 0^+}\left(-\frac{1}{2}s^2\ln s - \frac{1}{4} + \frac{s^2}{4}\right) = \lim_{s\to 0^+}\left(-\frac{\ln s}{2/s^2}\right) - \frac{1}{4}$

$\displaystyle \overset{h}{=} \lim_{s\to 0^+} -\frac{1/s}{-4/s^3} - \frac{1}{4} = \lim_{s\to 0^+}\frac{s^2}{4} - \frac{1}{4} = -\frac{1}{4}$

42. $\displaystyle\int_1^e \frac{1}{x\ln x}\,dx = \lim_{s\to 1^+}\int_s^e \frac{1}{\ln x}\left(\frac{1}{x}\,dx\right) = \lim_{s\to 1^+}\ln(\ln x)\Big]_s^e = \lim_{s\to 1^+}[0-\ln(\ln s)]$

The integral diverges.

43. $\displaystyle\int_0^{\pi/2}\tan t\,dt = \lim_{k\to\pi/2^-}\int_0^k \tan t\,dt = \lim_{k\to\pi/2^-}\ln\sec t\Big]_0^k = \lim_{k\to\pi/2^-}\ln\sec k$

The integral diverges.

44. $\displaystyle\int_0^{\pi/4}\frac{\sec^2\theta}{\sqrt{\tan\theta}}\,d\theta = \lim_{s\to 0^+}\int_s^{\pi/4}\frac{\sec^2\theta}{\sqrt{\tan\theta}}\,d\theta \qquad \boxed{u=\tan\theta,\ du=\sec^2\theta\,d\theta}$

$$= \lim_{s\to 0^+}\int_{\tan s}^1 \frac{1}{\sqrt{u}}\,du = \lim_{s\to 0^+}2\sqrt{u}\Big]_{\tan s}^1 = \lim_{s\to 0^+}(2-2\sqrt{\tan s}) = 2$$

45. $\displaystyle\int_0^{\pi}\frac{\sin x}{1+\cos x}\,dx = \lim_{t\to\pi^-}\int_0^t \frac{\sin x}{1+\cos x}\,dx \qquad \boxed{u=1+\cos x,\ du=-\sin x\,dx}$

$$= \lim_{t\to\pi^-}\int_2^{1+\cos t} -\frac{1}{u}\,du = \lim_{t\to\pi^-}(-\ln|u|)\Big]_2^{1+\cos t} = \lim_{t\to\pi^-}[\ln 2-\ln(1+\cos t)]$$

The integral diverges.

46. $\displaystyle\int_0^{\pi}\frac{\cos x}{\sqrt{1-\sin x}}\,dx = \lim_{t\to\pi/2^-}\int_0^t \frac{\cos x}{\sqrt{1-\sin x}}\,dx + \lim_{s\to\pi/2^+}\int_s^{\pi}\frac{\cos x}{\sqrt{1-\sin x}}\,dx$

$$= \lim_{t\to\pi/2^-}-2\sqrt{1-\sin x}\Big]_0^t + \lim_{s\to\pi/2^+}-2\sqrt{1-\sin x}\Big]_s^{\pi} = 2+(-2) = 0$$

47. $\displaystyle\int_{-1}^0 \frac{x}{\sqrt{1+x}}\,dx = \lim_{s\to -1^+}\int_s^0 \frac{x}{\sqrt{1+x}}\,dx \qquad \boxed{u^2=1+x,\ x=u^2-1,\ dx=2u\,du}$

$$= \lim_{s\to -1^+}\int_{\sqrt{1+s}}^1 \frac{u^2-1}{u}(2u\,du) = \lim_{s\to -1^+}\int_{\sqrt{1+s}}^1 (2u^2-2)\,du$$

$$= \lim_{s\to -1^+}\left(\frac{2}{3}u^3-2u\right)\Big]_{\sqrt{1+s}}^1 = \lim_{s\to -1^+}\left[-\frac{4}{3}-\frac{2}{3}(1+s)^{3/2}+2\sqrt{1+s}\right] = -\frac{4}{3}$$

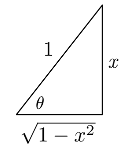
48. $\displaystyle\int_0^3 \frac{1}{x^2-1}\,dx = \lim_{t\to 1^-}\int_0^t \frac{1}{x^2-1}\,dx + \lim_{s\to 1^+}\int_s^3 \frac{1}{x^2-1}\,dx$

Since

$$\lim_{t\to 1^-}\int_0^t \frac{1}{x^2-1}\,dx = \lim_{t\to 1^-}\int_0^t\left(\frac{1/2}{x-1}-\frac{1/2}{x+1}\right)dx = \lim_{t\to 1^-}\left(\frac{1}{2}\ln|x-1|-\frac{1}{2}\ln|x+1|\right)\Big]_0^t$$

$$= \lim_{t\to 1^-}\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|\Big]_0^t = \lim_{t\to 1^-}\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|,$$

the integral diverges.



49. $$\int_0^1 \frac{x^2}{\sqrt{1-x^2}}\,dx = \lim_{t\to 1^-}\int_0^t \frac{x^2}{\sqrt{1-x^2}}\,dx \qquad \boxed{x=\sin\theta,\ dx=\cos\theta\,d\theta}$$
$$= \lim_{t\to 1^-}\int_0^{\sin^{-1}t} \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}}\cos\theta\,d\theta = \lim_{t\to 1^-}\int_0^{\sin^{-1}t}\sin^2\theta\,d\theta$$
$$= \lim_{t\to 1^-}\int_0^{\sin^{-1}t}\frac{1}{2}(1-\cos 2\theta)\,d\theta = \lim_{t\to 1^-}\left(\frac{1}{2}\theta-\frac{1}{4}\sin 2\theta\right)\Bigg]_0^{\sin^{-1}t}$$
$$= \lim_{t\to 1^-}\left(\frac{1}{2}\theta-\frac{1}{2}\sin\theta\cos\theta\right)\Bigg]_0^{\sin^{-1}t} = \lim_{t\to 1^-}\left(\frac{1}{2}\sin^{-1}t-\frac{1}{2}t\sqrt{1-t^2}\right) = \frac{\pi}{4}$$

50. $$\int_0^2 \frac{e^w}{\sqrt{e^w-1}}\,dw = \lim_{s\to 0^+}\int_s^2 \frac{e^w}{\sqrt{e^w-1}}\,dw \qquad \boxed{u=e^w-1,\ du=e^w\,dw}$$
$$= \lim_{s\to 0^+}\int_{e^s-1}^{e^2-1}\frac{1}{\sqrt{u}}\,du = \lim_{s\to 0^+} 2\sqrt{u}\,\Big]_{e^s-1}^{e^2-1} = \lim_{s\to 0^+}\left(2\sqrt{e^2-1}-2\sqrt{e^s-1}\right)$$
$$= 2\sqrt{e^2-1}$$

51. $$\int_1^3 \frac{1}{\sqrt{3+2x-x^2}}\,dx = \lim_{t\to 3^-}\int_1^t \frac{1}{\sqrt{4-(x-1)^2}}\,dx = \lim_{t\to 3^-}\sin^{-1}\frac{x-1}{2}\Big]_1^t$$
$$= \lim_{t\to 3^-}\sin^{-1}\frac{t-1}{2} = \sin^{-1}1 = \frac{\pi}{2}$$

52. $$\int_0^1\left(\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{1-x}}\right)dx = \lim_{s\to 0^+}\int_s^{1/2}\left(\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{1-x}}\right)dx + \lim_{t\to 1^-}\int_{1/2}^t\left(\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{1-x}}\right)dx$$
$$= \lim_{s\to 0^+}(2\sqrt{x}-2\sqrt{1-x})\Big]_s^{1/2} + \lim_{t\to 1^-}(2\sqrt{x}-2\sqrt{1-x})\Big]_{1/2}^t$$
$$= \lim_{s\to 0^+}(\sqrt{2}-\sqrt{2}-2\sqrt{s}+2\sqrt{1-s})$$
$$+ \lim_{t\to 1^-}(2\sqrt{t}-2\sqrt{1-t}-\sqrt{2}+\sqrt{2}) = 2+2=4$$

53. $$\int_{12}^{\infty}\frac{1}{\sqrt{x}(x+4)}\,dx = \lim_{t\to\infty}\int_{12}^t \frac{1}{\sqrt{x}(x+4)}\,dx \qquad \boxed{u=\sqrt{x},\ u^2=x,\ dx=2u\,du}$$
$$= 2\lim_{t\to\infty}\int_{2\sqrt{3}}^t \frac{u}{u(u^2+4)}\,du = 2\lim_{t\to\infty}\int_{2\sqrt{3}}^t\frac{1}{u^2+4}\,du$$
$$= 2\lim_{t\to\infty}\frac{1}{2}\tan^{-1}\frac{u}{2}\Big]_{2\sqrt{3}}^t = \lim_{t\to\infty}\left(\tan^{-1}\frac{t}{2}-\tan^{-1}\sqrt{3}\right) = \frac{\pi}{2}-\frac{\pi}{3}=\frac{\pi}{6}$$

54. $$\int_1^\infty \sqrt{x}e^{-\sqrt{x}}\,dx = \lim_{t\to\infty}\int_1^t \sqrt{x}e^{-\sqrt{x}}\,dx \quad \boxed{w=\sqrt{x},\ w^2=x,\ dx=2w\,dw}$$

$$= 2\lim_{t\to\infty}\int_1^t w^2e^{-w}\,dw \quad \boxed{u=w^2,\ du=2w\,dw;\quad dv=e^{-w}\,dw,\ v=-e^{-w}}$$

$$= \lim_{t\to\infty} 2\left(-w^2e^{-w}\Big]_1^t + 2\int_1^t we^{-w}\,dw\right)$$

$$\boxed{u=w,\ du=dw;\quad dv=e^{-w}\,dw,\ v=-e^{-w}}$$

$$= \lim_{t\to\infty} 2\left[-w^2e^{-w}\Big]_1^t + 2\left(-we^{-w}\Big]_1^t + \int_1^t e^{-w}\,dw\right)\right]$$

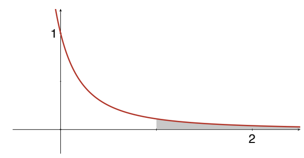
$$= \lim_{t\to\infty} 2\left[-w^2e^{-w}\Big]_1^t + 2\left(-we^{-w}\Big]_1^t - e^{-w}\Big]_1^t\right)\right]$$

$$= \lim_{t\to\infty} 2\left[-t^2e^{-t}+e^{-1}+2\left(-te^{-t}+e^{-1}-e^{-t}+e^{-1}\right)\right]$$

$$= 2[e^{-1}+2(e^{-1}+e^{-1})] = 10e^{-1}$$

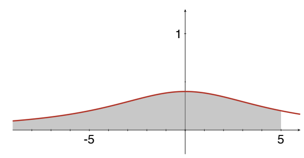
55. $$A = \int_1^\infty \frac{1}{(2x+1)^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{(2x+1)^2}\,dx$$

$$= \lim_{t\to\infty}\left(-\frac{1/2}{2x+1}\right)\Bigg]_1^t = \lim_{t\to\infty}\left(\frac{1}{6}-\frac{1}{4t+2}\right) = \frac{1}{6}$$



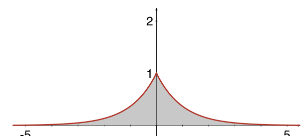
56. $$A = \int_{-\infty}^5 \frac{10}{x^2+25}\,dx = \lim_{s\to-\infty}\int_s^5 \frac{10}{x^2+25}\,dx = \lim_{s\to-\infty} 2\tan^{-1}\frac{x}{5}\Big]_s^5$$

$$= \lim_{s\to-\infty}\left(\frac{\pi}{2}-2\tan^1\frac{s}{5}\right) = \frac{\pi}{2}-2\left(-\frac{\pi}{2}\right) = \frac{3\pi}{2}$$



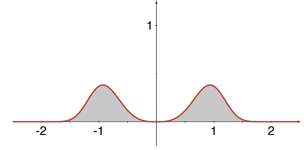
57. $$A = \int_{-\infty}^\infty e^{-|x|}\,dx = 2\int_0^\infty e^{-x}\,dx = 2\lim_{t\to\infty}\left(-e^{-x}\right)\Big]_0^t$$

$$= -2\lim_{t\to\infty}(e^{-t}-e^0) = -2(0-1) = 2$$



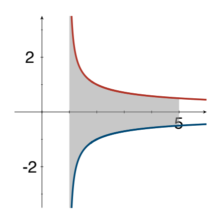
58. $$A = \int_{-\infty}^\infty |x|^3e^{-x^4}\,dx = 2\int_0^\infty x^3e^{-x^4}\,dx = -\frac{1}{2}\lim_{t\to\infty} e^{-x^4}\Big]_0^t$$

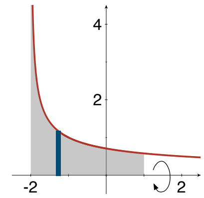
$$= -\frac{1}{2}\lim_{t\to\infty}(e^{-t^4}-e^0) = -\frac{1}{2}(0-1) = \frac{1}{2}$$



59. $$A = \int_1^5 \left[\frac{1}{\sqrt{x-1}} - \left(-\frac{1}{\sqrt{x-1}}\right)\right]dx = \lim_{s\to1^+}\int_s^5 \frac{2}{\sqrt{x-1}}\,dx$$

$$= \lim_{s\to1^+} 4\sqrt{x-1}\Big]_s^5 = \lim_{s\to1^+}(8-4\sqrt{s-1}) = 8$$





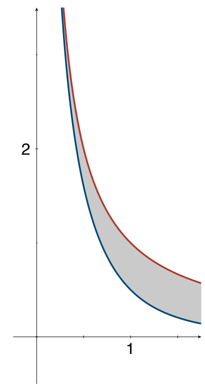
60. (a) $A = \displaystyle\int_{-2}^{1} \frac{1}{\sqrt{x+2}}\,dx = \lim_{s\to -2^+} \int_s^1 \frac{1}{\sqrt{x+2}}\,dx = \lim_{s\to -2^+} 2\sqrt{x+2}\,\Big]_s^1$

$= \displaystyle\lim_{s\to -2^+} (2\sqrt{3} - 2\sqrt{s+2}) = 2\sqrt{3}$

(b) $V = \pi \displaystyle\int_{-2}^{1} \frac{1}{x+2}\,dx = \pi \lim_{s\to -2^+} \int_s^1 \frac{1}{x+2}\,dx = \pi \lim_{s\to -2^+} \ln|x+2|\,\Big]_s^1$

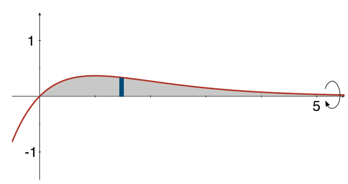
$= \pi \displaystyle\lim_{s\to -2^+} [\ln 3 - \ln(s+2)]$

The integral diverges, so the volume is infinite.



61. $A = \displaystyle\int_0^1 \left[\frac{1}{x} - \frac{1}{x(x^2+1)}\right] dx = \lim_{s\to 0^+} \int_s^1 \frac{x}{x^2+1}\,dx = \lim_{s\to 0^+} \frac{1}{2}\ln(x^2+1)\Big]_s^1$

$= \displaystyle\lim_{s\to 0^+} \left[\frac{1}{2}\ln 2 - \frac{1}{2}\ln(s^2+1)\right] = \frac{1}{2}\ln 2$



62. $V = \pi \displaystyle\int_0^\infty x^2 e^{-2x}\,dx = \pi \lim_{t\to\infty} \int_0^t x^2 e^{-2x}\,dx$

$\boxed{u = x^2,\ du = 2x\,dx; \quad dv = e^{-2x}\,dx,\ v = -\frac{1}{2}e^{-2x}}$

$= \pi \displaystyle\lim_{t\to\infty} \left(-\frac{1}{2}x^2 e^{-2x}\Big]_0^t + \int_0^t x e^{-2x}\,dx \right)$

$\boxed{u = x,\ du = dx; \quad dv = e^{-2x}\,dx,\ v = -\frac{1}{2}e^{-2x}}$

$= \pi \displaystyle\lim_{t\to\infty} \left(-\frac{1}{2}t^2 e^{-2t} - \frac{1}{2}x e^{-2x}\Big]_0^t + \int_0^t \frac{1}{2}e^{-2x}\,dx \right)$

$= \pi \displaystyle\lim_{t\to\infty} \left(-\frac{1}{2}t^2 e^{-2t} - \frac{1}{2}t e^{-2t} - \frac{1}{4}e^{-2x}\Big]_0^t \right)$

$= \pi \displaystyle\lim_{t\to\infty} \left(-\frac{t^2+t}{2e^{2t}} + \frac{1}{4} - \frac{1}{4}e^{-2t} \right) = -\pi \lim_{t\to\infty} \frac{t^2+t}{2e^{2t}} + \frac{\pi}{4} - \frac{\pi}{4}\lim_{t\to\infty} e^{-2t}$

$\overset{h}{=} -\pi \displaystyle\lim_{t\to\infty} \frac{2t+1}{4e^{2t}} + \frac{\pi}{4} - 0 \overset{h}{=} -\pi \lim_{t\to\infty} \frac{2}{8e^{2t}} + \frac{\pi}{4} = \frac{\pi}{4}$

63. $W = \int_{1.7\times10^6}^{\infty} (6.67\times10^{-11})(7.3\times10^{22})(10{,}000)\frac{1}{r^2}\,dr \approx 4.87\times10^{16}\lim_{t\to\infty}\int_{1.7\times10^6}^{t} r^{-2}\,dr$

$= 4.87\times10^{16}\lim_{t\to\infty}\left(-\frac{1}{r}\right)\bigg]_{1.7\times10^6}^{t} = 4.87\times10^{16}\lim_{t\to\infty}\left(\frac{1}{1.7\times10^6}-\frac{1}{t}\right)$

$= \frac{4.87\times10^{16}}{1.7\times10^6} \approx 2.86\times10^{10}$ joules

64. We use the formula $W = -\frac{qq_0}{4\pi e_0}\int_{r_A}^{r_B}\frac{1}{r^2}\,dr$.

(a) $W = -\frac{qq_0}{4\pi e_0}\int_{r_A}^{r_B}\frac{1}{r^2}\,dr = -\frac{qq_0}{4\pi e_0}\left(-\frac{1}{r}\right)\bigg]_{r_A}^{r_B} = \frac{qq_0}{4\pi e_0}\left(\frac{1}{r_B}-\frac{1}{r_A}\right)$

(b) $W = -\frac{qq_0}{4\pi e_0}\int_{\infty}^{r_B}\frac{1}{r^2}\,dr = \frac{qq_0}{4\pi e_0}\lim_{t\to\infty}\int_{r_B}^{t}\frac{1}{r^2}\,dr = \frac{qq_0}{4\pi e_0}\lim_{t\to\infty}\left(-\frac{1}{r}\right)\bigg]_{r_B}^{t}$

$= \frac{qq_0}{4\pi e_0}\lim_{t\to\infty}\left(\frac{1}{r_B}-\frac{1}{t}\right) = \frac{qq_0}{4\pi e_0 r_B}$

65. $\mathcal{L}\{1\} = \int_0^\infty e^{-st}\,dt = \lim_{k\to\infty}\int_0^k e^{-st}\,dt = \lim_{k\to\infty}\left(-\frac{1}{s}e^{-st}\right)\bigg]_0^k$

$= \lim_{k\to\infty}\left(\frac{1}{s}-\frac{1}{s}e^{-sk}\right) = \frac{1}{s},\ s>0$

66. $\mathcal{L}\{x\} = \int_0^\infty te^{-st}\,dt = \lim_{k\to\infty}\int_0^k te^{-st}\,dt$ $\boxed{u=t,\ du=dt;\quad dv=e^{-st}\,dt,\ v=-\frac{1}{s}e^{-st}}$

$= \lim_{k\to\infty}\left(-\frac{1}{s}te^{-st}\bigg]_0^k + \int_0^k\frac{1}{s}e^{-st}\,dt\right) = \lim_{k\to\infty}\left(-\frac{1}{s}ke^{-sk}-\frac{1}{s^2}e^{-st}\bigg]_0^k\right)$

$= \lim_{k\to\infty}\left[-\frac{1}{s}\left(\frac{k}{e^{sk}}\right)+\frac{1}{s^2}-\frac{1}{s^2}e^{-sk}\right] = -\frac{1}{s}\lim_{k\to\infty}\frac{k}{e^{sk}}+\frac{1}{s^2},\ s>0$

$\stackrel{h}{=} -\frac{1}{s}\lim_{k\to\infty}\frac{1}{se^{sk}}+\frac{1}{s^2} = \frac{1}{s^2},\ s>0$

67. $\mathcal{L}\{e^x\} = \int_0^\infty e^te^{-st}\,dt = \lim_{k\to\infty}\int_0^k e^{(1-s)t}\,dt = \lim_{k\to\infty}\frac{1}{1-s}e^{(1-s)t}\bigg]_0^k$

$= \lim_{k\to\infty}\left[\frac{1}{1-s}e^{(1-s)k}-\frac{1}{1-s}\right] = \frac{1}{1-s},\ s>1$

68. $\mathcal{L}\{e^{-5x}\} = \int_0^\infty e^{-5t}e^{-st}\,dt = \lim_{k\to\infty}\int_0^k e^{-(s+5)t}\,dt = \lim_{k\to\infty}\left[-\frac{1}{s+5}e^{-(s+5)t}\right]\bigg|_0^k$

$= \lim_{k\to\infty}\left[\frac{1}{s+5}-\frac{1}{s+5}e^{-(s+5)k}\right] = \frac{1}{s+5},\ s>-5$

69. $\mathcal{L}\{\sin x\} = \int_0^\infty e^{-st}\sin t\,dt = \lim_{k\to\infty}\int_0^k e^{-st}\sin t\,dt$

$$\boxed{u = e^{-st},\ du = -se^{-st}\,dt;\quad dv = \sin t\,dt,\ v = -\cos t}$$

$$= \lim_{k\to\infty}\left(-e^{-st}\cos t\Big]_0^k - \int_0^k se^{-st}\cos t\,dt\right) = 1 - \lim_{k\to\infty} s\int_0^k e^{-st}\cos t\,dt$$

$$\boxed{u = e^{-st},\ du = -se^{-st}\,dt;\quad dv = \cos t\,dt,\ v = \sin t}$$

$$= 1 - \lim_{k\to\infty} s\left(e^{-st}\sin t\Big]_0^k + s\int_0^k e^{-st}\sin t\,dt\right) = 1 - s^2\mathcal{L}\{\sin x\}$$

Solving for $\mathcal{L}\{\sin x\}$, we have $\mathcal{L}\{\sin x\} = \dfrac{1}{s^2+1}$, where $s > 0$.

70. $\mathcal{L}\{\cos 2x\} = \int_0^\infty e^{-st}\cos 2t\,dt = \lim_{k\to\infty}\int_0^k e^{-st}\cos 2t\,dt$

$$\boxed{u = e^{-st},\ du = -se^{-st};\quad dv = \cos 2t\,dt,\ v = \frac{1}{2}\sin 2t}$$

$$= \lim_{k\to\infty}\left(\frac{1}{2}e^{-st}\sin 2t\Big]_0^k + \frac{1}{2}\int_0^k se^{-st}\sin 2t\,dt\right) = 0 + \lim_{k\to\infty}\frac{s}{2}\int_0^k e^{-st}\sin 2t\,dt$$

$$\boxed{u = e^{-st},\ du = -se^{-st}\,dt;\quad dv = \sin 2t\,dt,\ v = -\frac{1}{2}\cos 2t}$$

$$= \lim_{k\to\infty}\frac{s}{2}\left(-\frac{1}{2}e^{-st}\cos 2t\Big]_0^k - \frac{1}{2}\int_0^k se^{-st}\cos 2t\,dt\right) = \frac{s}{4} - \frac{s^2}{4}\mathcal{L}\{\cos 2x\}$$

Solving for $\mathcal{L}\{\cos 2x\}$, we have $\mathcal{L}\{\cos 2x\} = \dfrac{s/4}{1+s^2/4} = \dfrac{s}{s^2+4}$, where $s > 0$.

71. $\mathcal{L}\{f(x)\} = \int_1^\infty e^{-st}\,dt = \lim_{k\to\infty}\left(-\frac{1}{s}e^{-st}\right)\Big]_1^k = \frac{1}{s}e^{-s},\ s > 0$

72. $\mathcal{L}\{f(x)\} = \int_3^\infty e^{-st}e^{-t}\,dt = \int_3^\infty e^{-(s+1)t}\,dt = \lim_{k\to\infty}\left[-\frac{1}{s+1}e^{-(s+1)t}\right]\Bigg|_3^k$

$$= \frac{1}{s+1}e^{-3(s+1)},\ s > 3$$

73. $\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^0 f(x)\,dx + \int_0^\infty f(x)\,dx = 0 + \lim_{t\to\infty}\int_0^t ke^{-kx}\,dx$

$$= \lim_{t\to\infty} -e^{-kx}\Big]_0^t = \lim_{t\to\infty}(e^0 - e^{-kx}) = 1$$

74. (a) $\Gamma(\alpha+1) = \int_0^\infty t^{(\alpha+1)-1}e^{-t}\,dt = \lim_{k\to\infty}\int_0^k t^\alpha e^{-t}\,dt$

$$\boxed{u = t^\alpha,\ du = \alpha t^{\alpha-1}\,dt; \quad dv = e^{-t}\,dt,\ v = -e^{-t}}$$

$$= \lim_{k\to\infty}\left(-t^\alpha e^{-t}\Big]_0^k + \int_0^k \alpha t^{\alpha-1}e^{-t}\,dt\right)$$

$$= \lim_{k\to\infty}(-k^\alpha e^{-k}) + \alpha\lim_{k\to\infty}\int_0^k t^{\alpha-1}e^{-t}\,dt = \lim_{k\to\infty}\left(-\frac{k^\alpha}{e^k}\right) + \alpha\Gamma(\alpha)$$

Using repeated applications of L'Hôpital's Rule on $\dfrac{k^\alpha}{e^k}$ until $\alpha \le 0$, we find that $\lim_{k\to\infty} -\dfrac{k^\alpha}{e^k} = 0$. Thus, $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$.

(b) Note that

$$\Gamma(1) = \int_0^\infty e^{-t}\,dt = \lim_{k\to\infty}\int_0^k e^{-t}\,dt = \lim_{k\to\infty}(-e^{-t})\Big]_0^k = \lim_{k\to\infty}(1-e^{-k}) = 1.$$

Then, using repeated applications of part (a),

$$\begin{aligned}\Gamma(n+1) &= n\Gamma(n) = n\Gamma(n-1+1) = n(n-1)\Gamma(n-1) = n(n-1)\Gamma(n-2+1)\\ &= n(n-1)(n-2)\Gamma(n-2) = \cdots = n(n-1)(n-2)\cdots 2\cdot 1\cdot\Gamma(1) = n!.\end{aligned}$$

75. $\displaystyle\int_1^\infty \frac{1}{x^k}\,dx = \lim_{t\to\infty}\int_1^t x^{-k}\,dx = \lim_{t\to\infty}\frac{x^{-k+1}}{-k+1}\Big]_1^t = \lim_{t\to\infty}\left(\frac{t^{-k+1}}{-k+1} - \frac{1}{-k+1}\right),\ k\neq 1$

$$= \lim_{t\to\infty}\left[\frac{1}{1-k}\left(\frac{1}{t^{k-1}}\right)\right] + \frac{1}{k-1},\ k\neq 1$$

The integral converges for $k > 1$ and diverges for $k < 1$. If $k = 1$ then

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty}\ln x\Big]_1^t = \lim_{t\to\infty}\ln t,$$

and the integral diverges.

76. Since $f(x) = x^{2k}$ has an infinite discontinuity at 0 when $k < 0$, we consider only $k \ge 0$.

$k = 0:\quad \displaystyle\int_{-\infty}^1 1\,dx = \lim_{s\to-\infty}\int_s^1 dx = \lim_{s\to-\infty} x\Big]_s^1 = \lim_{s\to-\infty}(1-s)$

This integral diverges.

$k > 0:\quad \displaystyle\int_{-\infty}^1 x^{2k}\,dx = \lim_{s\to-\infty}\int_s^1 x^{2k}\,dx = \lim_{s\to-\infty}\frac{x^{2k+1}}{2k+1}\Big]_s^1 = \lim_{s\to-\infty}\left(\frac{1}{2k+1} - \frac{x^{2k+1}}{2k+1}\right)$

This integral converges only when $2k+1 < 0$ or $k < -1/2$.

There are no non-negative values of k for which the integral converges.

77. $\displaystyle\int_0^\infty e^{kx}\,dx = \lim_{t\to\infty}\int_0^t e^{kx}\,dx = \lim_{t\to\infty}\frac{1}{k}e^{kx}\Big]_0^t = \lim_{t\to\infty}\left(\frac{e^{kt}}{k} - \frac{1}{k}\right),\ k\neq 0$

The integral converges for $k < 0$ and diverges for $k > 0$. If $k = 0$, $\int_0^\infty 1\,dx$ diverges.

78. Since $\dfrac{(\ln x)^k}{x}$ has an infinite discontinuity at 1 when $k < 0$, we consider only $k \geq 0$.

$k = 0$: $\quad \displaystyle\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \ln x\Big]_1^t = \lim_{t\to\infty} \ln t$

This integral diverges.

$k > 0$: $\quad \displaystyle\int_1^\infty \frac{(\ln x)^k}{x}\,dx = \lim_{t\to\infty}\int_1^t (\ln x)^k \frac{1}{x}\,dx = \lim_{t\to\infty} \frac{1}{k+1}(\ln x)^{k+1}\Big]_1^t = \lim_{t\to\infty}\frac{1}{k+1}(\ln t)^{k+1}$

This integral diverges.

There are no non-negative values of k for which the integral converges.

79. By Problem 75, $\int_1^\infty \frac{1}{x^2}\,dx$ converges. Since $0 \leq \frac{\sin^2 x}{x^2} \leq \frac{1}{x^2}$ for all x in $[1,\infty)$, $\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ converges.

80. By Example 1 in the text, $\int_2^\infty \frac{1}{x^3}\,dx$ converges. Since $0 < \frac{1}{x^3+4} < \frac{1}{x^3}$ for all x in $[2,\infty)$, $\int_2^\infty \frac{1}{x^3+4}\,dx$ converges.

81. By Problem 77, $\int_0^\infty \frac{1}{e^x}\,dx$ converges. Since $0 < \frac{1}{x+e^x} < \frac{1}{e^x}$ for all x in $[0,\infty)$, $\int_0^\infty \frac{1}{x+e^x}\,dx$ converges.

82. Let $g(x) = \begin{cases} 1, & 0 \leq x < 1 \\ e^{-x}, & 1 \leq x < \infty \end{cases}$. Then $0 < e^{-x^2} \leq g(x)$ for all x in $[0,\infty)$, and

$$\begin{aligned}\int_0^\infty g(x)\,dx &= \int_0^1 1\,dx + \int_1^\infty e^{-x}\,dx = 1 + \lim_{t\to\infty}\int_1^t e^{-x}\,dx = 1 + \lim_{t\to\infty}\left(-e^{-x}\right)\Big]_1^t \\ &= 1 + \lim_{t\to\infty}\left(e^{-1} - e^{-t}\right) = 1 + e^{-1}.\end{aligned}$$

Since $\int_0^\infty g(x)\,dx$ converges, $\int_0^\infty e^{-x^2}\,dx$ converges.

83. By Problem 75, $\int_1^\infty \frac{1}{\sqrt{x}}\,dx = \int_1^\infty \frac{1}{x^{1/2}}\,dx$ diverges.

Since $0 < \frac{1}{\sqrt{x}} < \frac{1+e^{-2x}}{\sqrt{x}}$ for all x in $[1,\infty)$, $\int_1^\infty \frac{1+e^{-2x}}{\sqrt{x}}\,dx$ diverges.

84. $0 < e^x \leq e^{x^2}$ for all x in $[1,\infty)$ since $x \leq x^2$ for $x \geq 1$. Considering $\int_1^\infty e^x\,dx$, we have

$$\int_1^\infty e^x\,dx = \lim_{t\to\infty}\int_1^t e^x\,dx = \lim_{t\to\infty} e^x\Big]_1^t = \lim_{t\to\infty}\left(e^t - e\right),$$

so $\int_1^\infty e^x\,dx$ diverges. Thus, $\int_1^\infty e^{x^2}\,dx$ diverges.

85. $$\int_1^\infty \frac{1}{x\sqrt{x^2-1}}\,dx = \lim_{s\to 1^+}\int_s^2 \frac{1}{x\sqrt{x^2-1}}\,dx + \lim_{t\to\infty}\int_2^t \frac{1}{x\sqrt{x^2-1}}\,dx$$
$$= \lim_{s\to 1^+} \sec^{-1} x\Big]_s^2 + \lim_{t\to\infty} \sec^{-1} x\Big]_2^t$$
$$= \lim_{s\to 1^+}(\sec^{-1} 2 - \sec^{-1} s) + \lim_{t\to\infty}(\sec^{-1} t - \sec^{-1} 2)$$
$$= \sec^{-1} 2 - 0 + \frac{\pi}{2} - \sec^{-1} 2 = \frac{\pi}{2}$$

86. $$\int_{-\infty}^4 \frac{1}{(x-1)^{2/3}}\,dx = \lim_{s\to -\infty}\int_s^0 \frac{1}{(x-1)^{2/3}}\,dx + \lim_{t\to 1^-}\int_0^t \frac{1}{(x-1)^{2/3}}\,dx$$
$$+ \lim_{r\to 1^+}\int_r^4 \frac{1}{(x-1)^{2/3}}\,dx$$

Since $\displaystyle\lim_{s\to -\infty}\int_s^0 \frac{1}{(x-1)^{2/3}}\,dx = \lim_{s\to -\infty} 3(x-1)^{1/3}\Big]_s^0 = \lim_{s\to -\infty}[-3-3(s-1)^{1/3}]$, the integral diverges.

87. $$\int_{-1}^1 \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{s\to -1^+}\int_s^0 \frac{1}{\sqrt{1-x^2}}\,dx + \lim_{t\to 1^-}\int_0^t \frac{1}{\sqrt{1-x^2}}\,dx$$
$$= \lim_{s\to -1^+} \sin^{-1} x\Big]_s^0 + \lim_{t\to 1^-} \sin^{-1} x\Big]_0^t$$
$$= \lim_{s\to -1^+}(0 - \sin^{-1} s) + \lim_{t\to 1^-}(\sin^{-1} t - 0) = -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = \pi$$

88. $$\int_0^2 \frac{2x-1}{\sqrt[3]{x^2-x}}\,dx = \lim_{s\to 0^+}\int_s^{1/2} \frac{2x-1}{\sqrt[3]{x^2-x}}\,dx + \lim_{t\to 1^-}\int_{1/2}^t \frac{2x-1}{\sqrt[3]{x^2-x}}\,dx + \lim_{r\to 1^+}\int_r^2 \frac{2x-1}{\sqrt[3]{x^2-x}}\,dx$$
$$= \lim_{s\to 0^+} \frac{3}{2}(x^2-x)^{2/3}\Big]_s^{1/2} + \lim_{t\to 1^-} \frac{3}{2}(x^2-x)^{2/3}\Big]_{1/2}^t + \lim_{r\to 1^+} \frac{3}{2}(x^2-x)^{2/3}\Big]_r^2$$
$$= \lim_{s\to 0^+}\left[\frac{3}{2}\left(-\frac{1}{4}\right)^{2/3} - \frac{3}{2}(s^2-s)^{2/3}\right] + \lim_{t\to 1^-}\left[\frac{3}{2}(t^2-t)^{2/3} - \frac{3}{2}\left(-\frac{1}{4}\right)^{2/3}\right]$$
$$+ \lim_{r\to 1^+}\left[\frac{3}{2}(2)^{2/3} - \frac{3}{2}(r^2-r)^{2/3}\right]$$
$$= \left[\frac{3}{2}\left(-\frac{1}{4}\right)^{2/3} - 0\right] + \left[0 - \frac{3}{2}\left(-\frac{1}{4}\right)^{2/3}\right] + \left(\frac{3}{\sqrt[3]{2}} - 0\right) = \frac{3}{\sqrt[3]{2}}$$

89. (a) $\displaystyle A = \int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \ln x\Big]_1^t = \lim_{t\to\infty} \ln t$

The integral diverges, so the area is not finite.

(b) $\displaystyle V = \pi\int_1^\infty \frac{1}{x^2}\,dx = \pi\lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx = \pi\lim_{t\to\infty}\left(-\frac{1}{x}\right)\Big]_1^t = \pi\lim_{t\to\infty}\left(1-\frac{1}{t}\right) = \pi$

(c) $\displaystyle y' = -\frac{1}{x^2};\quad S = 2\pi\int_1^\infty \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx = 2\pi\int_1^\infty \frac{\sqrt{x^4+1}}{x^3}\,dx$

For all x in $[1,\infty]$, $\dfrac{1}{x} = \dfrac{x^2}{x^3} = \dfrac{\sqrt{x^4}}{x^3} < \dfrac{\sqrt{x^4+1}}{x^3}$. By Problem 75, $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ diverges. Since $0 \le \dfrac{1}{x} \le \dfrac{\sqrt{x^4+1}}{x^3}$ for all x in $[1,\infty)$, $\displaystyle\int_1^\infty \frac{\sqrt{x^4+1}}{x^3}\,dx$ diverges.

90. (a) The peak death rate, occurring when $t = 17$, is $R = 890$.

(b) $\displaystyle\int_{-\infty}^{\infty} 890\,\mathrm{sech}^2(0.2t-3.4)\,dt$

$$= 890 \lim_{k\to-\infty} \frac{1}{0.2}\tanh(0.2t-3.4)\Big]_k^0 + 890 \lim_{p\to\infty} \frac{1}{0.2}\tanh(0.2t-3.4)\Big]_0^p$$
$$= 4450[0-(-1)] + 4450(1-0) = 8900$$

(c) $$\int_0^{34} 890\,\mathrm{sech}^2(0.2t-3.4)\,dt = 890\left[\frac{1}{0.2}\tanh(0.2t-3.4)\right]\Bigg|_0^{34}$$
$$= 4450[\tanh 3.4 - \tanh(-3.4)]$$
$$= 8900\tanh 3.4$$

The percentage of total deaths is $100\tanh 3.4 \approx 99.8\%$.

(d) To find the peak death rate we solve $\dfrac{dR_0}{dt} = -\dfrac{a(2t-2b)}{(t^2-2bt+c)^2} = 0$, obtaining $t = b$, at which time $R_0 = \dfrac{a}{c-b^2}$. Thus, we set $b = 17$ and $\dfrac{a}{c-b^2} = 890$. The total number of deaths is

$$\int_{-\infty}^{\infty} \frac{a}{t^2-2bt+c}\,dt = \int_{-\infty}^{\infty} \frac{a}{(t-b)^2+(c-b^2)}\,dt \qquad \boxed{c > b^2}$$
$$= \frac{a}{\sqrt{c-b^2}}\left(\lim_{k\to-\infty} \tan^{-1}\frac{t-b}{\sqrt{c-b^2}}\Big]_k^0 + \lim_{p\to\infty} \tan\frac{t-b}{\sqrt{c-b^2}}\Big]_0^p\right)$$
$$= \frac{a}{\sqrt{c-b^2}}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = \frac{a\pi}{\sqrt{c-b^2}}.$$

We then set $\dfrac{a\pi}{\sqrt{c-b^2}} = 8900$ or $\dfrac{a\pi\sqrt{c-b^2}}{c-b^2} = 8900$. Using $\dfrac{a}{c-b^2} = 890$ we obtain $\pi\sqrt{c-b^2} = 10$ or $c = \dfrac{100}{\pi^2} + b^2 = \dfrac{100}{\pi^2} + 17^2 \approx 299.13$. Then $a = 890(c-b^2) \approx 9017.59$.

(e) We note that $34 = 2b$. Then the number of deaths in the first 34 weeks is

$$\int_0^{2b} \frac{a}{t^2-2bt+c}\,dt = \int_0^{2b} \frac{a}{(t-b)^2+(c-b^2)}\,dt = \frac{a}{\sqrt{c-b^2}}\tan^{-1}\frac{t-b}{\sqrt{c-b^2}}\Big]_0^{2b}$$
$$= \frac{a}{\sqrt{c-b^2}}\left(\tan^{-1}\frac{b}{\sqrt{c-b^2}} - \tan^{-1}\frac{-b}{\sqrt{c-b^2}}\right)$$
$$= \frac{2a}{\sqrt{c-b^2}}\tan^{-1}\frac{b}{\sqrt{c-b^2}}$$

Using part (d) we see that the fraction of total deaths occurring in the first 34 weeks is

$$\frac{\displaystyle\int_0^{2b} \frac{a}{t^2-2bt+c}\,dt}{\displaystyle\int_{-\infty}^{\infty} \frac{a}{t^2-2bt+c}\,dt} = \frac{\dfrac{2a}{\sqrt{c-b^2}}\tan^{-1}\dfrac{b}{\sqrt{c-b^2}}}{\dfrac{a\pi}{\sqrt{c-b^2}}} = \frac{2}{\pi}\tan^{-1}\frac{b}{\sqrt{c-b^2}}.$$

With $b = 17$ and $c = 299.13$ we find the percentage of total deaths within the first 34 weeks is

$$\frac{2}{\pi}\tan^{-1}\frac{17}{\sqrt{299.13-17^2}} \times 100 = 88.22\%.$$

7.8 Approximate Integration

1. Midpoint Rule

k	1	2	3
x_k	3/2	5/2	7/2
$f(x_k)$	39/4	95/4	175/4

$$\int_1^4 (3x^2+2x)\,dx \approx \frac{4-1}{3}\left(\frac{39}{4}+\frac{95}{4}+\frac{175}{4}\right) = \frac{309}{4} = 77.25$$

$$\int_1^4 (3x^2+2x)\,dx = (x^3+x^2)\Big]_1^4 = 80-2 = 78$$

2. Midpoint Rule

k	1	2	3	4
x_k	$\pi/48$	$\pi/16$	$5\pi/48$	$7\pi/48$
$f(x_k)$	0.997859	0.980785	0.94693	0.896873

$$\int_0^{\pi/6} \cos x\,dx \approx \frac{\pi/6-0}{4}(0.997859+0.980785+0.94693+0.896873) \approx 0.500357$$

$$\int_0^{\pi/6} \cos x\,dx = \sin x\Big]_0^{\pi/6} = \frac{1}{2}-0 = \frac{1}{2}$$

3. Trapezoidal Rule

k	0	1	2	3	4
x_k	1	3/2	2	5/2	3
$f(x_k)$	2	35/8	9	133/8	28

$$\int_1^3 (x^3+1)\,dx \approx \frac{3-1}{8}\left[2+2\left(\frac{35}{8}\right)+2(9)+2\left(\frac{133}{8}\right)+28\right] = \frac{45}{2} \approx 22.5$$

$$\int_1^3 (x^3+1)\,dx = \left(\frac{x^4}{4}+x\right)\Big]_1^3 = \frac{93}{4}-\frac{5}{4} = 22$$

4. Trapezoidal Rule

k	0	1	2	3	4	5	6
x_k	0	1/3	2/3	1	4/3	5/3	2
$f(x_k)$	1	$2/\sqrt{3}$	$\sqrt{5}/\sqrt{3}$	$\sqrt{2}$	$\sqrt{7}/\sqrt{3}$	$2\sqrt{2}/\sqrt{3}$	$\sqrt{3}$

$$\int_0^2 \sqrt{x+1}\,dx \approx \frac{2-0}{12}(1+2\sqrt{4/3}+2\sqrt{5/3}+2\sqrt{2}+2\sqrt{7/3}+2\sqrt{8/3}+\sqrt{3}) \approx 2.7954$$

$$\int_0^2 \sqrt{x+1}\,dx = \left.\frac{2(x+1)^{3/2}}{3}\right]_0^2 = \frac{2(3^{3/2})}{3} - \frac{2}{3} \approx 2.79743$$

5. Midpoint Rule

k	1	2	3	4	5
x_k	3/2	5/2	7/2	9/2	11/2
$f(x_k)$	2/3	2/5	2/7	2/9	2/11

$$\int_1^6 \frac{1}{x}\,dx \approx \frac{6-1}{5}\left(\frac{2}{3}+\frac{2}{5}+\frac{2}{7}+\frac{2}{9}+\frac{2}{11}\right) = \frac{6086}{3465} \approx 1.75642$$

Trapezoidal Rule

k	0	1	2	3	4	5
x_k	1	2	3	4	5	6
$f(x_k)$	1	1/2	1/3	1/4	1/5	1/6

$$\int_1^6 \frac{1}{x}\,dx \approx \frac{6-1}{12}\left[1+2\left(\frac{1}{2}\right)+2\left(\frac{1}{3}\right)+2\left(\frac{1}{4}\right)+2\left(\frac{1}{5}\right)+\frac{1}{6}\right] = \frac{28}{15} \approx 1.86667$$

6. Midpoint Rule

k	1	2	3	4
x_k	1/4	3/4	5/4	7/4
$f(x_k)$	4/7	4/13	4/19	4/25

$$\int_0^2 \frac{1}{3x+1}\,dx \approx \frac{2-0}{4}\left(\frac{4}{7}+\frac{4}{13}+\frac{4}{19}+\frac{4}{25}\right) = \frac{27008}{43225} \approx 0.624824$$

Trapezoidal Rule

k	0	1	2	3	4
x_k	0	1/2	1	3/2	2
$f(x_k)$	1	2/5	1/4	2/11	1/7

$$\int_0^2 \frac{1}{3x+1}\,dx \approx \frac{2-0}{8}\left[1+2\left(\frac{2}{5}\right)+2\left(\frac{1}{4}\right)+2\left(\frac{2}{11}\right)+\frac{1}{7}\right] = \frac{2161}{3080} \approx 0.701623$$

7. Midpoint Rule

k	1	2	3	4	5
x_k	0.05	0.15	0.25	0.35	0.45
$f(x_k)$	1.00125	1.01119	1.03078	1.05948	1.09659

6	7	8	9	10
0.55	0.65	0.75	0.85	0.95
1.14127	1.19269	1.25000	1.31244	1.37931

$$\int_0^1 \sqrt{x^2+1}\,dx \approx \frac{1-0}{10}(1.00125+\cdots+1.37931) \approx 1.1475$$

Trapezoidal Rule

k	0	1	2	3	4	5
x_k	0	0.1	0.2	0.3	0.4	0.5
$f(x_k)$	1	1.00499	1.0198	1.04403	1.07703	1.11803

6	7	8	9	10
0.6	0.7	0.8	0.9	1.0
1.16619	1.22066	1.28062	1.34536	1.41421

$$\int_0^1 \sqrt{x^2+1}\,dx \approx \frac{1-0}{20}[1+2(1.00499)+\cdots+2(1.34536)+1.41421] \approx 1.14838$$

8. Midpoint Rule

k	1	2	3	4	5
x_k	1.1	1.3	1.5	1.7	1.9
$f(x_k)$	0.654981	0.559279	0.478091	0.411241	0.356711

$$\int_1^2 \frac{1}{\sqrt{x^3+1}}\,dx \approx \frac{2-1}{5}(0.654981+\cdots+0.356711) \approx 0.492061$$

Trapezoidal Rule

k	0	1	2	3	4	5
x_k	1.0	1.2	1.4	1.6	1.8	2.0
$f(x_k)$	0.707107	0.605449	0.516811	0.442981	0.382583	0.333333

$$\int_1^2 \frac{1}{\sqrt{x^3+1}}\,dx \approx \frac{2-1}{10}[0.707107+2(0.605449)+\cdots+2(0.382583)+0.333333] \approx 0.493609$$

9. Midpoint Rule

k	1	2	3	4	5	6
x_k	$\pi/12$	$\pi/4$	$5\pi/12$	$7\pi/12$	$3\pi/4$	$11\pi/12$
$f(x_k)$	0.0760474	0.180063	0.217033	0.194188	0.128617	0.0429833

$$\int_0^\pi \frac{\sin x}{x+\pi}\,dx \approx \frac{\pi-0}{6}(0.0760474+\cdots+0.0429833) \approx 0.439263$$

Trapezoidal Rule

k	0	1	2	3	4	5	6
x_k	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π
$f(x_k)$	0	0.136419	0.206748	0.212207	0.165399	0.0868118	0

$$\int_0^\pi \frac{\sin x}{x+\pi}\,dx \approx \frac{\pi-0}{12}[1+2(0.136419)+\cdots+2(0.0868118)+0] \approx 0.42285$$

10. Midpoint Rule

k	1	2	3
x_k	$\pi/24$	$\pi/8$	$5\pi/24$
$f(x_k)$	0.131652	0.414214	0.767327

$$\int_0^{\pi/4} \tan x \, dx \approx \frac{\pi/4 - 0}{3}(0.131652 + 0.414214 + 0.767327) \approx 0.343793$$

Trapezoidal Rule

k	0	1	2	3
x_k	0	$\pi/12$	$\pi/6$	$\pi/4$
$f(x_k)$	0	0.267949	0.57735	1

$$\int_0^{\pi/4} \tan x \, dx \approx \frac{\pi/4 - 0}{6}[0 + 2(0.267949) + 2(0.57735) + 1] \approx 0.352199$$

11. Midpoint Rule

k	1	2	3	4	5	6
x_k	1/6	1/2	5/6	7/6	3/2	11/6
$f(x_k)$	0.999614	0.968912	0.768409	0.208152	−0.628174	−0.976002

$$\int_0^{2} \cos x^2 \, dx \approx \frac{2 - 0}{6}(0.999614 + \cdots - 0.976002) \approx 0.446971$$

Trapezoidal Rule

k	0	1	2	3	4	5	6
x_k	0	1/3	2/3	1	4/3	5/3	2
$f(x_k)$	1	0.993834	0.90285	0.540302	−0.205507	−0.934546	−0.653644

$$\int_0^{2} \cos x^2 \, dx \approx \frac{2 - 0}{12}[1 + 2(0.993834) + \cdots + 2(0.934546) - 0.653644] \approx 0.490037$$

12. Midpoint Rule

k	1	2	3	4	5
x_k	1/10	3/10	1/2	7/10	9/10
$f(x_k)$	0.998334	0.985067	0.958851	0.920311	0.870363

$$\int_0^{1} \frac{\sin x}{x} \, dx \approx \frac{1 - 0}{5}(0.998334 + \cdots + 0.870363) \approx 0.946585$$

Trapezoidal Rule

k	0	1	2	3	4	5
x_k	0	1/5	2/5	3/5	4/5	1
$f(x_k)$	1	0.993347	0.973546	0.941071	0.896695	0.841471

$$\int_0^{1} \frac{\sin x}{x} \, dx \approx \frac{1 - 0}{10}[1 + 2(0.993347) + \cdots + 2(0.896695) + 0.841471] \approx 0.945079$$

13. Simpson's Rule

k	0	1	2	3	4
x_k	0	1	2	3	4
$f(x_k)$	1	$\sqrt{3}$	$\sqrt{5}$	$\sqrt{7}$	3

$$\int_0^{4} \sqrt{2x + 1} \, dx \approx \frac{4 - 0}{12}(1 + 4\sqrt{3} + 2\sqrt{5} + 4\sqrt{7} + 3) \approx 8.6611$$

$$\int_0^4 \sqrt{2x+1}\,dx \qquad \boxed{u = 2x+1,\ du = 2\,dx}$$

$$= \int_1^9 u^{1/2}\left(\frac{1}{2}\,du\right) = \frac{1}{2}\left(\frac{2}{3}u^{3/2}\right)\Big]_1^9 = \frac{26}{3} \approx 0.86667$$

14. Simpson's Rule

k	0	1	2
x_k	0	$\pi/4$	$\pi/2$
$f(x_k)$	0	$1/2$	1

$$\int_0^{\pi/2} \sin^2 x\,dx \approx \frac{\pi/2-0}{6}\left[0+4\left(\frac{1}{2}\right)+1\right] \approx 0.7854$$

$$\int_0^1 \sin^2 x\,dx = \int_0^{\pi/2} \frac{1}{2}(1-\cos 2x)\,dx = \left(\frac{x}{2}-\frac{\sin 2x}{2}\right)\Big]_0^{\pi/2} = \frac{\pi}{4} \approx 0.7854$$

15. Simpson's Rule

k	0	1	2	3	4
x_k	$1/2$	1	$3/2$	2	$5/2$
$f(x_k)$	2	1	$2/3$	$1/2$	$2/5$

$$\int_{1/2}^{5/2} \frac{1}{x}\,dx \approx \frac{5/2-1/2}{12}\left[2+4(1)+2\left(\frac{2}{3}\right)+4\left(\frac{1}{2}\right)+\frac{2}{5}\right] \approx 1.6222$$

16. Simpson's Rule

k	0	1	2	3	4	5	6
x_k	0	$5/6$	$5/3$	$5/2$	$10/3$	$25/6$	5
$f(x_k)$	$1/2$	$6/17$	$3/11$	$2/9$	$3/16$	$6/37$	$1/7$

$$\int_0^5 \frac{1}{x+2}\,dx \approx \frac{5-0}{18}\left[\frac{1}{2}+4\left(\frac{6}{17}\right)+2\left(\frac{3}{11}\right)+4\left(\frac{2}{9}\right)+2\left(\frac{3}{16}\right)+4\left(\frac{6}{37}\right)+\frac{1}{7}\right] \approx 1.2535$$

17. Simpson's Rule

k	0	1	2	3	4
x_k	0	$1/4$	$1/2$	$3/4$	1
$f(x_k)$	1	$16/17$	$4/5$	$16/25$	$1/2$

$$\int_0^1 \frac{1}{1+x^2}\,dx \approx \frac{1-0}{12}\left[1+4\left(\frac{16}{17}\right)+2\left(\frac{4}{5}\right)+4\left(\frac{16}{25}\right)+\frac{1}{2}\right] \approx 0.7854$$

18. Simpson's Rule

k	0	1	2
x_k	-1	0	1
$f(x_k)$	$\sqrt{2}$	1	$\sqrt{2}$

$$\int_{-1}^1 \sqrt{x^2+1}\,dx \approx \frac{1+1}{6}[\sqrt{2}+4(1)+\sqrt{2}] \approx 2.2761$$

19. Simpson's Rule

k	0	1	2	3	4	5	6
x_k	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π
$f(x_k)$	0	$3/7\pi$	$3\sqrt{8}/8\pi$	$2/3\pi$	$3\sqrt{3}/10\pi$	$3/11\pi$	0

$$\int_0^\pi \frac{\sin x}{x+\pi}\,dx \approx \frac{\pi - 0}{18}\left[0 + 4\left(\frac{3}{7\pi}\right) + 2\left(\frac{3\sqrt{3}}{8\pi}\right) + 4\left(\frac{2}{3\pi}\right) + 2\left(\frac{3\sqrt{3}}{10\pi}\right) + 4\left(\frac{3}{11\pi}\right) + 0\right]$$
$$\approx 0.4339$$

20. Simpson's Rule

k	0	1	2	3	4
x_k	0	1/4	1/2	3/4	1
$f(x_k)$	1	0.8776	0.7602	0.6479	0.5403

$$\int_0^1 \cos\sqrt{x}\,dx \approx \frac{1-0}{12}[1 + 4(0.8776) + 2(0.7602) + 4(0.6479) + 0.5403] \approx 0.7635$$

21. Simpson's Rule

k	0	1	2	3	4
x_k	2	2.5	3	3.5	4
$f(x_k)$	3.1623	4.2573	5.4772	6.8099	8.2462

$$\int_2^4 \sqrt{x^3+x}\,dx \approx \frac{4-2}{12}[3.1623 + 4(4.2573) + 2(5.4772) + 4(6.8099) + 8.2462] \approx 11.1053$$

22. Simpson's Rule

k	0	1	2
x_k	$\pi/4$	$3\pi/8$	$\pi/2$
$f(x_k)$	0.3694	0.3420	0.3333

$$\int_{\pi/4}^{\pi/2} \frac{1}{2+\sin x}\,dx \approx \frac{\pi/2 - \pi/4}{6}[0.3694 + 4(0.3420) + 0.3333] \approx 0.2711$$

23. $f(x) = \dfrac{1}{x+3}$; $f'(x) = -\dfrac{1}{(x+3)^2}$; $f''(x) = \dfrac{2}{(x+3)^3}$. Since $f''(x)$ decreases on $[-1,2]$, $f''(x) \le f''(-1) = 1/4$ on the interval. Taking $M = 1/4$, we want $\dfrac{(1/4)(2+1)^3}{24n^2} < 0.005$ or $n^2 > \dfrac{225}{4} = 56.25$. Take $n = 8$ to obtain the desired accuracy.

24. $f(x) = \sin^2 x$; $f'(x) = 2\sin x\cos x = \sin 2x$; $f''(x) = 2\cos 2x$. Since $|f''(x)| = 2|\cos 2x| \le 2$ for all x, we take $M = 2$. Then we want $\dfrac{2(1.5-0)^3}{12n^2} < 0.0001$ or $n^2 > 5625 = 75^2$. Take $n = 76$ to obtain the desired accuracy.

25. $f(x) = \dfrac{1}{1+x^2}$; $f'(x) = -\dfrac{2x}{(1+x^2)^2}$; $f''(x) = \dfrac{6x^2-2}{(1+x^2)^3}$. To determine an upper bound for $|f''(x)|$ on $[0,2]$, we compute $f'''(x) = \dfrac{24x(1-x^2)}{(1+x^2)^4}$. Setting $f'''(x) = 0$ we obtain the

critical numbers 0 and 1 on $[0, 2]$. Comparing $f''(0) = -2$, $f''(1) = \dfrac{1}{2}$, and $f''(2) = \dfrac{22}{125}$, we see that $|f''(x)| \le 2$ on the interval. Taking $M = 2$ we want $\dfrac{2(2-0)^3}{12n^2} < 0.005$ or $n^2 > \dfrac{800}{3} \approx 267$. For $n = 17$, the Trapezoidal Rule gives $\displaystyle\int_0^2 \frac{1}{1+x^2}\,dx \approx 1.11$ to two decimal places.

26. Trapezoidal Rule

k	0	1	2	3	4
x_k	-2	-1	0	1	2
$f(x_k)$	0.01	0.1	1	10	100

$$\int_{-2}^{2} 10^x\,dx \approx \frac{2+2}{8}[0.01 + 2(0.1) + 2(1) + 2(10) + 100] \approx 61.105$$

27. $f'(x) = -\dfrac{1}{x^2}$; $f''(x) = \dfrac{2}{x^3}$; $f'''(x) = -\dfrac{6}{x^4}$; $f^{(4)}(x) = \dfrac{24}{x^5}$. Since $f^{(4)}(x)$ is decreasing on $[1, 3]$, $|f^{(4)}(x)| \le f^{(4)}(1) = 24$ for all x in $[1, 3]$. Taking $M = 24$, we want $\dfrac{24(3-1)^5}{180n^4} < 10^{-5}$ or $n^4 > \dfrac{1,280,000}{3} \approx 426,667$. To have the desired accuracy with Simpson's Rule, we need $n \ge 26$. To obtain the required n for the Trapezoidal Rule, we note that $f''(x) = \dfrac{2}{x^3}$ is decreasing on $[1, 3]$. We thus take $M = f''(1) = 2$. We want $\dfrac{2(3-1)^3}{12n^2} < 10^{-5}$ or $n^2 > \dfrac{400,000}{3} \approx 133,333$. For the Trapezoidal Rule to have the desired accuracy, we need $n \ge 366$.

28. $f(x) = \dfrac{1}{2x+1}$; $f'(x) = -\dfrac{2}{(2x+1)^2}$; $f''(x) = \dfrac{8}{(2x+1)^3}$; $f'''(x) = -\dfrac{48}{(2x+1)^4}$; $f^{(4)}(x) = \dfrac{384}{(2x+1)^5}$. Since $f^{(4)}(x)$ is decreasing on $[0, 3]$, $|f^{(4)}(x)| \le f^{(4)}(0) = 384$ for all x in $[0, 3]$. Taking $M = 384$, we have $E_6 \le \dfrac{384(3-0)^5}{180(6^4)} = \dfrac{2}{5} = 0.4$.

29. Since $n = 5$ is odd we cannot use Simpson's Rule. Because the Midpoint Rule does not readily work with tabular data we use the Trapezoidal Rule:

$$\int_{2.05}^{2.30} f(x)\,dx \approx \frac{2.30 - 2.05}{10}[4.91 + 2(4.80) + 2(4.66) + 2(4.41) + 2(3.93) + 3.58] = 1.10225.$$

30. Since the subintervals are of unequal widths, Simpson's Rule as derived in the text cannot be

applied. We use trapezoids to approximate the integral.

$$\begin{aligned}\int_0^{1.20} f(x)\,dx &\approx (0.1-0)\frac{-0.72-0.55}{2} + (0.2-0.1)\frac{-0.55-0.16}{2} \\ &\quad + (0.4-0.2)\frac{-0.16+0.62}{2} + (0.6-0.4)\frac{0.62+0.78}{2} \\ &\quad + (0.8-0.6)\frac{0.78+1.34}{2} + (0.9-0.8)\frac{1.34+1.47}{2} \\ &\quad + (1.0-0.9)\frac{1.47+1.61}{2} + (1.2-1.0)\frac{1.61+1.51}{2} \\ &= 0.9055\end{aligned}$$

31. $\displaystyle\int_0^4 (2x+5)\,dx = (x^2+5x)\Big]_0^4 = 36$

$$n=2: \quad \int_0^4 (2x+5)\,dx \approx \frac{4-0}{2}[(2\cdot 1+5)+(2\cdot 3+5)] = 36$$

$$\begin{aligned}n=4: \quad \int_0^4 (2x+5)\,dx &\approx \frac{4-0}{4}[(2\cdot 0.5+5)+(2\cdot 1.5+5)+(2\cdot 2.5+5)+(2\cdot 3.5+5)] \\ &= 36\end{aligned}$$

32. $\displaystyle\int_0^4 (2x+5)\,dx = (x^2+5x)\Big]_0^4 = 36$

$$n=2: \quad \int_0^4 (2x+5)\,dx \approx \frac{4-0}{4}[(2\cdot 0+5)+2(2\cdot 2+5)+(2\cdot 4+5)] = 36$$

$$\begin{aligned}n=4: \quad \int_0^4 (2x+5)\,dx \approx \frac{4-0}{8}[&(2\cdot 0+5)+2(2\cdot 1+5)+2(2\cdot 2+5)+2(2\cdot 3+5) \\ &+(2\cdot 4+5)] = 36\end{aligned}$$

33. (a) $\displaystyle I = \int_{-1}^1 (x^3+x^2)\,dx = \left(\frac{1}{4}x^4+\frac{1}{3}x^3\right)\Big]_{-1}^1 = \frac{2}{3}$

(b) $$\begin{aligned}M_8 &= \frac{1-(-1)}{8}[f(-7/8)+f(-5/8)+\cdots+f(5/8)+f(7/8)] \\ &= \frac{1}{4}\left(\frac{49}{512}+\frac{75}{512}+\cdots+\frac{325}{512}+\frac{735}{512}\right) = \frac{21}{32}\end{aligned}$$

(c) $$\begin{aligned}T_8 &= \frac{1-(-1)}{16}[f(-1)+2f(-3/4)+\cdots+2f(3/4)+f(1)] \\ &= \frac{1}{8}\left[0+2\left(\frac{9}{64}\right)+\cdots+2\left(\frac{63}{64}\right)+2\right] = \frac{11}{16}\end{aligned}$$

(d) $\displaystyle E_8 = \left|\frac{2}{3}-M_8\right| = \frac{1}{96}; \qquad E_8 = \left|\frac{2}{3}-T_8\right| = \frac{1}{48}$

The error for the Midpoint Rule is one half the error for the Trapezoidal Rule.

34. $\displaystyle\int_{-1}^{3}(x^3-x^2)\,dx=\left(\frac{1}{4}x^4-\frac{1}{3}x^3\right)\bigg]_{-1}^{3}=\frac{32}{3}$

$$n=2:\quad \int_{-1}^{3}(x^3-x^2)\,dx\approx\frac{3+1}{6}\{[(-1)^3-(-1)^2]+4(1^3-1^2)+(3^3-3^2)\}=\frac{32}{3}$$

$$\begin{aligned}n=4:\quad \int_{-1}^{3}(x^3-x^2)\,dx\approx\frac{3+1}{12}\{&[(-1)^3-(-1)^2]+4(0^3-0^2)+2(1^3-1^2)\\&+4(2^3-2^2)+(3^3-3^2)\}=\frac{32}{3}\end{aligned}$$

35. The exact value is

$$\begin{aligned}\int_a^b(c_1x+c_0)\,dx&=\left(\frac{c_1}{2}x^2+c_0x\right)\bigg]_a^b=\frac{c_1}{2}(b^2-a^2)+c_0(b-a)\\&=\frac{b-a}{2}[(c_1a+c_0)+(c_1b+c_0)].\end{aligned}$$

The Trapezoidal Rule gives

$$\begin{aligned}&\frac{b-a}{2n}\left[(c_1a+c_0)+2\left[c_1\left(a+\frac{b-a}{n}\right)+c_0\right]+2\left\{c_1\left[a+\frac{2(b-a)}{n}\right]+c_0\right\}+\cdots\right.\\&\qquad\left.+2\left\{c_1\left[a+\frac{(n-1)(b-a)}{n}\right]+c_0\right\}+\left\{c_1\left[a+\frac{n(b-a)}{n}\right]+c_0\right\}\right]\\&=\frac{b-a}{2n}\left[2nc_1a+2nc_0+2c_1\frac{b-a}{n}(1+2+\cdots+n-1)+c_1(b-a)\right]\\&=\frac{b-a}{2n}\left\{2nc_1a+2nc_0+2c_1\left(\frac{b-a}{n}\right)\left[\frac{(n-1)n}{2}\right]+c_1(b-a)\right\}\\&=\frac{b-a}{2n}[2nc_1a+2nc_0+c_1(b-a)(n-1)+c_1(b-a)]\\&=\frac{b-a}{2n}[2nc_1a+2nc_0+nc_1b-nc_1a-c_1b+c_1a+c_1b-c_1a]\\&=\frac{b-a}{2n}(c_1a+c_0+c_1b+c_0).\end{aligned}$$

Since the graph of $f(x)=c_1x+c_0$ is a straight line and the Trapezoidal Rule uses straight line approximations to the curve, it will give the exact value.

36. From the derivation of Simpson's Rule in the text, it suffices to show that it will give the exact value of $\displaystyle\int_{-h}^{h}(c_3x^3+c_2x^2+c_1x+c_0)\,dx$ for $n=2$. In this case, the exact value is

$$\frac{c_3}{4}h^4+\frac{c_2}{3}h^3+\frac{c_1}{2}h^2+c_0h-\left[\frac{c_3}{4}(-h)^4+\frac{c_2}{3}(-h)^3+\frac{c_1}{2}(-h)^2+c_0(-h)\right]=\frac{2c_2}{3}h^3+2c_0h,$$

and Simpson's Rule gives

$$\frac{h+h}{6}\left\{[c_3(-h)^3+c_2(-h)^2+c_1(-h)+c_0]+4(c_0)+(c_3h^3+c_2h^2+c_1h+c_0)\right\}$$
$$=\frac{h}{3}(2c_2h^2+6c_0)=\frac{2c_2}{3}h^3+2c_0h.$$

37. $\displaystyle\int_1^4 f(x)\,dx \approx \frac{4-1}{3(6)}[1.3+4(1.5)+2(3)+4(3.3)+2(2.2)+4(2.4)+1.9] \approx 7.06667$

38. $\displaystyle A=\frac{9-0}{18}[0+2(3)+2(1)+2(3)+2(5)+2(3)+2(7)+2(4)+2(6)+3]=\frac{67}{2}$

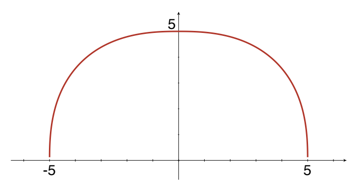
The Trapezoidal Rule gives the exact value of the area in this case because the curve segments are straight lines.

39. $$A \approx \frac{18.6-0}{3(10)}[0-4(5.8)+2(7.3)+4(6.9)+2(8.7)+4(8.8)+2(10.3)+4(14.5)$$
$$+2(15)+4(10.4)+0]$$
$$=166.284$$

The volume of the pond is $4(166.284)=665.136$ ft^3 and the number of gallons of water is $7.48(665.136)=4975.22$ gallons.

40. $$I \approx \frac{3(570\pi)}{2(32)}+\frac{3(570)}{32}\left(\frac{4.5-1}{14}\right)[1(0.3)+2(1.5)^2(0.5)+2(2)^2(0.62)+2(2.5)^2(0.7)$$
$$+2(3)^2(0.6)+2(3.5)^2(0.5)+2(4)^2(0.27)+(4.5)^2(0)]$$
$$\approx 83.94+53.44(11.99)\approx 724.69$$

41. Simpson's Rule

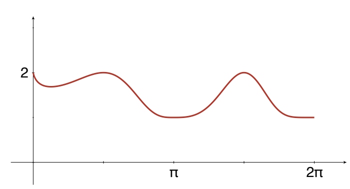


k	0	1	2	3	4	5
x_k	-5	-4	-3	-2	-1	0
$f(x_k)$	0	3.55936	4.38712	4.79112	4.96403	5

6	7	8	9	10
1	2	3	4	5
4.96403	4.79112	4.38712	3.55936	0

$$\int_{-5}^{5}\sqrt[5]{(5^{2.5}-|x|^{2.5})^2}\,dx \approx \frac{5-(-5)}{30}[0+4(3.55936)+2(4.38712)+\cdots$$
$$+2(4.38712)+4(3.55936)+0]\approx 41.4028$$

42. From the graph we define $f(0)=2$.



Simpson's Rule

k	0	1	2	3	4	5
x_k	0	$\pi/5$	$2\pi/5$	$3\pi/5$	$4\pi/5$	π
$f(x_k)$	2	1.71614	1.93889	1.90975	1.26302	1

6	7	8	9	10
$6\pi/5$	$7\pi/5$	$8\pi/5$	$9\pi/5$	2π
1.13489	1.80195	1.77706	1.04954	1

$$\int_0^{2\pi}(1+|\sin x|^x)\,dx \approx \frac{2\pi-0}{30}[2+4(1.71614)+2(1.93889)+\cdots$$
$$+2(1.77706)+4(1.04954)+1]\approx 9.45351$$

43. (a) $$\int_1^\infty \frac{e^{1/x}}{x^{5/2}}\,dx = \lim_{k\to\infty}\int_1^k \frac{e^{1/x}}{x^{5/2}}\,dx \qquad \boxed{x=\frac{1}{t},\ dx=-\frac{1}{t^2}\,dt}$$
$$=\lim_{k\to\infty}\int_1^{1/k}\frac{e^t}{(1/t)^{5/2}}\left(-\frac{1}{t^2}\,dt\right)=\int_1^0(-t^{1/2}e^t\,dt)=\int_0^1 t^{1/2}e^t\,dt$$

(b) $$\int_1^\infty \frac{e^{1/x}}{x^{5/2}}\,dx=\int_0^1 t^{1/2}e^t\,dt$$
$$\approx \frac{1-0}{12}\left[0+4\left(\frac{1}{2}e^{1/4}\right)+2\left(\sqrt{\frac{1}{2}}e^{1/2}\right)+4\left(\frac{\sqrt{3}}{2}e^{3/4}\right)+e\right]$$
$$\approx\frac{1}{12}(2.5681+2.3316+7.3335+2.7183)\approx 1.2460$$

44. $y'=x^2$; $\quad s=\int_0^2\sqrt{1+x^4}\,dx\approx\frac{2-0}{12}(1+4\sqrt{1.0625}+2\sqrt{2}+4\sqrt{6.0625}+\sqrt{17})\approx 3.6539$

45. $y'=2x$
$$s=\int_0^1\sqrt{1+4x^2}\,dx\approx\frac{1-0}{20}\Big\{[1+4(0)^2]^{1/2}+2[1+4(0.1)^2]^{1/2}+2[1+4(0.2)^2]^{1/2}+\cdots$$
$$+2[1+4(0.9)^2]^{1/2}+[1+4(1)^2]^{1/2}\Big\}$$
$$=\frac{1}{20}(1+2\sqrt{1.04}+2\sqrt{1.16}+\cdots+2\sqrt{4.24}+\sqrt{5})\approx 1.4804$$

46. $y'=\frac{1}{x}$; $\quad s=\int_1^2\sqrt{1+1/x^2}\,dx$. Using Simpson's Rule with $n=6$ we obtain
$$S_6=\frac{1}{18}\Big(\sqrt{1+1}+4\sqrt{1+36/49}+2\sqrt{1+9/16}+4\sqrt{1+4/9}+2\sqrt{1+9/25}$$
$$+4\sqrt{1+36/121}+\sqrt{1+1/4}\Big)$$
$$=\frac{1}{18}\left(\sqrt{2}+\frac{4}{7}\sqrt{85}+\frac{1}{2}\sqrt{25}+\frac{4}{3}\sqrt{13}+\frac{2}{5}\sqrt{34}+\frac{4}{11}\sqrt{157}+\frac{1}{2}\sqrt{5}\right)\approx 1.2220.$$

47. $y'=x$; $\quad S=2\pi\int_0^2\frac{1}{2}x^2\sqrt{1+x^2}\,dx=\pi\int_0^2 x^2\sqrt{1+x^2}\,dx$. Using the Midpoint Rule with $n=5$, we obtain
$$M_5=\frac{2}{5}\left(\frac{1}{25}\sqrt{1+\frac{1}{25}}+\frac{9}{25}\sqrt{1+\frac{9}{25}}+\sqrt{2}+\frac{49}{25}\sqrt{1+\frac{49}{25}}+\frac{81}{25}\sqrt{1+\frac{81}{25}}\right)\approx 4.76741.$$

Then $S \approx \pi M_5 \approx 14.9772$.

48. $\dfrac{dx}{dy} = 2y$; $\quad S = 2\pi \displaystyle\int_{-1}^{1} (y^2+1)\sqrt{1+4y^2}\,dy$. Using Simpson's Rule with $n = 6$, we obtain

$$\begin{aligned} S_6 = \frac{1-(-1)}{3.6}[&2\sqrt{5} + 4(4/9+1)\sqrt{1+16/9} + 2(1/9+1)\sqrt{1+4/9} \\ &+ 4(1)\sqrt{1} + 2(1/9+1)\sqrt{1+4/9} + 4(4/9+1)\sqrt{1+16/9} + 2\sqrt{5}] \approx 4.17168. \end{aligned}$$

Then $S \approx 2\pi S_6 \approx 26.2114$.

49. (a) Approximating the graph on each interval between integers with line segments and using the Pythagorean theorem, we have

$$\begin{aligned} s &< \sqrt{1^2+2^2} + \sqrt{1^2+4^2} + \sqrt{1^2+3^2} + \sqrt{1^2+1^2} + \sqrt{1^2+3^2} + \sqrt{1^2+1^2} + \sqrt{1^2+2^2} \\ &= \sqrt{5} + \sqrt{17} + \sqrt{10} + \sqrt{2} + \sqrt{10} + \sqrt{2} + \sqrt{5} \approx 17.75. \end{aligned}$$

(b) Since $y'(x_i) = 0$ for $x_i = 1, 2, \ldots, 8$, the formula for arc length gives $\displaystyle\int_1^8 \sqrt{1+(y')^2}\,dx = \int_1^8 dx = 7$ which is simply the length of the interval.

50. (a)
$$\begin{aligned} \lim_{x\to\infty} \frac{\mathrm{Li}(x)}{x/\ln x} &= \lim_{x\to\infty} \frac{\ln x \displaystyle\int_2^x \frac{1}{\ln t}\,dt}{x} \overset{h}{=} \lim_{x\to\infty} \frac{\ln x(1/\ln x) + \dfrac{1}{x}\displaystyle\int_2^x \frac{1}{\ln t}\,dt}{1} \\ &= \lim_{x\to\infty} \frac{x + \displaystyle\int_2^x \frac{1}{\ln t}\,dt}{x} \overset{h}{=} \lim_{x\to\infty} \frac{1 + 1/\ln x}{1} = 1 \end{aligned}$$

(b) Implementing Simpson's Rule with $n = 98$ on a computer, we obtain $\displaystyle\int_2^{100} \frac{1}{\ln t}\,dt \approx 29.09$. When $x = 100$, $x/\ln x \approx 21.71$. The number of primes less than 100 is 25.

Chapter 7 in Review

A. True/False

1. True
2. False; use $u = a\tan\theta$.
3. True
4. True
5. True
6. False; $\dfrac{x^2}{(x+1)^2} = 1 + \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2}$.

7. False; $\dfrac{1}{(x^2-1)^2} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x+1} + \dfrac{D}{(x+1)^2}$.

8. False; it is used n times.

9. False; substituting $u = 9 - x^2$ will work.

10. True

11. True

12. False; let $f(x) = x$ and $g(x) = -x$.

13. True

14. False; $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \lim_{s\to-\infty}\int_s^a f(x)\,dx + \lim_{t\to\infty}\int_a^t f(x)\,dx$.

15. False; the infinite discontinuity occurs at $x = 1/e \approx 0.3679$, which is outside of $[1/2, 1]$.

16. False; since $\displaystyle\int_0^1 x^{-3}\,dx = \lim_{s\to0^+}\int_s^1 x^{-3}\,dx = \lim_{s\to0^+}\left(-\frac{1}{2}x^{-2}\right)\Bigg]_s^1 = \lim_{s\to0^+}\left(-\frac{1}{2}+\frac{1}{2s^2}\right)$,
the integral diverges.

17. True

18. True

19. False; since

$$\begin{aligned}
\int_2^{\infty}\left(\frac{e^x}{e^x+1}-\frac{e^x}{e^x-1}\right)dx &= \lim_{t\to\infty}\int_2^t\left(\frac{e^x}{e^x+1}-\frac{e^x}{e^x-1}\right)dx \\
&= \lim_{t\to\infty}\left(\ln|e^x+1| - \ln|e^x-1|\right)\Big]_2^t \\
&= \lim_{t\to\infty}\ln\left|\frac{e^t+1}{e^t-1}\right| - \ln\frac{e^2+1}{e^2-1} = \ln\left(\lim_{t\to\infty}\frac{e^t+1}{e^t-1}\right) - \ln\frac{e^2+1}{e^2-1} \\
&= \ln\left(\lim_{t\to\infty}\frac{1+e^{-t}}{1-e^{-t}}\right) - \ln\frac{e^2+1}{e^2-1} = -\ln\frac{e^2+1}{e^2-1},
\end{aligned}$$

the integral converges.

20. False; see Example 9, Section 7.7 in the text.

B. Fill in the Blanks

1. 1/5 (see Problem 77 in Exercises 7.7)

2. $\sqrt{\pi}$, by symmetry.

3. $\displaystyle\int_0^\infty \frac{e^{-x}}{\sqrt{x}}\,dx$ $\boxed{u = \sqrt{x},\ x = u^2,\ dx = 2u\,du}$

$$= \int_0^\infty \frac{e^{-u^2}}{u}(2u\,du) = 2\int_0^\infty e^{-u^2}\,du = 2\left(\frac{\sqrt{\pi}}{2}\right) = \sqrt{\pi}$$

4. -1 (see Problem 75 in Exercises 7.7, noting $p = -k$)

5. Integrating, we obtain the equation $\displaystyle -\frac{1}{2}e^{-2t}\Big]_0^x = \lim_{k\to\infty}\left(-\frac{1}{2}e^{-2t}\right)\Big]_x^k = -\frac{1}{2}\lim_{k\to\infty} 2e^{-2t}\Big]_x^k.$

   Then, solving for x, $e^{-2x} - e^{-2(0)} = 0 - e^{-2x}$; $2e^{-2x} = 1$; $-2x = \ln\left(\dfrac{1}{2}\right)$; and $x = -\dfrac{1}{2}(0 - \ln 2) = \ln\sqrt{2}$.

6. $\displaystyle\int \sin x \ln(\sin x)\,dx$ $\boxed{u = \ln(\sin x),\ du = \cot x\,dx;\quad dv = \sin x\,dx,\ v = -\cos x}$

$$\begin{aligned}
&= -\cos x\ln(\sin x) + \int \cos x\cot x\,dx = -\cos x\ln(\sin x) + \int \frac{\cos^2 x}{\sin x}\,dx \\
&= -\cos x\ln(\sin x) + \int \frac{1-\sin^2 x}{\sin x}\,dx \\
&= -\cos x\ln(\sin x) + \int \csc x\,dx + \int(-\sin x)\,dx \\
&= \cos x - \cos x\ln(\sin x) + \ln|\csc x - \cot x| + C
\end{aligned}$$

C. Exercises

1. $\displaystyle\int \frac{1}{\sqrt{x}+9}\,dx$ $\boxed{u = \sqrt{x}+9,\ x = (u-9)^2,\ dx = 2(u-9)\,du}$

$$\begin{aligned}
&= \int \frac{2(u-9)}{u}\,du = 2\int du - 18\int \frac{1}{u}\,du = 2u = 18\ln|u| + C \\
&= 2\sqrt{x} + 18 - 18\ln(\sqrt{x}+9) + C = 2\sqrt{x} - 18\ln(\sqrt{x}+9) + C_1
\end{aligned}$$

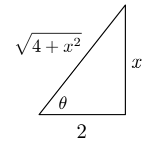
2. $\displaystyle\int e^{\sqrt{x+1}}\,dx$ $\boxed{t = \sqrt{x+1},\ x = t^2-1,\ dx = 2t\,dt}$

$$= \int e^t(2t\,dt) = \int 2te^t\,dt \qquad \boxed{u = 2t,\ du = 2\,dt;\quad dv = e^t\,dt,\ v = e^t}$$

$$= 2te^t - \int 2e^t\,dt = 2te^t - 2e^t + C = 2\sqrt{x+1}\,e^{\sqrt{x+1}} - 2e^{\sqrt{x+1}} + C$$

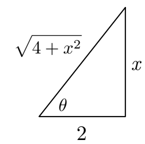
3. $\displaystyle\int \frac{x}{\sqrt{x^2+4}}\,dx = \frac{1}{2}\int (x^2+4)^{-1/2}(2x\,dx) = \frac{1}{2}\left[\frac{(x^2+4)^{1/2}}{1/2}\right] + C = \sqrt{x^2+4} + C$

4. $\displaystyle\int \frac{1}{\sqrt{x^2+4}}\,dx \qquad \boxed{x = 2\tan\theta,\ dx = 2\sec^2\theta\,d\theta} \qquad = \int \frac{2\sec^2\theta}{\sqrt{4\tan^2\theta+4}}\,d\theta$



$$= \int \frac{\sec^2\theta}{\sec\theta}\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C$$
$$= \ln\left|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right| + C = \ln\left|\sqrt{x^2+4} + x\right| + C_1$$

5. $\displaystyle\int \frac{1}{(x^2+4)^3}\,dx \qquad \boxed{x = 2\tan\theta,\ dx = 2\sec^2\theta\,d\theta} \qquad = \int \frac{2\sec^2\theta}{(4\tan^2\theta+4)^3}\,d\theta$



$$= \frac{1}{32}\int \frac{\sec^2\theta}{\sec^6\theta}\,d\theta = \frac{1}{32}\int \cos^4\theta\,d\theta$$
$$\boxed{\text{See Section 7.4, Example 5}}$$
$$= \frac{3}{256}\theta + \frac{1}{128}\sin 2\theta + \frac{1}{1024}\sin 4\theta + C$$
$$= \frac{3}{256}\tan^{-1}\frac{x}{2} + \frac{1}{64}\sin\theta\cos\theta + \frac{1}{256}\sin\theta\cos\theta(\cos^2\theta - \sin^2\theta) + C$$
$$= \frac{3}{256}\tan^{-1}\frac{x}{2} + \frac{x}{32(x^2+4)} + \frac{x}{128(x^2+4)}\left(\frac{4-x^2}{x^2+4}\right) + C$$
$$= \frac{3}{256}\tan^{-1}\frac{x}{2} + \frac{x}{32(x^2+4)} + \frac{x}{32(x^2+4)^2} - \frac{x^3}{128(x^2+4)^2} + C$$

6. $\displaystyle\int \frac{x^2}{x^2+4}\,dx = \int \frac{x^2+4-4}{x^2+4}\,dx = \int dx - 4\int \frac{1}{x^2+4}\,dx = x - 2\tan^{-1}\frac{x}{2} + C$

7. $\displaystyle\int \frac{x^2+4}{x^2}\,dx = \int dx + 4\int \frac{1}{x^2}\,dx = x - \frac{4}{x} + C$

8. Write $\dfrac{3x-1}{x(x^2-4)} = \dfrac{A}{x} + \dfrac{B}{x-2} + \dfrac{C}{x+2}$. Then $3x - 1 = A(x^2-4) + B(x^2+2x) + C(x^2-2x)$.
Setting $x = 0$, $x = 2$, and $x = -2$ gives $A = 1/4$, $B = 5/8$, and $C = -7/8$. Thus

$$\int \frac{3x-1}{x(x^2-4)}\,dx = \frac{1}{4}\int \frac{1}{x}\,dx + \frac{5}{8}\int \frac{1}{x-2}\,dx - \frac{7}{8}\int \frac{1}{x+2}\,dx$$
$$= \frac{1}{4}\ln|x| + \frac{5}{8}\ln|x-2| - \frac{7}{8}\ln|x+2| + C.$$

9. $\displaystyle\int \frac{x-5}{x^2+4}\,dx = \frac{1}{2}\int \frac{2x}{x^2+4}\,dx - 5\int \frac{1}{x^2+4}\,dx = \frac{1}{2}\ln(x^2+4) - \frac{5}{2}\tan^{-1}\frac{x}{2} + C$

10. $\displaystyle\int \frac{\sqrt[3]{x+27}}{x}\,dx \qquad \boxed{u = \sqrt[3]{x+27},\ x = u^3 - 27,\ dx = 3u^2\,du}$

$$= \int \frac{u}{u^3-27}(3u^2\,du) = 3\int \frac{u^3-27+27}{u^3-27}\,du = 3\int du + 81\int \frac{1}{u^3-27}\,du$$

Write $\dfrac{1}{u^3-27} = \dfrac{A}{u-3} + \dfrac{Bu+C}{u^2+3u+9}$. Then

$$1 = A(u^2+3u+9) + (Bu+C)(u-3) = (A+B)u^2 + (3A-3B+C)u + (9A-3C).$$

Solving $\boxed{A+B=0 \qquad 3A-3B+C=0 \qquad 9A-3C=1}$

gives $A = 1/27$, $B = -1/27$, and $C = -2/9$. Thus

$$\begin{aligned}
\int \frac{\sqrt[3]{x+27}}{x}\,dx &= 3\int du + 81\left(\frac{1}{27}\int \frac{1}{u-3}\,du - \frac{1}{27}\int \frac{u}{u^2+3u+9}\,du - \frac{2}{9}\int \frac{1}{u^2+3u+9}\,du\right) \\
&= 3u + 3\ln|u-3| - \frac{3}{2}\int \frac{2u+3-3}{u^2+3u+9}\,du - 18\int \frac{1}{u^2+3u+9}\,du \\
&= 3u + 3\ln|u-3| - \frac{3}{2}\int \frac{2u+3}{u^2+3u+9}\,du - \frac{27}{2}\int \frac{1}{(u+3/2)^2+27/4}\,du \\
&= 3u + 3\ln|u-3| - \frac{3}{2}\ln(u^2+3u+9) - 3\sqrt{3}\tan^{-1}\frac{2u+3}{3\sqrt{3}} + C \\
&= 3\sqrt[3]{x+27} + 3\ln|\sqrt[3]{x+27}-3| - \frac{3}{2}\ln[(x+27)^{2/3} + 3(x+27)^{1/3} + 9] \\
&\qquad - 3\sqrt{3}\tan^{-1}\frac{2\sqrt[3]{x+27}+3}{3\sqrt{3}} + C.
\end{aligned}$$

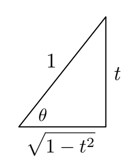
11. $\displaystyle\int \frac{(\ln x)^9}{x}\,dx$ $\quad\boxed{u=\ln x,\ du=\frac{1}{x}\,dx}\quad$ $\displaystyle = \int u^9\,du = \frac{1}{10}u^{10} + C = \frac{1}{10}(\ln x)^{10} + C$

12. $\displaystyle\int (\ln 3x)^2\,dx$ $\quad\boxed{u=(\ln 3x)^2,\ du=\frac{2\ln 3x}{x}\,dx; \quad dv=dx,\ v=x}$

$$= x(\ln 3x)^2 - 2\int \ln 3x\,dx \qquad \boxed{u=\ln 3x,\ du=\frac{1}{x}\,dx; \quad dv=dx,\ v=x}$$

$$= x(\ln 3x)^2 - 2\left(x\ln 3x - \int dx\right) = x(\ln 3x)^2 - 3x\ln 3x + 2x + C$$

13. $\displaystyle\int t\sin^{-1}t\,dt$ $\quad\boxed{u=\sin^{-1}t,\ du=\frac{1}{\sqrt{1-t^2}}\,dt; \quad dv=t\,dt,\ v=\frac{1}{2}t^2}$



$$\begin{aligned}
&= \frac{1}{2}t^2\sin^{-1}t - \frac{1}{2}\int \frac{t^2}{\sqrt{1-t^2}}\,dt \qquad \boxed{t=\sin\theta,\ dt=\cos\theta\,d\theta} \\
&= \frac{1}{2}t^2\sin^{-1}t - \frac{1}{2}\int \frac{\sin^2\theta}{\cos\theta}\cos\theta\,d\theta = \frac{1}{2}t^2\sin^{-1}t - \frac{1}{2}\int \sin^2\theta\,d\theta \\
&= \frac{1}{2}t^2\sin^{-1}t - \frac{1}{4}\int (1-\cos 2\theta)\,d\theta = \frac{1}{2}t^2\sin^{-1}t - \frac{1}{4}\left(\theta - \frac{1}{2}\sin 2\theta\right) + C \\
&= \frac{1}{2}t^2\sin^{-1}t - \frac{1}{4}\sin^{-1}t + \frac{1}{4}\sin\theta\cos\theta + C \\
&= \frac{1}{2}t^2\sin^{-1}t - \frac{1}{4}\sin^{-1}t + \frac{1}{4}t\sqrt{1-t^2} + C
\end{aligned}$$

14. $\displaystyle\int \frac{\ln x}{(x-1)^2}\,dx$ $\boxed{u = \ln x,\ du = \frac{1}{x}\,dx; \quad dv = \frac{1}{(x-1)^2}\,dx,\ v = -\frac{1}{x-1}}$

$$= -\frac{\ln x}{x-1} + \int \frac{1}{x(x-1)}\,dx$$

Write $\dfrac{1}{x(x-1)} = \dfrac{A}{x} + \dfrac{B}{x-1}$. Then $1 = A(x-1) + Bx$. Setting $x = 0$ and $x = 1$ gives $A = -1$ and $B = 1$. Thus

$$\int \frac{\ln x}{(x-1)^2}\,dx = -\frac{\ln x}{x-1} - \int \frac{1}{x}\,dx + \int \frac{1}{x-1}\,dx$$

$$= -\frac{\ln x}{x-1} - \ln|x| + \ln|x-1| + C = -\frac{\ln x}{x-1} + \ln\left|\frac{x-1}{x}\right| + C.$$

15. $\displaystyle\int (x+1)^3(x-2)\,dx$ $\boxed{u = x+1,\ x = u-1,\ dx = du}$

$$= \int u^3(u-3)\,du = \int (u^4 - 3u^3)\,du$$

$$= \frac{1}{5}u^5 - \frac{3}{4}u^4 + C = \frac{1}{5}(x+1)^5 - \frac{3}{4}(x+1)^4 + C$$

16. Write $\dfrac{1}{(x+1)^3(x-2)} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{(x+1)^3} + \dfrac{D}{x-2}$.

Then $1 = A(x+1)^2(x-2) + B(x+1)(x-2) + C(x-2) + D(x+1)^3$

$= (A+D)x^3 + (B+3D)x^2 + (-3A - B + C + 3D)x + (-2A - 2B - 2C + D)$.

Solving $\boxed{\begin{array}{rr} A + D = 0 & B + 3D = 0 \\ -3A - B + C + 3D = 0 & -2A - 2B - 2C + D = 1 \end{array}}$

gives $A = -1/27$, $B = -1/9$, $C = -1/3$, and $D = 1/27$. Thus

$$\int \frac{1}{(x+1)^3(x-2)}\,dx = -\frac{1}{27}\int \frac{1}{x+1}\,dx - \frac{1}{9}\int \frac{1}{(x+1)^2}\,dx - \frac{1}{3}\int \frac{1}{(x+1)^3}\,dx + \frac{1}{27}\int \frac{1}{x-2}\,dx$$

$$= -\frac{1}{27}\ln|x+1| + \frac{1}{9}\left(\frac{1}{x+1}\right) + \frac{1}{6}\left[\frac{1}{(x+1)^2}\right] + \frac{1}{27}\ln|x-2| + C$$

$$= \frac{1}{27}\ln\left|\frac{x-2}{x+1}\right| + \frac{1}{9(x+1)} + \frac{1}{6(x+1)^2} + C.$$

17. $\displaystyle\int \ln(x^2+4)\,dx$ $\boxed{u = \ln(x^2+4),\ du = \frac{2x}{x^2+4}\,dx; \quad dv = dx,\ v = x}$

$$= x\ln(x^2+4) - \int \frac{2x^2}{x^2+4}\,dx = x\ln(x^2+4) - 2\int \frac{x^2+4-4}{x^2+4}\,dx$$

$$= x\ln(x^2+4) - 2\int dx + 8\int \frac{1}{x^2+4}\,dx = x\ln(x^2+4) - 2x + 4\tan^{-1}\frac{x}{2} + C$$

18. $\displaystyle\int 8te^{2t^2}$ $\quad\boxed{u = 2t^2,\ du = 4t\,dt}$

$$= \int 2e^u\,du = 2e^u + C = 2e^{2t^2} + C$$

19. Write $\dfrac{1}{x^4+10x^3+25x^2} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+5} + \dfrac{D}{(x+5)^2}$.

Then $1 = Ax(x+5)^2 + B(x+5)^2 + Cx^2(x+5) + Dx^2$

$= (A+C)x^3 + (10A+B+5C+D)x^2 + (25A+10B)x + 25B.$

Solving $\boxed{\begin{array}{ll} A + C = 0 & 10A + B + 5C + D = 0 \\ 25A + 10B = 0 & 25B = 1 \end{array}}$

gives $A = -2/125$, $B = 1/25$, $C = 2/125$, and $D = 1/25$. Thus

$$\begin{aligned}\int \frac{1}{x^4+10x^3+25x^2}\,dx &= -\frac{2}{125}\int\frac{1}{x}\,dx + \frac{1}{25}\int\frac{1}{x^2}\,dx + \frac{2}{125}\int\frac{1}{x+5}\,dx + \frac{1}{25}\int\frac{1}{(x+5)^2}\,dx \\ &= -\frac{2}{125}\ln|x| - \frac{1}{25}\left(\frac{1}{x}\right) + \frac{2}{125}\ln|x+5| - \frac{1}{25}\left(\frac{1}{x+5}\right) + C\end{aligned}$$

20. $\displaystyle\int \frac{1}{x^2+8x+25}\,dx = \int\frac{1}{(x+4)^2+9}\,dx = \frac{1}{3}\tan^{-1}\frac{x+4}{3} + C$

21. Write $\dfrac{x}{x^3+3x^2-9x-27} = \dfrac{A}{x+3} + \dfrac{B}{(x+3)^2} + \dfrac{C}{x-3}$.

Then $x = A(x^2-9) + B(x-3) + C(x+3)^2 = (A+C)x^2 + (B+6C)x + (-9A-3B+9C)$.

Solving $\boxed{A + C = 0 \qquad B + 6C = 1 \qquad -9A - 3B + 9C = 0}$

gives $A = -1/12$, $B = 1/2$, and $C = 1/12$. Thus

$$\begin{aligned}\int\frac{x}{x^3+3x^2-9x-27}\,dx &= -\frac{1}{12}\int\frac{1}{x+3}\,dx + \frac{1}{2}\int\frac{1}{(x+3)^2}\,dx + \frac{1}{12}\int\frac{1}{x-3}\,dx \\ &= -\frac{1}{12}\ln|x+3| - \frac{1}{2}\left(\frac{1}{x+3}\right) + \frac{1}{12}\ln|x-3| + C \\ &= \frac{1}{12}\ln\left|\frac{x-3}{x+3}\right| - \frac{1}{2(x+3)} + C.\end{aligned}$$

22. Write $\dfrac{x+1}{(x^2-x)(x^2+3)} = \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{Cx+D}{x^2+3}$.

Then $x + 1 = A(x-1)(x^2+3) + Bx(x^2+3) + (Cx+D)(x^2-x)$

$= (A+B+C)x^3 + (-A-C+D)x^2 + (3A+3B-D)x - 3A.$

Solving $\boxed{\begin{array}{ll} A + B + C = 0 & -A - C + D = 0 \\ 3A + 3B - D = 1 & -3A = 1 \end{array}}$

gives $A = -1/3$, $B = 1/2$, $C = -1/6$, and $D = -1/2$. Thus

$$\begin{aligned}\int \frac{x+1}{(x^2-x)(x^2+3)}\,dx &= -\frac{1}{3}\int \frac{1}{x}\,dx + \frac{1}{2}\int \frac{1}{x-1}\,dx - \frac{1}{6}\int \frac{x}{x^2+3}\,dx - \frac{1}{2}\int \frac{1}{x^2+3}\,dx \\ &= -\frac{1}{3}\ln|x| + \frac{1}{2}\ln|x-1| - \frac{1}{12}\ln(x^2+3) - \frac{1}{2\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} + C.\end{aligned}$$

23. $\displaystyle\int \frac{\sin^2 t}{\cos^2 t}\,dt = \int \tan^2 t\,dt = \int (\sec^2 t - 1)\,dt = \tan t - t + C$

24. $$\begin{aligned}\int \frac{\sin^3\theta}{(\cos\theta)^{3/2}}\,d\theta &= \int \frac{1-\cos^2\theta}{(\cos\theta)^{3/2}}\sin\theta\,d\theta \qquad \boxed{u=\cos\theta,\ du=-\sin\theta\,d\theta} \\ &= -\int \frac{1-u^2}{u^{3/2}}\,du = \int (u^{1/2} - u^{-3/2})\,du = \frac{u^{3/2}}{3/2} + \frac{u^{-1/2}}{1/2} + C \\ &= \frac{2}{3}(\cos\theta)^{3/2} + \frac{2}{(\cos\theta)^{1/2}} + C\end{aligned}$$

25. $$\begin{aligned}\int \tan^{10} x\sec^4 x\,dx &= \int \tan^{10} x(\tan^2 x+1)\sec^2 x\,dx \qquad \boxed{u=\tan x,\ du=\sec^2 x\,dx} \\ &= \int u^{10}(u^2+1)\,du = \int (u^{12}+u^{10})\,du = \frac{1}{13}u^{13} + \frac{1}{11}u^{11} + C \\ &= \frac{1}{13}\tan^{13} x + \frac{1}{11}\tan^{11} x + C\end{aligned}$$

26. Write $\dfrac{x\tan x}{\cos x} = \dfrac{x\sin x}{\cos^2 x}$. Now

$$\begin{aligned}&\int \frac{x\sin x}{\cos^2 x}\,dx \qquad \boxed{u=\frac{x}{\cos^2 x},\ du=\frac{\cos^2 x+2x\cos x\sin x}{\cos^4 x}\,dx;\quad dv=\sin x\,dx,\ v=-\cos x} \\ &= -\frac{x}{\cos x} + \int \frac{\cos^2 x + 2x\cos x\sin x}{\cos^3 x}\,dx = -\frac{x}{\cos x} + \int \sec x\,dx + 2\int \frac{x\sin x}{\cos^2 x}\,dx \\ &= -\frac{x}{\cos x} + \ln|\sec x + \tan x| + 2\int \frac{x\sin x}{\cos^2 x}\,dx\end{aligned}$$

Solving for the integral, $\displaystyle\int \frac{x\sin x}{\cos^2 x}\,dx = \frac{x}{\cos x} - \ln|\sec x+\tan x| + C.$

27. $$\begin{aligned}&\int y\cos y\,dy \qquad \boxed{u=y,\ du=dy;\quad dv=\cos y\,dy,\ v=\sin y} \\ &= y\sin y - \int \sin y\,dy = y\sin y + \cos y + C\end{aligned}$$

28. $\displaystyle\int x^2\sin x^3\,dx = -\frac{1}{3}\cos x^3 + C$

29. $$\int (1+\sin^2 t)\cos^3 t\,dt = \int (1+\sin^2 t)(1-\sin^2 t)\cos t\,dt = \int (1-\sin^4 t)\cos t\,dt$$
$$\boxed{u=\sin t,\ du=\cos t\,dt}$$
$$= \int (1-u^4)\,du = u - \frac{1}{5}u^5 + C = \sin t - \frac{1}{5}\sin^5 t + C$$

30. $$\int \frac{\sec^3\theta}{\tan\theta}\,d\theta = \int \frac{\sec^2\theta}{\tan^2\theta}\sec\theta\tan\theta\,d\theta = \int \frac{\sec^2\theta}{\sec^2\theta-1}\sec\theta\tan\theta\,d\theta$$
$$\boxed{u=\sec\theta,\ du=\sec\theta\tan\theta\,d\theta}$$
$$= \int \frac{u^2}{u^2-1}\,du = \int \frac{u^2-1+1}{u^2-1}\,du = \int du + \int \frac{1}{u^2-1}\,du$$

Write $\dfrac{1}{u^2-1} = \dfrac{A}{u-1} + \dfrac{B}{u+1}$. Then $1 = A(u+1)+B(u-1)$. Setting $u=1$ and $u=-1$ gives $A=1/2$ and $B=-1/2$. Thus

$$\int \frac{\sec^3\theta}{\tan\theta}\,d\theta = \int du + \frac{1}{2}\int \frac{1}{u-1}\,du - \frac{1}{2}\int \frac{1}{u+1}\,du = u + \frac{1}{2}\ln|u-1| - \frac{1}{2}\ln|u+1| + C$$
$$= u + \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + C = \sec\theta + \frac{1}{2}\ln\left|\frac{\sec\theta-1}{\sec\theta+1}\right| + C.$$

31. $$\int e^w(1+e^w)^5\,dw \qquad \boxed{u=1+e^w,\ du=e^w\,dw}$$
$$= \int u^5\,du = \frac{1}{6}u^6 + C = \frac{1}{6}(1+e^w)^6 + C$$

32. $$\int (x-1)e^{-x}\,dx \qquad \boxed{u=x-1,\ du=dx;\quad dv=e^{-x}\,dx,\ v=-e^{-x}}$$
$$= -(x-1)e^{-x} + \int e^{-x}\,dx = -(x-1)e^{-x} - e^{-x} + C$$

33. $$\int \cot^3 4x\,dx = \int (\csc^2 4x - 1)\cot 4x\,dx = \int \csc 4x\csc 4x\cot 4x\,dx - \int \cot 4x\,dx$$
$$\boxed{u=\csc 4x,\ du=-4\csc 4x\cot 4x\,dx}$$
$$= -\frac{1}{4}\int u\,du - \frac{1}{4}\ln|\sin 4x| + C = -\frac{1}{8}u^2 - \frac{1}{4}\ln|\sin 4x| + C$$
$$= -\frac{1}{8}\csc^2 4x - \frac{1}{4}\ln|\sin 4x| + C$$

34. $$\int (3-\sec x)^2\,dx = \int (9 - 6\sec x + \sec^2 x)\,dx = 9 - 6\ln|\sec x + \tan x| + \tan x + C$$

35. $$\int_0^{\pi/4} \cos^2 x\tan x\,dx = \int_0^{\pi/4}\cos x\sin x\,dx = \frac{1}{2}\int_0^{\pi/4}\sin 2x\,dx = -\frac{1}{4}\cos 2x\Big]_0^{\pi/4}$$
$$= -\frac{1}{4}(0-1) = \frac{1}{4}$$

36. $$\int_0^{\pi/3} \sin^4 x \tan x\,dx = \int_0^{\pi/3} (1-\cos^2 x)^2 \frac{\sin x}{\cos x}\,dx \qquad \boxed{u = \cos x,\ du = -\sin x\,dx}$$
$$= -\int_1^{1/2} \frac{(1-u^2)^2}{u}\,du = -\int_1^{1/2} \left(\frac{1}{u} - 2u + u^3\right) du$$
$$= -\left(\ln|u| - u^2 + \frac{1}{4}u^4\right)\Bigg]_1^{1/2} = -\left[\left(\ln\frac{1}{2} - \frac{15}{64}\right) - \left(-\frac{3}{4}\right)\right] = \ln 2 - \frac{33}{64}$$

37. $$\int \frac{\sin x}{1+\sin x}\,dx = \int \frac{(\sin x)(1-\sin x)}{(1+\sin x)(1-\sin x)}\,dx = \int \frac{\sin x - \sin^2 x}{1-\sin^2 x}\,dx$$
$$= \int \left(\frac{\sin x}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x}\right) dx = \int \left[\frac{1}{\cos x}\left(\frac{\sin x}{\cos x}\right) - \tan^2 x\right] dx$$
$$= \int [\sec x \tan x - (\sec^2 x - 1)]\,dx = \sec x - \tan x + x + C$$

Alternatively, the substitution $u = \tan \dfrac{x}{2}$ leads to the equivalent solution

$$\int \frac{\sin x}{1+\sin x}\,dx = x + \frac{2}{1+\tan x/2} + C.$$

38. $$\int \frac{\cos x}{1+\sin x}\,dx \qquad \boxed{u = 1+\sin x,\ du = \cos x\,dx}$$
$$= \int \frac{1}{u}\,du = \ln|u| + C = \ln(1+\sin x) + C$$

39. Write $\dfrac{1}{(x+1)(x+2)(x+3)} = \dfrac{A}{x+1} + \dfrac{B}{x+2} + \dfrac{C}{x+3}$.

Then $1 = A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)$.

Setting $x = -1$, $x = -2$, and $x = -3$ gives $A = 1/2$, $B = -1$, and $C = 1/2$. Thus

$$\int_0^1 \frac{1}{(x+1)(x+2)(x+3)}\,dx = \frac{1}{2}\int_0^1 \frac{1}{x+1}\,dx - \int_0^1 \frac{1}{x+2}\,dx + \frac{1}{2}\int_0^1 \frac{1}{x+3}\,dx$$
$$= \frac{1}{2}\ln|x+1|\Big]_0^1 - \ln|x+2|\Big]_0^1 + \frac{1}{2}\ln|x+3|\Big]_0^1$$
$$= \frac{1}{2}(\ln 2 - \ln 1) - (\ln 3 - \ln 2) + \frac{1}{2}(\ln 4 - \ln 3) = \frac{5}{2}\ln 2 - \frac{3}{2}\ln 3.$$

40. $\displaystyle\int_{\ln 3}^{\ln 2} \sqrt{e^x+1}\,dx$ $\quad \boxed{u=\sqrt{e^x+1},\ x=\ln(u^2-1),\ dx=\dfrac{2u}{u^2-1}\,du}$

$$\begin{aligned}
&= \int_2^{\sqrt{3}} \frac{2u^2}{u^2-1}\,du = 2\int_2^{\sqrt{3}} \frac{u^2-1+1}{u^2-1}\,du = 2\int_2^{\sqrt{3}} du + 2\int_2^{\sqrt{3}} \frac{1}{u^2-1}\,du \\
&= 2u\Big]_2^{\sqrt{3}} + 2\int_2^{\sqrt{3}} \frac{1/2}{u-1}\,du + 2\int_2^{\sqrt{3}} \frac{-1/2}{u+1}\,du \\
&= 2\sqrt{3}-4+\ln|u-1|\Big]_2^{\sqrt{3}} - \ln|u+1|\Big]_2^{\sqrt{3}} \\
&= 2\sqrt{3}-4+\ln(\sqrt{3}-1)-[\ln(\sqrt{3}+1)-\ln 3] = 2\sqrt{3}-4+\ln\frac{\sqrt{3}-1}{\sqrt{3}+1}+\ln 3
\end{aligned}$$

41. $\displaystyle\int e^x\cos 3x\,dx$ $\quad \boxed{u=e^x,\ du=e^x\,dx;\quad dv=\cos 3x\,dx,\ v=\dfrac{1}{3}\sin 3x}$

$$= \frac{1}{3}e^x\sin 3x - \frac{1}{3}\int e^x\sin 3x\,dx$$

$$\boxed{u=e^x,\ du=e^x\,dx;\quad dv=\sin 3x\,dx,\ v=-\frac{1}{3}\cos 3x}$$

$$= \frac{1}{3}e^x\sin 3x - \frac{1}{3}\left(-\frac{1}{3}e^x\cos 3x + \frac{1}{3}\int e^x\cos 3x\,dx\right)$$

Solving for the integral, $\displaystyle\int e^x\cos 3x\,dx = \frac{3}{10}e^x\sin 3x + \frac{1}{10}e^x\cos 3x + C.$

42. $\displaystyle\int x(x-5)^9\,dx$ $\quad \boxed{u=x-5,\ du=dx}$

$$\begin{aligned}
&= \int (u+5)u^9\,du = \int (u^{10}+5u^9)\,du = \frac{1}{11}u^{11}+\frac{1}{2}u^{10}+C \\
&= \frac{1}{11}(x-5)^{11}+\frac{1}{2}(x-5)^{10}+C
\end{aligned}$$

43. $\displaystyle\int \cos(\ln t)\,dt$ $\quad \boxed{u=\cos(\ln t),\ du=-\dfrac{\sin(\ln t)}{t}\,dt;\quad dv=dt,\ v=t}$

$$= t\cos(\ln t) + \int \sin(\ln t)\,dt$$

$$\boxed{u=\sin(\ln t),\ du=\frac{\cos(\ln t)}{t}\,dt;\quad dv=dt,\ v=t}$$

$$= t\cos(\ln t) + t\sin(\ln t) - \int \cos(\ln t)\,dt$$

Solving for the integral, $\displaystyle\int \cos(\ln t)\,dt = \frac{1}{2}t\cos(\ln t) + \frac{1}{2}t\sin(\ln t) + C.$

44. $\displaystyle\int \sec^2 x \ln(\tan x)\,dx \qquad \boxed{u = \ln(\tan x),\ du = \frac{\sec^2 x}{\tan x}\,dx; \quad dv = \sec^2 x\,dx,\ v = \tan x}$

$$= \tan x \ln(\tan x) - \int \sec^2 x\,dx = \tan x \ln(\tan x) - \tan x + C$$

45. $\displaystyle\int \cos\sqrt{x}\,dx \qquad \boxed{t = \sqrt{x},\ x = t^2,\ dx = 2t\,dt}$

$$= 2\int t\cos t\,dt \qquad \boxed{u = t,\ du = dt; \quad dv = \cos t\,dt,\ v = \sin t}$$

$$= 2t\sin t - 2\int \sin t\,dt = 2t\sin t + 2\cos t + C$$

$$= 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$$

46. $\displaystyle\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx \qquad \boxed{u = \sqrt{x},\ du = \frac{1}{2\sqrt{x}}\,dx} \qquad = 2\int \cos u\,du = 2\sin u + C = 2\sin\sqrt{x} + C$

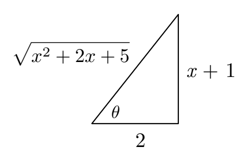
47. $\displaystyle\int \cos x \sin 2x\,dx = 2\int \cos^2 x \sin x\,dx \qquad \boxed{u = \cos x,\ du = -\sin x\,dx}$

$$= -2\int u^2\,du = -\frac{2}{3}u^3 + C = -\frac{2}{3}\cos^3 x + C$$

48. $\displaystyle\int (\cos^2 x - \sin^2 x)\,dx = \int \cos 2x\,dx = \frac{1}{2}\sin 2x + C$

49. $\displaystyle\int \sqrt{x^2 + 2x + 5}\,dx = \int \sqrt{(x+1)^2 + 4}\,dx \qquad \boxed{x + 1 = 2\tan\theta,\ dx = 2\sec^2\theta\,d\theta}$

$$= \int \sqrt{4\tan^2\theta + 4}(2\sec^2\theta\,d\theta) = 4\int \sec^3\theta\,d\theta$$



$\boxed{\text{See Section 7.3, Example 5}}$

$$= 2\sec\theta\tan\theta + 2\ln|\sec\theta + \tan\theta| + C$$

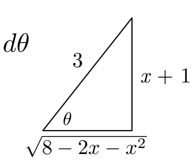
$$= \frac{1}{2}(x+1)\sqrt{x^2 + 2x + 5} + 2\ln\left|\frac{\sqrt{x^2 + 2x + 5}}{2} + \frac{x+1}{2}\right| + C$$

$$= \frac{1}{2}(x+1)\sqrt{x^2 + 2x + 5} + 2\ln\left|\sqrt{x^2 + 2x + 5} + x + 1\right| + C_1$$

50. $\displaystyle\int \frac{1}{(8 - 2x - x^2)^{3/2}}\,dx = \int \frac{1}{[9 - (x+1)^2]^{3/2}}\,dx \qquad \boxed{x + 1 = 3\sin\theta,\ dx = 3\cos\theta\,d\theta}$

$$= \int \frac{3\cos\theta}{(9 - 9\sin^2\theta)^{3/2}}\,d\theta = \frac{1}{9}\int \frac{\cos\theta}{\cos^3\theta}\,d\theta = \frac{1}{9}\int \sec^2\theta\,d\theta$$

$$= \frac{1}{9}\tan\theta + C = \frac{x+1}{9\sqrt{8 - 2x - x^2}} + C$$



51. $$\int \tan^5 x \sec^3 x\,dx = \int \tan^4 x \sec^2 x \sec x \tan x\,dx = \int (\sec^2 x - 1)^2 \sec^2 x \sec x \tan x\,dx$$
$$\boxed{u = \sec x,\ du = \sec x \tan x\,dx}$$
$$= \int (u^2-1)^2 u^2\,du = \int (u^6 - 2u^4 + u^2)\,du = \frac{1}{7}u^7 - \frac{2}{5}u^5 + \frac{1}{3}u^3 + C$$
$$= \frac{1}{7}\sec^7 x - \frac{2}{5}\sec^5 x + \frac{1}{3}\sec^3 x + C$$

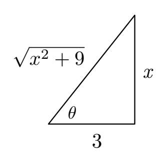
52. $$\int \cos^4 \frac{x}{2}\,dx = \frac{1}{4}\int (1+\cos x)^2\,dx = \frac{1}{4}\int (1 + 2\cos x + \cos^2 x)\,dx$$
$$= \frac{1}{4}\int \left(1 + 2\cos x + \frac{1+\cos 2x}{2}\right)dx = \frac{1}{4}\int \left(\frac{3}{2} + 2\cos x + \frac{1}{2}\cos 2x\right)dx$$
$$= \frac{3}{8}x + \frac{1}{2}\sin x + \frac{1}{16}\sin 2x + C$$

53. $$\int \frac{t^5}{1+t^2}\,dt = \int \left(t^3 - t + \frac{t}{1+t^2}\right)dt = \frac{1}{4}t^4 - \frac{1}{2}t^2 + \frac{1}{2}\ln(1+t^2) + C$$

54. $$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$$

55. $$\int \frac{5x^3 + x^2 + 6x + 1}{(x^2+1)^2}\,dx = \int \frac{(5x+1)(x^2+1) + x}{(x^2+1)^2}\,dx = \int \frac{5x+1}{x^2+1}\,dx + \int \frac{x}{(x^2+1)^2}\,dx$$
$$= \int \frac{5x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx + \int \frac{x}{(x^2+1)^2}\,dx$$
$$= \frac{5}{2}\ln(x^2+1) + \tan^{-1} x - \frac{1}{2(x^2+1)} + C$$

56. $$\int \frac{\sqrt{x^2+9}}{x^2}\,dx \qquad \boxed{x = 3\tan\theta,\ dx = 3\sec^2\theta\,d\theta}$$



$$= \int \frac{\sqrt{9\tan^2\theta + 9}}{9\tan^2\theta}(3\sec^2\theta\,d\theta) = \int \frac{\sec^3\theta}{\tan^2\theta}\,d\theta = \int \frac{\sec^3\theta}{\sec^2\theta - 1}\,d\theta$$
$$= \int \left(\sec\theta + \frac{\sec\theta}{\sec^2\theta - 1}\right)d\theta = \ln|\sec\theta + \tan\theta| + \int \frac{\sec\theta}{\tan^2\theta}\,d\theta$$
$$= \ln|\sec\theta + \tan\theta| + \int \frac{\cos\theta}{\sin^2\theta}\,d\theta \qquad \boxed{u = \sin\theta,\ du = \cos\theta\,d\theta}$$
$$= \ln|\sec\theta + \tan\theta| + \int \frac{1}{u^2}\,du = \ln|\sec\theta + \tan\theta| - \frac{1}{u} + C$$
$$= \ln|\sec\theta + \tan\theta| - \frac{1}{\sin\theta} + C = \ln\left|\frac{\sqrt{x^2+9}}{3} + \frac{x}{3}\right| - \frac{\sqrt{x^2+9}}{x} + C$$
$$= \ln\left|\sqrt{x^2+9} + x\right| - \frac{\sqrt{x^2+9}}{x} + C_1$$

57. $\displaystyle\int x\sin^2 x\,dx$ $\quad\boxed{u = x\sin x,\ du = (x\cos x + \sin x)\,dx;\quad dv = \sin x\,dx,\ v = -\cos x}$

$$\begin{aligned}
&= -x\sin x\cos x + \int x\cos^2 x\,dx + \int \sin x\cos x\,dx \\
&= -\frac{1}{2}x\sin 2x + \int x(1-\sin^2 x)\,dx + \frac{1}{2}\int \sin 2x\,dx \\
&= -\frac{1}{2}x\sin 2x + \frac{1}{2}x^2 - \int x\sin^2 x\,dx - \frac{1}{4}\cos 2x
\end{aligned}$$

Solving for the integral, $\displaystyle\int x\sin^2 x\,dx = \frac{1}{4}x^2 - \frac{1}{4}x\sin 2x - \frac{1}{8}\cos 2x + C.$

58. $\displaystyle\int (t+1)^2 e^{3t}\,dt$ $\quad\boxed{u = (t+1)^2,\ du = 2(t+1)\,dt;\quad dv = e^{3t}\,dt,\ v = \frac{1}{3}e^{3t}}$

$$= \frac{1}{3}(t+1)^2e^{3t} - \frac{2}{3}\int (t+1)e^{3t}\,dt$$

$$\boxed{u = t+1,\ du = dt;\quad dv = e^{3t}\,dt,\ v = \frac{1}{3}e^{3t}}$$

$$\begin{aligned}
&= \frac{1}{3}(t+1)^2e^{3t} - \frac{2}{3}\left[\frac{1}{3}(t+1)e^{3t} - \frac{1}{3}\int e^{3t}\,dt\right] \\
&= \frac{1}{3}(t+1)^2e^{3t} - \frac{2}{9}(t+1)e^{3t} + \frac{2}{27}e^{3t} + C
\end{aligned}$$

59. $\displaystyle\int e^{\sin x}\sin 2x\,dx = \int e^{\sin x}(2\sin x\cos x)\,dx = 2\int e^{\sin x}\sin x\cos x\,dx$

$$\boxed{u = \sin x,\ du = \cos x\,dx;\quad dv = e^{\sin x}\cos x,\ v = e^{\sin x}}$$

$$= 2e^{\sin x}\sin x - 2\int e^{\sin x}\cos x\,dx = 2e^{\sin x}\sin x - 2e^{\sin x} + C$$

60. $\displaystyle\int e^x\tan^2 e^x\,dx$ $\quad\boxed{u = e^x,\ du = e^x\,dx}$

$$= \int \tan^2 u\,du = \int (\sec^2 u - 1)\,du = \tan u - u + C = \tan e^x - e^x + C$$

61. $\displaystyle\int_0^{\pi/6} \frac{\cos x}{\sqrt{1+\sin x}}\,dx$ $\quad\boxed{u = 1+\sin x,\ du = \cos x\,dx}$

$$= \int_1^{3/2} \frac{1}{\sqrt{u}}\,du = 2\sqrt{u}\,\Big]_1^{3/2} = 2\sqrt{3/2} - 2 = \sqrt{6} - 2$$

62. Using *Mathematica* we find $\displaystyle\int_0^{\pi/2} \frac{1}{\sin x + \cos x}\,dx = \frac{1}{\sqrt{2}}\ln\frac{1+\sqrt{2}}{-1+\sqrt{2}}.$

63. $\displaystyle\int \sinh^{-1} t\,dt$ $\quad\boxed{u = \sinh^{-1} t,\ du = \frac{1}{\sqrt{t^2+1}}\,dt;\quad dv = dt,\ v = t}$

$$= t\sinh^{-1} t - \int \frac{t}{\sqrt{t^2+1}}\,dt = t\sinh^{-1} t - \sqrt{t^2+1} + C$$

64. $\displaystyle\int x\cot x^2\,dx = \frac{1}{2}\ln|\sin x^2| + C$

65. $\displaystyle\int_3^8 \frac{1}{x\sqrt{x+1}}\,dx \qquad \boxed{u^2 = x+1,\ 2u\,du = dx}$

$$= \int_2^3 \frac{2u}{(u^2-1)u}\,du = \int_2^3 \frac{2}{(u-1)(u+1)}\,du = \int_2^3 \frac{1}{u-1}\,du - \int_2^3 \frac{1}{u+1}\,du$$

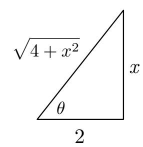
$$= \ln|u-1|\Big]_2^3 - \ln|u+1|\Big]_2^3 = \ln 2 - \ln\frac{4}{3} = \ln\frac{3}{2}$$

66. $\displaystyle\int \frac{t+3}{t^2+2t+1}\,dt = \int \frac{t+1+2}{(t+1)^2}\,dt = \int \frac{1}{t+1}\,dt + \int \frac{2}{(t+1)^2}\,dt = \ln|t+1| - \frac{2}{t+1} + C$

67. $\displaystyle\int \frac{\sec^4 3u}{\cot^{12} 3u}\,du = \int \tan^{12} 3u\sec^2 3u\sec^2 3u\,du = \int \tan^{12} 3u(1+\tan^2 3u)\sec^2 3u\,du$

$$= \frac{1}{39}\tan^{13} 3u + \frac{1}{45}\tan^{15} 3u + C$$

68. $\displaystyle\int_0^2 x^5\sqrt{x^2+4}\,dx \qquad \boxed{x = 2\tan\theta,\ dx = 2\sec^2\theta\,d\theta}$



$$= \int_0^{\pi/4} 32\tan^5\theta\sqrt{4\tan^2\theta+4}\;2\sec^2\theta\,d\theta$$

$$= 128\int_0^{\pi/4} \tan^5\theta\sec^3\theta\,d\theta = 128\int_0^{\pi/4} \tan^4\theta\sec^2\theta\tan\theta\sec\theta\,d\theta$$

$$= 128\int_0^{\pi/4} (\sec^2\theta-1)^2\sec^2\theta\tan\theta\sec\theta\,d\theta$$

$$= 128\int_0^{\pi/4} (\sec^6\theta - 2\sec^4\theta + \sec^2\theta)\tan\theta\sec\theta\,d\theta$$

$$= 128\left(\frac{1}{7}\sec^7\theta - \frac{2}{5}\sec^5\theta + \frac{1}{3}\sec^3\theta\right)\Bigg]_0^{\pi/4}$$

$$= 128\left\{\left[\frac{(\sqrt{2})^7}{7} - \frac{2(\sqrt{2})^5}{5} + \frac{(\sqrt{2})^3}{3}\right] - \left(\frac{1}{7} - \frac{2}{5} + \frac{1}{3}\right)\right\}$$

$$= 128\left(\frac{22}{105}\sqrt{2} - \frac{8}{105}\right) = \frac{256}{105}(11\sqrt{2}-4)$$

69. $\displaystyle\int \frac{3+\sin x}{\cos^2 x}\,dx = \int 3\sec^2 x\,dx + \int \tan x\sec x\,dx = 3\tan x + \sec x + C$

70. $\displaystyle\int \frac{\sin 2x}{5+\cos^2 x}\,dx = \int \frac{2\sin x\cos x}{5+\cos^2 x}\,dx \qquad \boxed{u = \cos x,\ du = -\sin x\,dx}$

$$= -\int \frac{2u}{5+u^2}\,du = -\ln(5+u^2) + C = -\ln(5+\cos^2 x) + C$$

71. $\displaystyle\int x(1+\ln x)^2\,dx \quad \boxed{u=(1+\ln x)^2,\ du=\frac{2(1+\ln x)}{x}\,dx;\quad dv=x\,dx,\ v=\frac{1}{x^2}}$

$$= \frac{1}{2}x^2(1+\ln x)^2 - \int x(1+\ln x)\,dx$$

$$\boxed{u=1+\ln x,\ du=\frac{1}{x}\,dx;\quad dv=x\,dx,\ v=\frac{1}{2}x^2}$$

$$= \frac{1}{2}x^2(1+\ln x)^2 - \left[\frac{1}{2}x^2(1+\ln x) - \int \frac{1}{2}x\,dx\right]$$

$$= \frac{1}{2}x^2(1+\ln x)\ln x - \frac{1}{2}x^2(1+\ln x) + \frac{1}{4}x^2 + C$$

72. $\displaystyle\int x\cos^2 x\,dx = \frac{1}{2}\int x(1+\cos 2x)\,dx = \frac{1}{4}x^2 + \frac{1}{2}\int x\cos 2x\,dx$

$$\boxed{u=x,\ du=dx;\quad dv=\cos 2x\,dx,\ v=\frac{1}{2}\sin 2x}$$

$$= \frac{1}{4}x^2 + \frac{1}{2}\left(\frac{1}{2}x\sin 2x - \frac{1}{2}\int \sin 2x\,dx\right) = \frac{1}{4}x^2 + \frac{1}{4}x\sin 2x + \frac{1}{8}\cos 2x + C$$

73. $\displaystyle\int e^x e^{e^x}\,dx \quad \boxed{u=e^x,\ du=e^x\,dx} \quad = \int e^u\,du = e^u + C = e^{e^x} + C$

74. $\displaystyle\int \frac{1}{\sqrt{x+1}-\sqrt{x}}\,dx = \int \frac{\sqrt{x+1}+\sqrt{x}}{x+1-x}\,dx = \int(\sqrt{x+1}-\sqrt{x})\,dx$

$$= \frac{2}{3}(x+1)^{3/2} + \frac{2}{3}x^{3/2} + C$$

75. $\displaystyle\int \frac{2t}{1+e^{t^2}}\,dt \quad \boxed{u=t^2,\ du=2t\,dt}$

$$= \int \frac{1}{1+e^u}\,du = \int \frac{e^{-u}}{e^{-u}+1}\,du \qquad \boxed{v=e^{-u},\ dv=e^{-u}\,du}$$

$$= \int \frac{1}{v+1}(-dv) = -\ln|v+1| + C = -\ln(e^{-u}+1) + C = -\ln(e^{-t^2}+1) + C$$

76. $\displaystyle\int \cos x\cos 2x\,dx = \int \cos x(1-2\sin^2 x)\,dx = \int \cos x\,dx - 2\int \sin^2 x\cos x\,dx$

$$= \sin x - \frac{2}{3}\sin^3 x + C$$

77. $\displaystyle\int \frac{1}{\sqrt{1-(5x+2)^2}}\,dx = \frac{1}{5}\int \frac{5}{\sqrt{1-(5x+2)^2}}\,dx = \frac{1}{5}\sin^{-1}(5x+2) + C$

78. $\displaystyle\int (\ln 2x)\ln x\,dx \quad \boxed{u=\ln 2x,\ du=\frac{1}{x}\,dx;\quad dv=\ln x\,dx,\ v=x\ln x - x}$

$$= (\ln 2x)(x\ln x - x) - \int(\ln x - 1)\,dx = (\ln 2x)(x\ln x - x) - \int \ln x\,dx + x$$

$$= (\ln 2x)(x\ln x - x) - x\ln x + 2x + C$$

79. $\displaystyle\int \cos x \ln|\sin x|\,dx$ $\quad \boxed{u = \ln|\sin x|,\ du = \frac{\cos x}{\sin x}\,dx;\quad dv = \cos x\,dx,\ v = \sin x}$

$$= \sin x \ln|\sin x| - \int \cos x\,dx = \sin x \ln|\sin x| - \sin x + C$$

80. $\displaystyle\int \ln\left(\frac{x+1}{x-1}\right)dx = \int \ln(x+1)\,dx - \int \ln(x-1)\,dx$

$$\boxed{u = \ln(x+1),\ du = \frac{1}{x+1}\,dx;\quad dv = dx,\ v = x}$$

$$\boxed{u = \ln(x-1),\ du = \frac{1}{x-1}\,dx;\quad dv = dx,\ v = x}$$

$$= x\ln(x+1) - \int \frac{x}{x+1}\,dx - x\ln(x-1) + \int \frac{x}{x-1}\,dx$$

$$= x\ln\left(\frac{x+1}{x-1}\right) - \int \frac{x+1-1}{x+1}\,dx + \int \frac{x-1+1}{x-1}\,dx$$

$$= x\ln\left(\frac{x+1}{x-1}\right) - \int \left(1 - \frac{1}{x+1}\right)dx + \int \left(1 + \frac{1}{x-1}\right)dx$$

$$= x\ln\left(\frac{x+1}{x-1}\right) - x + \ln(x+1) + x + \ln(x-1) + C$$

$$= x\ln\left(\frac{x+1}{x-1}\right) + \ln(x^2-1) + C$$

81. $\displaystyle\int_0^3 x(x^2-9)^{-2/3}\,dx = \lim_{t\to 3^-}\int_0^t x(x^2-9)^{-2/3}\,dx = \lim_{t\to 3^-}\frac{3}{2}(x^2-9)^{1/3}\Big]_0^t$

$$= \lim_{t\to 3^-}\left[\frac{3}{2}(t^2-9)^{1/3} - \frac{3}{2}(-9)^{1/3}\right] = \frac{3\sqrt[3]{9}}{2}$$

82. $\displaystyle\int_0^5 x(x^2-9)^{-2/3}\,dx = \lim_{t\to 3^-}\int_0^t x(x^2-9)^{-2/3}\,dx + \lim_{s\to 3^+}\int_s^5 x(x^2-9)^{-2/3}\,dx$

$$= \lim_{t\to 3^-}\frac{3}{2}(x^2-9)^{1/3}\Big]_0^t + \lim_{s\to 3^+}\frac{3}{2}(x^2-9)^{1/3}\Big]_s^5$$

$$= \lim_{t\to 3^-}\left[\frac{3}{2}(t^2-9)^{1/3} - \frac{3}{2}(-9)^{1/3}\right] + \lim_{s\to 3^+}\left[\frac{3}{2}(16)^{1/3} - \frac{3}{2}(s^2-9)^{1/3}\right]$$

$$= \frac{3\sqrt[3]{9}}{2} + 3\sqrt[3]{2}$$

83. $\displaystyle\int_{-\infty}^{0}(x+1)e^x\,dx = \lim_{s\to-\infty}\int_s^0 (x+1)e^x\,dx \qquad \boxed{u = x+1,\ du = dx;\quad dv = e^x\,dx,\ v = e^x}$

$$= \lim_{s\to-\infty}\left[(x+1)e^x\Big]_s^0 - \int_s^0 e^x\,dx\right] = \lim_{s\to-\infty}\left[1-(s+1)e^s - e^x\Big]_s^0\right]$$

$$= \lim_{s\to-\infty}[1-(s+1)e^s - 1 + e^s] = \lim_{s\to-\infty}(-se^s) \qquad \boxed{\text{Let } t = -s}$$

$$= \lim_{t\to\infty} te^{-t} = \lim_{t\to\infty}\frac{t}{e^t} \overset{h}{=} \lim_{t\to\infty}\frac{1}{e^t} = 0$$

84. $\displaystyle\int_0^\infty \frac{e^{2x}}{e^{4x}+1}\,dx = \lim_{t\to\infty}\int_0^t \frac{e^{2x}}{e^{4x}+1}\,dx \qquad \boxed{u = e^{2x},\ du = 2e^{2x}\,dx}$

$$= \lim_{t\to\infty}\int_1^{e^{2t}} \frac{1/2}{u^2+1}\,du = \lim_{t\to\infty}\frac{1}{2}\tan^{-1}u\Big]_1^{e^{2t}} = \lim_{t\to\infty}\left[\frac{1}{2}\tan^{-1}e^{2t} - \frac{1}{2}\left(\frac{\pi}{4}\right)\right]$$

$$= \frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{1}{2}\left(\frac{\pi}{4}\right) = \frac{\pi}{8}$$

85. $\displaystyle\int_3^\infty \frac{1}{1+5x}\,dx = \lim_{t\to\infty}\int_3^t \frac{1}{1+5x}\,dx = \lim_{t\to\infty}\frac{1}{5}\ln|1+5x|\Big]_3^t = \lim_{t\to\infty}\left(\frac{1}{5}\ln|1+5t| - \frac{1}{5}\ln 16\right)$

The integral diverges.

86. $\displaystyle\int_0^\infty \frac{x}{(x^2+4)^2}\,dx = \lim_{t\to\infty}\int_0^t \frac{x}{(x^2+4)^2}\,dx = \lim_{t\to\infty}\left(-\frac{1}{2}\cdot\frac{1}{x^2+4}\right)\Big]_0^t = \lim_{t\to\infty}\left(\frac{1}{8} - \frac{1}{2t^2+8}\right) = \frac{1}{8}$

87. $\displaystyle\int_0^e \ln\sqrt{x}\,dx = \lim_{s\to0^+}\int_s^e \frac{1}{2}\ln x\,dx = \lim_{s\to0^+}\frac{1}{2}(x\ln x - x)\Big]_s^e = \lim_{s\to0^+}\left[\left(\frac{e}{2}-\frac{e}{2}\right) - \frac{1}{2}(s\ln s - s)\right]$

$$= \lim_{s\to0^+}\left(\frac{1-\ln s}{2/s}\right) \overset{h}{=} \lim_{s\to0^+}\frac{-1/s}{-2/s^2} = \lim_{s\to0^+} s = 0$$

88. $\displaystyle\int_0^{\pi/2}\frac{\sec^2 t}{\tan^3 t}\,dt = \int_0^{\pi/2}\frac{\cos t}{\sin^3 t}\,dt = \lim_{s\to0^+}\int_s^{\pi/2}(\sin t)^{-3}\cos t\,dt = \lim_{s\to0^+}\left[-\frac{1}{2}(\sin t)^{-2}\right]\Bigg|_s^{\pi/2}$

$$= \lim_{s\to0^+}\left(-\frac{1}{2\sin^2 t}\right)\Big]_s^{\pi/2} = \lim_{s\to0^+}\left(\frac{1}{2\sin^2 s} - \frac{1}{2}\right)$$

The integral diverges.

89. $\displaystyle\int_0^{\pi/2}\frac{1}{1-\cos x}\,dx = \lim_{s\to0^+}\int_s^{\pi/2}\frac{1}{1-\cos x}\left(\frac{1+\cos x}{1+\cos x}\right)dx = \lim_{s\to0^+}\int_s^{\pi/2}\frac{1+\cos x}{\sin^2 x}\,dx$

$$= \lim_{s\to0^+}\int_s^{\pi/2}(\csc^2 x + \cot x\csc x)\,dx = \lim_{s\to0^+}(-\cot x - \csc x)\Big]_s^{\pi/2}$$

$$= \lim_{s\to0^+}(\cot s + \csc s - 0 - 1)$$

Since $\displaystyle\lim_{s\to0^+}\cot s = +\infty$ and $\displaystyle\lim_{s\to0^+}\csc s = +\infty$, the integral diverges.

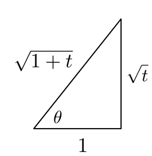
90. $$\int_0^\infty \frac{x}{x+1}\,dx = \lim_{t\to\infty}\int_0^t \left(1-\frac{1}{x+1}\right)dx = \lim_{t\to\infty}\left(x-\ln|x+1|\right)\Big]_0^t = \lim_{t\to\infty}[t-\ln(t+1)]$$
$$= \lim_{t\to\infty}[\ln e^t - \ln(t+1)] = \lim_{t\to\infty}\ln\frac{e^t}{t+1} = \ln\left(\lim_{t\to\infty}\frac{e^t}{t+1}\right) \overset{h}{=} \ln\left(\lim_{t\to\infty}\frac{e^t}{1}\right)$$

The integral diverges.

91. $$\int_0^1 \frac{1}{\sqrt{x}e^{\sqrt{x}}}\,dx = \lim_{s\to0^+}\int_s^1 x^{-1/2}e^{-x^{1/2}}\,dx \qquad \boxed{u=-x^{1/2},\ du=-\frac{1}{2}x^{-1/2}\,dx}$$
$$= \lim_{s\to0^+}\int_{-\sqrt{s}}^{-1} -2e^u\,du = \lim_{s\to0^+}\left(-2e^u\right)\Big]_{-\sqrt{s}}^{-1} = \lim_{s\to0^+}\left(2e^{-\sqrt{s}}-2e^{-1}\right) = 2-2e^{-1}$$

92. $$\int_0^\infty \frac{1}{\sqrt{x}e^{\sqrt{x}}}\,dx = \lim_{s\to0^+}\int_s^1 x^{-1/2}e^{-x^{1/2}}\,dx + \lim_{t\to\infty}\int_1^t x^{-1/2}e^{-x^{1/2}}\,dx$$
$$\boxed{u=-x^{1/2},\ du=-\frac{1}{2}x^{-1/2}\,dx} \quad = \lim_{s\to0^+}\int_{-\sqrt{s}}^{-1} -2e^u\,du + \lim_{t\to\infty}\int_{-1}^{-\sqrt{t}} -2e^u\,du$$
$$= \lim_{s\to0^+}\left(-2e^u\right)\Big]_{-\sqrt{s}}^{-1} + \lim_{t\to\infty}\left(-2e^u\right)\Big]_{-1}^{-\sqrt{t}}$$
$$= \lim_{s\to0^+}\left(2e^{-\sqrt{s}}-2e^{-1}\right) + \lim_{t\to\infty}\left(2e^{-1}-2e^{-\sqrt{t}}\right) = 2-2e^{-1}+2e^{-1} = 2$$

93. $$\int_1^\infty \frac{\sqrt{x}}{(1+x)^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{\sqrt{x}}{(1+x)^2}\,dx \qquad \boxed{x=u^2,\ dx=2u\,du}$$
$$= \lim_{t\to\infty}\int_1^{\sqrt{t}} \frac{u}{(1+u^2)^2}(2u\,du) = 2\lim_{t\to\infty}\int_1^{\sqrt{t}} \frac{u^2}{(1+u^2)^2}\,du$$
$$= 2\lim_{t\to\infty}\int_1^{\sqrt{t}} \frac{1+u^2-1}{(1+u^2)^2}\,du = 2\lim_{t\to\infty}\int_1^{\sqrt{t}}\left[\frac{1}{1+u^2}-\frac{1}{(1+u^2)^2}\right]du$$
$$= 2\lim_{t\to\infty}\left[\tan^{-1}u\Big]_1^{\sqrt{t}} - \int_1^{\sqrt{t}}\frac{1}{(1+u^2)^2}\,du\right]$$
$$= 2\lim_{t\to\infty}\left[\tan^{-1}\sqrt{t}-\frac{\pi}{4}-\int_1^{\sqrt{t}}\frac{1}{(1+u^2)^2}\,du\right] \qquad \boxed{u=\tan\theta,\ du=\sec^2\theta\,d\theta}$$
$$= 2\left[\frac{\pi}{2}-\frac{\pi}{4}-\lim_{t\to\infty}\int_{\pi/4}^{\tan^{-1}\sqrt{t}}\frac{\sec^2\theta\,d\theta}{(1+\tan^2\theta)^2}\right] = \frac{\pi}{2}-2\lim_{t\to\infty}\int_{\pi/4}^{\tan^{-1}\sqrt{t}}\frac{\sec^2\theta\,d\theta}{\sec^4\theta}$$
$$= \frac{\pi}{2}-2\lim_{t\to\infty}\int_{\pi/4}^{\tan^{-1}\sqrt{t}}\cos^2\theta\,d\theta = \frac{\pi}{2}-2\lim_{t\to\infty}\int_{\pi/4}^{\tan^{-1}\sqrt{t}}\frac{1}{2}(1+\cos2\theta)\,d\theta$$
$$= \frac{\pi}{2}-2\lim_{t\to\infty}\left(\frac{1}{2}\theta+\frac{1}{4}\sin2\theta\right)\Big]_{\pi/4}^{\tan^{-1}\sqrt{t}}$$



$$= \frac{\pi}{2}-2\lim_{t\to\infty}\left(\frac{1}{2}\theta+\frac{1}{2}\sin\theta\cos\theta\right)\Big]_{\pi/4}^{\tan^{-1}\sqrt{t}}$$

$$= \frac{\pi}{2} - 2\lim_{t\to\infty}\left[\frac{1}{2}\tan^{-1}\sqrt{t} + \frac{1}{2}\left(\frac{\sqrt{t}}{\sqrt{1+t}}\right)\frac{1}{\sqrt{1+t}} - \frac{\pi}{8} - \frac{1}{4}\right]$$

$$= \frac{\pi}{2} - 2\lim_{t\to\infty}\left[\frac{1}{2}\tan^{-1}\sqrt{t} + \frac{1}{2}\left(\frac{\sqrt{t}}{1+t}\right) - \frac{\pi}{8} - \frac{1}{4}\right]$$

$$= \frac{\pi}{2} - 2\left[\frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{\pi}{8} - \frac{1}{4}\right] - \lim_{t\to\infty}\frac{1}{1/\sqrt{t}+\sqrt{t}} = \frac{\pi}{4} + \frac{1}{2}$$

94. $$\int_0^\infty \frac{1}{\sqrt{x}(x+1)}\,dx = \lim_{s\to0^+}\int_s^1 \frac{1}{\sqrt{x}(x+1)}\,dx + \lim_{t\to\infty}\int_1^t \frac{1}{\sqrt{x}(x+1)}\,dx \qquad \boxed{x = u^2,\ dx = 2u\,du}$$

$$= \lim_{s\to0^+}\int_{\sqrt{s}}^1 \frac{2u}{u(u^2+1)}\,du + \lim_{t\to\infty}\int_1^{\sqrt{t}} \frac{2u}{u(u^2+1)}\,du$$

$$= \lim_{s\to0^+} 2\tan^{-1}u\Big]_{\sqrt{s}}^1 + \lim_{t\to\infty} 2\tan^{-1}u\Big]_1^{\sqrt{t}}$$

$$= 2\left(\frac{\pi}{4}\right) - 0 + 2\left(\frac{\pi}{2}\right) - 2\left(\frac{\pi}{4}\right) = \pi$$

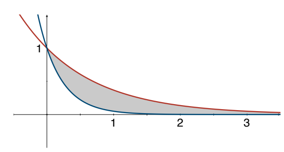
95. $$\lim_{x\to\infty}\frac{x\displaystyle\int_0^x e^{t^2}\,dt}{e^{x^2}} \overset{h}{=} \lim_{x\to\infty}\frac{xe^{x^2} + \displaystyle\int_0^x e^{t^2}\,dt}{2xe^{x^2}} = \lim_{x\to\infty}\left(\frac{1}{2} + \frac{\displaystyle\int_0^x e^{t^2}\,dt}{2xe^{x^2}}\right)$$

$$\overset{h}{=} \frac{1}{2} + \lim_{x\to\infty}\frac{e^{x^2}}{2x(2xe^{x^2}) + 2e^{x^2}} = \frac{1}{2} + \lim_{x\to\infty}\frac{1}{4x^2+2} = \frac{1}{2}$$

96. $$\lim_{x\to\infty}\frac{\displaystyle\int_0^x e^{t^2}\,dt}{xe^{x^2}} \overset{h}{=} \lim_{x\to\infty}\frac{e^{x^2}}{2x^2e^{x^2} + e^{x^2}} = \lim_{x\to\infty}\frac{1}{2x^2+1} = 0$$

97. $$A = \int_0^\infty (e^{-x} - e^{-3x})\,dx = \lim_{t\to\infty}\int_0^t (e^{-x} - e^{-3x})\,dx$$

$$= \lim_{t\to\infty}\left(-e^{-x} + \frac{1}{3}e^{-3x}\right)\Big]_0^t = \lim_{t\to\infty}\left(\frac{1}{3}e^{-3t} - e^{-t} + 1 - \frac{1}{3}\right) = \frac{2}{3}$$



98. (a) $$A = \int_0^1 \frac{1}{\sqrt[3]{1-x}}\,dx = \lim_{t\to1^-}\int_0^t (1-x)^{-1/3}\,dx$$

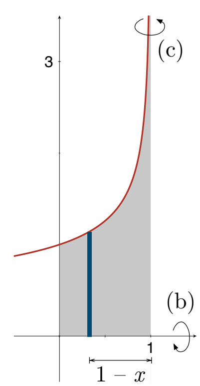
$$= \lim_{t\to1^-}\left[-\frac{3}{2}(1-x)^{2/3}\right]\Big|_0^t = \lim_{t\to1^-}\left[\frac{3}{2} - \frac{3}{2}(1-t)^{2/3}\right] = \frac{3}{2}$$

(b) $$V = \pi\int_0^1 \frac{1}{(\sqrt[3]{1-x})^2}\,dx = \pi\lim_{t\to1^-}\int_0^t (1-x)^{-2/3}\,dx$$

$$= \pi\lim_{t\to1^-}\left[-3(1-x)^{1/3}\right]\Big|_0^t = \pi\lim_{t\to1^-}[3 - 3(1-t)^{1/3}] = 3\pi$$

(c) $$V = 2\pi\int_0^1 (1-x)\frac{1}{\sqrt[3]{1-x}}\,dx = 2\pi\lim_{t\to1^-}\int_0^t (1-x)^{2/3}\,dx$$

$$= 2\pi\lim_{t\to1^-}\left[-\frac{3}{5}(1-x)^{5/3}\right]\Big|_0^t = 2\pi\lim_{t\to1^-}\left[\frac{3}{5} - \frac{3}{5}(1-t)^{5/3}\right] = \frac{6\pi}{5}$$



99. (a) The horizontal asymptote is $y = 1$. Using symmetry,

$$\begin{aligned}\text{Area}(R_1) &= 2\int_0^\infty \left(1 - \frac{x^2-1}{x^2+1}\right) dx = 2\int_0^\infty \frac{2}{x^2+1}\, dx = 2\lim_{t\to\infty}\int_0^t \frac{2}{x^2+1}\, dx \\ &= 2\lim_{t\to\infty} 2\tan^{-1} x\Big]_0^t = 2\lim_{t\to\infty} 2\tan^{-1} t = 2\pi.\end{aligned}$$

(b) If the area of R_3 were finite, then, since R_1 has finite area, the area of the infinite strip bounded by $x = 0$, $y = 0$, and $y = 1$ would be finite. The area of the strip is infinite, so the area of R_3 must be infinite. By symmetry, the area of R_2 is infinite.

100. The total area under the graph of $y = xe^{-x}$ on $[0, \infty)$ is $\int_0^\infty xe^{-x}\, dx = \lim_{t\to\infty}(-xe^{-x} - e^{-x})\Big]_0^t = 1$. The area of the shaded region is $\int_0^{x^*} xe^{-x}\, dx = 1-(x^*+1)e^{-x^*}$. We need to solve $1-(x^*+1)e^{-x^*} = 0.99(1) = 0.99$ or $(x^*+1)e^{-x^*} - 0.01 = 0$. Letting $f(x) = (x+1)e^{-x} - 0.01$, Newton's formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{(x+1)e^{-x} - 0.01}{xe^x}.$$

Since $f(6) > 0$ and $f(7) < 0$, we take $x_0 = 6.5$. Then $x_1 \approx 6.63055$, $x_2 \approx 6.63833$, $x_3 \approx 6.63835$, $x_4 \approx 6.63835$. Thus, $x^* \approx 6.63835$.

101. Using the Trapezoidal Rule,

$$W = \int_0^1 F(x)\, dx \approx \frac{1-0}{10}[0 + 2(50) + 2(90) + 2(150) + 2(210) + 260] = 126 \text{ joules}.$$

102. (a) To compute the rectangular elements of area, we use the midpoints of each of the four subintervals.

$$\begin{aligned}W &= \int_1^5 F(x)\, dx \approx F(1.5)\cdot 1 + F(2.5)\cdot 1 + F(3.5)\cdot 1 + F(4.5)\cdot 1 \\ &\approx 5.3 + 3.7 + 2.8 + 2.3 = 14.1 \text{ joules}\end{aligned}$$

(b) Using the Trapezoidal Rule, we find

$$\begin{aligned}W &= \int_1^5 F(x)\, dx \approx \frac{5-1}{8}[F(1) + 2F(2) + 2F(3) + 2F(4) + F(5)] \\ &= \frac{1}{2}[7 + 2(4.3) + 2(3.2) + 2(2.5) + 2.0] = 14.5 \text{ joules}.\end{aligned}$$

Chapter 8

First-Order Differential Equations

8.1 Separable Equations

In many of the following problems we will encounter an expression of the form $\ln|g(g)| = f(x) + c$. To solve for $g(y)$ we exponentiate both sides of the equation. This yields $|g(y)| = e^{f(x)+c} = e^c e^{f(x)}$ which implies $g(y) = \pm e^c e^{f(x)}$. Letting $c_1 = \pm e^c$ we obtain $g(y) = c_1 e^{f(x)}$.

1. From $dy = \sin 5x dx$ we obtain $y = -\dfrac{1}{5}\cos 5x + c$.

2. From $dy = (t+1)^2 dt$ we obtain $y = \dfrac{1}{3}(t+1)^3 + c$.

3. $$\begin{aligned} \int y^{-3}\, dy &= \int x^{-2}\, dx \\ -\frac{1}{2}y^{-2} &= -x^{-1} + C \\ y^{-2} &= 2x^{-1} + C_1 \end{aligned}$$

4. $$\begin{aligned} \int 5y^4\, dy &= \int dx \\ y^5 &= x + C \end{aligned}$$

5. $$\begin{aligned} \int (1 + 2y + y^2)\, dy &= \int (1 + 2x + x^2)\, dx \\ y + y^2 + \frac{1}{3}y^3 &= x + x^2 + \frac{1}{3}x^3 + C \end{aligned}$$

6. $$\begin{aligned} \int y^{-1/2}\, dy &= \int x^{1/2}\, dx \\ 2y^{1/2} &= \frac{2}{3}x^{3/2} + C \\ y &= \left(\frac{1}{3}x^{3/2} + C_1\right)^2 \end{aligned}$$

7. $\displaystyle\int \sin y\,dy = \int (x^{-2}+5)\,dx$

$$-\cos y = -x^{-1} + 5x + C$$

$$\cos y = \frac{1}{x} - 5x + C_1$$

8. $\displaystyle\int y^{-3}\,dy = \int \cos x\,dx$

$$-\frac{1}{2}y^{-2} = \sin x + C$$

$$y^{-2} = -2\sin x + C_1$$

9. From $\dfrac{1}{y}dy = \dfrac{4}{x}dx$ we obtain $\ln|y| = 4\ln|x| + c$ or $y = c_1x^4$.

10. From $\dfrac{1}{y}dy = -2xdx$ we obtain $\ln|y| = -x^2 + c$ or $y = c_1e^{-x^2}$.

11. From $e^{-2y}dy = e^{3x}dx$ we obtain $3e^{-2y} + 2e^{3x} = c$.

12. From $ye^y dy = \left(e^{-x} + e^{-3x}\right)dx$ we obtain $ye^y - e^y + e^{-x} + \dfrac{1}{3}e^{-3x} = c$.

13. From $\left(y + 2 + \dfrac{1}{y}\right)dy = x^2\ln x dx$ we obtain $\dfrac{y^2}{2} + 2y + \ln|y| = \dfrac{x^3}{3}\ln|x| - \dfrac{1}{9}x^3 + c$.

14. From $\dfrac{1}{(2y+3)^2}dy = \dfrac{1}{(4x+5)^2}$ we obtain $\dfrac{2}{2y+3} = \dfrac{1}{4x+5} + c$.

15. From $\dfrac{1}{N}dN = (te^{t+2} - 1)dt$ we obtain $\ln|N| = te^{t+2} - e^{t+2} - t + c$ or $N = c_1e^{te^{t+2} - e^{t+2} - t}$.

16. From $\dfrac{1}{Q-70}dQ = kdt$ we obtain $\ln|Q-70| = kt + c$ or $Q - 70 = c_1e^{kt}$.

17. From $\dfrac{1}{5P-P^2}dP = \left(\dfrac{1}{5P} - \dfrac{1}{5(P-5)}\right)dP = dt$ we obtain $\dfrac{1}{5}\ln|P| - \dfrac{1}{5}\ln|P-5| = t + c$ so that $\ln\left|\dfrac{P}{(P-5)}\right| = 5t + c_1$ or $\dfrac{P}{P-5} = c_2e^{5t}$. Solving for P we have $P = \dfrac{5c_2e^{5t}}{c_2e^{5t}-1}$

18. From $\dfrac{1}{(10-x)(50-x)}dx = \left(\dfrac{1}{40(x-50)} - \dfrac{1}{40(x-10)}\right)dx = dt$ we obtain $\dfrac{1}{40}\ln|x-50| - \dfrac{1}{40}\ln|x-10| = t + c$ so that $\ln\left|\dfrac{x-50}{x-10}\right| = 40t + c_1$ or $\dfrac{x-50}{x-10} = c_2e^{40t}$. Solving for x we have $x = \dfrac{10(c_2e^{40t}-5)}{c_2e^{40t}-1}$

19. From $\dfrac{y-2}{y+3}dy = \dfrac{x-1}{x+4}dx$ or $\left(1 - \dfrac{5}{y+3}\right)dy = \left(1 - \dfrac{5}{x+4}\right)dx$ we obtain $y - 5\ln|y+3| = x - 5\ln|x+4| + c$ or $\left(\dfrac{x+4}{y+3}\right)^5 = c_1e^{x-y}$.

20. From $\dfrac{y+2}{y-1}dy = \dfrac{x+2}{x-3}dx$ or $\left(1+\dfrac{2}{y-1}\right)dy = \left(1+\dfrac{5}{x-3}\right)dx$ we obtain $y + 2\ln|y-1| = x + 5\ln|x-3| + c$ or $\dfrac{(y-1)^2}{(x-3)^5} = c_1 e^{x-y}$.

21. $$\int y^2\,dy = \int x^{-2}\,dx$$
$$\frac{1}{3}y^3 = -x^{-1} + C$$
$$y^3 = -3x^{-1} + C_1$$
Setting $x = 1$ and $y = 3$, we obtain $27 = -3 + C_1$ or $C_1 = 30$. Thus, $y^3 = -3x^{-1} + 30$.

22. $$\int 2y\,dy = \int (2x + \sec^2 x)\,dx$$
$$y^2 = x^2 + \tan x + C$$
Setting $x = 0$ and $y = -2$, we obtain $4 = 0 + C$ or $C = 4$. Thus $y^2 = x^2 + \tan x + 4$ or $y = -\sqrt{x^2 + \tan x + 4}$.

23. From $\dfrac{1}{x^2+1}dx = 4dt$ we obtain $\tan^{-1} x = 4t + c$. Using $x(\pi/4) = 1$ we find $c = -3/\pi$. The solution of the initial-value problem is $\tan^{-1} x = 4t - \dfrac{3\pi}{4}$ or $x = \tan\left(4t - \dfrac{3\pi}{4}\right)$.

24. From $\dfrac{1}{y^2+1}dy = \dfrac{1}{x^2-1}dx$ or $\dfrac{1}{2}\left(\dfrac{1}{y-1} - \dfrac{1}{y+1}\right)dx$ we obtain $\ln|y-1| - \ln|y+1| = \ln|x-1| - \ln|x+1| + \ln c$ or $\dfrac{y-1}{y+1} = \dfrac{c(x-1)}{x+1}$. Using $y(2) = 2$ we find $c = 1$. A solution of the initial-value problem is $\dfrac{y-1}{y+1} = \dfrac{(x-1)}{x+1}$ or $y = x$.

25. From $\dfrac{1}{y}dy = \dfrac{1-x}{x^2}dx = \left(\dfrac{1}{x^2} - \dfrac{1}{x}\right)dx$ we obtain $\ln|y| = -\dfrac{1}{x} - \ln|x| = c$ or $xy = c_1 e^{-1/x}$. Using $y(-1) = -1$ we find $c_1 = e^{-1}$. The solution of the initial-value problem is $xy = e^{-1-1/x}$ or $y = e^{-(1+1/x)}$**3.**.

26. From $\dfrac{1}{1-2y}dy = dt$ we obtain $-\frac{1}{2}\ln|1-2y| = t + c$ or $1 - 2y = c_1 e^{-2t}$. Using $y(0) = 5/2$ we find $c_1 = -4$. The solution of the initial-value problem is $1 - 2y = -4e^{-2t}$ or $y = 2e^{-2t} + \dfrac{1}{2}$.

27. Separating variables and integrating we obtain
$$\frac{dx}{\sqrt{1-x^2}} - \frac{dy}{\sqrt{1-y^2}} = 0 \quad \text{and} \quad \sin^{-1} x - \sin^{-1} y = c.$$
Setting $x = 0$ and $y = \sqrt{3}/2$ we obtain $c = -\pi/3$. Thus, an implicit solution of the initial-value problem is $\sin^{-1} x - \sin^{-1} y = \pi/3$. Solving for y and using an addition formula from

trigonometry, we get

$$y = \sin\left(\sin^{-1} x + \frac{\pi}{3}\right) = x\cos\frac{\pi}{3} + \sqrt{1-x^2}\sin\frac{\pi}{3} = \frac{x}{2} + \frac{\sqrt{3}\sqrt{1-x^2}}{2}.$$

28. From $\dfrac{1}{1+(2y)^2}dy = \dfrac{-x}{1+(x^2)^2}dx$ we obtain

$$\frac{1}{2}\tan^{-1} 2y = -\frac{1}{2}\tan^{-1} x^2 + c \qquad \text{or} \qquad \tan^{-1} 2y + \tan^{-1} x^2 = c_1.$$

Using $y(1) = 0$ we find $c_1 = \pi/4$. Thus, an implicit solution of the initial-value problem is $\tan^{-1} 2y + \tan^{-1} x^2 = \pi/4$. Solving for y and using a trigonometric identity we get

$$\begin{aligned} 2y &= \tan\left(\frac{\pi}{4} - \tan^{-1} x^2\right) \\ y &= \frac{1}{2}\tan\left(\frac{\pi}{4} - \tan^{-1} x^2\right) \\ &= \frac{1}{2}\frac{\tan\frac{\pi}{4} - \tan(\tan^{-1} x^2)}{1+\tan\frac{\pi}{4}\tan(\tan^{-1} x^2)} \\ &= \frac{1}{2}\frac{1-x^2}{1+x^2}. \end{aligned}$$

29. Substituting $y = k$ and $\dfrac{dy}{dx} = 0$, we get $6k = 18$ or $k = 3$ so that $y = 3$.

30. Substituting $y = k$ and $\dfrac{dy}{dx} = 0$, we get $0 = 5k + 40$ or $k = -8$ so that $y = -8$.

31. Substituting $y = k$ and $\dfrac{dy}{dx} = 0$, we get $0 = k^2 - k - 20$ or $0 = (k-5)(k+4)$ so that $y = 5$ or $y = -4$.

32. Substituting $y = k$ and $\dfrac{dy}{dx} = 0$, we get $0 = k^2 + 2k + 4$ which is false for all real k. Thus, there is no constant real solution.

33. Substituting $y = k$ and $\dfrac{dy}{dx} = 0$ and then solving for k, we see that there are two constant solutions: $y = 0$ and $y = 1$. Separating variables, we have

$$\frac{dy}{y^2 - y} = \frac{dx}{x} \qquad \text{or} \qquad \int \frac{dy}{y(y-1)} = \ln|x| + c.$$

Using partial fractions, we obtain

$$\int\left(\frac{1}{y-1} - \frac{1}{y}\right)dy = \ln|x| + c \ln|y-1| - \ln|y| = \ln|x| + c\ln\left|\frac{y-1}{xy}\right| = c\frac{y-1}{xy} = e^c = c_1.$$

Solving for y we get $y = 1/(1 - c_1 x)$.

(a) Setting $x = 0$ and $y = 1$ we have $1 = 1/(1/0)$, which is true for all values of c_1. Thus, solutions passing through $(0, 1)$ are $y = 1/(1 - c_1 x)$.

(b) Setting $x = 0$ and $y = 0$ in $y = 1/(1 - c_1 x)$ we get $0 = 1$. Thus, the only solution passing through $(0, 0)$ is $y = 0$.

(c) Setting $x = \dfrac{1}{2}$ and $y = \dfrac{1}{2}$ we have $\dfrac{1}{2} = 1/(1 - \frac{1}{2}c_1)$, so $c_1 = -2$ and $y = 1/(1 + 2x)$.

34. By substituting $y = k$ and $\dfrac{dy}{dx} = 0$, and then solving for k we see that there are two constant solutions: $y = 3$ and $y = -3$. Separating variables we have $\dfrac{dy}{y^2 - 9} = dx$ or $\dfrac{dy}{(y-3)(y+3)} = dx$. Using partial fractions, we obtain

$$\int \left(\frac{1}{6(y-3)} - \frac{1}{6(y+3)} \right) dy = x + c \frac{1}{6} \ln|y-3| - \frac{1}{6} \ln|y+3| = x + c \ln\left|\frac{y-3}{y+3}\right| = 6c + c_1 \frac{y-3}{y+3} = c_2 e^{6x}$$

Solving for y we get $y = \dfrac{-3(c_2 e^{6x} + 1)}{c_2 e^{6x} - 1}$.

(a) Setting $x = 0$ and $y = 0$. we have $0 = \dfrac{-3(c_2 + 1)}{c_2 - 1}$ so that $c_2 = -1$ and $y = \dfrac{-3(-6e^{6x} + 1)}{-e^{6x} - 1}$.

(b) The constant solution $y = 3$ passes through $(0, 3)$.

(c) Setting $x = \frac{1}{3}$ and $y = 1$, we have

$$\begin{aligned} 1 &= \frac{-3(c^2 e^2 + 1)}{c_2 e^2 - 1} \\ c_2 e^2 - 1 &= -3c_2 e^2 - 3 \\ 4c_2 e^2 &= -2 \\ c_2 &= \frac{-1}{2e^2} \end{aligned}$$

so that $y = \dfrac{-3\left(\dfrac{-e^{6x+2}}{2} + 1\right)}{\dfrac{-e^{6x-2}}{2} - 1}$

35. In order for $y'(x_0) = \sqrt{y + 0}$ to be defined, we must have $y + 0 \geq 0$.

36. In Problem 31, we have

$$\frac{dy}{(y-5)(y+4)} = dx \text{ or } \left(\frac{1}{9(y-5)} - \frac{1}{9(y+4)} \right) dy = dx$$

Integration yields

$$\frac{1}{9}\ln|y-5| - \frac{1}{9}\ln|y+4| = x + c$$
$$\ln\left|\frac{y-5}{y+4}\right| = 9x + c_1$$
$$\frac{y-5}{y+4} = c_2 e^{9x}$$

Solving for y we get the family of solution $y = \dfrac{-(4c_2e^{9x}+5)}{c_2e^{9x}-1}$.
The constant solution $y = -4$ is not a member of this family and is therefore singular.
In Problem 33, we found the family of solutions $y = \dfrac{1}{1-c_1x}$. The constant solution $y = 0$ is not a member of this family and is therefore singular.
In Problem 34, we found the family of solutions $y = \dfrac{-3(c_2e^{6x}+1)}{c_2e^{6x}-1}$. The constant solution $y = -3$ is not a member of this family and is therefore singular.

37. The right side of the differential equation $\dfrac{dy}{dx} = -\dfrac{x}{y}$ is not defined for $y = 0$. At $x = \pm 5$, however, the function $y = -\sqrt{25-x^2}$ is zero.

8.2 Linear Equations

1. For $y' - 4y = 0$, an integrating factor is $e^{-\int 4\,dx} = e^{-4x}$ so that $\dfrac{d}{dx}\left[e^{-4x}y\right] = 0$ and $y = ce^{4x}$ for $-\infty < x < \infty$.

2. For $y' + 2y = 0$, an integrating factor is $e^{\int 2\,dx} = e^{2x}$ so that $\dfrac{d}{dx}\left[e^{2x}y\right] = 0$ and $y = ce^{-2x}$ for $-\infty < x < \infty$.

3. Writing the equation as $y' + 5y = \dfrac{1}{2}$, the integrating factor is $e^{\int 5\,dx} = e^{5x}$. Then $\dfrac{d}{dx}\left[e^{5x}y\right] = \dfrac{1}{2}e^{5x}$, so $e^{5x}y = \dfrac{1}{10}e^{5x} + C$, and $y = \dfrac{1}{10} + Ce^{-5x}$. (This equation can also be solved by separation of variables.)

4. For $y' + \dfrac{2}{x}y = \dfrac{3}{x}$, an integrating factor is $e^{-\int (2/x)\,dx} = x^2$ so that $\dfrac{d}{dx}\left[x^2y\right] = 3x$ and $y = \dfrac{3}{2} + cx^{-2}$ for $0 < x < \infty$.

5. For $y' + y = e^{3t}$, an integrating factor is $e^{\int dt} = e^t$ so that $\dfrac{d}{dt}\left[e^ty\right] = e^{4t}$ and $y = \dfrac{1}{4}e^{3t} + ce^{-t}$ for $-\infty < t < \infty$.

6. For $y' - y = e^t$, an integrating factor is $e^{-\int dt} = e^{-t}$ so that $\dfrac{d}{dt}\left[e^{-t}y\right] = 1$ and $y = te^t + ce^t$ for $-\infty < t < \infty$.

7. For $y' + 3x^2y = x^2$, an integrating factor is $e^{\int 3x^2\,dx} = e^{x^3}$ so that $\dfrac{d}{dx}\left[e^{x^3}y\right] = x^2e^{x^3}$ and $y = \dfrac{1}{3} + ce^{-x^3}$ for $-\infty < x < \infty$.

8. For $y' + 2xy = x^3$ an integrating factor is $e^{\int 2x dx} = e^{x^2}$ so that $\frac{d}{dx}\left[e^{x^2}y\right] = x^3e^{x^2}$ and $y = \frac{1}{2}x^2 - \frac{1}{2} + ce^{-x^2}$ for $-\infty < x < \infty$. The transient term is ce^{-x^2}.

9. For $y' + \frac{1}{x}y = \frac{1}{x^2}$, an integrating factor is $e^{\int (1/x)\, dx} = x$ so that $\frac{d}{dx}[xy] = \frac{1}{x}$ and $y = \frac{1}{x}\ln x + \frac{c}{x}$ for $0 < x < \infty$.

10. For $y' + \frac{x}{1+x^2}y = \frac{2x}{1+x^2}$ an integrating factor is $e^{\int \frac{x}{1+x^2}dx} = e^{\frac{1}{2}\ln(1+x^2)} = (1+x^2)^{1/2}$ so that $\frac{d}{dx}\left[(1+x^2)^{1/2}y\right] = \frac{2x(1+x^2)^{1/2}}{1+x^2} = \frac{2x}{(1+x^2)^{1/2}}$ and $y = 2 + \frac{c}{(1+x^2)^{1/2}}$ for $-\infty < x < \infty$.

11. For $y' + \frac{e^x}{1+e^x}y = 0$, an integrating factor is $e^{\int e^x/(1+e^x)\, dx} = 1 + e^x$ so that $\frac{d}{dx}[1+e^xy] = 0$ and $y = \frac{c}{1+e^x}$ for $-\infty < x < \infty$.

12. For $y' + \frac{3x^2}{x^3-1}y = 0$, an integrating factor is $e^{\int 3x^2/(x^3-1)\, dx} = x^3 - 1$ so that $\frac{d}{dx}\left[(x^3-1)y\right] = 0$ and $y = \frac{c}{x^3-1}$ for $1 < x < \infty$.

13. For $y' - \frac{1}{x}y = x\sin x$ an integrating factor is $e^{-\int (1/x)dx} = \frac{1}{x}$ so that $\frac{d}{dx}\left[\frac{1}{x}y\right] = \sin x$ and $y = cx - x\cos x$ for $0 < x < \infty$.

14. For $y' + y = \cos(e^x)$ an integrating factor is $e^{\int 1 dx} = e^x$ so that $\frac{d}{dx}[e^xy] = e^x\cos(e^x)$ and $y = e^{-x}\sin(e^x) + ce^{-x}$.

15. For $y' + (\tan x)y = \sec x$, an integrating factor is $e^{\int \tan x\, dx} = \sec x$ so that $\frac{d}{dx}[(\sec x)y] = \sec^2 x$ and $y = \sin x + c\cos x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

16. For $y' + (\cot x)y = \sec^2 x\csc x$ and integrating factor is $e^{\int \cot x dx} = e^{\ln|\sin x|} = \sin x$ so that $\frac{d}{dx}[(\sin x)y] = \sec^2 x$ and $y = \sec x + c\csc x$ for $0 < x < \pi/2$.

17. For $y' + (\cot x)y = 2\cos x$, an integrating factor is $e^{\int \cot x\, dx} = \sin x$ so that $\frac{d}{dx}[(\sin x)y] = 2\sin x\cos x$ and $y = \sin x + c\csc x$ for $0 < x < \pi$.

18. For $\frac{dr}{d\theta} + r\sec\theta = \cos\theta$, an integrating factor is $e^{\int \sec\theta\, d\theta} = \sec\theta + \tan\theta$ so that $\frac{d}{d\theta}[r(\sec\theta + \tan\theta)] = 1 + \sin\theta$ and $r(\sec\theta + \tan\theta) = \theta - \cos\theta + c$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

19. For $y' + \frac{4}{x+2}y = \frac{5}{(x+2)^2}$ an integrating factor is $e^{\int [4/(x+2)]dx} = (x+2)^4$ so that $\frac{d}{dx}[(x+2)^2y] = 5(x+2)^2$ and $y = \frac{5}{3}(x+2)^{-1} + c(x+2)^{-4}$ for $-2 < x < \infty$. The entire solution is transient.

20. For $\dfrac{dP}{dt}+(2t-1)P=4t-2$, an integrating factor is $e^{\int(2t-1)\,dt}=e^{t^2-t}$ so that $\dfrac{d}{dt}\left[Pe^{t^2-t}\right]=(4t-2)e^{t^2-t}$ and $P=2+ce^{t-t^2}$ for $-\infty<t<\infty$.

21. For $y'+\left(1+\dfrac{2}{x}\right)y=\dfrac{e^x}{x^2}$ an integrating factor is $e^{\int[1+(2/x)]dx}=x^2e^x$ so that $\dfrac{d}{dx}[x^2e^xy]=e^{2x}$ and $y=\dfrac{1}{2}\dfrac{e^x}{x^2}+\dfrac{ce^{-x}}{x^2}$ for $0<x<\infty$. The transient term is $\dfrac{ce^{-x}}{x^2}$.

22. For $y'+\left(1+\dfrac{1}{x}\right)y=\dfrac{1}{x}e^{-x}\sin 2x$ and integrating factor is $e^{\int[1+(1/x)]dx}=xe^x$ so that $\dfrac{d}{dx}[xe^xy]=\sin 2x$ and $y=-\dfrac{1}{2x}\cos 2x+\dfrac{ce^{-x}}{x}$ for $0<x<\infty$. The entire solution is transient.

23. For $y'-y=x$ and integrating factor is $e^{\int -1dx=e^{-x}}$ so that $\dfrac{d}{dx}[e^{-x}y]=e^{-x}x$ and $y=-x-1+ce^x$. Substituting $x=0$ and $y=-4$, we have $-4=-1+c$ or $c=-3$ so that $y=-x-1-3e^x$.

24. For $y'+3y=2x$, an integrating factor is $e^{\int 3\,dx}=e^{3x}$ so that $\dfrac{d}{dx}\left[e^{3x}y\right]=2xe^{3x}$, $e^{3x}y=2\left(\dfrac{1}{3}xe^{3x}-\dfrac{1}{9}e^{3x}+C_1\right)$, and $y=\dfrac{2}{3}x-\dfrac{2}{9}+ce^{-3x}$ for $-\infty<x<\infty$. If $y(0)=\dfrac{1}{3}$ then $c=\dfrac{5}{9}$ and $y=\dfrac{2}{3}x-\dfrac{2}{9}+\dfrac{5}{9}e^{-3x}$.

25. For $y'+\dfrac{1}{x}y=\dfrac{1}{x}e^x$, an integrating factor is $e^{\int(1/x)\,dx}=x$ so that $\dfrac{d}{dx}[xy]=e^x$ and $y=\dfrac{1}{x}e^x+\dfrac{c}{x}$ for $0<x<\infty$. If $y(1)=2$ then $c=2-e$ and $y=\dfrac{1}{x}e^x+\dfrac{2-e}{x}$.

26. For $y'+\dfrac{1}{x}y=4+\dfrac{1}{x}$ an integrating factor is $e^{\int\frac{1}{x}dx}=e^{\ln x}=x$ so that $\dfrac{d}{dx}[xy]=4x+1$ and $y=2x+1+\dfrac{c}{x}$. Substituting $x=1$ and $y=8$, we have $8=3+c$ or $c=5$ so that $y=2x+1+\dfrac{3}{x}$.

27. For $y'-\dfrac{1}{x}y=2x$ an integrating factor is $e^{\int-\frac{1}{x}dx}=e^{-\ln x}=\dfrac{1}{x}$ so that $\dfrac{d}{dx}\left[\dfrac{y}{x}\right]=2$ and $y=2x^2+cx$. Substituting $x=5$ and $y=1$, we have $1=50+5c$ or $c=\dfrac{-49}{5}$ so that $y=2x^2-\dfrac{49}{5}x$.

28. For $y'+\dfrac{1}{x+1}y=\dfrac{1}{x(x+1)}$, an integrating factor is $e^{\int 1/(x+1)\,dx}=x+1$ so that $\dfrac{d}{dx}[(x+1)y]=\dfrac{1}{x}$ and $y=\dfrac{\ln x}{x+1}+\dfrac{c}{x+1}$ for $-1<x<\infty$. If $y(1)=10$ then $c=20$ and $y=\dfrac{\ln x}{x+1}+\dfrac{20}{x+1}$.

29. For $x'+\dfrac{1}{t+1}x=\dfrac{\ln t}{t+1}$ and integrating factor is $e^{\int[1/(t+1)]dt}=t+1$ so that $\dfrac{d}{dt}[(t+1)x]=\ln t$

and $x = \dfrac{t}{t+1}\ln t - \dfrac{t}{t+1} + \dfrac{c}{t+1}$ for $0 < t < \infty$. If $x(1) = 10$ then $c = 21$ and $x = \dfrac{t}{t+1}\ln t - \dfrac{t}{t+1} + \dfrac{21}{t+1}$.

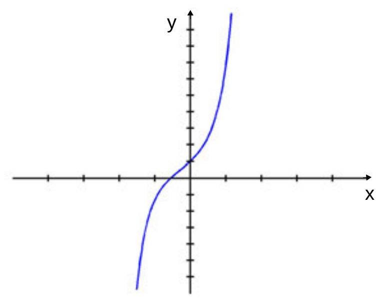
30. For $y' + (\tan t)y = \cos^2 t$ and integrating factor is $e^{\tan t dt} = e^{\ln|\sec t| = \sec t}$ so that $\dfrac{d}{dt}[(\sec t)y] = \cos t$ and $y = \sin t \cos t + c\cos t$ for $-\pi/2 < t < \pi/2$. If $y(0) = -1$ then $c = -1$ and $y = \sin t \cos t - \cos t$.

31. For $\dfrac{di}{dt} + \dfrac{R}{L}i = \dfrac{E}{L}$ and integrating factor is $e^{\int (R/L)dt} = e^{Rt/L}$ so that $\dfrac{d}{dt}\left[e^{Rt/L}i\right] = \dfrac{E}{L}e^{Rt/L}$ and $i = \dfrac{E}{R} + ce^{-Rt/L}$ for $-\infty < t < \infty$. If $i(0) = i_0$ then $c = i_0 - E/R$ and $i = \dfrac{E}{R} + \left(i_0 - \dfrac{E}{R}\right)e^{-Rt/L}$.

32. For $\dfrac{dT}{dt} - kT = -T_m k$ an integrating factor is $e^{\int(-k)dt} = e^{-kt}$ so that $\dfrac{d}{dt}[e^{-kt}T] = -T_m k e^{-kt}$ and $T = T_m + ce^{kt}$ for $\infty < t < \infty$. If $T(0) = T_0$ then $c = T_0 - T_m$ and $T = T_m + (T_0 - T_m)e^{kt}$.

33. (a) An integrating factor for $y' - 2xy = 2$ is $e^{\int -2x dx} = e^{-x^2}$. Thus $\dfrac{d}{dx}[e^{-x^2}y] = 2e^{-x^2}$
$e^{-x^2}y = 2\int_0^x e^{-t^2}dt + c = \sqrt{\pi}erf(x) + c$.

(b) Using a CAS, we find $y(2) \approx 150.92$.



34. (a) An integrating factor for

$$y' + \frac{2}{x}y = \frac{10\sin x}{x^3}$$

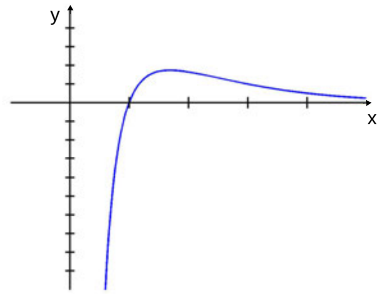
is x^2. Thus

$$\begin{aligned} \frac{d}{dx}[x^2y] &= 10\frac{\sin x}{x} \\ x^2y &= 10\int_0^x \frac{\sin t}{t}dt + c \\ y &= 10x^{-2}Si(x) + cx^{-2}. \end{aligned}$$

From $y(1) = 0$ we get $c = -10Si(1)$. Thus

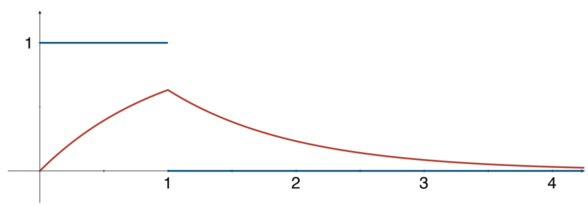
$$y = 10x^{-2}Si(x) - 10x^{-2}Si(1) = 10x^{-2}(Si(x) - Si(1)).$$

(b) Using a CA, we find $y(2) \approx 1.65$.



35. Note that f is discontinuous at $x = 1$. We solve the problem in two parts. For $0 \leq x \leq 1$, $f(x) = 1$ and an integrating factor is $e^{\int dx} = e^x$. Then $\dfrac{d}{dx}[e^x y] = e^x$, $e^x y = e^x + c$, and $y = 1 + ce^{-x}$. Since $y(0) = 0$, $c = -1$ and $y = 1 - e^{-x}$ for $0 \leq x \leq 1$. For $x > 1$, the differential equation is $\dfrac{dy}{dx} = -y$. By separation of variables, we obtain $y = c_1 e^{-x}$. Thus, $y = \begin{cases} 1 - e^{-x}, & 0 \leq x \leq 1 \\ c_1 e^{-x}, & x > 1 \end{cases}$. In order to make y a continuous function, we require $\lim_{x\to 1^-} y(x) = \lim_{x\to 1^+} y(x)$ or $1 - e^{-1} = c_1 e^{-1}$. Then $c_1 = e - 1$ and $y = \begin{cases} 1 - e^{-x}, & 0 \leq x \leq 1 \\ (e-1)e^{-x}, & x > 1 \end{cases}$.

In the graph, f is shown in blue, and the solution of the IVP is shown in red.



36. Since $e^{\int P(x)dx + c} = e^c e^{\int P(x)dx} = c_1 e^{\int P(x)dx}$, we would have

$$c_1 e^{\int P(x)dx} y = c_2 + \int c_1 e^{\int P(x)dx} f(x)dx \quad \text{and} \quad e^{\int P(x)dx} y = c_3 + \int e^{\int P(x)dx} f(x)dx,$$

which is the same result achieved upon integration of (4) from the text.

37. On the interval $(-3, 3)$ the integrating factor is

$$e^{\int x\,dx/(x^2-9)} = e^{-\int x\,dx/(9-x^2)} = e^{\frac{1}{2}\ln(9-x^2)} = \sqrt{9-x^2}$$

and so

$$\frac{d}{dx}\left[\sqrt{9-x^2}\,y\right] = 0 \text{ and } y = \frac{c}{\sqrt{9-x^2}}.$$

38. The rate at which the animal population changes is proportional to the current animal population.

39. The solution of the first equation is $x = c_1 e^{-\lambda} t$. From $x(0) = x_0$, we obtain $c_1 = x_9$ and so $x = x_0 e^{-\lambda_1 t}$. The equation equation then becomes

$$\frac{dy}{dt} = -x_0\lambda_1 e^{-\lambda_1 t} - \lambda_2 y \quad \text{or} \quad y' + \lambda_2 y = -x_0\lambda_1 e^{-\lambda_1 t}$$

which is linear. An integrating factor is $e^{\int \lambda_2 dt} = e^{\lambda_2 t}$. Thus

$$\begin{aligned} \frac{d}{dt}\left[e^{\lambda_2 t} y\right] &= -x_0\lambda_1 e^{-\lambda_1 t} e^{\lambda_2 t} = x_0\lambda_1 e^{(\lambda_2-\lambda_1)t} \\ e^{\lambda_2 t} y &= \frac{-x_0\lambda_1}{\lambda_2-\lambda_1} e^{-\lambda_1 t} + c_2 e^{-\lambda_2 t}. \end{aligned}$$

From $y(0) = y_0$, we obtain

$$y_0 = \frac{-x_0\lambda_1}{\lambda_2-\lambda_1} + c_2 \qquad \text{or } c_2 = \frac{y_0\lambda_2 - y_0\lambda_1 + x_0\lambda_1}{\lambda_2-\lambda_1}.$$

The solution is then $y = \dfrac{-x_0\lambda_1}{\lambda_2-\lambda_1} e^{-\lambda_1 t} + \dfrac{y_0\lambda_2 - y_0\lambda_1 + x_0\lambda_1}{\lambda_2-\lambda_1} e^{-\lambda_2 t}$.

40. (a) Letting $y = u^{-1}$, we can use the chain rule to obtain $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = \dfrac{-1}{u^2}\dfrac{du}{dx}$. Substituting into the DE $\dfrac{dy}{dx} + \dfrac{1}{x}y = xy^2$ yields $\dfrac{-1}{u^2}\dfrac{du}{dx} = \dfrac{1}{x}u^{-1} = x(u^{-1})^2$. Upon multiplication by $-u^2$, we have $\dfrac{du}{dx} - \dfrac{1}{x}u = -x$.

(b) An integrating factor is $e^{\int -\frac{1}{x}dx} = e^{\ln x} = x^{-1}$. Thus

$$\begin{aligned} \frac{d}{dx}[x^{-1}u] &= -1 \\ x^{-1}u &= -x + c \\ u &= -x^2 + cx \end{aligned}$$

Since $y = u^{-1}$ we have the solution $y = \dfrac{1}{-x^2+cx}$.

41. The new DE is $\dfrac{dx}{dy} = -x - y$ or $\dfrac{dx}{dy} + x = -y$ which is linear in the variable x and can therefore be solved.

42. After substituting $Y = y'$, we get $Y' + Y = x$. An integrating factor is e^x so that

$$\begin{aligned} \frac{d}{dx}\left[e^x Y\right] &= xe^x \\ e^x Y &= (x-1)e^x + c \\ Y &= x - 1 + ce^{-x} \\ y' &= x - 1 + ce^{-x} \\ y = x^2 - x - ce^{-x} + c \end{aligned}$$

43. (a) For $y' + \frac{3}{x}y = 6x$, an integrating factor is $e^{\int \frac{3}{x}\,dx} = e^{3\ln x} = x^3$ so that

$$\begin{aligned}\frac{d}{dx}[x^3y] &= 6x^4\\ x^3y &= \frac{6}{5}x^5 + c\\ y &= \frac{6}{5}x^2 + cx^{-3}\end{aligned}$$

(b) Substituting $x = -1$ and $y = 2$, we have

$$\begin{aligned}2 &= \frac{6}{5}(-1)^2 + c(-1)^{-3}\\ 2 &= \frac{6}{5} - c\\ c &= -\frac{4}{5}\end{aligned}$$

The solution is $y = \frac{6}{5}x^3 - \frac{4}{5}x^{-3}$. The solution is valid for x in $(-\infty, 0)$.

(c) Substituting $x = 1$ and $y = 2$, we have

$$\begin{aligned}2 &= \frac{6}{5}(1)^2 + c(1)^{-3}\\ 2 &= \frac{6}{5} + c\\ c &= \frac{4}{5}\end{aligned}$$

The solution is $y = \frac{6}{5}x^3 + \frac{4}{5}x^{-3}$. The solution is valid for $x$in $(0, \infty)$.

(d) We need $c = 0$ so that $y = \frac{6}{5}x^2$. Let the initial condition be $y(1) = \frac{6}{5}$.

8.3 Mathematical Models

1. Let $P = P(t)$ be the population at time t, and P_0 the initial population. From $\frac{dP}{dt} = kP$, we obtain $P = P_0e^{kt}$. Using $P(5) = 2P_0$, we find $k = \frac{1}{5}\ln 2$ and $P = P_0e^{(\ln 2)t/5}$. Setting $P(t) = 3P_0$, we have $3 = e^{(\ln 2)t/5}$, so $\ln 3 = (\ln 2)\frac{t}{5}$, and $t = \frac{5\ln 3}{\ln 2} \approx 7.9$ years. Setting $P(t) = 4P_0$, we have $4 = e^{(\ln 2)t/5}$, so $\ln 4 = (\ln 2)\frac{t}{5}$, and $t = 10$ years.

2. Setting $P = 10,000$ and $t = 3$ in Problem 1, we obtain $10,000 = P_0e^{(\ln 2)3/5}$, so $P_0 = 10,000e^{-0.6\ln 2} \approx 6597.5$. Then $P(10) = P_0e^{2\ln 2} = 4P_0 \approx 26,390$.

3. Let $P = P(t)$ be the population at time t. Then $dP/dt = kP$ and $P + ce^k t$. From $P(0) = c = 500$ we see that $P = 500e^{5t}$. Since 15% of 500 is 75, we have $P(10) = 500e^{10k} = 575$. Solving for k, we get $k = \frac{1}{10} \ln \frac{575}{500} = \frac{1}{10} \ln 1.15$. When $t = 30$,

$$P(30) = 500e^{(1/10)(\ln 1.15)^3 0} = 500e^{3 \ln 1.15} = 760 \text{ years.}$$

4. Let $P = P(t)$ be bacteria population at time t and P_0 the initial number. From $dP/dt = kP$ we obtain $P = P_0 e^{kt}$. Using $P(3) = 400$ and $P(10) = 2000$ we find $400 = P - 0e^{3k}$ or $e^k = (400/P_0)^{1/3}$. From $P(10) = 2000$ we then have $2000 = P_0 e^{10k} = P_0(400/P_0)^{10/3}$, so

$$\frac{2000}{400e^{10/3}} = P_0^{-7/3} \quad \text{and} \quad P_0 = \left(\frac{2000}{400^{10/3}}\right)^{-3/7} \approx 201.$$

5. Let $A = A(t)$ be the amount of lead present at time t. From $dA/dt = kA$ and $A(0) = 1$ we obtain $A = e^{kt}$. Using $A(3.3) = 1/2$ we find $k = \frac{1}{3.3} \ln(1/2)$. When 90% of the lead has decayed, 0.1 grams will remain. Setting $A(t) = 0.1$ we have $e^{t(1/3.3) \ln(1/2)} = 0.1$, so

$$\frac{t}{3.3} \ln \frac{1}{2} = \ln 0.1 \quad \text{and} \quad t = \frac{3.3 \ln 0.1}{\ln(1/2)} \approx 10.96 \text{ hours.}$$

6. Let $N = N(t)$ be the amount at time t. From $\dfrac{dN}{dt} = kt$ and $N(0) = 100$, we obtain $N = 100e^{kt}$. Using $N(6) = 97$, we find $k = \dfrac{1}{6} \ln 0.97$. Then $N(24) = 100e^{(1/6)(\ln 0.97)24} = 100(0.97)^4 \approx 88.5$ mg.

7. Setting $N(t) = 50$ in Problem 6, we obtain $50 = 100e^{kt}$, so $kt = \ln \dfrac{1}{2}$, and $t = \dfrac{\ln 1/2}{(1/6) \ln 0.97} \approx$ 136.5 hours.

8. Let h be the half-life and let $A(t) = A_0 e^{kt}$. Then

$$\begin{aligned} \frac{1}{2} A_0 &= A_0 e^{kh} \\ \ln \frac{1}{2} &= kh \\ \frac{-\ln 2}{h} &= k \end{aligned}$$

Using $A(t_1) = A_1$ and $A(t_2) = A_2$, we have

$$\begin{aligned} A_1/A_2 = \frac{A_0 e^{\frac{-\ln 2}{h} t_1}}{A_0 e^{\frac{-\ln 2}{h} t_2}} &= e^{\frac{-\ln 2}{h} t_1 + \frac{\ln 2}{h} t_2} \\ &= e^{\frac{\ln 2 (t_2 - t_1)}{h}} \end{aligned}$$ Hence $\ln(A_1/A_2) = \dfrac{\ln 2 (t_2 - t_1)}{h}$ and $h = \dfrac{\ln 2(t_2 - t_1)}{\ln(A_1/A_2)}$.

9. Let $I = I(t)$ be the intensity, t the thickness, and $I(0) = I_0$. If $\dfrac{dI}{dt} = kI$ and $I(3) = 0.25 I_0$, then $I = I_0 e^{kt}$, $k = \dfrac{1}{3} \ln 0.25$, and $I(15) = 0.00098 I_0$.

10. From $\dfrac{dS}{dt} = rS$, we obtain $S = S_0 e^{rt}$ where $S(0) = S_0$.

    (a) If $S_0 = \$5000$ and $r = 5.75\%$, then $S(5) = \$6665.45$.

    (b) If $S(t) = \$10,000$, then $t = 12$ years.

    (c) $S \approx \$6651.82$

11. Assume that $A = A_0 e^{kt}$ and $k = -0.00012378$. If $A(t) = 0.145A_0$, then $t \approx 15,600$ years.

12. Assume that $\dfrac{dT}{dt} = k(T-5)$ so that $T = 5 + ce^{kt}$. If $T(1) = 55°$ and $T(5) = 30°$, then $k = -\dfrac{1}{4}\ln 2$ and $c = 59.4611$ so that $T(0) = 64.4611°$.

13. Assume that $\dfrac{dT}{dt} = k(T-10)$ so that $T = 10 + ce^{kt}$. If $T(0) = 70°$ and $T(1/2) = 50°$, then $c = 60$ and $k = 2\ln\dfrac{2}{3}$ so that $T(1) = 36.67°$. If $T(t) = 15°$, then $t = 3.06$ minutes.

14. Assume that $\dfrac{dT}{dt} = k(T-100)$ so that $T = 100 + ce^{kt}$. If $T(0) = 20°$ and $T(1) = 22°$, then $c = -80$ and $k = \ln\dfrac{39}{40}$ so that $T(t) = 90°$ implies $t = 82.1$ seconds. If $T(t) = 98°$, then $t = 145.7$ seconds.

15. From $\dfrac{dA}{dt} = 4 - \dfrac{A}{50}$, we obtain $A = 200 + ce^{-t/50}$. If $A(0) = 30$ then $c = -170$ and $A = 200 - 170e^{-t/50}$.

16. From $\dfrac{dA}{dt} = 0 - \dfrac{A}{50}$, we obtain $A = ce^{-t/50}$. If $A(0) = 30$ then $c = 30$ and $A = 30e^{-t/50}$.

17. From $\dfrac{dA}{dt} = 10 - \dfrac{A}{50}$, we obtain $A = 1000 + ce^{-t/100}$. If $A(0) = 0$ then $c = -1000$ and $A = 1000 - 1000e^{-t/100}$.

18. For $\dfrac{dA}{dt} + \dfrac{2}{300+t}A = 6$, an integrating factor is $e^{\int \frac{2}{300+t}dt} = e^{2\ln|t+300|} = (t+300)^2$ so we have

$$\begin{aligned}
\frac{d}{dt}[(t+300)^2 A] &= 6(t+300)^2 \\
(t+300)^2 A &= 2(t+300)^3 = c \\
A &= 2t + 300 + \frac{c}{(t+300)^2}
\end{aligned}$$

Substituting $t = 0$ and $A = 50$, we get

$$\begin{aligned}
50 &= 600 + \frac{c}{300^2} \\
(-550)300^2 &= c \\
-49,500,000 &= c
\end{aligned}$$

Hence $A(t) = 2t + 600 - \dfrac{49,500,000}{(t+300)^2}$.

19. From $\dfrac{dA}{dt} = 10 - \dfrac{10A}{500-(10-5)t} = 10 - \dfrac{2A}{100-t}$, we obtain $A = 1000 - 10t + c(100-t)^2$. If $A(0) = 0$, then $c = -\dfrac{1}{10}$. The tank is empty in 100 minutes.

20. From $\dfrac{dA}{dt} = 3 - \dfrac{4A}{100+(6-4)t} = 3 - \dfrac{2A}{50+t}$, we obtain $A = 50 + t + c(50+t)^{-2}$. If $A(0) = 10$ then $c = -100,000$ and $A(30) = 64.38$ pounds.

21. From $\dfrac{ds}{dt} = v(t)$, we have $ds = \left[\dfrac{mg}{k} + \left(v_0 - \dfrac{mg}{k}\right)e^{-kt/m}\right]dt$. Integrating, we obtain

$$s = \frac{mg}{k}t + \frac{v_0 - mg/k}{-k/m}e^{-kt/m} + C = \frac{mg}{k}t + \left(\frac{m^2g}{k^2} - \frac{mv_0}{k}\right)e^{-kt/m} + C.$$

Setting $s(0) = 0$, we obtain $0 = \dfrac{m^2g}{k^2} - \dfrac{mv_0}{k} + C$ or $C = -\dfrac{m^2g}{k^2} + \dfrac{mv_0}{k}$. Thus,

$$s(t) = \frac{mg}{k}t + \left(\frac{m^2g}{k^2} - \frac{mv_0}{k}\right)(e^{-kt/m} - 1).$$

22. $\dfrac{dv}{dt} = g - \dfrac{k}{m}v^2 = g\left(1 - \dfrac{k}{mg}v^2\right)$. Letting $c = \sqrt{\dfrac{k}{mg}}$, we have $\dfrac{dv}{dt} = g(1-c^2v^2)$ or $\dfrac{dv}{(1-c^2v^2)} =$ gdt. Integrating both sides, we have $\dfrac{\ln\left(\dfrac{1+cv}{1-cv}\right)}{2c} = gt + c_1$ But $\tan^{-1}(cv) = \ln\left(\dfrac{1+cv}{1-cv}\right)$ so we have

$$\begin{aligned}
\frac{\tan^{-1}(cv)}{c} &= gt + c_1 \\
\tan^{-1}(cv) &= cgt + c_2 \\
cv &= \tanh(cgt + c_2) \\
v &= \frac{\tanh\left(\sqrt{\frac{kg}{m}}t + c_2\right)}{\sqrt{\frac{k}{mg}}}
\end{aligned}$$

Plugging in $t = 0$ and $v = v_0$, we have $\sqrt{\frac{k}{mg}}v_0 = \tanh(c_2)$ or $c_2 = \tanh^{-1}\left(\sqrt{\frac{k}{mg}}v_0\right)$. This yields $v = \sqrt{\dfrac{mg}{k}}\tanh\left(\sqrt{\dfrac{kg}{m}}t + \tanh^{-1}\left(\sqrt{\frac{k}{mg}}v_0\right)\right)$. As t approaches infinity, the hyperbolic tangent tends to 1. Hence the terminal velocity is $v_{ter} = \sqrt{\dfrac{mg}{k}}$.

23. From $\dfrac{dX}{dt} = A - BX$ and $X(0) = 0$, we obtain $X = \dfrac{A}{B} - \dfrac{A}{B}e^{-Bt}$ so that $X \to \dfrac{A}{B}$ as $t \to \infty$. If $X(t) = \dfrac{A}{2B}$, then $t = \dfrac{\ln 2}{B}$.

24. From $V\dfrac{dC}{dt} = kA(C_s - C)$ and $C(0) = C_0$, we obtain $C = C_s + (C_0 - C_s)e^{-kAt/V}$.

25. From $\dfrac{dE}{dt} = -\dfrac{E}{RC}$ and $E(t_1) = E_0$, we obtain $E = E_0 e^{(t_1 - t)/RC}$.

26. From the differential equation for the current, we see that we must solve $\left(\dfrac{1}{2}\right)\dfrac{di}{dt} + 10i = 12$. Separating variables, we obtain $\dfrac{di}{24 - 20i} = dt$. Integrating, we find $-\dfrac{1}{20}\ln|24 - 20i| = t + C$. Solving for i, we get $i = \dfrac{6}{5} + ce^{-20t}$. Now $i(0) = 0$ implies $0 = 6/5 + c$ or $c = -6/5$. Therefore, the current is $i(t) = \dfrac{6}{5} - \dfrac{6}{5}e^{-20t}$.

27. Assume $L\dfrac{di}{dt} + Ri = E(t)$, $L = 0.1$, $R = 50$, and $E(t) = 50$ so that $i = \dfrac{3}{5} + ce^{-500t}$. If $i(0) = 0$, then $c = -3/5$ and $\lim_{t\to\infty} i(t) = 3/5$.

28. (a) Setting $A_h = \dfrac{1}{4}$, $A_w = 50$, and $g = 32$, the differential equation is $\dfrac{dh}{dt} = -c\dfrac{\sqrt{h}}{25}$. Separating variables, we have $\displaystyle\int 25h^{-1/2}\,dh = -c\int dt$. Then $50\sqrt{h} = -ct + C$ or $h = \dfrac{1}{2500}(-ct + C)^2$. From $h(0) = 20$ we find $C = 100\sqrt{5}$. Thus $h(t) = \dfrac{1}{2500}(-ct + 100\sqrt{5})^2$.

    (b) Setting $h = 0$, $c = 1$, and solving for t, we obtain $t = 100\sqrt{5}$ s.

    (c) Setting $h = 0$, $c = 0.6$, and solving for t, we obtain $t = \dfrac{500\sqrt{5}}{3}$ s.

29. We note first that $P(0) = \dfrac{aP_0}{bP_0 + (a - bP_0)} = \dfrac{aP_0}{a} = P_0$ so that the initial condition is satisfied. To verify that $P(t)$ satisfies the differential equation, compute $P'(t) = \dfrac{a^2P_0(a - bP_0)e^{-at}}{(bP_0 + (a - bP_0)e^{-at})^2}$. We now compute $P(a - bP) = aP - bP^2 = \dfrac{a^2P_0}{bP_0 + (a - bP_0)e^{-at}} - \dfrac{a^2bP_0^2}{(bP_0 + (a - bP_0)e^{-at})^2}$ From

$$= \frac{a^2P_0(bP_0 + (a - bP_0)e^{-at}) - a^2bP_0^2}{(bP_0 + (a - bP_0)e^{-at})^2}$$

$$= \frac{a^2P_0(a - b)P_0}{e}^{-at} (bP_0 + (a - bP_0)e^{-at})^2$$

these calculations, we note that the differential equation $\dfrac{dP}{dt} = P(a - bP)$ is satisfied.

30. Using Problem 29, we have $a = 10^{-1}$, $b = 10^{-7}$, $P_0 = 5000$. The solution is therefore

$$P(t) = \frac{10^{-1}(5000)}{10^{-7}(5000) + (10^{-1} - 10^{-7}(5000))e^{-.1t}} \quad \lim_{t\to\infty} P(t) = \frac{500}{0.0005} = 1,000,000$$

$$= \frac{500}{0.0005 + 0.0995e^{-.1t}}$$

To find when the population is half the limiting value, we must solve

$$\begin{aligned}
500,000 &= \frac{500}{0.0005 + 0.0995e^{-.1t}} \\
e^{-.1t} &= \frac{\left(\frac{500}{500,000} - 0.0005\right)}{0.0995} \\
t &= \frac{\ln\left[\frac{\left(\frac{500}{500,000}\right) - 0.0005}{0.0995}\right]}{-0.1} \\
t &= 52.93 \text{ years}
\end{aligned}$$

31. $\dfrac{dx}{dt} = kx(1000 - x) = x(1000k - kx)$ Using the result from Problem 29 with $a = 1000k$, $b = k$, and $x_0 = 1$,

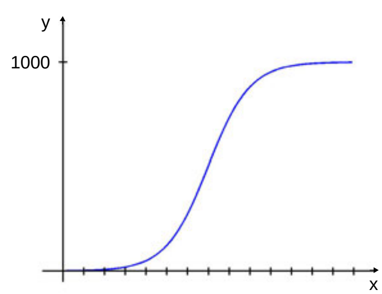
$$x(t) = \frac{1000k}{k + 999ke^{-1000kt}} = \frac{1000}{1 + 999e^{-1000kt}}$$

Substituting $t = 4$ and $x = 50$ yields

$$\begin{aligned}
50 &= \frac{1000}{1 + 999e^{-4000k}} \\
e^{-4000k} &= \frac{\frac{1000}{50} - 1}{999} = \frac{19}{999} \\
k &= \frac{\ln\left(\frac{19}{999}\right)}{-4000} = 0.00099
\end{aligned}$$

Hence $x(t) = \dfrac{1000}{1 + 999e^{-.99t}}$.

After 6, days, the number of infections is $x(6) = \dfrac{1000}{1 + 999e^{-99(6)}} \approx 276$.



32. (a)

$$\frac{dX}{(250-X)(40-X)} = kdt$$
$$\frac{dX}{210(X-250)} - \frac{dX}{210(X-40)} = kdt$$
$$\frac{1}{210}\left(\ln|X-250| - \ln|X-40|\right) = kt + c$$
$$\ln\left|\frac{X-250}{X-40}\right| = 210kt + c_1$$
$$\frac{X-250}{X-40} = c_2e^{210kt}$$
$$X = \frac{40c_2e^{210kt}-250}{c_2e^{210kt}-1}$$

Substituting $t=0$ and $X=0$, we have

$$0 = \frac{400c_2 - 250}{c_2 - 1}$$
$$250 = 40c_2$$
$$\frac{25}{4} = c_2$$

This gives $X(t) = \dfrac{250e^{210kt}-250}{\frac{25}{4}e^{210kt}-1}$. Substituting $t=10$ and $X=30$, we have

$$30 = \frac{250e^{2100k}-250}{\frac{25}{4}e^{2100k}-1}$$
$$30\left(\frac{25}{4}e^{2100k}-1\right) = 250e^{2100k}-250$$
$$\frac{30(25)}{4}e^{2100k} - 250e^{2100k} = -220$$
$$-\frac{125}{2}e^{2100k} = -220$$
$$e^{2100k} = \frac{88}{25}$$
$$k = \frac{\ln\left(\frac{88}{25}\right)}{2100}$$

Hence, $X(t) = \dfrac{250e^{.1t}-250}{\frac{25}{4}e^{01t}-1}$

(b) $X(15) = \dfrac{250e^{.1(15)}-250}{\frac{25}{4}e^{.1(15)}-1} \approx 32.23$ g

(c) $\lim_{t\to\infty} X(t) = \dfrac{250}{\left(\frac{25}{4}\right)} = 40$ so 40 g of C are formed as $t \to \infty$. $50 - \frac{1}{5}(40) = 42$ g of A and $32 - \frac{4}{5}(40) = 0$ g of B remain as $t \to \infty$.

33. (a) Writing the equation as $v\,dv = -ky^{-2}\,dy$ and integrating, we obtain $\dfrac{1}{2}v^2 = \dfrac{k}{y} + C$ or $v^2 = \dfrac{2k}{y} + C_1$. Since $v(R) = v_0$, we have $v_0^2 = \dfrac{2k}{R} + C_1$. Thus $v^2 = \dfrac{2k}{y} + v_0^2 - \dfrac{2k}{R}$.

(b) For the rocket to escape, v must remain positive. That is, v can never be zero; for then, the rocket stops and begins falling back to earth. Since $\dfrac{2k}{y}$ approaches zero as y increases, we can guarantee that v^2 will never be zero by requiring $v_0^2 - \dfrac{2k}{R} > 0$ or $v_0 > \sqrt{2kR}$. Since $k = gR^2$, $\dfrac{2k}{R} = 2gR$ and we take the escape velocity to be

$$\begin{aligned} v_0 = \sqrt{2gR} &\approx \sqrt{2(32\text{ ft/s}^2)(4000\text{ mi})} = \sqrt{\frac{2(32)4000}{5280}\text{ mi}^2/\text{s}^2} \\ &\approx 6.96\text{ mi/s} = 6.96(60)^2\text{ mi/h} \approx 25,000\text{ mi/h}. \end{aligned}$$

34. Since the surface area of a sphere is $S = 4\pi r^2$, we have $\dfrac{dV}{dt} = 4\pi k r^2$. From $V = \dfrac{4}{3}\pi r^3$, we find $r = (3V/4\pi)^{1/3}$. Then $\dfrac{dV}{dt} = 4\pi k\,(3V/4\pi)^{2/3} = (4\pi)^{1/3}k3^{2/3}V^{2/3}$ and

$$\int V^{-2/3}\,dV = (36\pi)^{1/3}k\int dt; \qquad 3V^{1/3} = (36\pi)^{1/3}kt + C; \qquad V^{1/3} = \left(\frac{4}{3}\pi\right)^{1/3}kt + C_1.$$

Then $V(t) = \left[\left(\dfrac{4}{3}\pi\right)^{1/3}kt + C_1\right]^3$.

35. (a) $\dfrac{dA}{\sqrt{A}(M-A)} = kdt$

$\displaystyle\int \frac{dA}{\sqrt{A}(M-A)} = kt + c$

With the substitution $x = \sqrt{A}$, the integral on the left becomes

$$\begin{aligned} 2\int \frac{1}{M-x^2}dx &= 2\int \frac{1}{(\sqrt{M}+x)(\sqrt{M}-x)}dx \\ &= \int \frac{1}{\sqrt{M}(x+\sqrt{M})}dx - \int \frac{1}{\sqrt{M}(x-\sqrt{M})}dx \\ &= \frac{1}{\sqrt{M}}\ln\left|\frac{x+\sqrt{M}}{x-\sqrt{M}}\right| \end{aligned}$$

Hence

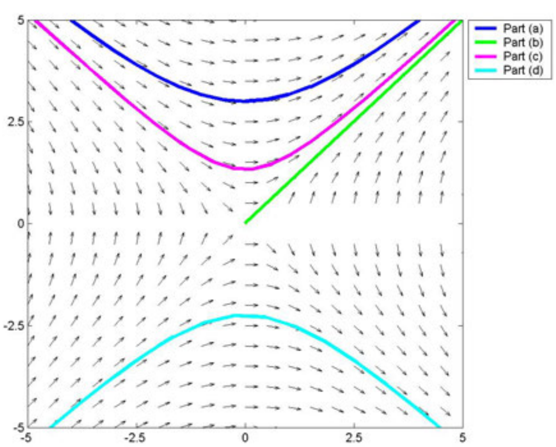
$$\begin{aligned}
\int \frac{dA}{\sqrt{A}(M-A)} &= \frac{1}{\sqrt{M}}\ln\left|\frac{\sqrt{A}+\sqrt{M}}{\sqrt{A}-\sqrt{M}}\right| = kt + c\\
\ln\left|\frac{\sqrt{A}+\sqrt{M}}{\sqrt{A}-\sqrt{M}}\right| &= k\sqrt{M}t + c\\
\frac{\sqrt{A}+\sqrt{M}}{\sqrt{A}-\sqrt{M}} &= c_1 e^{k\sqrt{M}t}\\
\sqrt{A} &= \frac{\sqrt{M}(c_1 e^{k\sqrt{M}t}+1)}{c_1 e^{k\sqrt{M}t}-1}
\end{aligned}$$

So $A(t) = \left(\dfrac{\sqrt{M}(c_1 e^{k\sqrt{M}t}+1)}{c_1 e^{k\sqrt{m}t}-1}\right)^2$

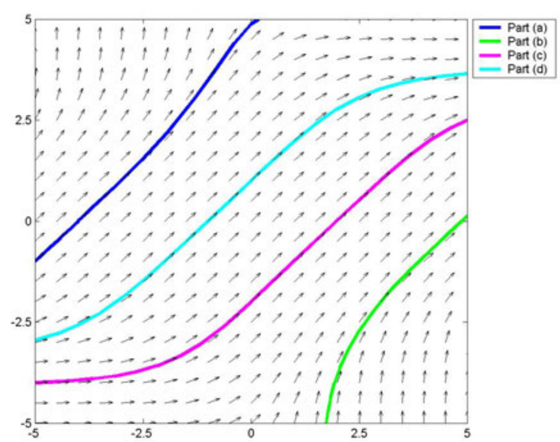
(b) $\lim_{t\to\infty} A(t) = M$.

8.4 Solution Curves without a Solution

1.



2.



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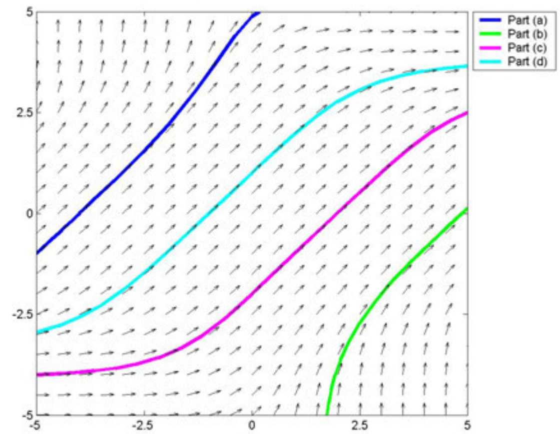
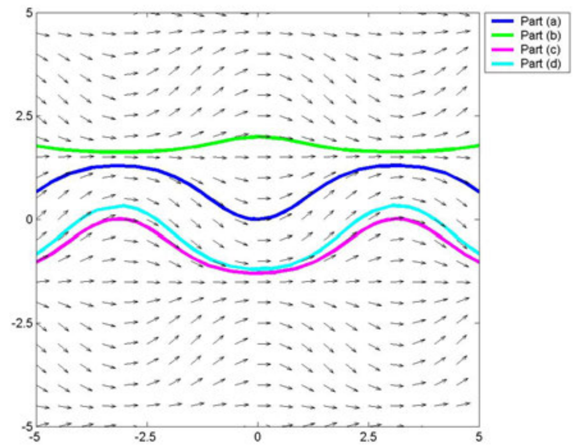
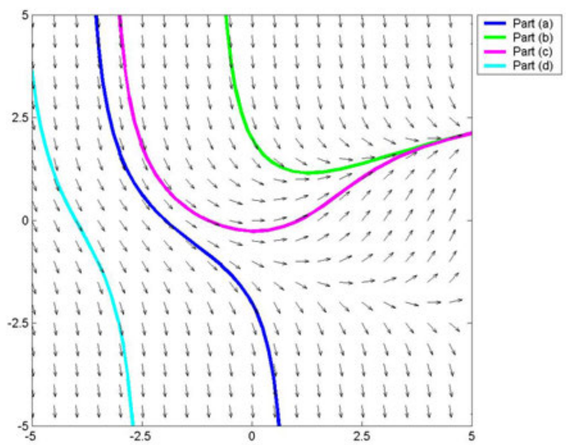

Part (a)
Part (b)
Part (c)
Part (d)
5
2.5
0
-2.5
-5
-2.5
0
2.5
5

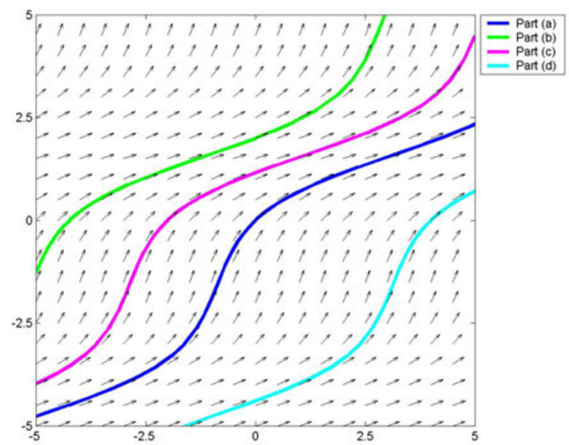
4.


Part (a)
Part (b)
Part (c)
Part (d)
5
2.5
0
-2.5
-5
-2.5
0
2.5
5

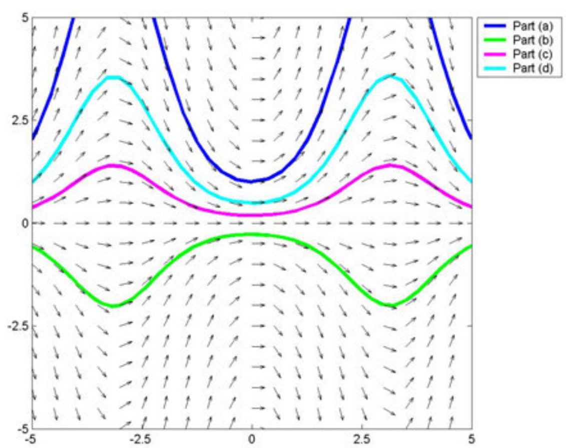
5.


Part (a)
Part (b)
Part (c)
Part (d)
5
2.5
0
-2.5
-5
-2.5
0
2.5
5

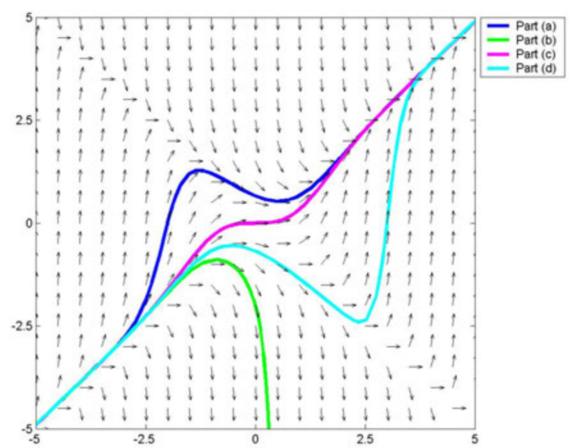
6.


Part (a)
Part (b)
Part (c)
Part (d)
5
2.5
0
-2.5
-5
-2.5
0
2.5
5

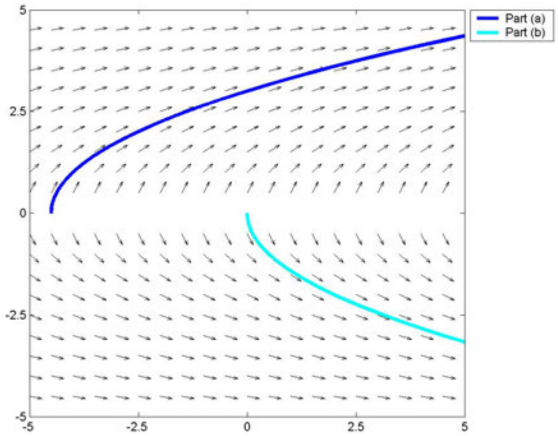
7.


Part (a)
Part (b)
Part (c)
Part (d)
5
2.5
0
-2.5
-5
-2.5
0
2.5
5

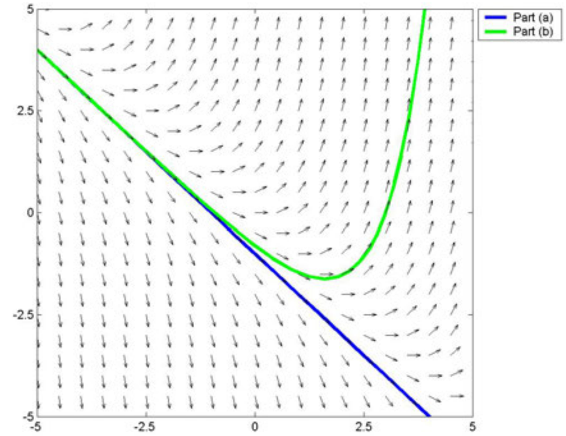
8.


Part (a)
Part (b)
Part (c)
Part (d)
5
2.5
0
-2.5
-5
-2.5
0
2.5
5

9.



10.



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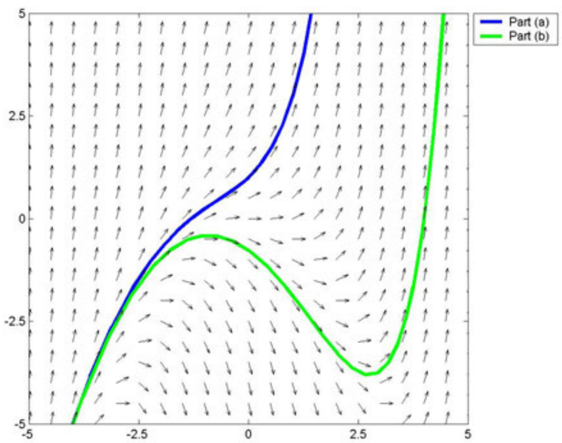
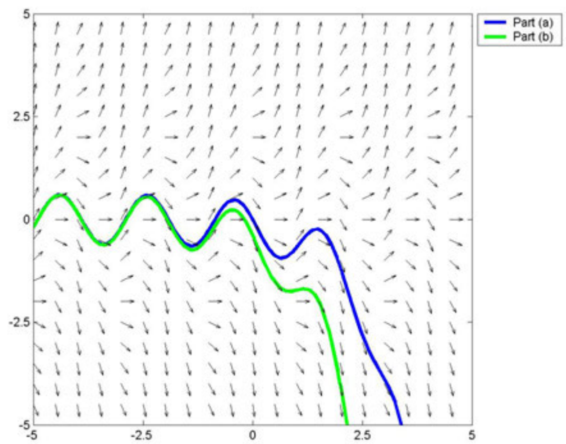
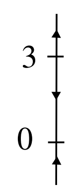


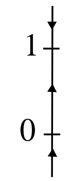
12.



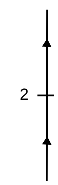
13. Solving $y^2 - 3y = y(y-3) = 0$ we obtain the critical points 0 and 3. From the phase portrait we see that 0 is asymptotically stable (attractor) and 3 is unstable (repeller).



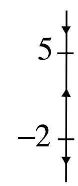
14. Solving $y^2 - y^3 = y^2(1 - y) = 0$ we obtain the critical points 0 and 1. From the phase portrait we see that 1 is asymptotically stable (attractor) and 0 is semi-stable.



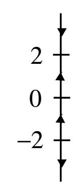
15. Solving $(y - 2)^4 = 0$ we obtain the critical point 2. From the phase portrait we see that 2 is semi-stable.



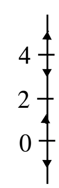
16. Solving $10 + 3y - y^2 = (5 - y)(2 + y) = 0$ we obtain the critical points -2 and 5. From the phase portrait we see that 5 is asymptotically stable (attractor) and -2 is unstable (repeller).



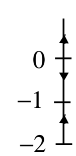
17. Solving $y^2(4 - y^2) = y^2(2 - y)(2 + y) = 0$ we obtain the critical points -2, 0, and 2. From the phase portrait we see that 2 is asymptotically stable, 0 is semi-stable, and -2 is unstable (repeller).



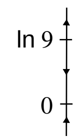
18. Solving $y(2 - y)(4 - y) = 0$ we obtain the critical points 0, 2, and 4. From the phase portrait we see that 2 is asymptotically stable (attractor) and 0 and 3 are unstable (repellers).



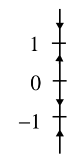
19. Solving $y\ln(y + 2) = 0$ we obtain the critical points -1 and 0. From the phase portrait we see that -1 is asymptotically stable (attractor) and 0 is unstable (repeller).

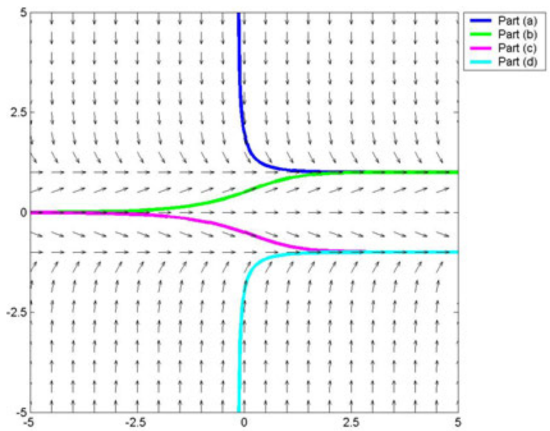


20. Solving $ye^y - 9y = y(e^y - 9) = 0$ we obtain the critical point 0 and $\ln 9$. From the phase portrait we see that 0 is asymptotically stable (attractor) and $\ln 9$ is unstable (repeller).

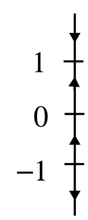


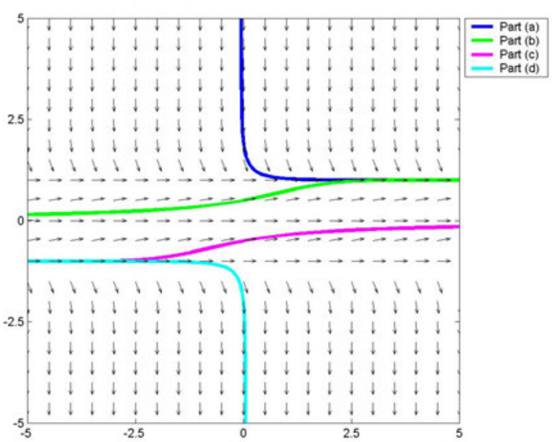
21. Writing the differential equation in the form $dy/dx = y(1-y)(1+y)$ we see that critical points are located at $y = -1$, $y = 0$, and $y = 1$. The phase portrait is shown at the right.



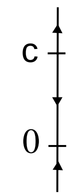


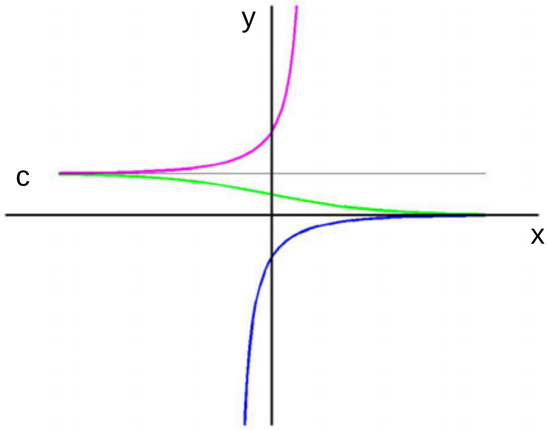
22. Writing the differential equation in the form $dy/dx = y^2(1-y)(1+y)$ we see that critical points are located at $y = -1$, $y = 0$, and $y = 1$. The phase portrait is shown at the right.



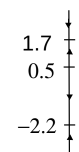


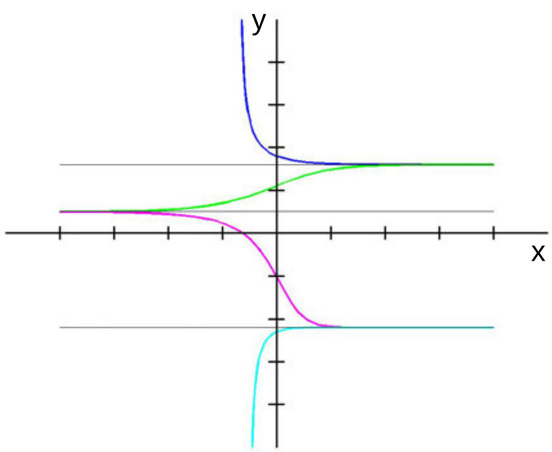
23. The critical points are 0 and c because the graph of $f(y)$ is 0 at these points. Since $f(y) > 0$ for $y < 0$ and $y > c$, the graph of the solution is increasing on $(-\infty, 0)$ and (c, ∞). Since $f(y) < 0$ for $0 < y < c$, the graph of the solution is decreasing on $(0, c)$.





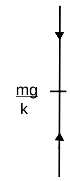
24. The critical points are approximately at -2,2,0.5, and 1.7. Since $f(y) > 0$ for $y < -2.2$ and $0.5 < y < 1.7$, the graph of the solution is increasing on $(-\infty, -2.2)$ and $(0.5, 1.7)$. Since $f(y) < 0$ for $-2.2 < y < 0.5$ and $y > 1.7$, the graph is decreasing on $(-2.2, 0.5)$ and $(1.7, \infty)$.





25. Writing the differential equation in the form

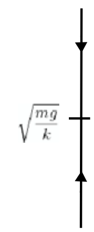
$$\frac{dv}{dt} = \frac{k}{m}\left(\frac{mg}{k} - v\right)$$



we see that a critical point is mg/k.
From the phase portrait we see that mg/k is an asymptotically stable critical point. Thus, $\lim_{t\to\infty} v = mg/k$.

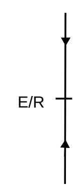
26. Writing the differential equation in the form

$$\frac{dv}{dt} = \frac{k}{m}\left(\frac{mg}{k} - v^2\right) = \frac{k}{m}\left(\sqrt{\frac{mg}{k}} - v\right)\left(\sqrt{\frac{mg}{k}} + v\right)$$

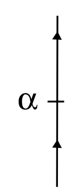


we see that the only physically meaningful critical point is $\sqrt{mg/k}$.
From the phase portrait we see that $\sqrt{mg/k}$ is an asymptotically stable critical point. Thus, $\lim_{t\to\infty} v = \sqrt{mg/k}$.

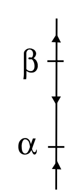
27. From an inspection of the autonomous differential equation $\frac{di}{dt} = \frac{1}{L}(E - Ri)$ we see that $i = E/R$ is the equilibrium solution. If $i_0 < E/R$, then $E - Ri > 0$ and hence $\frac{di}{dt} > 0$. If $i_0 > E/R$, then $E - Ri < 0$ and hence $\frac{di}{dt} < 0$. Upon exaimination of the resulting phase portrait, we see that $i \to E/R$ as $t \to \infty$. Thus Ohm's Law $E = iR$ is satisfied as $t \to \infty$.



28. (a) From the phase portrait we see that critical points are α and β. Let $X(0) = X_0$. If $X_0 < \alpha$, we see that $X \to \alpha$ as $t \to \infty$. If $\alpha < X_0 < \beta$, we see that $X(t)$ increases in an unbounded manner, but more specific behavior of $X(t)$ as $t \to \infty$ is not known.



(b) When $\alpha = \beta$ the phase portrait is as shown. If $X_0 < \alpha$, then $X(t) \to \alpha$ as $t \to \infty$. If $X_0 > \alpha$, then $X(t)$ increases in an unbounded manner. This could happen in a finite amount of time. That is, the phase portrait does not indicate that X becomes unbounded as $t \to \infty$.



(c) When $k = 1$ and $\alpha = \beta$ the differential equation is $dX/dt = (\alpha - X)^2$. For $X(t) = \alpha - 1/(t+c)$ we have $dX/dt = 1/(t+c)^2$ and

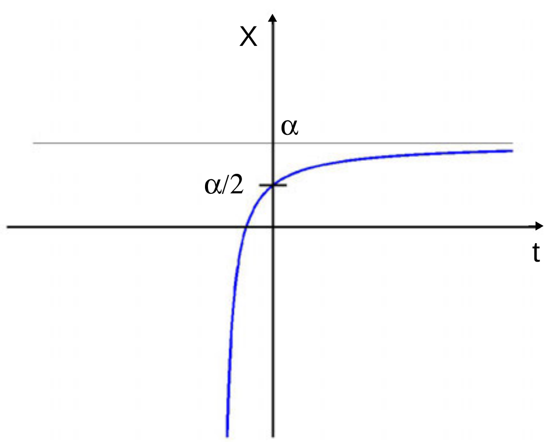
$$(\alpha - X)^2 = \left[\alpha - \left(\alpha - \frac{1}{t+c}\right)\right]^2 = \frac{1}{(t+c)^2} = \frac{dX}{dt}.$$

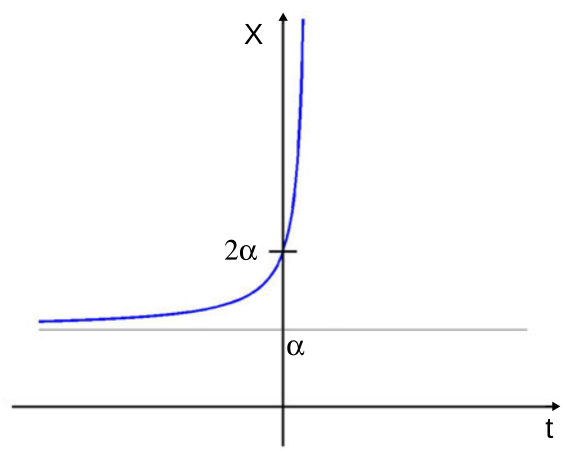
For $X(0) = \alpha/2$ we obtain

$$X(t) = \alpha - \frac{1}{t + 2/\alpha}.$$

For $X(0) = 2\alpha$ we obtain

$$X(t) = \alpha - \frac{1}{t - 1/\alpha}.$$

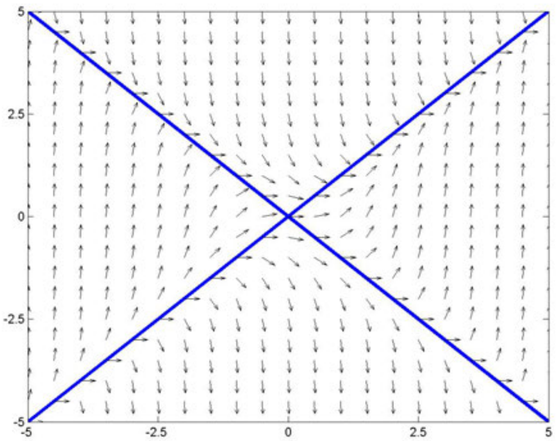




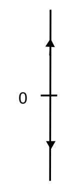
For $X(0) = \alpha/2$, we have $X \to \alpha$ as $t \to \infty$ which is what we expected from the phase portrait. For $X(0) = 2\alpha$, we have $X \to \infty$ as t increases to $1/\alpha$. At this point, the solution has a vertical asymptote. Thus, the solution approaches infinity before t grows very large.

29. At points on the isocline $\frac{dy}{dx} = x^2 + y^2 = 1$, the line segments should each have a slope of 1. Isoclines of the differential equation $\frac{dy}{dx} = x + y$ have the form $x + y = c$ or $y = -x + c$. Thus, the isoclines are all lines in the plane with slope -1.

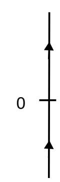
30. At points on the nullcline $\frac{dy}{dx} = x^2 + y^2 - 1 = 0$, the line segments should each have slope 0. Nullcines of the differential equation $\frac{dy}{dx} = x^2 - y^2$ have the form $x^2 - y^2 = 0$ or $y = \pm x$. Thus, the nullcline are the lines $y = x$ and $y = -x$.



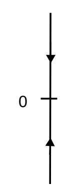
31. Consider the differential equation $\dfrac{dy}{dt} = y^n$.
    If n is odd, then the phase portrait shows that 0 is unstable.



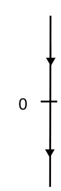
If n is even, then the phase portrait shows that 0 is semi-stable.



Consider the differential equation $\dfrac{dy}{dt} = -y^n$.
If n is odd, then the phase portrait shows that 0 is stable.



If n is even, then the phase portrait shows that 0 is semi-stable.



8.5 Euler's Method

1. We identify $f(x, y) = 2x - 3y + 1$. Then, for $h = 0.1$,

$$y_{n+1} = y_n + 0.1(2x_n - 3y_n + 1) = 0.2x_n + 0.7y_n + 0.1,$$

and

$$y(1.1) \approx y_1 = 0.2(1) + 0.7(5) + 0.1 = 3.8$$
$$y(1.2) \approx y_2 = 0.2(1.1) + 0.7(3.8) + 0.1 = 2.98.$$

For $h = 0.05$,

$$y_{n+1} = y_n + 0.05(2x_n - 3y_n + 1) = 0.1x_n + 0.85y_n + 0.1,$$

and

$$y(1.05) \approx y_1 = 0.1(1) + 0.85(5) + 0.1 = 4.4$$
$$y(1.1) \approx y_2 = 0.1(1.05) + 0.85(4.4) + 0.1 = 3.895$$
$$y(1.15) \approx y_3 = 0.1(1.1) + 0.85(3.895) + 0.1 = 3.47075$$
$$y(1.2) \approx y_4 = 0.1(1.15) + 0.85(3.47075) + 0.1 = 3.11514.$$

2. We identify $f(x, y) = x + y^2$. Then, for $h = 0.1$,

$$y_{n+1} = y_n + 0.1(x_n + y_n^2) = 0.1x_n + y_n + 0.1y_n^2,$$

and

$$y(0.1) \approx y_1 = 0.1(0) + 0 + 0.1(0)^2 = 0$$
$$y(0.2) \approx y_2 = 0.1(0.1) + 0 + 0.1(0)^2 = 0.01.$$

For $h = 0.05$,

$$y_{n+1} = y_n + 0.05(x_n + y_n^2) = 0.05x_n + y_n + 0.05y_n^2,$$

and

$$y(0.05) \approx y_1 = 0.05(0) + 0 + 0.05(0)^2 = 0$$
$$y(0.1) \approx y_2 = 0.05(0.05) + 0 + 0.05(0)^2 = 0.0025$$
$$y(0.15) \approx y_3 = 0.05(0.1) + 0.0025 + 0.05(0.0025)^2 = 0.0075$$
$$y(0.2) \approx y_4 = 0.05(0.15) + 0.0075 + 0.05(0.0075)^2 = 0.0150.$$

3. Separating variables and integrating, we have

$$\frac{dy}{y} = dx \text{ and } \ln|y| = x + c.$$

Thus $y = c_1e^x$ and, using $y(0) = 1$, we find $c = 1$, so $y = e^x$ is the solution of the initial-value problem.

$h = 0.1$

x_n	y_n	Actual Value	Abs. Error	% Rel. Error
0.00	1.0000	1.0000	0.0000	0.00
0.10	1.1000	1.1052	0.0052	0.47
0.20	1.2100	1.2214	0.0114	0.93
0.30	1.3310	1.3499	0.0189	1.40
0.40	1.4641	1.4918	0.0277	1.86
0.50	1.6105	1.6487	0.0382	2.32
0.60	1.7716	1.8221	0.0506	2.77
0.70	1.9487	2.0138	0.0650	3.23
0.80	2.1436	2.2255	0.0820	3.68
0.90	2.3579	2.4596	0.1017	4.13
1.00	2.5937	2.7183	0.1245	4.58

$h = 0.05$

x_n	y_n	Actual Value	Abs. Error	% Rel. Error
0.00	1.0000	1.0000	0.0000	0.00
0.05	1.0500	1.0513	0.0013	0.12
0.10	1.1025	1.1052	0.0027	0.24
0.15	1.1576	1.1618	0.0042	0.36
0.20	1.2155	1.2214	0.0059	.48
0.25	1.2763	1.2840	0.0077	0.60
0.30	1.3401	1.3499	0.0098	0.72
0.35	1.4071	1.4191	0.01320	0.84
0.40	1.4775	1.4918	0.0144	0.96
0.45	1.5513	1.5683	0.0170	1.08
0.50	1.6289	1.6487	0.0198	1.20
0.55	1.7103	1.7333	0.0229	1.32
0.60	1.7959	1.8221	0.0263	1.44
0.65	1.8856	1.9155	0.299	1.56
0.70	1.9799	2.0138	0.0338	1.68
0.75	2.0789	2.1170	0.0381	1.80
0.80	2.1829	2.2255	0.0427	1.92
0.85	2.2920	2.3396	0.0476	2.04
0.90	2.4066	2.4596	0.0530	2.15
0.95	2.5270	2.5857	0.0588	2.27
1.00	2.6533	2.7183	0.0650	2.39

4. An integrating factor for $y' + 2y = 4x$ is e^{2x}. Thus we have

$$\begin{aligned} \frac{d}{dx}[e^{2x}y] &= 4xe^{2x} \\ e^{2x}y &= (2x-1)e^{2x} + c \\ y &= 2x - 1 + ce^{-2x} \end{aligned}$$

Substituting $x = 0$ and $y = 2$, we have $2 = -1 + c$ or $c = 3$. Therefore, the actual solution is $y = 2x - 1 + 3e^{-2x}$.

$h = 0.1$

x_n	y_n	Actual Value	Abs. Error	% Rel. Error
0.00	2.0000	2.0000	0.0000	0.00
0.10	1.6000	1.6562	0.0562	3.39
0.20	1.3200	1.4110	0.0910	6.45
0.30	1.1360	1.2464	0.1104	8.86
0.40	1.0288	1.1480	0.1192	10.38
0.50	0.9830	1.1036	0.1206	10.93

$h = 0.05$

x_n	y_n	Actual Value	Abs. Error	% Rel. Error
0.00	2.0000	2.0000	0.0000	0.00
0.05	1.8000	1.8145	0.0145	0.80
0.10	1.6300	1.6562	0.0262	1.58
0.15	1.4870	1.5225	0.0355	2.33
0.20	1.3683	1.4110	0.0427	3.02
0.25	1.2715	1.3196	0.0481	3.65
0.30	1.1943	1.2464	0.0521	4.18
0.35	1.1349	1.1898	0.0549	4.61
0.40	1.0914	1.1480	0.0566	4.93
0.45	1.0623	1.1197	0.0574	5.13
0.50	1.0460	1.1036	0.0576	5.22

5.

$h = 0.1$

x_n	y_n
0.00	0.0000
0.10	0.1000
0.20	0.1905
0.30	0.2731
0.40	0.3492
0.50	0.4198

$h = 0.1$

x_n	y_n
0.00	0.0000
0.10	0.1000
0.20	0.1905
0.30	0.2731
0.40	0.3492
0.50	0.4198

6.

$h = 0.1$

x_n	y_n
0.00	1.0000
0.10	1.1000
0.20	1.2220
0.30	1.3753
0.40	1.5735
0.50	1.8371

$h = 0.1$

x_n	y_n
0.00	1.0000
0.05	1.0500
0.10	1.1053
0.15	1.1668
0.20	1.2360
0.25	1.3144
0.30	1.4039
0.35	1.5070
0.40	1.6267
0.45	1.7670
0.50	1.9332

7.

$h = 0.1$

x_n	y_n
0.00	0.5000
0.10	0.5250
0.20	0.5431
0.30	0.5548
0.40	0.5613
0.50	0.5639

$h = 0.1$

x_n	y_n
0.00	0.5000
0.05	0.5125
0.10	0.5232
0.15	0.5322
0.20	0.5395
0.25	0.5452
0.30	0.5496
0.35	0.5527
0.40	0.5547
0.45	0.5559
0.50	0.5565

8.

$h = 0.1$

x_n	y_n
0.00	1.0000
0.10	1.1000
0.20	1.2159
0.30	1.3505
0.40	1.5072
0.50	1.6902

$h = 0.1$

x_n	y_n
0.00	1.0000
0.05	1.0500
0.10	1.1039
0.15	1.1619
0.20	1.2245
0.25	1.2921
0.30	1.3651
0.35	1.4440
0.40	1.5293
0.45	1.6217
0.50	1.7219

9.

$h = 0.1$

x_n	y_n
1.00	1.0000
1.10	1.0000
1.20	1.0191
1.30	1.0588
1.40	1.1231
1.50	1.2194

$h = 0.1$

x_n	y_n
1.00	1.0000
1.05	1.0000
1.10	1.0049
1.15	1.0147
1.20	1.0298
1.25	1.0506
1.30	1.0775
1.35	0.1115
1.40	1.1538
1.45	1.2057
1.50	1.2696

10.

$h = 0.1$

x_n	y_n
0.00	0.5000
0.10	0.5250
0.20	0.5499
0.30	0.5747
0.40	0.5991
0.50	0.6231

$h = 0.1$

x_n	y_n
0.00	0.5000
0.05	0.5125
0.10	0.5250
0.15	0.5375
0.20	0.5499
0.25	0.5623
0.30	0.5746
0.35	0.5868
0.40	0.5989
0.45	0.6109
0.50	0.6228

Chapter 8 in Review

A. True/False

1. True

2. True

3. True

4. True

B. Fill in the Blanks

1. $y = x - 3x^2 + 36e^{3x} + C$

2. two

3. $e^{\int -1dx} = e^{-x}$

4. -12

5. half-life

6. $2^{16}P_0$

7. $\dfrac{dP}{dt} = 0.16P,\ P(0) = P_0$

8. $\dfrac{dy}{dx} = x + xy$

C. Exercises

1. Separating variables, we obtain

$$\frac{1}{y}\,dy = -\cot x\,dx \quad\Longrightarrow\quad \ln|y| = -\ln|\sin x| + C \quad\Longrightarrow\quad y = C_1\csc x.$$

2. The equation is linear and an integrating factor is e^t, so

$$\frac{d}{dt}\left[e^t x\right] = \cos 2t \quad\Longrightarrow\quad e^t x = \frac{1}{2}\sin 2t + C \quad\Longrightarrow\quad x = \frac{1}{2}e^{-t}\sin 2t + Ce^{-t}.$$

3. Write the equation in the form $\dfrac{dy}{dt} = -\dfrac{5}{t}y = 1$. An integrating factor is $\dfrac{1}{t^5}$, so

$$\frac{d}{dt}\left[\frac{1}{t^5}y\right] = \frac{1}{t^5} \quad\Longrightarrow\quad \frac{1}{t^5}y = -\frac{1}{4t^4} + C \quad\Longrightarrow\quad y = -\frac{1}{4}t + Ct^5.$$

4. Separating variables, we obtain

$$ye^{-y^2}\,dy = -x^2e^{2x^3}\,dx \quad\Longrightarrow\quad -\frac{1}{2}e^{-y^2} = -\frac{1}{6}e^{2x^3} + C \quad\Longrightarrow\quad 3e^{-y^2} = e^{2x^3} + C_1.$$

5. Write the equation in the form $\dfrac{dy}{dx} + \dfrac{8x}{x^2+4}y = \dfrac{2x}{x^2+4}$. An integrating factor is $(x^2+4)^4$, so

$$\begin{aligned}\frac{d}{dx}\left[(x^2+4)^4y\right] = 2x(x^2+4)^3 \quad&\Longrightarrow\quad (x^2+4)^4y = \frac{1}{4}(x^2+4)^4 + C\\ &\Longrightarrow\quad y = \frac{1}{4} + C(x^2+4)^{-4}.\end{aligned}$$

6. Separating variables, we obtain

$$\begin{aligned}\cos^2 x\,dx = \frac{y}{y^2+1}\,dy \quad&\Longrightarrow\quad \frac{1}{2}x + \frac{1}{4}\sin 2x = \frac{1}{2}\ln(y^2+1) + C\\ &\Longrightarrow\quad 2x + \sin 2x = 2\ln(y^2+1) + C.\end{aligned}$$

7. From $\dfrac{1}{\sqrt{1-y^2}}dy = 2xdx$, we have $\sin^{-1}(y) = x^2 + c$ or $y = \sin(x^2+c)$

8. From $\dfrac{1}{y^2}ey = \dfrac{1}{e^x+e^{-x}}dx = \dfrac{e^x}{(e^x)^2+1}dx$ we obtain $-\dfrac{1}{y} = \tan^{-1}e^x + c$ or $y = -\dfrac{1}{\tan^{-1}e^x + c}$.

9. An integrating factor is e^{-2x} so that

$$\begin{aligned}\frac{d}{dx}[e^{-2x}y] &= e^{-2x}y\left(e^{3x} - e^{2x}\right) = x\left(e^x - 1\right)\\ e^{-2x}y &= (x-1)e^x - \frac{x^2}{2} + C\\ y &= (x-1)e^{3x} - \frac{x^2}{2}e^{2x} + Ce^{2x}\end{aligned}$$

10. An integrating factor is e^x so that

$$\begin{aligned}\frac{d}{dx}[e^x y] &= e^x\sqrt{1-e^x}\\ e^x y &= \frac{2}{3}(e^x+1)^{3/2}+C\\ y &= \frac{2}{3}e^{-x}(e^x+1)^{3/2}+Ce^{-x}\end{aligned}$$

11. The general solution is $P = ce^{0.05t}$. Substituting $t = 0$ and $P = 1000$, we have $1000 = c$ so that $P = 1000e^{0.05t}$.

12. The general solution is $A = ce^{-0.015t}$. Substituting $t = 0$ and $A = 5$, we have $5 = c$ so that $A = 5e^{-0.015t}$.

13. Write the equation in the form

$$\frac{dy}{dt}+\frac{1}{t}y = t^3\ln t.$$

An integrating factor is $e^{\ln t} = t$, so

$$\begin{aligned}\frac{d}{dt}[ty] &= t^4\ln t\\ ty &= -\frac{1}{25}t^5+\frac{1}{5}t^5\ln t+c\\ y &= -\frac{1}{25}t^4+\frac{1}{5}t^4\ln t+\frac{c}{t}\end{aligned}$$

Substituting $y = 1$ and $y = 0$, we have $c = \frac{1}{25}$ so that $y = -\frac{1}{25}y^4+\frac{1}{5}t^4\ln t+\frac{1}{25t}$

14. From $\dfrac{dy}{10y} = \dfrac{dx}{x}$, we have

$$\begin{aligned}\frac{1}{10}\ln|y| &= \ln|x|+c\\ \ln|y| &= 10\ln|x|+c_1\\ y &= c_2x^{10}\end{aligned}$$

Substituting $x = 1$ and $y = -3$, we have $-3 = c_2$ so that $y = -3x^{10}$.

15.

$$
\begin{aligned}
\frac{dy}{2y+y^2} &= dx \\
\int \frac{dy}{(2+y)y} &= \int dx \\
\int \frac{1}{2y} - \frac{1}{2(y+2)} dy &= \int dx \\
\frac{1}{2}\ln|y| - \frac{1}{2}\ln|y+2| &= x + c \\
\frac{1}{2}\ln\left|\frac{y}{y+2}\right| &= x + c \\
\ln\left|\frac{y}{y+2}\right| &= 2x + c_1 \\
\frac{y}{y+2} &= c_2 e^{2x}
\end{aligned}
$$

Solving for y we have $y = \dfrac{-2c_2 e^{2x}}{c_2 e^{2x} - 1}$. Substituting $x = 0$ and $y = 3$, we have $3 = \dfrac{-3c_2}{c_2 - 1}$ which yields $3c_2 - 3 = -2c_2$ or $c_2 = \frac{3}{5}$. Thus the solution is $y = \dfrac{-\frac{6}{5}e^{2x}}{\frac{3}{5}e^{2x} - 1}$.

16.

$$
\begin{aligned}
\frac{dy}{y(10-2y)} &= dx \\
\int \frac{1}{10y} - \frac{1}{10(y-5)} dy &= \int dx \\
\frac{1}{10}\ln\left|\frac{y}{y-5}\right| &= x + c \\
\ln\left|\frac{y}{y-5}\right| &= 10x + c_1 \\
\frac{y}{y-5} &= c_2 e^{10x} \\
y &= \frac{5c_2 e^{10x}}{c_2 e^{10x} - 1}
\end{aligned}
$$

Substituting $x = 0$ and $x = 7$, we have $7 = \dfrac{5c_2}{c_2 - 1}$ which yields $7c_2 - 7 = 5c_2$ or $c_2 = \dfrac{7}{2}$. Thus the solution is $y = \dfrac{\frac{35}{2}e^{10x}}{\frac{7}{2}e^{10x} - 1}$.

17. $\dfrac{dy}{1+y^2} = dx$ or $\tan^{-1}(y) = x + c$. Substituting $x = \frac{\pi}{3}$ and $y = -1$, we have

$$\begin{aligned}
\tan^{-1}(-1) &= \frac{\pi}{3} + c \\
\frac{-\pi}{4} &= \frac{\pi}{3} + c \\
\frac{-7\pi}{12} &= c
\end{aligned}$$

Thus $\tan^{-1}(t) = x - \frac{7\pi}{12}$ or $y = \tan\left(x - \frac{7\pi}{12}\right)$.

18.

$$\begin{aligned}
\frac{dy}{y^2-1} &= \frac{dx}{x} \\
\int \frac{1}{2(y-1)} - \frac{1}{2(y+1)} dy &= \int \frac{1}{x} dx \\
\frac{1}{2}\ln\left|\frac{y-1}{y+1}\right| &= \ln|x| + c \\
\ln\left|\frac{y-1}{y+1}\right| &= 2\ln|x| + c_1 \\
\frac{y-1}{y+1} &= c_2 x^2 \\
y &= \frac{-(c_2x^2+1)}{c_2x^2-1}
\end{aligned}$$

Substituting $x = 2$ and $y = 2$, we have $2 = \dfrac{-(4c_2+1)}{4c_2-1}$ or $c_2 = \dfrac{1}{12}$. Thus, the solution is $y = \dfrac{-\left(\frac{1}{12}x^2+1\right)}{\frac{1}{12}x^2-1}$.

19.

$$\begin{aligned}
\frac{dy}{y^2} &= -8x^2 dx \\
\frac{-1}{y} &= -\frac{8}{3}x^3 + c \\
y &= \frac{1}{\frac{8}{3}x^3 + c_1}
\end{aligned}$$

Substituting $x = 0$ and $y = \frac{1}{2}$, we have $\frac{1}{2} = \frac{1}{c_1}$ or $c_1 = 2$. Thus the solution is $y = \dfrac{1}{\frac{8}{3}x^3+2}$.

20.

$$\begin{aligned}
\frac{dy}{dx} &= e^x e^{-y} \\
e^y dy &= e^x dx \\
e^y &= e^x + c \\
y &= \ln(e^x + c)
\end{aligned}$$

Substituting $x = 0$ and $y = 1$, we have

$$\begin{aligned} 1 &= \ln(1+c) \\ e &= 1+c \\ e-1 &= c \end{aligned}$$

Thus, the solution is $y = \ln(e^x + e - 1)$.

21.

$$\begin{aligned} \frac{dy}{dx} &= \frac{2x}{3y^3} \\ 3y^3 dy &= 2x dx \\ \frac{3}{4}y^4 &= x^2 + c \\ y^4 &= \frac{4}{3}x^2 + c_1 \\ y &= \sqrt[4]{\frac{4}{3}x^2 + c_1} \end{aligned}$$

Substituting $x = 0$ and $y = 2$, we have $2 = \sqrt[4]{c_1}$ or $c_1 = 16$. Thus the solution is $y = \sqrt[4]{\frac{4}{3}x^2 + 16}$.

22.

$$\begin{aligned} \frac{dy}{dx} &= x + y \\ \frac{dy}{dx} - y &= x \end{aligned}$$

An integrating factor is e^{-x}. Therefore we have

$$\begin{aligned} \frac{d}{dx}[e^{-x}y] &= x \\ e^{-x}y &= \frac{x^2}{2} + c \\ y &= \frac{x^2}{2}e^x + ce^x \end{aligned}$$

Substituting $x = 0$ and $y = 1$, we have $1 = c$ so the solution is $y = \dfrac{x^2}{2}e^x + e^x$.

23. Writing $\dfrac{dP}{dt} = kP$ as $\dfrac{1}{P}\,dP = k\,dt$ and integrating, we obtain $\ln P = kt + C$ or $P = C_1e^{kt}$. Since $P(0) = P_0$, $P_0 = C_1$ and $P = P_0e^{kt}$. From $P(t_1) = P_1$ and $P(t_2) = P_2$, we have $P_1 = P_0e^{kt_1}$ and $P_2 = P_0e^{kt_2}$ or $\dfrac{P_1}{P_0} = e^{kt_1}$ and $\dfrac{P_2}{P_0} = e^{kt_2}$. Then $e^k = (P_1/P_0)^{1/t_1}$ and $e^k = (P_2/P_0)^{1/t_2}$ or $(P_1/P_0)^{1/t_1} = (P_2/P_0)^{1/t_2}$. Thus, $(P_1/P_0)^{t_2} = (P_2/P_0)^{t_1}$.

24. The temperature of the surrounding medium is $T_0 = 30°$ C and the initial temperature is $T(0) = 150°$ C. Solving $dT/dt = k(T - 30)$, we obtain $T = 30 + Ce^{kt}$. Since $T(0) = 150$,

$C = 120$ and $T = 30 + 120e^{kt}$. Using $T(1/4) = 90$, we have $90 = 30 + 120e^{k/4}$ or $1/2 = e^{k/4}$. Thus, $e^k = (1/2)^4 = 1/16$ and

$$\begin{aligned} T(1/2) &= 30 + 120e^{k/2} = 30 + 120(e^k)^{1/2} = 30 + 120(1/16)^{1/2} = 60° \text{ C} \\ T(1) &= 30 + 120e^k = 30 + 120(1/16) = 37.5° \text{ C}. \end{aligned}$$

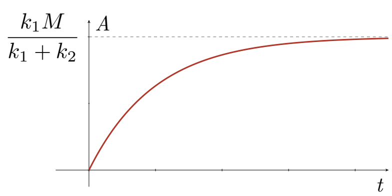
25. (a) Write the differential equation in the form $\dfrac{dA}{dt} + (k_1 + k_2)A = k_1 M$. Then an integrating factor is $e^{(k_1+k_2)t}$, and

$$\begin{aligned} \frac{d}{dt}\left[e^{(k_1+k_2)t}A\right] = k_1 M e^{(k_1+k_2)t} \quad &\Longrightarrow \quad e^{(k_1+k_2)t}A = \frac{k_1 M}{k_1+k_2}e^{(k_1+k_2)t} + C \\ &\Longrightarrow \quad A = \frac{k_1 M}{k_1+k_2} + Ce^{-(k_1+k_2)t}. \end{aligned}$$

Using $A(0) = 0$, we find $C = -\dfrac{k_1 M}{k_1+k_2}$ and $A = \dfrac{k_1 M}{k_1+k_2}\left[1 + e^{-(k_1+k_2)t}\right]$.

(b) As $t \to \infty$, $A \to \dfrac{k_1 M}{k_1+k_2}$. If $k_2 > 0$, the material will never be completely memorized.

(c)



26. Separating variables, we obtain

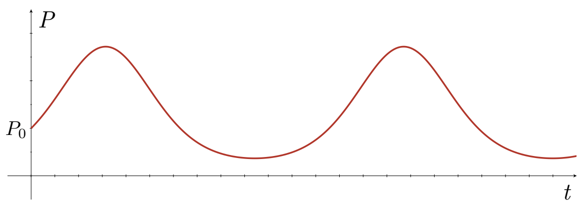
$$\begin{aligned} \frac{1}{E_0 - q/C}\,dq = \frac{1}{k_1 + k_2 t}\,dt \quad &\Longrightarrow \quad -C\ln\left|E_0 - \frac{q}{C}\right| = \frac{1}{k_2}\ln|k_1 + k_2 t| + c_1 \\ &\Longrightarrow \quad \frac{(E_0 - q/C)^{-C}}{(k_1 + k_2 t)^{1/k_2}} = c_2. \end{aligned}$$

Setting $q(0) = q_0$, we find $c_2 = \dfrac{(E_0 - q_0/C)^{-C}}{k_1^{1/k_2}}$, so

$$\begin{aligned} \frac{(E_0 - q/C)^{-C}}{(k_1 + k_2 t)^{1/k_2}} &= \frac{(E_0 - q_0/C)^{-C}}{k_1^{1/k_2}} \\ \left(E_0 - \frac{q}{C}\right)^{-C} &= \left(E_0 - \frac{q_0}{C}\right)^{-C}\left(\frac{k_1}{k + k_2 t}\right)^{-1/k_2} \\ E_0 - \frac{q}{C} &= \left(E_0 - \frac{q_0}{C}\right)\left(\frac{k_1}{k + k_2 t}\right)^{1/Ck_2} \\ q &= E_0 C + (q_0 - E_0 C)\left(\frac{k_1}{k + k_2 t}\right)^{1/Ck_2}. \end{aligned}$$

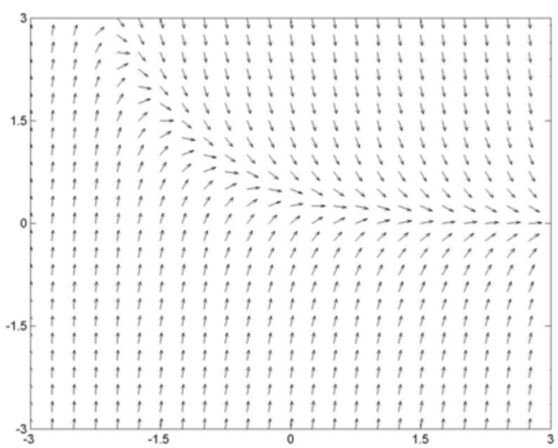
27. (a) Separating variables, we obtain $\dfrac{1}{P}\,dP = k\cos t\,dt$, so $\ln|P| = k\sin t + c$, and $P = c_1 e^{k\sin t}$. If $P(0) = P_0$ then $c_1 = P_0$ and $P = P_0 e^{k\sin t}$.

(b)

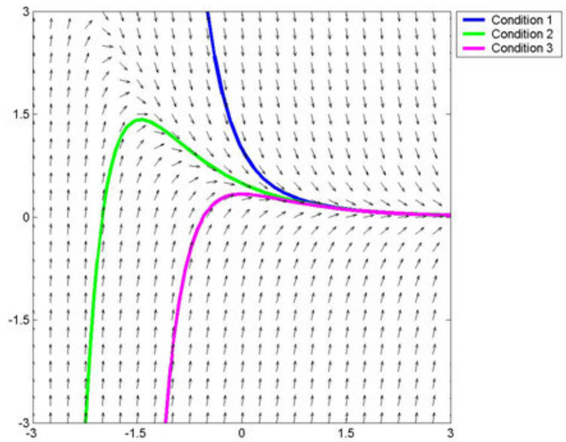


28. Let $\dfrac{dv}{dt} = v\dfrac{dv}{dy}$ so that $m\dfrac{dv}{dt} = -mg - kv^2$ becomes $mv\dfrac{dv}{dy} = -mg - kv^2$. Using $y(0) = 0$ and $v(0) = v_0$, it follows that $v^2 = \dfrac{mg + kv_0^2}{k}e^{-2ky/m} - \dfrac{mg}{k}$. If $v = 0$, then the maximum height is $h = \dfrac{m}{2k}\ln\dfrac{mg + kv_0^2}{mg}$. From $mv\dfrac{dv}{dy} = mg - kv^2$, $v(0) = 0$, and $y(0) = 0$, we find that $v^2 = \dfrac{mg}{k}(1 - e^{-2ky/m})$. Setting $y = h$, we see that the velocity at impact is $v = \dfrac{v_0}{\sqrt{1 + \dfrac{k}{mg}v_0^2}}$.

29. (a)



(b)



(c) As $x \to \infty$, $y(x) \to \infty$.
As $x \to -\infty$, $y(x) \to -\infty$.

30. (a) $\dfrac{dy}{dx} = (y-1)^2(y-3)^2$

(b) $\dfrac{dy}{dx} = y(y-2)^2(y-4)$

31. Using a CAS we find that the zero of f occurs at approximately $y = 1.3214$. From the graph we observe that $dy/dx > 0$ for $y < 1.3214$ and $dy/dt < 0$ for $y > 1.3214$, so $y = 1.3214$ is an asymptotically stable critical point. Thus, $\lim\limits_{t\to\infty} y(x) = 1.3214$.

32. The first step of Euler's method gives $y(1.1) \approx 9 + 0.1(1+3) = 9.4$. Applying Euler's method one more time gives $y(1.2) \approx 9.4 + 0.1(1 + 1.1\sqrt{9.4}) \approx 9.8373$.

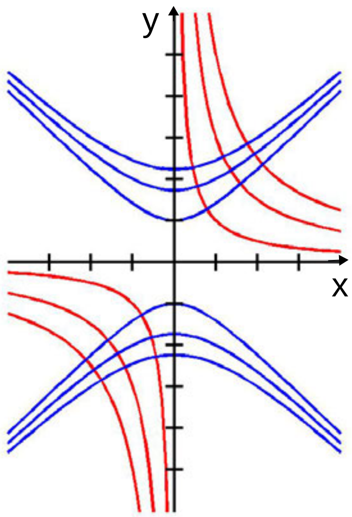
33. For the curve $y = x^3$, we have $\dfrac{dy}{dx} = 3x^2 = 3$ at both $(-1,-1)$ and $(1,1)$. For the curve $= x^2 + 3y^2 = 4$, we can differentiate implicitly to get $2x + 6y\dfrac{dy}{dx} = 0$ or $\dfrac{dy}{dx} = \dfrac{-x}{3y} = \dfrac{-1}{3}$ at both $(-1,-1)$ and $(1,1)$. Therefore, the slopes of the tangent lines are perpendicular at $(-1,-1)$ and $(1,1)$.

34. (a) For the curve $xy = c_1$, we differentiate implicitly to get $y + xy' = 0$ or $y' = \dfrac{-y}{x}$. For the curve $y^2 - x^2 = c_2$, we differentiate implicitly to get

$$\begin{aligned} 2yy' - 2x0 \\ yy' &= x \\ y' &= \frac{x}{y} \end{aligned}$$

Since the slopes are negative reciprocals of each other, we see that the families are orthogonal trajectories.

(b)



Chapter 9

Sequences and Series

9.1 Sequences

In this exercise set, the symbol "$\overset{h}{=}$" is used to denote the fact that L'Hôpital's Rule was applied to obtain the equality.

1. $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}$

2. $\frac{3}{2}, \frac{1}{2}, \frac{3}{10}, \frac{3}{14}$

3. $-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}$

4. $-\frac{1}{2}, \frac{4}{3}, -\frac{9}{4}, \frac{16}{5}$

5. 10, 100, 1000, 10000

6. $\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \frac{1}{10000}$

7. 2, 4, 12, 48

8. 2, 24, 720, 40320

9. $1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}$

10. $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}$

11. Let $\epsilon > 0$. Then $\left|\frac{1}{n} - 0\right| < \epsilon$ implies $\frac{1}{n} < \epsilon$ or $n > \frac{1}{\epsilon}$. Take N to be the smallest integer greater than $\frac{1}{\epsilon}$.

12. Let $\epsilon > 0$. Then $\left|\frac{1}{n^2} - 0\right| < \epsilon$ implies $\frac{1}{n^2} < \epsilon$ or $n > \frac{1}{\sqrt{\epsilon}}$. Take N to be the smallest integer greater than $\frac{1}{\sqrt{\epsilon}}$.

13. Let $\epsilon > 0$. Then $\left|\frac{n}{n+1} - 1\right| = \left|-\frac{1}{n+1}\right| < \epsilon$ implies $\frac{1}{n+1} < \epsilon$ or $n > \frac{1}{\epsilon} - 1$. Take N to be the smallest integer greater than $\frac{1}{\epsilon} - 1$.

14. Let $\epsilon > 0$. Then $\left|\frac{e^n+1}{e^n} - 1\right| = \left|\frac{1}{e^n}\right| < \epsilon$ implies $\frac{1}{e^n} < \epsilon$ or $n > \ln\frac{1}{\epsilon}$. Take N to be the smallest integer greater than $\ln\frac{1}{\epsilon}$.

15. $\lim\limits_{n\to\infty} \frac{10}{\sqrt{n+1}} = 0$

16. $\lim\limits_{n\to\infty} \frac{1}{n^{3/2}} = 0$

17. $\lim\limits_{n\to\infty} \frac{1}{5n+6} = 0$

18. $\lim\limits_{n\to\infty} \frac{4}{2n+7} = 0$

19. $\lim\limits_{n\to\infty} \frac{3n-2}{6n+1} = \lim\limits_{n\to\infty} \frac{3-2/n}{6+1/n} = \frac{1}{2}$

20. $\lim\limits_{n\to\infty} \frac{n}{1-2n} = \lim\limits_{n\to\infty} \frac{1}{1/n-2} = -\frac{1}{2}$

21. The terms alternate between 20 and -20. The sequence diverges.

22. $\lim\limits_{n\to\infty} \left(-\frac{1}{3}\right)^n = \lim\limits_{n\to\infty} \frac{(-1)^n}{3^n} = 0$

23. $\lim\limits_{n\to\infty} \frac{n^2-1}{2n} = \lim\limits_{n\to\infty} \frac{n-1/n}{2} = \infty$. The sequence diverges.

24. $\lim\limits_{n\to\infty} \frac{7n}{n^2+1} = \lim\limits_{n\to\infty} \frac{7}{n+1/n} = 0$

25. $\lim\limits_{n\to\infty} ne^{-n} = \lim\limits_{n\to\infty} \frac{n}{e^n} \stackrel{h}{=} \lim\limits_{n\to\infty} \frac{1}{e^n} = 0$

26. $\lim\limits_{n\to\infty} n^3e^{-n} = \lim\limits_{n\to\infty} \frac{n^3}{e^n} \stackrel{h}{=} \lim\limits_{n\to\infty} \frac{3n^2}{e^n} \stackrel{h}{=} \lim\limits_{n\to\infty} \frac{6n}{e^n} \stackrel{h}{=} \lim\limits_{n\to\infty} \frac{6}{e^n} = 0$

27. $\lim\limits_{n\to\infty} \frac{\sqrt{n+1}}{n} = \lim\limits_{n\to\infty} \frac{\sqrt{1+1/n}}{\sqrt{n}} = 0$

28. $\displaystyle\lim_{n\to\infty} \frac{n}{\sqrt{n+1}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{1+1/n}} = \infty$. The sequence diverges.

29. Since the terms alternate between -1 and 1, the sequence diverges.

30. Since $\sin n\pi = 0$ for all integers n, the sequence converges to 0.

31. $\displaystyle\lim_{n\to\infty} \frac{\ln n}{n} \stackrel{h}{=} \lim_{n\to\infty} \frac{1/n}{1} = 0$

32. $\displaystyle\lim_{n\to\infty} \frac{e^n}{\ln(n+1)} \stackrel{h}{=} \lim_{n\to\infty} \frac{e^n}{1/(n+1)} = \lim_{n\to\infty} e^n(n+1) = \infty$. The sequence diverges.

33. $\displaystyle\lim_{n\to\infty} \frac{5-2^{-n}}{7+4^{-n}} = \frac{5}{7}$

34. $\displaystyle\lim_{n\to\infty} \frac{2^n}{3^n+1} = \lim_{n\to\infty} \frac{1}{(3/2)^n + 1/2^n} = 0$

35. $\displaystyle\lim_{n\to\infty} \frac{e^n+1}{e^n} = \lim_{n\to\infty} \frac{1+1/e^n}{1} = 1$

36. $\displaystyle\lim_{n\to\infty} \left(4 + \frac{3^n}{2^n}\right) = \lim_{n\to\infty} \left[4 + \left(\frac{3}{2}\right)^n\right] = \infty$. The sequence diverges.

37. $\displaystyle\lim_{n\to\infty} n\sin\left(\frac{6}{n}\right) = \lim_{n\to\infty} \frac{\sin(6/n)}{1/n} \stackrel{h}{=} \lim_{n\to\infty} \frac{[\cos(6/n)](-6/n^2)}{-1/n^2} = \lim_{n\to\infty} 6\cos\left(\frac{6}{n}\right) = 6$

38. Let $y = (1-5/x)^x$. Then $\ln y = x\ln(1-5/x)$ and

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln(1-5/x)}{1/x} \stackrel{h}{=} \lim_{x\to\infty} \frac{\dfrac{5/x^2}{1-5/x}}{-1/x^2} = \lim_{x\to\infty} -\frac{5}{1-5/x} = -5.$$

Thus, $\displaystyle\lim_{n\to\infty} (1-5/n)^n = e^{-5}$.

39. $\displaystyle\lim_{n\to\infty} \frac{e^n - e^{-n}}{e^n + e^{-n}} = \lim_{n\to\infty} \frac{1-e^{-2n}}{1+e^{-2n}} = 1$

40. $\displaystyle\lim_{n\to\infty} \left(\frac{\pi}{4} - \arctan n\right) = \frac{\pi}{4} - \frac{\pi}{2} = -\frac{\pi}{4}$

41. Let $y = x^{2/(x+1)}$. Then $\ln y = \dfrac{2}{x+1}\ln x$ and $\displaystyle\lim_{x\to\infty} \frac{2\ln x}{x+1} \stackrel{h}{=} \lim_{x\to\infty} \frac{2/x}{1} = 0$. Thus, $\displaystyle\lim_{n\to\infty} n^{2/(n+1)} = e^0 = 1$.

42. $\displaystyle\lim_{n\to\infty} 10^{(n+1)/n} = \lim_{n\to\infty} 10^{(1+1/n)} = 10$

43. $\displaystyle\lim_{n\to\infty} \ln\left(\frac{4n+1}{3n-1}\right) = \ln\left(\lim_{n\to\infty} \frac{4+1/n}{3-1/n}\right) = \ln\frac{4}{3}$

44. $\displaystyle\lim_{n\to\infty}\frac{\ln n}{\ln 3n} \overset{h}{=} \lim_{n\to\infty}\frac{1/n}{1/n} = \lim_{n\to\infty} 1 = 1$

45. $$\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right) = \lim_{n\to\infty}\left[\left(\sqrt{n+1}-\sqrt{n}\right)\left(\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\right)\right] = \lim_{n\to\infty}\frac{n+1-n}{\sqrt{n+1}+\sqrt{n}}$$
$$= \lim_{n\to\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}} = 0$$

46. $$\lim_{n\to\infty}\sqrt{n}\left(\sqrt{n+1}-\sqrt{n}\right) = \lim_{n\to\infty}\left[\sqrt{n}\left(\sqrt{n+1}-\sqrt{n}\right)\left(\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\right)\right]$$
$$= \lim_{n\to\infty}\frac{\sqrt{n}(n+1-n)}{\sqrt{n+1}+\sqrt{n}} = \lim_{n\to\infty}\frac{1}{\sqrt{1+1/n}+1} = \frac{1}{2}$$

47. $\displaystyle\left\{\frac{2n}{2n-1}\right\} \qquad \lim_{n\to\infty}\frac{2n}{2n-1} = \lim_{n\to\infty}\frac{2n-1+1}{2n-1} = \lim_{n\to\infty}\left(1+\frac{1}{2n-1}\right) = 1$

48. $\displaystyle\left\{\frac{1}{n}+\frac{1}{n+1}\right\} \qquad \lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n+1}\right) = 0$

49. $\left\{(-1)^{n+1}(2n+1)\right\}$

    Since the terms alternate between values that are increasingly greater than and less than 0, the sequence diverges.

50. $\{(-1)^n 2\}$

    Since the terms alternate between -2 and 2, the sequence diverges.

51. $\displaystyle\left\{\frac{2}{3^{n-1}}\right\} \qquad \lim_{n\to\infty}\frac{2}{3^{n-1}} = 0$

52. $\displaystyle\left\{\frac{1}{n2^{n+1}}\right\} \qquad \lim_{n\to\infty}\frac{1}{n2^{n+1}} = 0$

53. $\displaystyle -\frac{1}{2}, -\frac{1}{4}, -\frac{1}{8}, -\frac{1}{16}, \cdots$

54. $3, 5, 9, 17, \cdots$

55. $\displaystyle 3, 1, \frac{1}{3}, \frac{1}{3}, \cdots$

56. $2, -8, -22, -20, \cdots$

57. $\displaystyle a_{n+1} = \frac{1}{4}a_n + 6 \quad\Longrightarrow\quad \lim_{n\to\infty} a_{n+1} = \frac{1}{4}\lim_{n\to\infty} a_n + 6 \quad\Longrightarrow\quad L = \frac{1}{4}L + 6 \quad\Longrightarrow\quad L = 8$

58. $$a_{n+1} = \frac{1}{2}\left(a_n + \frac{5}{a_n}\right) \quad\Longrightarrow\quad \lim_{n\to\infty} a_{n+1} = \frac{1}{2}\left(\lim_{n\to\infty} a_n + \frac{5}{\lim\limits_{n\to\infty} a_n}\right)$$
$$\Longrightarrow\quad L = \frac{1}{2}\left(L + \frac{5}{L}\right) \quad\Longrightarrow\quad L = \sqrt{5}$$

59. For $a_n = \dfrac{5^n}{n!}$, we have $a_{n+1} = \dfrac{5^{n+1}}{(n+1)!}$. Expanding, we get $a_{n+1} = \dfrac{5 \cdot 5^n}{(n+1) \cdot n!} = \dfrac{5}{n+1}\left(\dfrac{5^n}{n!}\right)$. The last factor of the last term is a_n, so $a_{n+1} = \dfrac{5}{n+1} a_n$.

60. Starting with $a_1 = \sqrt{3}$, we are given $a_2 = \sqrt{3 + a_1}$, $a_3 = \sqrt{3 + a_2}$, and so on. Thus, the recursion formula is $a_{n+1} = \sqrt{3 + a_n}$.

61. Let $a_n = 0$, $b_n = \dfrac{\sin^2 n}{4^n}$, and $c_n = \dfrac{1}{4^n}$. Then $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = 0$, so $\lim\limits_{n\to\infty} \dfrac{\sin^2 n}{4^n} = 0$.

62. Let $a_n = 4$, $b_n = \sqrt{16 + \dfrac{1}{n^2}}$, and $c_n = 4 + \dfrac{1}{n}$. [To see that $b_n \le c_n$, note that for $x, y \ge 0$, $x^2 + y^2 \le x^2 + 2xy + y^2 = (x+y)^2$. Thus, $\sqrt{x^2 + y^2} \le x + y$.] Then $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = 4$, so $\lim\limits_{n\to\infty} \sqrt{16 + \dfrac{1}{n^2}} = 4$.

63. Let $a_n = 0$, $b_n = \dfrac{\ln n}{n(n+2)}$, and $c_n = \dfrac{n}{n(n+2)} = \dfrac{1}{n+2}$. Then, for $n \ge 1$, $a_n \le b_n \le c_n$. Since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = 0$, we have $\lim\limits_{n\to\infty} \dfrac{\ln n}{n(n+2)} = 0$.

64. Let $a_n = 0$, $b_n = \dfrac{n!}{n^n} = \dfrac{1}{n}\left(\dfrac{2}{n} \cdot \dfrac{3}{n} \cdot \dfrac{4}{n} \cdots \dfrac{n}{n}\right)$, and $c_n = \dfrac{1}{n}$. [To see that $b_n \le c_n$, note that $\dfrac{1}{n} \le 1$ and for any $y \le 1$, $\dfrac{1}{n} y \le 1$.] Then $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = 0$, so $\lim\limits_{n\to\infty} \dfrac{n!}{n^n} = 0$.

65. Let $y = \left(1 + \dfrac{x}{t}\right)^t$. Then $\ln y = t \ln\left(1 + \dfrac{x}{t}\right)$ and using L'Hôpital's Rule,

$$\lim_{t\to\infty} \ln y = \lim_{t\to\infty} \frac{\ln(1 + x/t)}{1/t} = \lim_{t\to\infty} \frac{\dfrac{-x/t^2}{1 + x/t}}{-1/t^2} = \lim_{t\to\infty} \frac{x}{1 + x/t} = x.$$

Thus, $\lim\limits_{n\to\infty} \left(1 + \dfrac{x}{n}\right)^n = e^x$.

66. 1, 0.80685, 0.73472, 0.69704, 0.67390, 0.65824, 0.64695, 0.63842, 0.63174, 0.62638

(To compute the terms of this sequence, use the fact that it can be recursively defined by $a_{n+1} = a_n + \ln n + \dfrac{1}{n+1} - \ln(n+1)$, with $a_1 = 1$.)

67. Let a_n be the height of the ball on the n-th bounce. Then $a_0 = 15$, $a_1 = \dfrac{2}{3}(15) = 10$, $a_2 = \dfrac{2}{3}(10) = \dfrac{20}{3}$, $a_3 = \dfrac{2}{3}\left(\dfrac{20}{3}\right) = \dfrac{40}{9}$ ft, $a_n = 15\left(\dfrac{2}{3}\right)^n$.

68. Let a_n be the distance travelled during the n-th second. Then $a_1 = 16$, $a_2 = 48 = 16(3)$, $a_3 = 80 = 16(5)$, $a_4 = 16(7) = 112$, $a_5 = 16(9) = 144$, $a_6 = 16(11) = 176$ ft.

69. $A_1 = 15$, $A_2 = 15(0.2) + 15 = 18$, $A_3 = 18(0.2) + 15 = 18.6$, $A_4 = 18.6(0.2) + 15 = 18.72$, $A_5 = 18.72(0.2) + 15 = 18.744$, $A_6 = 18.744(0.2) + 15 = 18.7488$

70. First year: $1 + 1r = 1 + r$; second year: $(1 + r) + (1 + r)r = (1 + r)^2$; third year: $(1 + r)^2 + (1 + r^2)r = (1 + r)^3$

71. Parents: 2; grandparents: $2 \cdot 2 = 4$; great-grandparents: $2 \cdot 4 = 8$; great-great-grandparents: $2 \cdot 8 = 16$; great-great-great-grandparents: $2 \cdot 16 = 32$

72. $\lim_{n\to\infty} p_{n+1} = 3 \lim_{n\to\infty} p_n - \frac{1}{400} \lim_{n\to\infty} p_n^2$, so $K = 3K - \frac{1}{400}K^2$. Solving for K, we get

$$2K - \frac{1}{400}K^2 = 0 \quad \Longrightarrow \quad K\left(2 - \frac{1}{400}K\right) = 0 \quad \Longrightarrow \quad K = 800.$$

With $p_0 = 450$, we have $p_1 = 844$, $p_2 = 751$, $p_3 = 843$, $p_4 = 753$, $p_5 = 842$, $p_6 = 754$, $p_7 = 841$, $p_8 = 755$, $p_9 = 840$.

73. (a) $a_{n+1} = 1 + \dfrac{1}{1 + a_n}$

(b) $a_5 = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}}}$ $\qquad a_6 = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}}}}$

(c) Letting $L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$, we have $L = 1 + \dfrac{1}{1 + L}$, $L^2 + L = 1 + L + 1$, and $L^2 = 2$. Thus, $L = \sqrt{2} = \lim_{n\to\infty} a_n$, and the sequence converges to $\sqrt{2}$.

74. $a_1 \approx 1.7321$, $a_2 \approx 2.2795$, $a_3 \approx 2.6151$, $a_4 \approx 2.8009$, $a_5 \approx 2.8988$, $a_6 \approx 2.9489$, $a_7 \approx 2.9744$, $a_8 \approx 2.9872$, $a_9 \approx 2.9936$, $a_{10} \approx 2.9968$, ...

The sequence appears to converge to 3.

75. Since $\{a_n\}$ converges, then $\lim_{n\to\infty} a_n$ is some value L. Since the limit of a product is the same as the product of the factors' limits, we have $\lim_{n\to\infty} a_n^2 = \lim_{n\to\infty} (a_n \cdot a_n) = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} a_n) = L^2$. Thus, $\{a_n^2\}$ actually does converge.

76. (a) The area of each of the four triangles between the $(n-1)$st and n-th squares is $\frac{1}{8}A_n - 1$. Thus, the area of the n-th square is $A_{n-1} - 4\left(\frac{1}{8}\right)A_{n-1} = \frac{1}{2}A_{n-1}$. That is, $A_1 = 1$, $A_2 = \frac{1}{2}$, $A_3 = \frac{1}{4}$, $A_4 = \frac{1}{8}$, ..., and so on. In general, $A_n = \frac{1}{2^{n-1}}$.

(b) $S_1 = 1$, $S_2 = 1 + \frac{1}{2} = \frac{3}{2}$, $S_3 = \frac{3}{2} + \frac{1}{4} = \frac{7}{4}$, $S_4 = \frac{7}{4} + \frac{1}{8} = \frac{15}{8}$, $S_5 = \frac{15}{8} + \frac{1}{16} = \frac{31}{16}$, $S_6 = \frac{31}{16} + \frac{1}{32} = \frac{63}{32}$, $S_7 = \frac{63}{32} + \frac{1}{64} = \frac{127}{64}$, $S_8 = \frac{127}{64} + \frac{1}{128} = \frac{255}{128}$, $S_9 = \frac{255}{128} + \frac{1}{256} = \frac{511}{256}$, $S_{10} = \frac{511}{256} + \frac{1}{512} = \frac{1023}{512}$

(c) The S_n's appear to converge to 2.

77. (a) $P_1 = 3$, $P_2 = 3\left(\frac{4}{3}\right)$, $P_3 = 3\left(\frac{4}{3}\right)^2$, $P_4 = 3\left(\frac{4}{3}\right)^3$

(b) $P_n = 3\left(\frac{4}{3}\right)^{n-1}$

(c) $\lim\limits_{n\to\infty} P_n = \lim\limits_{n\to\infty}\left[3\left(\frac{4}{3}\right)^{n-1}\right] = \infty$

78.

end of month	8	9	10	11	12
adult pairs	34	55	89	144	233
baby pairs	21	34	55	89	144
total pairs	55	89	144	233	377

In the bottom three rows of the table, each number (after the second one) is the sum of the two preceding numbers in that row.

79. 2, 3, 5, 8, 13, $\cdots$

The recursion formula matches the pattern in Problem 78. Not surprisingly, the resulting sequence is called the *Fibonacci sequence*.

80. (a) $$a_n = 1 + \frac{1}{a_{n-1}} \implies \lim_{n\to\infty} a_n = 1 + \frac{1}{\lim\limits_{n\to\infty} a_{n-1}}$$

$$\implies \phi = 1 + \frac{1}{\phi} \implies \phi^2 - \phi - 1 = 0$$

By the quadratic formula, the solutions are $\dfrac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} = \dfrac{1 \pm \sqrt{5}}{2}$.

The limit must be positive, so $\phi = \dfrac{1+\sqrt{5}}{2}$.

(b) Since this portion of the exercise involves a research report, no solution is given. The relationship between ϕ and the shape of the multi-chambered nautilus shell lies in how the shell's successive spirals grow at a rate that approximates ϕ for every quarter turn that they make.

9.2 Monotonic Sequences

1. $a_{n+1} - a_n = \dfrac{n+1}{3n+4} - \dfrac{n}{3n+1} = \dfrac{1}{(3n+4)(3n+1)} > 0$. The sequence is increasing.

2. $a_{n+1} - a_n = \dfrac{11+n}{n+1} - \dfrac{10+n}{n} = \dfrac{-10}{n(n+1)} < 0$. The sequence is decreasing.

3. $a_1 = -1$, $a_2 = \sqrt{2}$, $a_3 = -\sqrt{3}$. The sequence is not monotonic.

4. $a_1 = 0$, $a_2 = 0$. Let $f(x) = (x-1)(x-2) = x^2 - 3x + 2$. Then $f'(x) = 2x - 3 > 0$ for $x \le 2$. The sequence is nondecreasing.

5. Let $f(x) = \dfrac{e^x}{x}$. Then $f'(x) = \dfrac{(x-1)e^x}{x^2} > 0$ for $x > 1$. The sequence is increasing.

6. Let $f(x) = \dfrac{e^x}{x^5}$. Then $f'(x) = \dfrac{(x-5)e^x}{x^6}$. Since $f'(x) < 0$ for $x < 5$ and $f'(x) > 0$ for $x > 5$, the sequence is not monotonic.

7. $a_1 = 2$, $a_2 = 2$, $\dfrac{a_{n+1}}{a_n} = \dfrac{2^{n+1}/(n+1)!}{2^n/n!} = \dfrac{2}{n+1} < 1$ for $n > 1$.

   The sequence is nonincreasing.

8. $\dfrac{a_{n+1}}{a_n} = \dfrac{2^{2n+2}[(n+1)!]^2/(2n+2)!}{2^{2n}(n!)^2/(2n)!} = \dfrac{4(n+1)^2}{(2n+2)(2n+1)} = \dfrac{4n^2+8n+4}{4n^2+6n+2} > 1$

   The sequence is increasing.

9. Let $f(x) = x + \dfrac{1}{x}$. Then $f'(x) = 1 - \dfrac{1}{x^2} > 0$ for $x > 1$. The sequence is increasing.

10. $a_1 = 0$, $a_2 = 6$, $a_3 = 6$

$$\begin{aligned} a_{n+1} - a_n &= (n+1)^2 + (-1)^{n+1}(n+1) - n^2 - (-1)^n n \\ &= n^2 + 2n + 1 - (-1)^n n - (-1)^n - n^2 - (-1)^n n \\ &= 2n + 1 - 2n(-1)^n - (-1)^n = (2n+1)[1 - (-1)^n] \ge 0 \end{aligned}$$

   The sequence is nondecreasing.

11. Since $n < \pi$ for $n = 1, 2, 3$, we have $\sin n > 0$ for $n = 1, 2, 3$. Since $\pi < n < 2\pi$ for $n = 4, 5$, we have $\sin n < 0$ for $n = 4, 5$. Thus, $a_3 > 0$, $a_4 < 0$, and $a_5 > 0$. The sequence is not monotonic.

12. $a_{n+1} - a_n = \ln\left(\dfrac{n+3}{n+2}\right) - \ln\left(\dfrac{n+2}{n+1}\right) = \ln\left[\dfrac{(n+3)(n+1)}{(n+2)^2}\right] = \ln\left(\dfrac{n^2+4n+3}{n^2+4n+4}\right)$

   Since $\dfrac{n^2+4n+3}{n^2+4n+4} < 1$, $\ln\left(\dfrac{n^2+4n+3}{n^2+4n+4}\right) < 0$ and the sequence is decreasing.

13. Since $a_{n+1} - a_n = \dfrac{4n+3}{5n+7} - \dfrac{4n-1}{5n+2} = \dfrac{13}{(5n+7)(5n+2)} > 0$, the sequence is monotonic. Using $\dfrac{4n-1}{5n+2} > 0$ and $\dfrac{4n-1}{5n+2} < \dfrac{4n}{5n+2} < \dfrac{4n}{5n} = \dfrac{4}{5}$, we see that the sequence is bounded. Thus, the sequence converges.

14. Since $a_{n+1} - a_n = \dfrac{6-4(n+1)^2}{1+(n+1)^2} - \dfrac{6-4n^2}{1+n^2} = \dfrac{10[n^2-(n+1)^2]}{[1+(n+1)^2][1+n^2]} < 0$, the sequence is monotonic. Using $\dfrac{6-4n^2}{1+n^2} < 6$ and $\dfrac{6-4n^2}{1+n^2} > \dfrac{-4-4n^2}{1+n^2} = \dfrac{-4(1+n^2)}{1+n^2} = -4$, we see that the sequence is bounded. Thus, the sequence converges.

15. Since $\dfrac{a_{n+1}}{a_n} = \dfrac{3^{n+1}/(1+3^{n+1})}{3^n/(1+3^n)} = \dfrac{3+3^{n+1}}{1+3^{n+1}} = 1 + \dfrac{2}{1+3^{n+1}} > 1$, the sequence is monotonic. Using $\dfrac{3^n}{1+3^n} > 0$ and $\dfrac{3^n}{1+3^n} = 1 - \dfrac{1}{1+3^n} < 1$, we see that the sequence is bounded. Thus, the sequence converges.

16. Since $\dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)/5^{n+1}}{n/5^n} = \dfrac{n+1}{5n} = \dfrac{1}{5}\left(1+\dfrac{1}{n}\right) < 1$, the sequence is monotonic. Using $\dfrac{n}{5^n} > 0$ and $\dfrac{n}{5^n} \le \dfrac{1}{5}$ (since the sequence is decreasing), we see that the sequence is bounded. Thus, the sequence converges.

17. Let $f(x) = e^{1/x}$. Then $f'(x) = \dfrac{-e^{1/x}}{x^2} < 0$ and the sequence is monotonic. Using $e^{1/n} > 0$ and $e^{1/n} \le e$ (since the sequence is decreasing), we see that the sequence is bounded. Thus, the sequence converges.

18. Since $\dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)!/(n+1)^{n+1}}{n!/n^n} = \dfrac{(n+1)n^n}{(n+1)^{n+1}} = \left(\dfrac{n}{n+1}\right)^n < 1$, the sequence is monotonic. Using $\dfrac{n!}{n^n} > 0$ and $\dfrac{n!}{n^n} \le 1$ (since the sequence is decreasing), we see that the sequence is bounded. Thus, the sequence converges.

19. Since $\dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)!/[1\cdot 3\cdot 5\cdots(2n+1)]}{n!/[1\cdot 3\cdot 5\cdots(2n-1)]} = \dfrac{n+1}{2n+1} < 1$, the sequence is monotonic. Using $\dfrac{n!}{1\cdot 3\cdot 5\cdots(2n-1)} > 0$ and $\dfrac{n!}{1\cdot 3\cdot 5\cdots(2n-1)} \le 1$ (since the sequence is decreasing), we see that the sequence is bounded. Thus, the sequence converges.

20. Since $\dfrac{a_{n+1}}{a_n} = \dfrac{[2\cdot 4\cdot 6\cdots(2n+2)]/[1\cdot 3\cdot 5\cdots(2n+3)]}{[2\cdot 4\cdot 6\cdots 2n]/[1\cdot 3\cdot 5\cdots(2n+1)]} = \dfrac{2n+2}{2n+3} < 1$, the sequence is monotonic. Using $\dfrac{2\cdot 4\cdot 6\cdots 2n}{1\cdot 3\cdot 5\cdots(2n+1)} > 0$ and $\dfrac{2\cdot 4\cdot 6\cdots 2n}{1\cdot 3\cdot 5\cdots(2n+1)} \le \dfrac{2}{3}$ (since the sequence is decreasing), we see that the sequence is bounded. Thus, the sequence converges.

21. Let $f(x) = \tan^{-1} x$. Then $f'(x) = \dfrac{1}{1+x^2} > 0$ and the sequence is monotonic. Since $|\tan^{-1} n| < \dfrac{\pi}{2}$, we see that the sequence is bounded. Thus, the sequence converges.

22. Let $f(x) = \dfrac{\ln(x+3)}{x+3}$. Then $f'(x) = \dfrac{1-\ln(x+3)}{(x+3)^2} < 0$ for $x \ge 0$ and the sequence is monotonic. Using $\dfrac{\ln(n+3)}{n+3} > 0$ and $\dfrac{\ln(n+3)}{n+3} < \dfrac{\ln 4}{4}$ (since the sequence is decreasing), we see that the sequence is bounded. Thus, the sequence converges.

23. The sequence is $\{(0.8)^n\}$. Since $\dfrac{a_{n+1}}{a_n} = \dfrac{(0.8)^{n+1}}{(0.8)^n} = 0.8 < 1$, the sequence is monotonic. Using $(0.8)^n > 0$ and $(0.8)^n \le 0.8$ (since the sequence is decreasing), we see that the sequence is bounded. Thus, the sequence converges.

24. The sequence is $\{3^{1/2^n}\}$. When $x > 1$, $\sqrt{x} > 1$, so $\dfrac{a_{n+1}}{a_n} = \dfrac{3^{1/2^{n+1}}}{3^{1/2^n}} = \dfrac{1}{3^{1/2^{n+1}}} < 1$ and the sequence is monotonic. Using $3^{1/2^n} > 0$ and $3^{1/2^n} \leq \sqrt{3}$ (since the sequence is decreasing), we see that the sequence is bounded. Thus, the sequence converges.

25. $a_{n+1} = \dfrac{1}{2}a_n + 5$, $a_1 = 1$. We will show that $a_n < 10$ for all n.

    For $n = 1$, we have $a_2 = \dfrac{11}{2} < 10$. Assume that $a_k < 10$. Then $a_{k+1} = \dfrac{1}{2}a_k + 5 < \dfrac{1}{2}(10) + 5 = 10$; that is, $a_{k+1} < 10$ whenever $a_k < 10$. The sequence is bounded because $0 < a_n < 10$.

    Next, we will show that the sequence $\{a_n\}$ is monotonic. Because $a_n < 10$, necessarily $\dfrac{1}{2}a_n < \dfrac{1}{2} \cdot 10 = 5$. Therefore, from the recursion formula,

$$a_{n+1} = \frac{1}{2}a_n + 5 > \frac{1}{2}a_n + \frac{1}{2}a_n = a_n.$$

    This shows that $a_{n+1} > a_n$ for all n, and so the sequence is increasing.

    Since $\{a_n\}$ is bounded and monotonic, it follows from Theorem 9.2.1 that the sequence converges. Because we must have $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{n\to\infty} a_{n+1} = L$, the limit of the sequence can be determined from the recursion formula:

$$\lim_{n\to\infty} a_{n+1} = \frac{1}{2}\lim_{n\to\infty} a_n + 5 \quad \Longrightarrow \quad L = \frac{1}{2}L + 5 \quad \Longrightarrow \quad L = 10.$$

26. $a_{n+1} = \sqrt{2 + a_n}$, $a_1 = 0$. We will show that $a_n < 2$ for all n.

    For $n = 1$, we have $a_2 = \sqrt{2} < 2$. Assume that $a_k < 2$. Then $a_{k+1} = \sqrt{2 + a_k} < \sqrt{2 + 2} = 2$; that is, $a_{k+1} < 2$ whenever $a_k < 2$. The sequence is bounded because $0 < a_n < 2$.

    Also, the sequence is increasing:

$$a_{n+1} = \sqrt{2 + a_n} > \sqrt{a_n + a_n} = \sqrt{2a_n} > \sqrt{a_n a_n} = a_n.$$

    That is, $a_{n+1} > a_n$.

    By Theorem 9.2.1, the sequence is bounded and monotonic, and so it is convergent. From

$$\lim_{n\to\infty} a_{n+1} = \frac{1}{2}\lim_{n\to\infty} \sqrt{2 + a_n} \quad \Longrightarrow \quad L = \sqrt{2 + L} \quad \Longrightarrow \quad (L-2)(L+1) = 0,$$

    we have $L = 2$.

27. $a_{n+1} = \sqrt{7a_n}$, $a_1 = \sqrt{7}$. Now $0 < a_n < 7 \implies \sqrt{a_n} < \sqrt{7}$, and so

$$a_{n+1} = \sqrt{7a_n} = \sqrt{7}\sqrt{a_n} > \sqrt{a_n}\sqrt{a_n} = a_n.$$

    Thus, $a_{n+1} > a_n$ for all n. The sequence is therefore monotonic (increasing) and bounded. By Theorem 9.2.1, the sequence converges. From

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{7a_n} \quad \Longrightarrow \quad L = \sqrt{7L} \quad \Longrightarrow \quad L^2 = 7L \quad \Longrightarrow \quad L(L-7) = 0,$$

    we have $L = 7$.

28. Since $\dfrac{a_{n+1}}{a_n} = 1 - \dfrac{1}{n^2} < 1$ for $n \geq 2$, the sequence is monotonic. Using $a_n > 0$ and $a_n \leq 2$ (since the sequence is decreasing), we see that the sequence is bounded. Thus, the sequence converges. Taking the limit of the recursion formula shows nothing, since

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\left(1 - \frac{1}{n^2}\right)a_n = \lim_{n\to\infty} a_n \qquad \Longrightarrow \qquad L = L.$$

29. (a) If $\lim_{n\to\infty} p_n = L$, then $\lim_{n\to\infty} p_{n+1} = L$ and $\lim_{n\to\infty} \dfrac{bp_n}{a+p_n} = \dfrac{bL}{a+L}$. Thus, $L = \dfrac{bL}{a+L}$ or $(a+L-b)L = 0$ and $L = 0$ or $L = b-a$.

(b) Since $p_n > 0$, $p_{n+1} = \dfrac{bp_n}{a+p_n} = \dfrac{b}{a+p_n}p_n < \dfrac{b}{a}p_n$.

(c) If $a > b$ then $\dfrac{b}{a} < 1$ and by part (b), $p_{n+1} < p_n$. The sequence is thus monotonically decreasing. Now $p_1 < \dfrac{b}{a}p_0$ implies $p_2 < \dfrac{b}{a}p_1 < \dfrac{b}{a}\left(\dfrac{b}{a}p_0\right) = \left(\dfrac{b}{a}\right)^2 p_0$, which in turn implies $p_3 < \dfrac{b}{a}p_2 < \dfrac{b}{a}\left[\left(\dfrac{b}{a}\right)^2 p_0\right] = \left(\dfrac{b}{a}\right)^3 p_0$. In general, $p_{n+1} < \left(\dfrac{b}{a}\right)^{n+1} p_0$. Since $\dfrac{b}{a} < 1$, $\lim_{n\to\infty} p_n = \lim_{n\to\infty} p_{n+1} \leq \lim_{n\to\infty}\left(\dfrac{b}{a}\right)^{n+1} p_0 = 0$. Since $p_n > 0$ for all n, $\lim_{n\to\infty} p_n = 0$.

(d) We first note that

$$|b-a-p_{n+1}| = \left|b-a-\frac{bp_n}{a+p_n}\right| = \left|\frac{ab+bp_n-a^2-ap_n-bp_n}{a+p_n}\right| = \frac{a}{a+p_n}|b-a-p_n|.$$

Since $0 < \dfrac{a}{a+p_n} < 1$, the distance from p_{n+1} to $b-a$ is less than the distance from p_n to $b-a$. This means that p_{n+1} is between $b-a$ and p_n. Thus, if $0 < p_0 < b-a$, the sequence $\{p_n\}$ is increasing and bounded above by $b-a$. If $0 < b-a < p_0$, the sequence $\{p_n\}$ is decreasing and bounded below by $b-a$. In either case, it follows from part (a) that the sequence converges to $b-a$.

30. $\{(-1)^n\}$ is bounded but not convergent.

31. Since $\{a_n\}$ is convergent, it follows from Definition 9.1.2 that there exists an N such that $|a_n - L| < 1$ whenever $n > N$. Adding $|L|$ to both sides, we have $|a_n - L| + |L| < 1 + |L|$. By the triangle inequality, $|(a_n - L) + L| = |a_n| \leq |a_n - L| + |L|$, and so $|a_n| \leq |a_n - L| + |L| < 1 + |L|$ for all $n > N$. For $n \leq N$, we have a finite set of numbers that therefore has a maximum value M and minimum value m; thus, $\{a_n\}$ is bounded.

32. Since $a_{n+1} - a_n = \displaystyle\int_1^{n+1} e^{-t^2}\,dt - \int_1^{n} e^{-t^2}\,dt = \int_n^{n+1} e^{-t^2}\,dt > 0$, the series is monotonic.

Using $\int_1^n e^{-t^2}\,dt \geq 0$ and

$$\int_1^n e^{-t^2}\,dt < \int_1^\infty e^{-t^2}\,dt \leq \int_1^\infty e^{-t}\,dt = \lim_{k\to\infty}\int_1^k e^{-t}\,dt = \lim_{k\to\infty}(-e^{-t})\Big]_1^k$$
$$= \lim_{k\to\infty}(e^{-1} - e^{-k}) = e^{-1},$$

we see that the sequence is bounded. Thus, the sequence converges.

33. Note that $a_1 = 1$ and assume $n \geq 2$. Then the area under the graph of $y = \dfrac{1}{x}$ on $[1, n]$ is $A = \int_1^n \dfrac{1}{x}\,dx = \ln x\Big]_1^n = \ln n$.

Partitioning $[1, n]$ at 1, 2, 3, …, n, the upper sum is $U = 1 + \dfrac{1}{2} + \cdots + \dfrac{1}{n-1}$ and the lower sum is $L = \dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{n}$. Since $L < A < U$, we have, for $n \geq 2$, $\dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{n} < \ln n < 1 + \dfrac{1}{2} + \cdots + \dfrac{1}{n-1}$. Now, $a_n = 1 + \dfrac{1}{2} + \cdots + \dfrac{1}{n} - \ln n$, so $\dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{n} = a_n - 1 + \ln n$ and $1 + \dfrac{1}{2} + \cdots + \dfrac{1}{n-1} = a_n - \dfrac{1}{n} + \ln n$. Thus, for $n \geq 2$, $a_n - 1 + \ln n < \ln n < a_n - \dfrac{1}{n} + \ln n$ and $a_n - 1 < 0 < a_n - \dfrac{1}{n}$, or $a_n < 1$ and $a_n > \dfrac{1}{n} > 0$. Since $a_1 = 1$, the sequence is bounded below by 0 and above by 1. To see that the sequence is monotonic, note that $a_n - 1 + \ln n < \ln n$ implies $a_{n+1} - 1 + \ln(n+1) < \ln(n+1)$. Subtracting, we have $[a_{n+1} - 1 + \ln(n+1)] - (a_n - 1 + \ln n) < \ln(n+1) - \ln n$ or $a_{n+1} - a_n < 0$. Since the sequence is bounded and monotonic, it is convergent.

9.3 Series

1. $3 + \dfrac{5}{2} + \dfrac{7}{3} + \dfrac{9}{4} + \cdots$

2. $2 + 2 + \dfrac{8}{3} + 4 + \cdots$

3. $\dfrac{1}{2} - \dfrac{1}{6} + \dfrac{1}{12} - \dfrac{1}{20} + \cdots$

4. $\dfrac{1}{3} - \dfrac{1}{8} + \dfrac{1}{81} - \dfrac{1}{324} + \cdots$

5. $1 + 2 + \dfrac{3}{2} + \dfrac{2}{3} + \cdots$

6. $1 + \dfrac{24}{5} + 72 + \dfrac{40,320}{17} + \cdots$

7. $2 + \dfrac{8}{3} + \dfrac{16}{5} + \dfrac{128}{35} + \cdots$

8. $1+\frac{3}{2}+\frac{5}{2}+\frac{35}{8}\cdots$

9. $-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\cdots$

10. $5+0-7+0-\cdots$

11. Write $a_k=\frac{1}{k}-\frac{1}{k+1}$. Then $S_n=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)=1-\frac{1}{n+1}$ and $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k(k+1)}=\lim_{n\to\infty}S_n=1$.

12. Write $a_k=\frac{1}{k+1}-\frac{1}{k+2}$. Then
$$S_n=\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n+1}-\frac{1}{n+2}\right)=\frac{1}{2}-\frac{1}{n+2}$$
and $\displaystyle\sum_{k=1}^{\infty}\frac{1}{(k+1)(k+2)}=\lim_{n\to\infty}S_n=\frac{1}{2}$.

13. Write $a_k=\frac{1/2}{2k-1}-\frac{1/2}{2k+1}$. Then
$$S_n=\frac{1}{2}\left[\left(1-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\cdots+\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)\right]=\frac{1}{2}\left(1-\frac{1}{2n+1}\right)$$
and $\displaystyle\sum_{k=1}^{\infty}\frac{1}{4k^2-1}=\lim_{n\to\infty}S_n=\frac{1}{2}$.

14. Write $a_k=\frac{1}{(k+3)(k+4)}=\frac{1}{k+3}-\frac{1}{k+4}$. Then
$$S_n=\left(\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\cdots+\left(\frac{1}{n+3}-\frac{1}{n+4}\right)=\frac{1}{4}-\frac{1}{n+4}$$
and $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2+7k+12}=\lim_{n\to\infty}S_n=\frac{1}{4}$.

15. Identify $r=\frac{1}{5}$ and $a=3$. The series converges to $\frac{3}{1-1/5}=\frac{15}{4}$.

16. Identify $r=\frac{3}{4}$ and $a=10$. The series converges to $\frac{10}{1-3/4}=40$.

17. Identify $r=-\frac{1}{2}$ and $a=1$. The series converges to $\frac{1}{1+1/2}=\frac{2}{3}$.

18. $\displaystyle\sum_{k=1}^{\infty}\pi^k\left(\frac{1}{3}\right)^{k-1}=\sum_{k=1}^{\infty}\pi\left(\frac{\pi}{3}\right)^{k-1}$. Identify $a=\pi$ and $r=\frac{\pi}{3}>1$. The series diverges.

19. The common ratio is $\dfrac{5}{4} > 1$. The series diverges.

20. Identify $t = -\dfrac{3}{7}$ and $a = -\dfrac{3}{7}$. The series converges to $\dfrac{-3/7}{1+3/7} = -\dfrac{3}{10}$.

21. Identify $r = 0.9$ and $a = 900$. The series converges to $\dfrac{900}{1-0.9} = 9000$.

22. Identify $r = 1.1 > 1$. The series diverges.

23. Identify $r = \dfrac{1}{\sqrt{3}-\sqrt{2}} > 1$. The series diverges.

24. Identify $r = \dfrac{\sqrt{5}}{1+\sqrt{5}}$ and $a = 1$. The series converges to $\dfrac{1}{1-\sqrt{5}/(1+\sqrt{5})} = 1+\sqrt{5}$.

25. $0.222\ldots = 0.2+0.02+0.002+\cdots$. Identify $r = 0.1$ and $a = 0.2$. Then $0.222\ldots = \dfrac{0.2}{1-0.1} = \dfrac{2}{9}$.

26. $0.555\ldots = 0.5+0.05+0.005+\cdots$. Identify $r = 0.1$ and $a = 0.5$. Then $0.555\ldots = \dfrac{0.5}{1-0.1} = \dfrac{5}{9}$.

27. $0.616161\ldots = 0.61 + 0.0061 + 0.000061 + \cdots$. Identify $r = 0.01$ and $a = 0.61$. Then $0.616161\ldots = \dfrac{0.61}{1-0.01} = \dfrac{61}{99}$.

28. $0.393939\ldots = 0.39 + 0.0039 + 0.000039 + \cdots$. Identify $r = 0.01$ and $a = 0.39$. Then $0.393939\ldots = \dfrac{0.39}{1-0.01} = \dfrac{39}{99} = \dfrac{13}{33}$.

29. $1.314314\ldots = 1 + (0.314 + 0.000314 + \cdots)$. Identify $r = 0.01$ and $a = 0.314$. Then $1.314314\ldots = 1 + \dfrac{0.314}{1-0.001} = 1 + \dfrac{314}{999} = \dfrac{1313}{999}$.

30. $0.5262626\ldots = 0.5 + (0.026 + 0.00026 + \cdots$. Identify $r = 0.01$ and $a = 0.026$. Then $0.5262626\ldots = \dfrac{5}{10} + \dfrac{0.26}{1-0.01} = \dfrac{5}{10} + \dfrac{26}{990} = \dfrac{521}{990}$.

31. $\displaystyle\sum_{k=1}^{\infty}\left[\left(\frac{1}{3}\right)^{k-1} + \left(\frac{1}{4}\right)^{k-1}\right] = \sum_{k=1}^{\infty}\left(\frac{1}{3}\right)^{k-1} + \sum_{k=1}^{\infty}\left(\frac{1}{4}\right)^{k-1} = \frac{1}{1-1/3} + \frac{1}{1-1/4} = \frac{3}{2} + \frac{4}{3} = \frac{17}{6}$

32. $\displaystyle\sum_{k=1}^{\infty}\frac{2^k-1}{4^k} = \sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^{k} - \sum_{k=1}^{\infty}\left(\frac{1}{4}\right)^{k} = \frac{1/2}{1-1/2} - \frac{1/4}{1-1/4} = 1 - \frac{1}{3} = \frac{2}{3}$

33. $\displaystyle\lim_{k\to\infty} a_k = \lim_{k\to\infty} 10 \neq 0$, so the series diverges.

34. $\displaystyle\lim_{k\to\infty} a_k = \lim_{k\to\infty} (5k+1) = \infty$, so the series diverges.

35. $\displaystyle\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k}{2k+1} = \lim_{k\to\infty} \frac{1}{2+1/k} = \frac{1}{2} \neq 0$, so the series diverges.

36. $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \dfrac{k^2+1}{k^2+2k+3} = \lim_{k\to\infty} \dfrac{1+1/k^2}{1+2/k+3/k^2} = 1 \neq 0$, so the series diverges.

37. $\lim_{k\to\infty} a_k = \lim_{k\to\infty} (-1)^k$ does not exist, so the series diverges.

38. $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \ln\left(\dfrac{k}{3k+1}\right) = \ln\left(\lim_{k\to\infty} \dfrac{k}{3k+1}\right) = \ln\left(\lim_{k\to\infty} \dfrac{1}{3+1/k}\right) = \ln\dfrac{1}{3} \neq 0$, so the series diverges.

39. $10\sum_{k=1}^{\infty} \dfrac{1}{k}$ diverges because the harmonic series diverges. Thus $\sum_{k=1}^{\infty} \dfrac{10}{k}$ diverges.

40. $\dfrac{1}{6}\sum_{k=1}^{\infty} \dfrac{1}{k}$ diverges because the harmonic series diverges. Thus, $\sum_{k=1}^{\infty} \dfrac{1}{6k}$ diverges.

41. Since $\sum_{k=1}^{\infty} \dfrac{1}{2^{k-1}}$ is a geometric series with $r = \dfrac{1}{2}$, it converges. Since $\sum_{k=1}^{\infty} \dfrac{1}{k}$ is the harmonic series, it diverges. Thus, $\sum_{k=1}^{\infty} \left(\dfrac{1}{2^{k-1}} + \dfrac{1}{k}\right)$ diverges.

42. Let $f(x) = x\sin\dfrac{1}{x}$. Then, using L'Hôpital's Rule,

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\sin(1/x)}{1/x} \overset{h}{=} \lim_{x\to\infty} \frac{(-1/x^2)\cos(1/x)}{-1/x^2} = \lim_{x\to\infty} \cos\frac{1}{x} = 1 \neq 0,$$

so the series diverges.

43. This is a geometric series with $r = \dfrac{x}{2}$ and will converge for $\left|\dfrac{x}{2}\right| < 1$ or $|x| < 2$.

44. This is a geometric series with $r = \dfrac{1}{x}$ and will converge for $\left|\dfrac{1}{x}\right| < 1$ or $|x| > 1$.

45. This is a geometric series with $r = x+1$ and will converge for $|x+1| < 1$ or $-2 < x < 0$.

46. This is a geometric series with $r = 2x^2$ and will converge for $|2x^2| < 1$ or $|x| < \dfrac{\sqrt{2}}{2}$.

47. The total distance is $15 + 2(15)\left(\dfrac{2}{3}\right) + 2(15)\left(\dfrac{2}{3}\right)^2 + \cdots = 15 + \sum_{k=1}^{\infty} 30\left(\dfrac{2}{3}\right)^k$. The sum is a geometric series with $a = 20$ and $r = \dfrac{2}{3}$, so $\sum_{k=1}^{\infty} 30\left(\dfrac{2}{3}\right)^k = \dfrac{20}{1-2/3} = 60$. The total distance is $15 + 60 = 75$ ft.

48. Using the formula in Example 6 of Section 9.3 in the text with $s = 15$, $f = \dfrac{2}{3}$, and $g = 32$, we have $T = \sqrt{\dfrac{2(15)}{32}}\left(\dfrac{1+\sqrt{2/3}}{1-\sqrt{2/3}}\right) \approx 9.58$ s.

49. $N_0 + N_0 s + N_0 s^2 + \cdots = \sum_{k=0}^{\infty} N_0 s^k = \dfrac{N_0}{1-s}$. Solving $10,000 = \dfrac{N_0}{1-0.9}$, we obtain $N_0 = 1000$.

50. $A_0 + A_0 e^{-k} + A_0 e^{-2k} + \cdots = \sum_{n=0}^{\infty} A_0 (e^{-k})^n$. This is a geometric series with $a = A_0$ and $r = e^{-k}$. The sum is $\dfrac{A_0}{1-e^{-k}} = \dfrac{A_0 e^k}{e^k - 1}$.

51. The total amount of the drug immediately after the n-th dose is $A_n = 15 + 15(0.2) + 15(0.2)^2 + \cdots + 15(0.2)^{n-1}$. As $n \to \infty$, the total accumulation of the drug will be

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} \sum_{k=1}^{n} 15(0.2)^{k-1} = \sum_{k=1}^{\infty} 15(0.2)^{k-1} = \frac{15}{1-0.2} = \frac{75}{4} = 18.75 \text{ mg}.$$

52. The total distance is $20 + 10 + 5 + 5/2 + \cdots = \sum_{k=0}^{\infty} 20 \left(\dfrac{1}{2}\right)^k = \dfrac{20}{1-1/2} = 40$ cm.

53. By Theorem 9.3.3 in the text, if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{k=1}^{\infty} a_k$ diverges.

54. For the series $\dfrac{1}{1.1} + \dfrac{1}{1.11} + \dfrac{1}{1.111} + \cdots$,

$$a_1 = \frac{1}{1+\frac{1}{10}}, a_2 = \frac{1}{1+\frac{1}{10}+\frac{1}{10^2}}, \cdots, a_n = \frac{1}{1+\frac{1}{10}+\cdots+\frac{1}{10^{n-1}}+\frac{1}{10^n}} = \frac{1}{\sum_{k=0}^{n} \frac{1}{10^k}}$$

As $n \to \infty$, $a_n \to \dfrac{1}{\sum_{k=0}^{\infty} \frac{1}{10^k}} = \dfrac{1}{10/9} = \dfrac{9}{10} \neq 0$. The series diverges by the n-th term test.

55. The series $\sum_{k=1}^{\infty} k$ and $\sum_{k=1}^{\infty} (-k)$ both diverge, but their sum $\sum_{k=1}^{\infty} (k-k) = \sum_{k=1}^{\infty} 0 = 0$ converges.

56. For $S_n = \dfrac{1}{1\cdot 1} + \dfrac{1}{2\cdot 2} + \dfrac{1}{3\cdot 3} + \cdots + \dfrac{1}{n\cdot n}$, the inequality $0 < S_n < 1 + \dfrac{1}{1\cdot 2} + \dfrac{1}{2\cdot 3} + \cdots + \dfrac{1}{(n-1)\cdot n}$ is true because, for all $n \geq 1$, $n > n-1$, $n \cdot n > (n-1)\cdot n$, and therefore $\dfrac{1}{n\cdot n} < \dfrac{1}{(n-1)\cdot n}$. The second inequality is true because each individual term $\dfrac{1}{(n-1)\cdot n} = \dfrac{(n-n)+1}{n(n-1)} = \dfrac{n}{n(n-1)} - \dfrac{n-1}{n(n-1)} = \dfrac{1}{n-1} - \dfrac{1}{n}$. Simplifying this, we get

$$\begin{aligned} 1 + \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{(n-1)\cdot n} &= 1 + \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) \\ &= 1 + 1 + \left(-\frac{1}{2} + \frac{1}{2}\right) + \left(-\frac{1}{3} + \frac{1}{3}\right) + \cdots + \left(-\frac{1}{n-1} + \frac{1}{n-1}\right) - \frac{1}{n} = 2 - \frac{1}{n}. \end{aligned}$$

Thus, $0 < S_n < 2 - \dfrac{1}{n}$. Since $n > 0$, then $0 < S_n < 2$ for all n and so $\{S_n\}$ is bounded. Because S_n is a partial sum whose addends are all positive, $\{S_n\}$ is also monotonic. Thus, it converges, and so $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges.

57. $$\begin{aligned}\frac{1+9}{25} + \frac{1+27}{125} + \frac{1+81}{625} + \cdots &= \frac{1+3^2}{5^2} + \frac{1+3^3}{5^3} + \frac{1+3^4}{5^4} + \cdots \\ &= \frac{1}{5^2} + \left(\frac{3}{5}\right)^2 + \frac{1}{5^3} + \left(\frac{3}{5}\right)^3 + \frac{1}{5^4} + \left(\frac{3}{5}\right)^4 + \cdots \\ &= \frac{1}{5^2} + \frac{1}{5^3} + \frac{1}{5^4} + \cdots + \left(\frac{3}{5}\right)^2 + \left(\frac{3}{5}\right)^3 + \cdots \\ &= \left[\left(\sum_{k=1}^{\infty} \frac{1}{5^{k-1}}\right) - 1 - \frac{1}{5}\right] + \left[\left(\sum_{k=1}^{\infty} \frac{3^{k-1}}{5^{k-1}}\right) - 1 - \frac{3}{5}\right] \\ &= \sum_{k=1}^{\infty} \frac{1}{5^{k-1}} + \sum_{k=1}^{\infty} \left(\frac{3}{5}\right)^{k-1} - 1 - 1 - \frac{1}{5} - \frac{3}{5} \\ &= \sum_{k=1}^{\infty} \frac{1}{5^{k-1}} + \sum_{k=1}^{\infty} \left(\frac{3}{5}\right)^{k-1} - 2 - \frac{4}{5} \\ &= \frac{1}{1-1/5} + \frac{1}{1-3/5} - 2 - \frac{4}{5} = \frac{19}{20}\end{aligned}$$

58. Integrating by parts, we obtain $\displaystyle\int xe^{-x}\,dx = -xe^{-x} - e^{-x}$. Then

$$\begin{aligned}\sum_{k=1}^{\infty} \left(\int_k^{k+1} xe^{-x}\,dx\right) &= \sum_{k=1}^{\infty} (-xe^{-x} - e^{-x})\Big]_k^{k+1} \\ &= \sum_{k=1}^{\infty} \left[-(k+1)e^{-(k+1)} - e^{-(k+1)} + ke^{-k} + e^{-k}\right] \\ &= \sum_{k=1}^{\infty} \left[(k+1)e^{-k} - (k+2)e^{-(k+1)}\right]\end{aligned}$$

and

$$S_n = (2e^{-1} - 3e^{-2}) + (3e^{-2} - 4e^{-3}) + \cdots + [(n+1)e^{-n} - (n+2)e^{-(n+1)}] = 2e^{-1} - \frac{n+2}{e^{n+1}}.$$

Then, using L'Hôpital's Rule,

$$\sum_{k=1}^{\infty} \left(\int_k^{k+1} xe^{-x}\,dx\right) = \lim_{n\to\infty} S_n = 2e^{-1} - \lim_{n\to\infty} \frac{n+2}{e^{n+1}} \overset{h}{=} 2e^{-1} - \lim_{n\to\infty} \frac{1}{e^{n+1}} = 2e^{-1}.$$

59. $\displaystyle\sum_{k=0}^{n} \tan^k x$ is a geometric series with $a = 1$ and $r = \tan x$. The infinite series $\displaystyle\sum_{k=0}^{\infty} \tan^k x$ will

converge to $\dfrac{1}{1-\tan x}$ when $|\tan x| < 1$ or $|x| < \dfrac{\pi}{4}$. Thus, $\displaystyle\lim_{n\to\infty}\left(\frac{1}{1-\tan x} - \sum_{k=0}^{n} \tan^k x\right) = 0$ for $|x| < \dfrac{\pi}{4}$.

60. Suppose $\displaystyle\lim_{n\to\infty} f(n+1) = L$ and let $\displaystyle S_n = \sum_{k=1}^{n}[f(k+1) - f(k)]$. Then

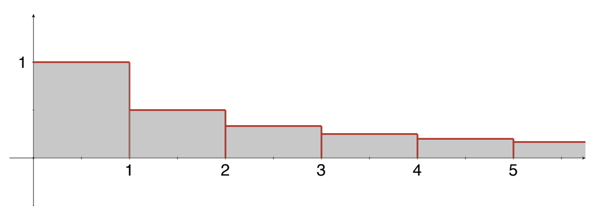
$$S_n = [f(2) - f(1)] + [f(3) - f(2)] + \cdots + [f(n+1) - f(n)] = f(n+1) - f(1)$$

and $\displaystyle\sum_{k=1}^{\infty}[f(k+1) - f(k)] = \lim_{n\to\infty} S_n = \lim_{n\to\infty}[f(n+1) - f(1)] = L - f(1)$.

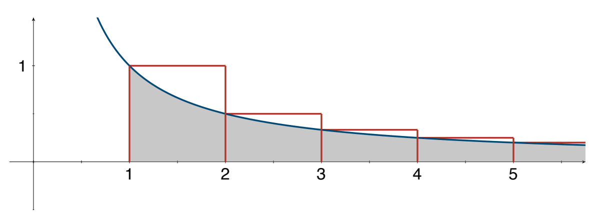
61. The general term of the series is $\displaystyle a_n = \sum_{k=1}^{n}\frac{1}{k}$. Since $\displaystyle\lim_{n\to\infty} a_n = \sum_{k=1}^{\infty}\frac{1}{k}$ and the harmonic series diverges, then by Theorem 9.3.3, the series $\displaystyle\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{1}{k}\right)$ diverges.

62. Since $\displaystyle S_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \geq \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \cdots + \frac{1}{\sqrt{n}} = \frac{n}{\sqrt{n}} = \sqrt{n}$, $\displaystyle\lim_{n\to\infty} S_n = \infty$ and the series diverges.

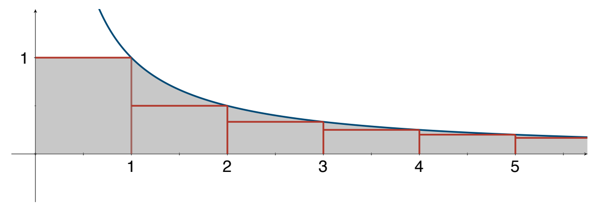
63. (a) The partial sum $1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \cdots + \dfrac{1}{n}$, $n > 1$ can be represented as an area. The graph below shows an area of $1 + \dfrac{1}{2}$ on $[0, 2]$, $1 + \dfrac{1}{2} + \dfrac{1}{3}$ on $[0, 3]$, and so on:



In addition, the area under the graph of $f(x) = \dfrac{1}{x}$ from $x = 1$ up to any $n > 1$ is $\displaystyle\int_1^n \frac{1}{x}\,dx = \ln x\Big]_1^n = \ln n - \ln 1 = \ln n$. Shifting the partial sum areas to the right allows us to compare $\displaystyle\int_1^{n+1}\frac{1}{x}\,dx = \ln(n+1)$ with their corresponding partial sums. This shows that $\ln(n+1) < S_n = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \cdots + \dfrac{1}{n}$:



Similarly, the graphs can be aligned to show that $1 + \displaystyle\int_1^n \frac{1}{x}\,dx = 1 + \ln n > S_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$. Recall that $n > 1$ is stipulated, so the area on $[0, 1]$ consists solely of the 1×1 square (thus corresponding to the first term in $1 + \ln n$):



This yields the overall inequality $\quad \ln(n+1) < 1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \cdots + \dfrac{1}{n} < 1 + \ln n.$

(b) Using the inequality in part (a), we know that $S_n \geq 10$ when $\ln(n+1) \geq 10$. Solving $\ln(n+1) = 10$, we get $n+1 = e^{10}$, and $n = e^{10} - 1$. Thus, $S_n \geq 10$ for $n > e^{10} - 1 \approx 22026$. Using a calculator yields $\displaystyle\sum_{k=1}^{22026} \frac{1}{k} \approx 10.5772$ (it should be noted that the smallest n for which $S_n \geq 10$ is 12367). Similarly, to estimate the value of n for which $S_n \geq 100$, we solve $\ln(n+1) = 100$, getting $n = e^{100} - 1 \approx 2.6811 \times 10^{43}$ (the smallest n for which $S_n \geq 100$ is approximately 1.509×10^{43}).

64. (a) $A_1 = \dfrac{\sqrt{3}}{4}$; $\quad A_2 = \dfrac{\sqrt{3}}{4} + 3\left(\dfrac{\sqrt{3}}{4} \cdot \dfrac{1}{3^2}\right) = \dfrac{\sqrt{3}}{4} + \dfrac{\sqrt{3}}{4}\left(\dfrac{1}{3}\right)$

$$A_3 = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4}\left(\frac{1}{3}\right) + 12\left(\frac{\sqrt{3}}{4} \cdot \frac{1}{9^2}\right) = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4}\left(\frac{1}{3}\right) + \frac{\sqrt{3}}{4}\left(\frac{1}{3}\right)\left(\frac{4}{9}\right)$$

$$\begin{aligned} A_4 &= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4}\left(\frac{1}{3}\right) + \frac{\sqrt{3}}{4}\left(\frac{1}{3}\right)\left(\frac{4}{9}\right) + 48\left(\frac{\sqrt{3}}{4} \cdot \frac{1}{27^2}\right) \\ &= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4}\left(\frac{1}{3}\right) + \frac{\sqrt{3}}{4}\left(\frac{1}{3}\right)\left(\frac{4}{9}\right) + \frac{\sqrt{3}}{4}\left(\frac{1}{3}\right)\left(\frac{4}{9}\right)^2 \end{aligned}$$

(b)
$$\begin{aligned} A_n &= \frac{\sqrt{3}}{4} + 3\left(\frac{\sqrt{3}}{4}\right)\left(\frac{1}{3^2}\right) + 12\left(\frac{\sqrt{3}}{4}\right)\left(\frac{1}{9^2}\right) + 48\left(\frac{\sqrt{3}}{4}\right)\left(\frac{1}{27^2}\right) + \cdots \\ &\qquad \cdots + 3(4^{n-2})\left(\frac{\sqrt{3}}{4}\right)\left(\frac{1}{3^{2n-2}}\right) \\ &= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4}\sum_{k=1}^{n-1}\frac{3(4^{k-1})}{3^{2k}} = \frac{\sqrt{3}}{4}\left(1 + \sum_{k=1}^{n-1}\frac{4^{k-1}}{3^{2k-1}}\right) = \frac{\sqrt{3}}{4}\left[1 + \sum_{k=1}^{n-1}\frac{1}{3}\left(\frac{4}{3^2}\right)^{k-1}\right] \\ &= \frac{\sqrt{3}}{4}\left[1 + \frac{1}{3} \cdot \frac{1 - (4/9)^{n-1}}{1 - 4/9}\right] = \frac{\sqrt{3}}{4}\left\{1 + \frac{3}{5}\left[1 - \left(\frac{4}{9}\right)^{n-1}\right]\right\} \\ &= \frac{\sqrt{3}}{4}\left[\frac{8}{5} - \frac{3}{5}\left(\frac{4}{9}\right)^{n-1}\right] = \frac{\sqrt{3}}{20}\left[8 - 3\left(\frac{4}{9}\right)^{n-1}\right] \end{aligned}$$

(c) $\lim_{n\to\infty} A_n = \frac{\sqrt{3}}{20}(8) = \frac{2\sqrt{3}}{5}$

65. (a) We observe $L_1 = d$, $L_2 = d + L_1 - pL_1 = d + d(1-p)$, $L_3 = d + L_2 - pL_2 = d + (1-p)L_2 = d + d(1-p) + d(1-p)^2$. In general, $L_n = d + d(1-p) + d(1-p)^2 + \cdots + d(1-p)^{n-1}$. This is a geometric series with first term d and common ratio $(1-p)$. Thus

$$L_n = \frac{d[1-(1-p)^n]}{1-(1-p)} = \frac{d}{p}[1-(1-p)^n].$$

Since $0 < p < 1$, $\lim_{n\to\infty} L_n = d/p$.

(b) With $d = 1.4$ and $p = 0.009$, $\lim_{n\to\infty} L_n = 1.4/0.009 \approx 155.56 \approx 155.56$ mg. To determine when the various symptoms occur, we solve $L_n = \frac{d}{p}[1-(1-p)^n]$ for n, obtaining $n = \frac{\ln(1-pL_n/d)}{\ln(1-p)}$. With $d = 1.4$ and $p = 0.009$, this becomes $n = \frac{\ln(1-9L_n/1400)}{\ln 0.991}$.

$$\begin{aligned} \text{parasthesia}: &\quad L_n = 25, \quad n \approx 19.4 \\ \text{ataxia}: &\quad L_n = 55, \quad n \approx 48.3 \\ \text{dysarthia}: &\quad L_n = 90, \quad n \approx 95.6 \end{aligned}$$

Neither deafness nor death can occur at these values of d and p.

(c) Solving $200 = \frac{d}{0.009}[1-(1-0.009)^{100}]$ for d, we obtain $d \approx 3.02$ mg.

66. Suppose the tortoise starts 10 feet (120 inches) in front of Achilles, the tortoise travels at 1 inch per second, and Achilles runs at 5 feet (60 inches) per second. Assuming constant speeds, conventional reasoning states that Achilles and the tortoise will have reached the same point at some time t such that $(1)t + 120 = 60t$, $59t = 120$, and $t = 120/59 \approx 2.1864$ seconds. Achilles then passes the tortoise after approximately 2.1864 seconds.

As phrased by Zeno, it will take 2 seconds for Achilles to reach the tortoise's starting point of 120 inches. By this time, the tortoise will be 2 inches in front of Achilles. It will then take Achilles 1/30 second to reach that point, by which time the tortoise will be 1/30 inches in front of Achilles. The distance travelled by Achilles can thus be written as a geometric series:

$$120 + 2 + 1/30 + \cdots = 120 + 120(60^{-1}) + 120(60^{-2}) + \cdots + 120(60^{1-n}) + \cdots$$

In this example, we have $a = 120$ and $r = 1/60$; in general, a is the tortoise's head start, and r is the reciprocal of Achilles's speed. $|r| < 1$, so the series converges. At a constant speed, Achilles will reach this finite distance in a finite amount of time.

The issue with Zeno's statement of the problem is that Achilles reaches the tortoise's previous position in less and less time — infinitely less, in fact. This infinite division of time constitutes the "trick," so to speak, behind the paradox. Time in the story never exceeds the time that it takes to reach the sum of the infinite series, thus resulting in the tortoise's apparent victory.

67. This exercise involves a research report, and thus a preset solution is not applicable. It should be noted, however, that the series of the reciprocal of primes (i.e., the *harmonic series of primes*) does diverge, with multiple proofs available in the literature.

68. $$\begin{aligned} &AP_1 + P_1P_2 + P_2P_3 + P_3P_4 + P_4P_5 + P_5P_6 + \cdots \\ &= \sqrt{2} + 1 + \frac{\sqrt{2}}{2} + \frac{1}{2} + \frac{\sqrt{2}}{4} + \frac{1}{4} + \cdots \\ &= \sqrt{2}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right) + \left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right) \\ &= (1+\sqrt{2})\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right) = (1+\sqrt{2})\left(\frac{1}{1-1/2}\right) = 2 + 2\sqrt{2} \end{aligned}$$

69. $$\begin{aligned} &AP_1 + P_1P_2 + P_2P_3 + P_3P_4 + P_4P_5 + P_5P_6 + \cdots \\ &= \sin 30^\circ + (\cos 30^\circ)\sin 30^\circ + (\cos 30^\circ)^2 \sin 30^\circ + (\cos 30^\circ)^3 \sin 30^\circ + \cdots \\ &= \frac{1}{2} + \left(\frac{\sqrt{3}}{2}\right)\frac{1}{2} + \left(\frac{\sqrt{3}}{2}\right)^2\frac{1}{2} + \left(\frac{\sqrt{3}}{2}\right)^3\frac{1}{2} + \cdots \\ &= \frac{1}{2}\left[1 + \frac{\sqrt{3}}{2} + \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^3 + \cdots\right] \\ &= \frac{1}{2}\left(\frac{1}{1-\sqrt{3}/2}\right) = \frac{1}{2-\sqrt{3}} = 2 + \sqrt{3} \end{aligned}$$

70. The function f is nonzero on the intervals

$$(1 - 1/2, 1 + 1/2), (2 - 1/4, 2 + 1/4), (3 - 1/8, 3 + 1/8), \ldots, (n - 1/2^n, n + 1/2^n), \ldots$$

so that

$$\int_0^\infty f(x)\,dx = \lim_{n\to\infty} \int_0^{n+1/2^n} f(x)\,dx = \lim_{n\to\infty} [A(1) + A(2) + A(3) + \cdots + A(n) + \cdots]$$

where $A(n)$ is the area of the isosceles triangles whose base is centered at n. Thus,

$$\begin{aligned} \int_0^\infty f(x)\,dx &= \lim_{n\to\infty} [A(1) + A(2) + A(3) + \cdots + A(n) + \cdots] \\ &= \lim_{n\to\infty} \left[\frac{1}{2}\left(\frac{2}{2}\right)1 + \frac{1}{2}\left(\frac{2}{4}\right)1 + \frac{1}{2}\left(\frac{2}{8}\right)1 + \cdots + \frac{1}{2}\left(\frac{2}{2^n}\right)1 + \cdots\right] \\ &= \lim_{n\to\infty} \left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} + \cdots\right) \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2^k} = \sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1/2}{1-1/2} = 1. \end{aligned}$$

71. (a) The overhangs from the edge of the table are:

$$\begin{aligned} d_2 &= \frac{L}{2} + \frac{L}{4} = \frac{L}{2}\left(1 + \frac{1}{2}\right) = \frac{L}{2}H_2 = \frac{1}{2}\left(\frac{3}{2}\right) = 0.75 \\ d_3 &= \frac{L}{2} + \frac{L}{4} + \frac{L}{6} = \frac{L}{2}\left(1 + \frac{1}{2} + \frac{1}{3}\right) = \frac{L}{2}H_3 = \frac{1}{2}\left(\frac{11}{6}\right) \approx 0.917 \\ d_4 &= \frac{L}{2} + \frac{L}{4} + \frac{L}{6} + \frac{L}{8} = \frac{L}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) = \frac{L}{2}H_4 = \frac{1}{2}\left(\frac{25}{12}\right) \approx 1.0417, \end{aligned}$$

where $H_n = \sum_{k=1}^{n} \frac{1}{k}$ are the harmonic numbers. If m denotes the mass of a book and x_k denotes the x-coordinate of the center of mass of the kth book, then the centers of mass are defined by $\overline{x}_n = \dfrac{mx_1 + mx_2 + \cdots + mx_n}{nm} = \dfrac{1}{n}\sum_{k=1}^{n} x_k$.

Therefore, $$\overline{x}_2 = \frac{\frac{L}{2} + \left(\frac{L}{2} + \frac{L}{2}\right)}{2} = \frac{L\left(\frac{2}{2} + \frac{1}{2}\right)}{2} = \frac{3}{4}L$$

$$\overline{x}_3 = \frac{\frac{L}{2} + \left(\frac{L}{2} + \frac{L}{4}\right) + \left(\frac{L}{2} + \frac{L}{4} + \frac{L}{2}\right)}{3} = \frac{L\left(\frac{3}{2} + \frac{2}{4} + \frac{1}{2}\right)}{3} = \frac{5}{6}L$$

$$\overline{x}_4 = \frac{\frac{L}{2} + \left(\frac{L}{2} + \frac{L}{6}\right) + \left(\frac{L}{2} + \frac{L}{6} + \frac{L}{4}\right) + \left(\frac{L}{2} + \frac{L}{6} + \frac{L}{4} + \frac{L}{2}\right)}{4}$$

$$= \frac{L\left(\frac{4}{2} + \frac{3}{6} + \frac{2}{4} + \frac{1}{2}\right)}{4} = \frac{7}{8}L.$$

In other words, the center of mass for each stack of books is at the edge of the table.

(b) $d_4 = \frac{1}{2}H_4 = \frac{1}{2}\left(\frac{25}{12}\right) \approx 1.0417 > 1$ means that the fourth book is completely beyond the edge of the table.

(c) The overhang of n books from the edge of the table is

$$d_n = \frac{L}{2} + \frac{L}{4} + \frac{L}{6} + \cdots + \frac{L}{2(n-1)} + \frac{L}{2n} = \frac{L}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{n}\right) = \frac{L}{2}H_n,$$

where $H_n = \sum_{k=1}^{n} \frac{1}{k}$ are the harmonic numbers. The x-coordinate of the center of mass is

$$\begin{aligned}
\overline{x}_n &= \frac{1}{n}\left\{\frac{L}{2} + \left[\frac{L}{2} + \frac{L}{2(n-1)}\right] + \left[\frac{L}{2} + \frac{L}{2(n-1)} + \frac{L}{2(n-2)}\right] + \cdots\right. \\
&\qquad \left.\cdots + \left[\frac{L}{2} + \frac{L}{2(n-1)} + \frac{L}{2(n-2)} + \cdots + \frac{L}{4} + \frac{L}{2} + \frac{L}{2}\right]\right\} \\
&= \frac{L}{n}\left[\frac{n}{2} + \frac{n-1}{2(n-1)} + \frac{n-2}{2(n-2)} + \cdots + \frac{3}{6} + \frac{2}{4} + \frac{1}{2}\right] \\
&= \frac{L}{n}\left[\frac{n}{2} + (n-1)\frac{1}{2}\right] = \frac{L}{n}\left(\frac{2n-1}{2}\right) = L\left(\frac{2n-1}{2n}\right) = L - \frac{L}{2n}
\end{aligned}$$

Since the overhang of the first (or bottom) book in the stack from the edge of the table is $\frac{L}{2n}$, $\overline{x}_n$ is the distance to the edge of the table. That is, the center of mass of n books is again at the edge of the table.

(d) For $n = 30$ and $n = 31$, *Mathematica* gives

$$\begin{aligned} d_{30} &= \frac{9304682830147}{465817912560} \approx 1.99749L \\ d_{31} &= \frac{L}{2}\left(\frac{290774257297357}{72201776446800}\right) \approx 2.01362L, \end{aligned}$$

which means that for a stack of 31 books, the overhang of the top book from the edge of the table is over twice the length of the book.

(e) There is no theoretical limit to the number of books that can be stacked in this manner because the overhang $d_n = \dfrac{L}{2}H_n$ for large n behaves as the divergent harmonic series. Namely, $d_n \to \infty$ as $n \to \infty$.

72. The total distance is:

$$\begin{aligned} 20\left(\frac{2}{3}\right) + 20\left(\frac{2}{9}\right) + 20\left(\frac{2}{27}\right) + \cdots &= 40\left(\frac{1}{3}\right) + 40\left(\frac{1}{3^2}\right) + 20\left(\frac{1}{3^3}\right) + \cdots \\ &= 40\left(\frac{1/3}{1-1/3}\right) = 40\left(\frac{1/3}{2/3}\right) = 20 \text{ miles} \end{aligned}$$

Alternatively, note that the fly flies at a constant rate of 20 mph for 1 hour. Thus, the distance covered is $20(1) = 20$ miles.

9.4 Integral Test

1. The function $f(x) = \dfrac{1}{x^{1.1}}$ is continuous and decreasing on $[1, \infty)$. Since

$$\int_1^\infty \frac{1}{x^{1.1}}\,dx = \lim_{t\to\infty}\int_1^t x^{-1.1}\,dx = \lim_{t\to\infty}\left(-\frac{1}{0.1}x^{-0.1}\right)\Bigg]_1^t = -10(0-1) = 10,$$

the integral converges and $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^{1.1}}$ converges.

2. The function $f(x) = \dfrac{1}{x^{0.99}}$ is continuous and decreasing on $[1, \infty)$. Since

$$\int_1^\infty \frac{1}{x^{0.99}}\,dx = \lim_{t\to\infty}\int_1^t x^{-0.99}\,dx = \lim_{t\to\infty}\left(\frac{1}{0.01}x^{0.01}\right)\Bigg]_1^t = 100\lim_{t\to\infty}(t^{0.01}-1) = \infty,$$

the integral diverges and $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^{0.99}}$ diverges.

3. Rewriting the series, we have

$$1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \cdots = \sum_{k=1}^{\infty}\frac{1}{x\sqrt{x}} = \sum_{k=1}^{\infty}\frac{1}{x^{3/2}}.$$

The function $f(x) = \dfrac{1}{x^{3/2}}$ is continuous and decreasing on $[1, \infty)$. Since

$$\int_1^\infty \frac{1}{x^{3/2}}dx = \lim_{t\to\infty}\int_1^t x^{-3/2}dx = \lim_{t\to\infty}\left(-2x^{-1/2}\right)\Big|_1^t = -2(0-1) = 2,$$

the integral converges and the series $1 + \dfrac{1}{2\sqrt{2}} + \dfrac{1}{3\sqrt{3}} + \cdots$ converges.

4. Rewriting the series, we have

$$\begin{aligned}\frac{1}{100} + \frac{1}{100\sqrt{2}} + \frac{1}{100\sqrt{3}} + \cdots &= \frac{1}{100}\left(1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots\right)\\ &= \frac{1}{100}\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}\end{aligned}$$

The function $f(x) = \dfrac{1}{\sqrt{x}}$ is continuous and decreasing on $[1, \infty)$. Since

$$\int_1^\infty \frac{1}{\sqrt{x}}dx = \lim_{t\to\infty}\int_1^t x^{-1/2}dx = \lim_{t\to\infty}\left(2x^{1/2}\right)\Big|_1^t = \infty,$$

the integral diverges and the series $\dfrac{1}{100} + \dfrac{1}{100\sqrt{2}} + \dfrac{1}{100\sqrt{3}} + \cdots$ diverges.

5. The function $f(x) = \dfrac{1}{2x+7}$ is continuous and decreasing on $[1, \infty)$. Since

$$\int_1^\infty \frac{1}{2x+7}dx = \lim_{t\to\infty}\int_1^t \frac{1}{2x+7}dx = \lim_{t\to\infty}\left(\frac{1}{2}\ln(2x+7)\right)\Bigg|_1^t = \infty,$$

the integral diverges and $\displaystyle\sum_{k=1}^{\infty}\frac{1}{2k+7}$ diverges.

6. Since $\displaystyle\lim_{n\to\infty}\frac{n}{3n+1} = \frac{1}{3} \neq 0$, the series diverges by the n-th term test.

7. The function $f(x) = \dfrac{1}{1+5x^2}$ is continuous and decreasing on $[1, \infty)$. Since

$$\begin{aligned}\int_1^\infty \frac{1}{1+5x^2}dx = \lim_{t\to\infty}\int_1^t \frac{1}{1+5x^2}dx &= \lim_{t\to\infty}\left(\frac{\sqrt{5}}{5}\tan^{-1}(\sqrt{5}x)\right)\Bigg|_1^t\\ &= \frac{\sqrt{5}}{5}\left(\frac{\pi}{2} - \tan^{-1}(\sqrt{5})\right),\end{aligned}$$

the integral converges and $\displaystyle\sum_{k=1}^{\infty}\frac{1}{1+5k^2}$ converges.

8. The function $f(x) = \dfrac{x}{x^2+5}$ is continuous and decreasing on $[1, \infty)$. Since

$$\begin{aligned} \int_1^\infty \frac{x}{x^2+5} dx &= \lim_{t\to\infty} \int_1^t \frac{x}{x^2+5} \\ &= \lim_{t\to\infty} \left(\frac{\ln(x^2+5)}{2} \right)\Bigg|_1^t = \infty, \end{aligned}$$

the integral diverges and $\displaystyle\sum_{k=1}^\infty \frac{k}{k^2+5}$ diverges.

9. Using the limit comparison test with $a_n = ne^{-n^2}$ and $b_n = 1/n^2$, we have (using L'Hôpital's Rule)

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n/e^{n^2}}{1/n^2} = \lim_{n\to\infty} \frac{n^3}{e^{n^2}} \overset{h}{=} \lim_{n\to\infty} \frac{3n^2}{2ne^{n^2}} = \lim_{n\to\infty} \frac{3n}{2e^{n^2}} \overset{h}{=} \lim_{n\to\infty} \frac{3}{4ne^{n^2}} = 0.$$

Since $\displaystyle\sum_{n=1}^\infty b_n$ is a p-series with $p = 2 > 1$, it converges and $\displaystyle\sum_{n=1}^\infty ke^{-k^2}$ converges.

10. Since $\dfrac{e^{1/k}}{k^2} \le \dfrac{e}{j^2}$, the series converges by comparison with the p-series $\displaystyle\sum_{k=1}^\infty \frac{e}{k^2}$.

11. The function $f(x) = \dfrac{x}{e^x}$ is continuous and decreasing on $[1, \infty)$. Since

$$\begin{aligned} \int_1^\infty \frac{x}{e^x} dx &= \lim_{t\to\infty} \int_1^t \frac{x}{e^x} dx \\ &= \lim_{t\to\infty} \left((-x-1)e^{-x} \right)\Big|_1^t = \frac{2}{e}, \end{aligned}$$

the integral converges and $\displaystyle\sum_{k=1}^\infty \frac{k}{e^k}$ converges.

12. The function $f(x) = x^2e^{-x}$ is continuous and decreasing on $[2, \infty)$. Since

$$\int_2^\infty x^2e^{-x}dx = \lim_{t\to\infty} \int_2^t x^2e^{-x}dx \quad = \lim_{t\to\infty} \left((-x^2-2x-2)e^{-x} \right)\Big|_2^t = \frac{10}{e},$$

the integral converges and $\displaystyle\sum_{k=2}^\infty k^2e^{-k}$ converges.

13. The function $f(x) = \dfrac{1}{x\ln x}$ is continuous and decreasing on $[2, \infty)$. Since

$$\int_2^\infty \frac{1}{x\ln x}\, dx = \lim_{t\to\infty} \int_2^t \frac{1}{x\ln x}\, dx = \lim_{t\to\infty} \ln(\ln x)\Big]_2^t = \lim_{t\to\infty} [\ln(\ln t) - \ln(\ln 2)] = \infty,$$

the integral diverges and $\displaystyle\sum_{k=2}^\infty \frac{1}{k\ln k}$ diverges.

14. Since $k > \ln k$ for $k \geq 2$, we have $\dfrac{k}{\ln k} > 1$ for $k \geq 2$. Therefore, since the terms do not converge to zero, the series $\displaystyle\sum_{k=2}^{\infty} \frac{k}{\ln k}$ diverges.

15. The function $f(x) = \dfrac{10}{x(\ln x)^2}$ is continuous and decreasing on $[2, \infty)$. Since

$$\begin{aligned}\int_2^{\infty} \frac{10}{x(\ln x)^2}\,dx &= 10 \lim_{t\to\infty} \int_2^t \frac{(\ln x)^{-2}}{x}\,dx = 10 \lim_{t\to\infty} \left[-(\ln x)^{-1}\right]_2^t \\ &= 10 \lim_{t\to\infty} [(\ln 2)^{-1} - (\ln t)^{-1}] = 10/\ln 2,\end{aligned}$$

the integral converges and $\displaystyle\sum_{k=2}^{\infty} \frac{10}{k(\ln k)^2}$ converges.

16. The function $f(x) = \dfrac{1}{x\sqrt{\ln x}}$ is continuous and decreasing on $[2, \infty)$. Since

$$\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}\,dx = \lim_{t\to\infty} \int_2^t \frac{(\ln x)^{-1/2}}{x}\,dx = \lim_{t\to\infty} 2(\ln x)^{1/2}\Big]_2^t = \lim_{t\to\infty} (2\sqrt{\ln t} - 2\sqrt{\ln 2}) = \infty,$$

the integral diverges and $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k\sqrt{\ln k}}$ diverges.

17. The function $f(x) = \dfrac{\arctan x}{1+x^2}$ is continuous and decreasing on $[1, \infty)$. Since

$$\begin{aligned}\int_1^{\infty} \frac{\arctan x}{1+x^2}\,dx &= \lim_{t\to\infty} \int_1^t \frac{\arctan x}{1+x^2}\,dx = \lim_{t\to\infty} \frac{(\arctan x)^2}{2}\Bigg]_1^t = \lim_{t\to\infty}\left[\frac{(\arctan t)^2}{2} - \frac{\pi^2}{32}\right] \\ &= \frac{\pi^2}{8} - \frac{\pi^2}{32} = \frac{3\pi^2}{32},\end{aligned}$$

the integral converges and $\displaystyle\sum_{k=1}^{\infty} \frac{\arctan k}{1+k^2}$ converges.

18. The function $f(x) = \dfrac{x}{1+x^4}$ is continues and decreasing on $[1, \infty)$. Since

$$\begin{aligned}\int_1^{\infty} \frac{x}{1+x^4}dx &= \lim_{t\to\infty} \int_1^t \frac{x}{1+x^4}dx \\ &= \lim_{t\to\infty} \left(\frac{\tan^{-1}(x^2)}{2}\right)\Bigg|_1^t = \frac{\pi}{4} - \frac{\pi}{8} = \frac{\pi}{8},\end{aligned}$$

The integral converges and $\displaystyle\sum_{k=1}^{\infty} \frac{k}{1+k^4}$ converges.

19. The function $f(x) = \dfrac{1}{1+\sqrt{x}}$ is continuous and decreasing on $[1, \infty)$. Since

$$\begin{aligned}\int_1^\infty \frac{1}{1+\sqrt{x}}dx &= \lim_{t\to\infty}\int_1^t \frac{1}{1+\sqrt{x}}dx \\ &= \lim_{t\to\infty} \left(2\sqrt{x+1}\right)\Big|_1^t = \infty,\end{aligned}$$

the integral diverges and $\displaystyle\sum_{k=1}^\infty \frac{1}{\sqrt{1+k}}$ diverges.

20. The function $f(x) = \dfrac{1}{\sqrt{1+x^2}}$ is continuous and decreasing on $[1, \infty)$. Since

$$\begin{aligned}\int_1^\infty \frac{1}{\sqrt{1+x^2}}dx &= \lim_{t\to\infty}\int_1^t \frac{1}{\sqrt{1+x^2}}dx \\ &= \lim_{t\to\infty} \ln(\sqrt{x^2+1}+x)\Big|_1^t = \infty,\end{aligned}$$

the integral diverges and $\displaystyle\sum_{k=1}^\infty \frac{1}{\sqrt{1+k^2}}$ diverges.

21. The function $f(x) = \dfrac{x}{(x^2+1)^3}$ is continuous and decreasing on $[1, \infty)$. Since

$$\begin{aligned}\int_1^\infty \frac{x}{(x^2+1)^3}dx &= \lim_{t\to\infty}\int_1^t \frac{x}{(x^2+1)^3}dx \\ &= \lim_{t\to\infty} \left(\frac{-1}{4(x^2+1)^2}\right)\Bigg|_1^t = \frac{1}{16},\end{aligned}$$

the integral converges and $\displaystyle\sum_{k=1}^\infty \frac{k}{(k^2+1)^3}$ diverges.

22. The function $f(x) = \dfrac{1}{(4x+1)^{3/2}}$ is continuous and decreasing on $[1, \infty)$. Since

$$\begin{aligned}\int_1^\infty \frac{1}{(4x+1)^{3/2}}dx &= \lim_{t\to\infty}\int_1^t \frac{1}{(4x+1)^{3/2}}dx \\ &= \lim_{t\to\infty} \left(\frac{-1}{2\sqrt{4x+1}}\right)\Bigg|_1^t = \frac{1}{2\sqrt{5}},\end{aligned}$$

the integral converges and $\displaystyle\sum_{k=1}^\infty \frac{1}{(kx+1)^{3/2}}$ converges.

23. Since $\displaystyle\lim_{x\to\infty} x\sin\frac{1}{x} = \lim_{t\to 0}\frac{\sin t}{t} = 1$, the series diverges by the n-th term test.

24. Since the terms $\ln(1+3k)$ are increasing and positive, we see that the terms do not converge to zero. Therefore, the series diverges.

25. The function $f(x) = \dfrac{1}{x(x+1)}$ is continuous and decreasing on $[1, \infty)$. Since

$$\begin{aligned}\int_1^\infty \frac{1}{x(x+1)}dx &= \lim_{t\to\infty} \int_1^t \frac{1}{x(x+1)}dx \\ &= \lim_{t\to\infty} \ln\left(\frac{x}{x+1}\right)\Big|_1^t = 0 - \ln\left(\frac{1}{2}\right) = \ln(2),\end{aligned}$$

the integral converges and $\displaystyle\sum_{k=1}^\infty \frac{1}{k(k+1)}$ converges.

26. The function $f(x) = \dfrac{2x+1}{x(x+1)}$ is continuous and decreasing on $[1, \infty)$. Since

$$\begin{aligned}\int_1^\infty \frac{2x+1}{x(x+1)}dx &= \lim_{t\to\infty} \int_1^t \frac{2x+1}{x(x+1)}dx \\ &= \lim_{t\to\infty} \ln\left(x(x+1)\right)\big|_1^t = \infty,\end{aligned}$$

the integral diverges and $\displaystyle\sum_{k=1}^\infty \frac{2k+1}{k(k+1)}$ diverges .

27. The function $f(x) = \dfrac{1}{(x+1)(x+7)}$ is continuous and decreasing on $[1, \infty)$. Since

$$\begin{aligned}\int_1^\infty \frac{1}{(x+1)(x+7)}dx &= \lim_{t\to\infty} \int_1^t \frac{1}{(x+1)(x+7)}dx \\ &= \lim_{t\to\infty} \left(\frac{1}{6}\ln\left(\frac{x+1}{x+7}\right)\right)\Big|_1^t \\ &= 0 - \frac{1}{6}\ln\left(\frac{2}{8}\right) = \frac{1}{6}\ln(4),\end{aligned}$$

the integral converges and $\displaystyle\sum_{k=1}^\infty \frac{1}{(k+1)(k+7)}$ converges.

28. The function $f(x) = \dfrac{1}{x(x^2+1)}$ is continuous and decreasing on $[1, \infty)$. Since

$$\begin{aligned}\int_1^\infty \frac{1}{x(x^2+1)}dx &= \lim_{t\to\infty} \int_1^t \frac{1}{x(x^2+1)}dx \\ &= \lim_{t\to\infty} \left(\frac{1}{2}\ln\left(\frac{x^2}{x^2+1}\right)\right)\Big|_1^t = 0 - \frac{1}{2}\ln\left(\frac{1}{2}\right) = \frac{1}{2}\ln(2),\end{aligned}$$

the integral converges and $\displaystyle\sum_{k=1}^\infty \frac{1}{k(k^2+1)}$ converges.

29. The function $f(x) = \dfrac{2}{e^x + e^{-x}}$ is continuous and decreasing on $[1, \infty)$. Since

$$\begin{aligned}\int_1^\infty \frac{2}{e^x + e^{-x}} dx &= \lim_{t\to\infty} \int_1^t \frac{2}{e^x + e^{-x}} dx \\ &= \lim_{t\to\infty} 2\tan^{-1}(e^x)\Big|_1^t = \pi - 2\tan^{-1}(e),\end{aligned}$$

the integral converges and $\displaystyle\sum_{k=1}^{\infty} \frac{2}{e^k + e^{-k}}$ converges.

30. The function $f(x) = \dfrac{1}{\sqrt{e^{3x}}}$ is continuous and decreasing on $[0, \infty)$. Since

$$\begin{aligned}\int_0^\infty \frac{1}{\sqrt{e^{3x}}} dx &= \lim_{t\to\infty} \int_0^t \frac{1}{\sqrt{e^{3x}}} dx \\ &= \lim_{t\to\infty} \left(\frac{-2e^{\frac{-3x}{2}}}{3} \right)\Bigg|_0^t = \frac{2}{3},\end{aligned}$$

the integral converges and $\displaystyle\sum_{k=0}^{\infty} \frac{1}{\sqrt{e^{3k}}}$ converges. Therefore, $\displaystyle\sum_{k=0}^{\infty} \frac{-1}{\sqrt{e^{3k}}}$ converges.

31. Since $\displaystyle\sum_{k=1}^{\infty} \frac{2}{k}$ diverges while $\displaystyle\sum_{k=1}^{\infty} \frac{3}{k^2}$ converges, the entire series $\displaystyle\sum_{k=1}^{\infty} \frac{2}{k} + \frac{3}{k^2}$ diverges.

32. Since $\displaystyle\sum_{k=1}^{\infty} 5k^{-1.6}$ and $\displaystyle\sum_{k=1}^{\infty} 10k^{-1.1}$ both converge by the p-series test, the entire series $\displaystyle\sum_{k=1}^{\infty} 5k^{-1.6} - 10k^{-1.1}$ converges.

33. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ donverges by the p-series test, $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2^k}$ is a geometric series with $r = \dfrac{1}{2} < 1$. Therefore, $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} + \frac{1}{2^k}$ converges.

34. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges by the p-series test. $\displaystyle\sum_{k=1}^{\infty} \frac{4\sqrt{k}}{k^2} = \sum_{k=1}^{\infty} \frac{4}{k^{3/2}}$ also converges by the p-series test. Therefore, $\displaystyle\sum_{k=1}^{\infty} \frac{1 + 4\sqrt{k}}{k^2}$ converges.

35. For $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^p}$, we have

$$\int_2^\infty (\ln x)^{-p} \left(\frac{1}{x}\, dx \right) = \lim_{b\to\infty} (\ln x)^{-p+1}\Big]_2^b = \lim_{b\to\infty} \left[\frac{1}{(\ln b)^{p-1}} - \frac{1}{(\ln 2)^{p-1}} \right].$$

For $p > 1$, this limit converges to $-\dfrac{1}{(\ln 2)^{p-1}}$. For $p = 1$, $\displaystyle\int_2^\infty \frac{1}{\ln x}\left(\frac{1}{x}\right)dx = \lim_{b\to\infty}[\ln(\ln b) - \ln(\ln 2)] = \infty$. For $p < 1$, the integral also diverges.

36. For $\displaystyle\sum_{k=3}^{\infty} \frac{1}{k \ln k[\ln(\ln k)]^p}$, we have

$$\int_3^\infty [\ln(\ln x)]^{-p}\left(\frac{1}{x\ln x}\,dx\right) = \lim_{b\to\infty}[\ln(\ln x)]^{-p+1}\Big]_3^b = \lim_{b\to\infty}\left\{\frac{1}{[\ln(\ln b)]^{p-1}} - \frac{1}{[\ln(\ln 3)]^{p-1}}\right\}.$$

By the same reasoning as in Exercise 35, the series converges for $p > 1$ and diverges for $p \le 1$.

37. For $p \ge 0$, $f(x) = x^p \ln x$ is not decreasing. So for $p < 0$, integration by parts gives

$$\begin{aligned}\sum_{k=2}^{\infty} k^p \ln k \implies \int_2^\infty x^p \ln x\,dx &= \lim_{b\to\infty}\frac{x^{p+1}}{(p+1)^2}[1-(p+1)\ln x]\Big]_2^b \\ &= \lim_{b\to\infty}\left\{\frac{b^{p+1}}{(p+1)^2}[1-(p+1)\ln b] - \frac{2^{p+1}}{(p+1)^2}[1-(p+1)\ln 2]\right\} \\ &= -\frac{2^{p+1}}{(p+1)^2}[1-(p+1)\ln 2] \text{ for } p+1<0 \text{ or } p<-1.\end{aligned}$$

The integral converges for $p < -1$ and diverges for $-1 \le p < 0$ and for $p \ge 0$ (that is, the integral diverges for $p \ge -1$).

38. Since f is decreasing and $f(k) = a_k$, we have $f(x) \le f(k) = a_k$ on $[k, k+1]$. This implies $\int_k^{k+1} f(x)dx \le \int_k^{k+1} a_k dx = a_k$. Therefore, $\displaystyle\int_1^{n+1} f(x)dx = \sum_{k=1}^{n}\int_k^{k+1} f(x)dx \le \int_k^{k+1} a_k$.
Also, note that $f(x) \ge f(k+1) = a_{k+1}$ on .$[k, k+1]$. This implies $\int_k^{k+1} f(x)dx \ge \int_k^{k+1} a_{k+1}dx = a_{k+1}$. Therefore,

$$\begin{aligned}\sum_{k=1}^{n} a_k = a_1 + \sum_{k=2}^{n} a_k &= a_1 + \sum_{k=1}^{n-1}\int_k^{k+1} a_{k+1}dx \\ &\le a_1 + \sum_{k=1}^{n-1}\int_k^{k+1} f(x)dx = a_1 + \int_1^n f(x)dx\end{aligned}$$

39. The function $f(x) = \dfrac{1}{1+x^2}$ is continuous, positive, and decreasing on $[1, \infty)$. Using the result from Problem 38, we have

$$\int_1^{n+1}\frac{1}{1+x^2}dx \le \sum_{k=1}^{n}\frac{1}{1+k^2} \le \frac{1}{2} + \int_1^n \frac{1}{1+x^2}dx$$

Letting $n \to \infty$, we have

$$\int_1^{\infty}\frac{1}{1+x^2}dx \le \sum_{k=1}^{\infty}\frac{1}{1+k^2} \le \frac{1}{2} + \int_1^\infty \frac{1}{1+x^2}dx$$

After integrating, this becomes

$$\frac{\pi}{4} \leq \sum_{k=1}^{\infty} \frac{1}{1+k^2} \leq \frac{1}{2} + \frac{\pi}{4}$$

40. (a) The function $f(x) = \dfrac{1}{x}$ is continuous, positive, and decreasing on $[1, \infty)$. Using the result from Problem 38, we have

$$\int_1^{10^{10}+1} \frac{1}{x} dx \leq \sum_{k=1}^{10^10} \leq 1 + \int_1^{10^10} \frac{1}{x} dx.$$

Performing the integration, we have

$$23.02585 \leq \sum_{k=1}^{10^{10}} \leq 24.02585$$

(b) To fin d the number of terms necessary to guarantee $S_n \geq 100$, we need to find n such that

$$100 \leq \int_1^{n+1} \frac{1}{x} dx = \ln(x)|_1^{n+1} = \ln(n+1).$$

Therefore, $e^{100} \leq n+1$ or $n \geq e^{100} - 1$.

41. Since f is decreasing and $f(k) = a_k$, we have $f(x) \leq f(k) = a_k$ on $[k, k+1]$. This implies $\int_k^{k+1} f(x)dx \leq \int_k^{k+1} a_k dx = a_k$. Therefore, $\displaystyle\int_{n+1}^{n+p} f(x)dx = \sum_{k=n+1}^{n+p-1} \int_k^{k+1} f(x)dx \leq \sum_{k=n+1}^{n+p-1} a_k$. Since this is true for every integer p, we can let $p \to \infty$ to get $\displaystyle\int_{n+1}^{\infty} f(x)dx \leq \sum_{k=n+1}^{\infty} a_k$ or $\displaystyle\int_{n+1}^{\infty} f(x)dx \leq R_n$. Also, note that $f(x) \geq f(k+1) = a_{k+1}$ on $[k, k+1]$. This implies $\int_k^{k+1} f(x)dx \geq \int_k^{k+1} a_{k+1} dx = a_{k+1}$. Therefore,

$$\sum_{k=n+1}^{n+p} a_k = \sum_{k=n}^{n+p-1} a_{k+1} \leq \sum_{k=n}^{n+p-1} \int_k^{k+1} f(x)dx = \int_n^{n+p} f(x)dx.$$

Since this is true for every integer p, we can let $p \to \infty$ to get $\displaystyle\sum_{k=n+1}^{\infty} a_k \leq \int_n^{\infty} f(x)dx$ or $R_n \leq \int_n^{\infty} f(x)dx$. Hence $\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_n^{\infty} f(x)dx$.

42. Letting $f(x) = \dfrac{1}{x^2}$ and using the result from Problem 41, we need to find n such that $\int_n^{\infty} \dfrac{1}{x^2} dx < 0.001$. Performing the integration, $\dfrac{1}{n} < 0.001$ or $n > 1000$.

9.5 Comparison Tests

1. Since $\dfrac{1}{(k+1)(k+2)} < \dfrac{1}{k^2}$, the series converges by comparison with the p-series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$.

2. Since $\dfrac{1}{k^2+5} < \dfrac{1}{k^2}$, the series converges by comparison with the p-series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$.

3. Since $\dfrac{1}{\sqrt{k}-1} \geq \dfrac{1}{\sqrt{k}}$ for $k \geq 2$, the series diverges by comparison with the p-series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{1/2}}$.

4. Since $\dfrac{2k^2+1}{k^3-k} \geq \dfrac{2k^2+1}{k^3} \geq \dfrac{2k^2}{k^3} = \dfrac{2}{k}$ for $k \geq 2$, the series diverges by comparison with the harmonic series $\displaystyle\sum_{k=1}^{\infty} \frac{2}{k}$.

5. Since $\dfrac{1}{\ln k} > \dfrac{1}{k}$ for $k \geq 2$, the series diverges by comparison with the harmonic series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$.

6. Since $\dfrac{\ln k}{k^5} < \dfrac{k}{k^5} = \dfrac{1}{k^4}$, the series converges by comparison with the p-series $\displaystyle\sum_{k=3}^{\infty} \frac{1}{k^4}$.

7. Since $\dfrac{1+3^k}{2^k} = \dfrac{1}{2^k} + \left(\dfrac{3}{2}\right)^k > 1$ for all $k \geq 1$ the series diverges by comparison with the series $\displaystyle\sum_{k=1}^{\infty} 1$.

8. Since $\dfrac{1+8^k}{3+10^k} \leq \dfrac{1+8^k}{10^k} \leq \dfrac{9^k}{10^k} = \left(\dfrac{9}{10}\right)^k$ for all $k \geq 1$, the series converges by comparison with the geometric series $\displaystyle\sum_{k=1}^{\infty} \left(\frac{9}{10}\right)^k$.

9. Since $\dfrac{2+\sin k}{\sqrt[3]{k^4+1}} < \dfrac{3}{k^{4/3}}$, the series converges by comparison with the p-series $\displaystyle\sum_{k=1}^{\infty} \frac{3}{k^{4/3}}$.

10. Since $2k+1 > \ln k$, then $\dfrac{2k+1}{k\ln k} > \dfrac{1}{k}$, and the series diverges by comparison with the harmonic series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$.

11. Since $j + e^{-j} < j + 9$, then $\dfrac{j+e^{-j}}{5^j(j+9)} < \dfrac{1}{5^j}$, and the series converges by comparison with the geometric series $\displaystyle\sum_{j=1}^{\infty} \left(\frac{1}{5}\right)^j$.

12. Since $\dfrac{ie^{-i}}{i+1} = \left(\dfrac{i}{i+1}\right)\dfrac{1}{e^i} < \dfrac{1}{e^i}$, the series converges by comparison with the geometric series $\displaystyle\sum_{i=1}^{\infty}\left(\frac{1}{e}\right)^i$.

13. Since $\dfrac{\sqrt{k+1}-\sqrt{k}}{k} = \dfrac{1}{k(\sqrt{k+1}+\sqrt{k})} \le \dfrac{1}{k(\sqrt{k}+\sqrt{k})} = \dfrac{1}{2k^{3/2}}$, the series converges by comparison with the p-series $\displaystyle\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k^{3/2}}$.

14. Since $\dfrac{1}{n3^n} < \dfrac{1}{3^n}$, the series converges by comparison with the geometric series $\displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{3}\right)^n$.

15. Using the limit comparison test with $a_n = \dfrac{1}{2n+7}$ and $b_n = \dfrac{1}{n}$, we have

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{1/(2n+7)}{1/n} = \lim_{n\to\infty}\frac{n}{2n+7} = \frac{1}{2}.$$

Since $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n+7}$ diverges.

16. Using the limit comparison test with $a_n = \dfrac{1}{10+\sqrt{n}}$ and $b_n = \dfrac{1}{\sqrt{n}}$, we have

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{1/(10+\sqrt{n})}{1/\sqrt{n}} = \lim_{n\to\infty}\frac{\sqrt{n}}{10+\sqrt{n}} = 1.$$

Since $\displaystyle\sum_{n=1}^{\infty} b_n$ is a p-series with $p = \dfrac{1}{2} < 1$, it diverges and $\displaystyle\sum_{k=1}^{\infty}\frac{1}{10+\sqrt{k}}$ diverges.

17. Using the limit comparison test with $a_n = \dfrac{1}{n\sqrt{n^2-1}}$ and $b_n = \dfrac{1}{n^2}$, we have

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{1/n\sqrt{n^2-1}}{1/n^2} = \lim_{n\to\infty}\frac{n^2}{n\sqrt{n^2-1}} = \lim_{n\to\infty}\frac{1}{\sqrt{1-1/n^2}} = 1.$$

Since $\displaystyle\sum_{n=1}^{\infty} b_n$ is a p-series with $p = 2 > 1$, it converges and $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\sqrt{n^2-1}}$ converges.

18. Using the limit comparison test with $a_n = \dfrac{1}{\sqrt{(n+1)(n+2)}}$ and $b_n = \dfrac{1}{n}$, we have

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{1/\sqrt{(n+1)(n+2)}}{1/n} = \lim_{n\to\infty}\frac{n}{\sqrt{n^2+3n+2}} = \lim_{n\to\infty}\frac{1}{\sqrt{1+3/n+2/n^2}} = 1.$$

Since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{(n+1)(n+2)}}$ diverges.

19. Using the limit comparison test with $a_n = \dfrac{n^2 - n + 2}{3n^5 + n^2}$ and $b_n = \dfrac{1}{n^3}$, we have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(n^2 - n + 2)/(3n^5 + n^2)}{1/n^3} = \lim_{n\to\infty} \frac{n^5 - n^4 + 2n^3}{3n^5 + n^2} = \lim_{n\to\infty} \frac{1 - 1/n + 2/n^2}{3 + 1/n^3} = \frac{1}{3}.$$

Since $\displaystyle\sum_{n=1}^{\infty} b_n$ is a p-series with $p = 3 > 1$, it converges and $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 - n + 2}{3n^5 + n^2}$ converges.

20. Using the limit comparison test with $a_n = \dfrac{n}{(4n+1)^{3/2}}$ and $b_n = \dfrac{1}{n^{1/2}}$, we have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n/(4n+1)^{3/2}}{1/n^{1/2}} = \lim_{n\to\infty} \frac{n^{3/2}}{(4n+1)^{3/2}} = \lim_{n\to\infty} \left(\frac{1}{4 + 1/n}\right)^{3/2} = \frac{1}{8}.$$

Since $\displaystyle\sum_{n=1}^{\infty} b_n$ is a p-series with $p = \dfrac{1}{2} < 1$, it diverges and $\displaystyle\sum_{n=2}^{\infty} \frac{n}{(4n+1)^{3/2}}$ diverges.

21. Using the Limit Comparison Test with $a_k = \dfrac{\sqrt{k+1}}{\sqrt[3]{64k^9 + 40}}$ and $b_k = \dfrac{\sqrt{k}}{\sqrt[3]{64k^9}}$, we have

$$\begin{aligned}
\lim_{k\to\infty} \frac{a_k}{b_k} &= \lim_{k\to\infty} \frac{\left(\dfrac{\sqrt{k+1}}{\sqrt[3]{64k^9 + 40}}\right)}{\left(\dfrac{\sqrt{k}}{\sqrt[3]{64k^9}}\right)} \\
&= \lim_{k\to\infty} \frac{\sqrt{k+1}}{\sqrt{k}} \cdot \frac{\sqrt[3]{64k^9}}{\sqrt[3]{64k^9 + 40}} \\
&= \lim_{k\to\infty} \frac{\sqrt{k+1}}{\sqrt{k}} \cdot \lim_{k\to\infty} \frac{\sqrt[3]{64k^9}}{\sqrt[3]{64k^9 + 40}} = 1 \cdot 1 = 1.
\end{aligned}$$

Since $\displaystyle\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{\sqrt{k}}{\sqrt[3]{64k^9}} = \sum_{k=1}^{\infty} \frac{1}{4k^{5/2}}$ is a p-series with $p = 5/2 > 1$, it converges and $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt{k+1}}{\sqrt[3]{64k^9 + 40}}$ converges.

22. Using the Limit Comparison Test with $a_k = \dfrac{5k^2 - k}{2k^3 + 2k^2 - 8}$ and $b_k = \dfrac{1}{k}$, we have

$$\begin{aligned}
\lim_{k\to\infty} \frac{a_k}{b_k} &= \lim_{k\to\infty} \frac{\dfrac{5k^2 - k}{2k^3 + 2k^2 - 8}}{\dfrac{1}{k}} \\
&= \lim_{k\to\infty} \frac{5k^3 - k^2}{2k^3 + 2k^2 - 8} = \frac{5}{2}.
\end{aligned}$$

Since $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k}$ diverges, $\displaystyle\sum_{k=2}^{\infty} \frac{5k^2 - k}{2k^3 + 2k^2 - 8}$ diverges.

23. Using the limit comparison test with $a_n = \dfrac{n+\ln n}{n^3+2n-1}$ and $b_n = \dfrac{1}{n^2}$, we have

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{(n+\ln n)/(n^3+2n-1)}{1/n^2} = \lim_{n\to\infty}\frac{n^3+n^2\ln n}{n^3+2n-1} = \lim_{n\to\infty}\frac{1+(\ln n)/n}{1+2/n-1/n} = 1.$$

(By L'Hôpital's Rule, $\displaystyle\lim_{n\to\infty}\frac{\ln n}{n} = \lim_{n\to\infty}\frac{1/n}{1} = 0$.) Since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a p-series with $p = 2 > 1$, it converges and $\displaystyle\sum_{n=2}^{\infty}\frac{k+\ln k}{k^3+2k-1}$ converges.

24. Using the Limit Comparison Test with $a_k = \dfrac{10}{e^k-2}$ and $b_k = \dfrac{1}{e^k}$, we have

$$\lim_{k\to\infty}\frac{a_k}{b_k} = \lim_{k\to\infty}\frac{\left(\dfrac{10}{e^k-2}\right)}{\left(\dfrac{1}{e^k}\right)} = \lim_{k\to\infty}\frac{10e^k}{e^k-2} = 10.$$

Since $\displaystyle\sum_{k=1}^{\infty} b_k$ is a geometric series with $r = \dfrac{1}{e} < 1$, it converges and $\displaystyle\sum_{k=1}^{\infty}\frac{10}{e^k-2}$ converges.

25. Using the limit comparison test with $a_n = \sin\dfrac{1}{n}$ and $b_n = \dfrac{1}{n}$, we have

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\sin(1/n)}{1/n} \overset{h}{=} \lim_{n\to\infty}\frac{\cos(1/n)(-1/n^2)}{-1/n^2} = \lim_{n\to\infty}\cos\frac{1}{n} = 1.$$

Since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, the series $\displaystyle\sum_{k=1}^{\infty}\sin\frac{1}{k}$ diverges.

26. Using the limit comparison test with $a_n = 1-\cos\dfrac{1}{n}$ and $b_n = \dfrac{1}{n^2}$, we have

$$\begin{aligned}\lim_{n\to\infty}\frac{a_n}{b_n} &= \lim_{n\to\infty}\frac{1-\cos(1/n)}{1/n^2} \overset{h}{=} \lim_{n\to\infty}\frac{\sin(1/n)(-1/n^2)}{-2/n^3} = \frac{1}{2}\lim_{n\to\infty}\left(n\sin\frac{1}{n}\right)\\ &= \frac{1}{2}\lim_{n\to\infty}\frac{\sin(1/n)}{1/n} = \frac{1}{2}\lim_{t\to 0}\frac{\sin t}{t} = \frac{1}{2}.\end{aligned}$$

Since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a p-series with $p = 2 > 1$, it converges and $\displaystyle\sum_{k=1}^{\infty}\left(1-\cos\frac{1}{k}\right)$ converges.

27. Using the limit comparison test with $a_n = \left(\dfrac{1}{2}+\dfrac{1}{2n}\right)^n$ and $b_n = \left(\dfrac{1}{2}\right)^n$, we have

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\left(\dfrac{1}{2}+\dfrac{1}{2n}\right)^n}{(1/2)^n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

Since $\sum_{n=1}^{\infty} b_n$ is a geometric series with $r = 1/2 < 1$, it converges and the series $\sum_{k=1}^{\infty}\left(\frac{1}{2}+\frac{1}{2k}\right)^k$ converges.

28. Using the limit comparison test with $a_n = \frac{n}{(n+1)(n+2)}$ and $b_n = \frac{1}{n}$, we have

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n/(n+1)(n+2)}{1/n} = \lim_{n\to\infty}\frac{n^2}{n^2+3n+2} = 1.$$

Since $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, the series $\sum_{n=1}^{\infty}\frac{n}{(n+1)(n+2)}$ diverges.

29. Since $\lim_{k\to\infty}\frac{k}{100\sqrt{k^2+1}} = \frac{1}{100}$, the series diverges by the n-th term test.

30. Using the limit comparison test with $a_n = \frac{1}{n+\sqrt{n}}$ and $b_n = \frac{1}{n}$, we have

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{1/(n+\sqrt{n})}{1/n} = \lim_{n\to\infty}\frac{n}{n+\sqrt{n}} = \lim_{n\to\infty}\frac{1}{1+1/\sqrt{n}} = 1.$$

Since $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, the series $\sum_{k=1}^{\infty}\frac{1}{k+\sqrt{k}}$ diverges.

31. Since $\lim_{k\to\infty}\ln\left(5+\frac{k}{5}\right) = \infty$, the series diverges by the n-th term test.

32. Using the limit comparison test with $a_n = \ln\left(1+\frac{1}{3^n}\right)$ and $b_n = \frac{1}{3^n}$, we have (using L'Hôpital's Rule)

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\ln(1+3^{-n})}{3^{-n}} \overset{h}{=} \lim_{n\to\infty}\frac{(-3^{-n}\ln 3)/(1+3^{-n})}{-3^n\ln 3} = \lim_{n\to\infty}\frac{1}{1+3^{-n}} = 1.$$

Since $\sum_{n=1}^{\infty} b_n$ is a geometric series with $r = 1/3 < 1$, it converges and the series $\sum_{k=1}^{\infty}\ln\left(1+\frac{1}{3^k}\right)$ converges.

33. The function $f(x) = \frac{x}{(x^2+1)^2}$ is continuous and decreasing on $[1,\infty)$. Since

$$\int_1^{\infty}\frac{x}{(x^2+1)^2}\,dx = \lim_{t\to\infty}\int_1^t\frac{x}{(x^2+1)^2}\,dx = -\frac{1}{2}\lim_{t\to\infty}\frac{1}{x^2+1}\Bigg]_1^t = -\frac{1}{2}\lim_{t\to\infty}\left(\frac{1}{t^2+1}-\frac{1}{2}\right) = \frac{1}{4},$$

the integral converges and $\sum_{k=1}^{\infty}\frac{k}{(k^2+1)^2}$ converges.

(The direct comparison test and limit comparison test can also be used.)

34. Since $\dfrac{k}{\sqrt{k-1}\sqrt[3]{k^2-2}} > \dfrac{1}{k^{1/6}}$ for $k \geq 2$, the series diverges by comparison with the p-series $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^{1/6}}$.

35. Since $\dfrac{1}{9+\sin^2 k} > \dfrac{1}{10}$ for $k \geq 1$, the series diverges by comparison with the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{10}$.

36. Using the limit comparison test with $a_n = \dfrac{3^n}{3^{2n}-1}$ and $b_n = \dfrac{1}{3^n}$, we have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{3^n/(3^{2n}-1)}{1/3^n} = \lim_{n\to\infty} \frac{3^{2n}}{3^{2n}-1} = \lim_{n\to\infty} \frac{1}{1-1/3^{2n}} = 1.$$

Since $\displaystyle\sum_{n=1}^{\infty} b^n$ is a geometric series with $r = 1/3 < 1$, it converges and the series $\displaystyle\sum_{k=1}^{\infty} \frac{3^k}{3^{2k}-1}$ converges.

37. Since $\dfrac{2}{2+k2^k} < \dfrac{2}{k2^k} < \dfrac{2}{2^k} = \dfrac{1}{2^{k-1}}$, the series converges by comparison with the geometric series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2^{k-1}}$.

38. Since $\displaystyle\lim_{k\to\infty} \frac{2}{2+k2^{-k}} = 1$, the series diverges by the n-th term test.

39. Since $\ln\left(1+\dfrac{1}{k}\right) > \ln\dfrac{1}{k} = -\ln k$, the series diverges by comparison with the series $\displaystyle -\sum_{k=2}^{\infty} \ln k$.

40. Since $\dfrac{(0.9)^k}{k} \leq (0.9)^k$, the series converges by comparison with the geometric series $\displaystyle\sum_{k=1}^{\infty} (0.9)^k$.

41. Since $\sum a_k$ converges, then $\displaystyle\lim_{k\to\infty} a_k = 0$. Therefore, for a sufficiently large n and $k \geq n$, we can say that $0 < a_k < 1$ and so $a_k^2 < a_k$. Thus, $\sum a_k^2$ converges by the direct comparison test.

42. Assuming $q(k) \neq 0$ for $k \geq 1$, the series will converge if $m \geq n+2$.

43. The statement is false. The condition of a positive-term series is missing: as a counterexample, consider the convergent series $\sum b_k = \sum 0 = 0$ and the divergent series $a_k = -\sum \dfrac{1}{k}$.

44. For $a_k > 0$ for all k, the limit comparison test

$$\lim_{n\to\infty} \frac{\ln(1+a_n)}{a_n} = \lim_{x\to 0} \frac{\ln(1+x)}{x} \overset{h}{=} \lim_{x\to 0} \frac{1/(1+x)}{1} = 1$$

shows that if $\sum a_k$, then $\sum \ln(1+a_k)$ converges.

45. The series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{1+1/k}}$ diverges by the limit comparison test with $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/n^{1+1/n}}{1/n} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = 1.$$

46. Since $1+2+3+\cdots+n = \frac{1}{2}n(n+1)$, we have $\displaystyle\sum_{k=1}^{\infty} \frac{1}{1+2+3+\cdots+k} = \sum_{k=1}^{\infty} \frac{2}{k(k+1)}$, which converges by either the direct comparison test or the limit comparison test with $\displaystyle\sum_{k=1}^{\infty} \frac{2}{k^2}$.

47. Of the set of integers a_i used in the decimal representation of the number, let a_B be the biggest integer. Then for the series $\displaystyle\sum_{k=1}^{\infty} \frac{a_k}{10^k}$, we have $\dfrac{a_k}{10^k} \le \dfrac{a_B}{10^k}$ for all k. Since $\displaystyle\sum_{k=1}^{\infty} \frac{a_B}{10^k} = a_B \sum_{k=1}^{\infty} \frac{1}{10^k}$ and $\displaystyle\sum_{k=1}^{\infty} \frac{1}{10^k} = \sum_{k=1}^{\infty} \left(\frac{1}{10}\right)^k$ is a convergent geometric series, then $\displaystyle\sum_{k=1}^{\infty} \frac{a_k}{10^k}$ converges by the direct comparison test.

48. This exercise involves a research report, and thus a preset solution is not given.

9.6 Ratio and Root Tests

1. Since $$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1/(n+1)!}{1/n!} = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1,$$ the series converges by the ratio test.

2. Since $$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+1}/(n+1)!}{2^n/n!} = \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1,$$ the series converges by the ratio test.

3. Since $$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!/1000^{n+1}}{n!/1000^n} = \lim_{n\to\infty} \frac{n+1}{1000} = \infty,$$ the series diverges by the ratio test.

4. Since $$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)(2/3)^{n+1}}{n(2/3)^n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)\left(\frac{2}{3}\right) = \frac{2}{3} < 1,$$ the series converges by the ratio test.

5. Since $$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^{10}/(1.1)^{n+1}}{n^{10}/(1.1)^n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{10} \frac{1}{1.1} = \frac{1}{1.1} < 1,$$ the series converges by the ratio test.

6. Since $$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1/(n+1)^5(0.99)^{n+1}}{1/n^5(0.99)^n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^5 \frac{1}{0.99} = \frac{1}{0.99} > 1,$$ the series diverges by the ratio test.

7. Since $$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{4^n/(n+1)3^{n-1}}{4^{n-1}/n3^{n-2}}=\lim_{n\to\infty}\frac{4}{3}\left(\frac{n}{n+1}\right)=\frac{4}{3}>1,$$
the series diverges by the ratio test.

8. Since $$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(n+1)^3 2^{n+4}/7^n}{n^3 2^{n+3}/7^{n-1}}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^3\left(\frac{2}{7}\right)=\frac{2}{7}<1,$$
the series converges by the ratio test.

9. Since
$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(n+1)!/(2n+2)!}{n!/(2n)!}=\lim_{n\to\infty}\frac{n+1}{(2n+1)(2n+2)}=\lim_{n\to\infty}\frac{n+1}{4n^2+6n+2}=0<1,$$
the series converges by the ratio test.

10. Since
$$\begin{aligned}\lim_{k\to\infty}\frac{a_{k+1}}{a_k}&=\lim_{k\to\infty}\frac{(2k+2)!/(k+1)!2^{k+1}(k+1)^{k+1}}{(2k)!/k!2^k k^k}=\lim_{k\to\infty}\frac{(2k+1)(2k+2)}{2(k+1)(k+1)}\left(\frac{k}{k+1}\right)^k\\&=\lim_{k\to\infty}\frac{(2k+1)(2k+2)}{2(k+1)^2\left(\frac{k+1}{k}\right)^k}=\lim_{k\to\infty}\frac{4k^2+6k+2}{(2k^2+4k+2)\left(1+\frac{1}{k}\right)^k}=\frac{2}{e}<1,\end{aligned}$$
the series converges by the ratio test.

11. Since
$$\begin{aligned}\lim_{n\to\infty}\frac{a_{n+1}}{a_n}&=\lim_{n\to\infty}\frac{99^{n+1}((n+1)^3+1)}{99^n(n^3+1)}\cdot\frac{n^2 100^n}{(n+1)^2 100^{n+1}}\\&=\lim_{n\to\infty}\left(\frac{99}{100}\right)\cdot\left(\frac{(n+1)^3+1}{n^3+1}\right)\cdot\left(\frac{n^2}{(n+1)^2}\right)\\&=\left(\frac{99}{100}\right)\left(\lim_{n\to\infty}\frac{(n+1)^3+1}{n^3+1}\right)\left(\lim_{n\to\infty}\frac{n^2}{(n+1)^2}\right)\\&=\frac{99}{100}\cdot 1\cdot 1=\frac{99}{100}<1,\end{aligned}$$
the series converges by the ratio test.

12. Since
$$\begin{aligned}\lim_{n\to\infty}\frac{a_{n+1}}{a_n}&=\lim_{n\to\infty}\frac{(n+1)!}{n!}\cdot\frac{e^{n^2}}{e^{(n+1)^2}}\\&=\lim_{n\to\infty}(n+1)\cdot\frac{e^{n^2}}{e^{n^2\cdot e^{2n}}\cdot e}\\&=\lim_{n\to\infty}\frac{n+1}{e^{2n+1}}=0<1,\end{aligned}$$
the series converges by the ratio test.

13. Since

$$\begin{aligned}\lim_{n\to\infty}\frac{a_{n+1}}{a_n}&=\lim_{n\to\infty}\frac{5^{n+1}}{5^n}\cdot\frac{n^n}{(n+1)^{n+1}}\\&\le\lim_{n\to\infty}5\cdot\frac{n^n}{n^{n+1}}\\&=\lim_{n\to\infty}\frac{5}{n}=0<1,\end{aligned}$$

the series converges by the ratio test.

14. Since

$$\lim_{k\to\infty}\frac{a_{k+1}}{a_k}=\lim_{k\to\infty}\frac{(k+1)!3^{k+1}/(k+1)^{k+1}}{k!3^k/k^k}=\lim_{k\to\infty}3\left(\frac{k}{k+1}\right)^k=\lim_{k\to\infty}\frac{3}{(1+1/k)^k}=\frac{3}{e}>1,$$

the series diverges by the ratio test.

15. Since $$\lim_{k\to\infty}\frac{a_{k+1}}{k_n}=\lim_{k\to\infty}\frac{1\cdot3\cdot5\cdots(2k+1)/(k+1)!}{1\cdot3\cdot5\cdots(2k-1)/k!}=\lim_{k\to\infty}\frac{2k+1}{k+1}=2>1,$$

the series diverges by the ratio test.

16. Since $$\lim_{k\to\infty}\frac{a_{k+1}}{k_n}=\lim_{k\to\infty}\frac{(k+1)!/[2\cdot4\cdot6\cdots(2k+2)]}{k!/[2\cdot4\cdot6\cdots(2k)]}=\lim_{k\to\infty}\frac{k+1}{2k+2}=\frac{1}{2}<1,$$

the series converges by the ratio test.

17. Since $\displaystyle\lim_{n\to\infty}\left(\frac{1}{n^n}\right)^{1/n}=\lim_{n\to\infty}\frac{1}{n}=0<1$, the series converges by the root test.

18. Since $$\lim_{k\to\infty}\left[\left(\frac{ke}{k+1}\right)^k\right]^{1/k}=\lim_{k\to\infty}\frac{ke}{k+1}=\lim_{k\to\infty}\frac{e}{1+1/k}=e>1,$$

the series diverges by the root test.

19. Since $\displaystyle\lim_{n\to\infty}\left[\left(\frac{n}{\ln n}\right)^n\right]^{1/n}=\lim_{n\to\infty}\frac{n}{\ln n}=\infty$, the series diverges by the root test.

20. Since $\displaystyle\lim_{n\to\infty}\left[\frac{1}{(\ln n)^n}\right]^{1/n}=\lim_{n\to\infty}\frac{1}{\ln n}=0<1$, the series converges by the root test.

21. Since $\displaystyle\lim_{n\to\infty}\left[\left(\frac{n}{n+1}\right)^{n^2}\right]^{1/n}=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n=\lim_{n\to\infty}\frac{1}{\left(\frac{n+1}{n}\right)^n}=\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n}=\frac{1}{e}<1,$

the series converges by the root test.

22. Using the root test, $\displaystyle\lim_{n\to\infty}\left[\left(1-\frac{2}{n}\right)^{n^2}\right]^{1/n}=\lim_{n\to\infty}\left(1-\frac{2}{n}\right)^n$. Let $y=\left(1-\frac{2}{x}\right)^x$.

Then $\ln y = x\ln\left(1-\dfrac{2}{x}\right) = \dfrac{\ln(1-2/x)}{1/x}$ and by L'Hôpital's Rule,

$$\lim_{x\to\infty}\ln y = \lim_{x\to\infty}\frac{\ln(1-2/x)}{1/x} \overset{h}{=} \lim_{x\to\infty}\frac{2/x^2(1-2/x)}{-1/x^2} = \lim_{x\to\infty}\left(-\frac{2}{1-2/x}\right) = -2.$$

Thus, $\lim\limits_{n\to\infty}\left(1-\dfrac{2}{n}\right)^{n^2} = e^{-2} < 1$, and the series converges.

23. Since $\lim\limits_{n\to\infty}\left(\dfrac{6^{2n+1}}{n^n}\right)^{1/n} = \lim\limits_{n\to\infty}\dfrac{6^{2+1/n}}{n} = 0 < 1$, the series converges by the root test.

24. Since $\lim\limits_{k\to\infty}\left(\dfrac{k^k}{e^{k+1}}\right)^{1/k} = \lim\limits_{k\to\infty}\dfrac{k}{e^{1+1/k}} = \infty$, the series diverges by the root test.

25. Since $\dfrac{k^2+k}{k^3+2k+1} \geq \dfrac{k^2}{k^3+2k+1} \geq \dfrac{k^2}{k^3+2k^3+k^3} = \dfrac{1}{4k}$ for $k \geq 1$, the series diverges by comparison with the series $\dfrac{1}{4}\sum\limits_{k=1}^{\infty}\dfrac{1}{k}$.

26. Since $\lim\limits_{n\to\infty}\left[\left(\dfrac{3n}{2n+1}\right)^n\right]^{1/n} = \lim\limits_{n\to\infty}\dfrac{3n}{2n+1} = \dfrac{3}{2} > 1$, the series diverges by the root test.

27. Since $\dfrac{e^{1/n}}{n^2} \leq \dfrac{e}{n^2}$ for $n \geq 1$, the series converges by the comparison with the series $\sum\limits_{n=1}^{\infty}\dfrac{e}{n^2}$.

28. Since
$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{(n+1)^2+(n+1)}{n^2+n}\cdot\frac{e^n}{e^{n+1}}$$
$$= \lim_{n\to\infty}\left(\left(\frac{n^2+3n+2}{n^2+n}\right)\right)\cdot\lim_{n\to\infty}\left(\frac{1}{e}\right) = \frac{1}{e} < 1,$$
the series converges by the ratio test.

29. Since
$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{(5^{n+1}(n+1)!}{5^n\cdot n!}\cdot\frac{(n+1)!}{(n+2)!}$$
$$= \lim_{n\to\infty}\frac{5(n+1)}{n+2} = 5 > 1,$$
the series diverges by the ratio test.

30. Since $\dfrac{3}{2^k+k} \leq \dfrac{3}{2^k} = 3\left(\dfrac{1}{2}\right)^k$ for $k \geq 1$, the series converges by comparison with the geometric series, $\sum\limits_{k=1}^{\infty}3\left(\dfrac{1}{2}\right)^k$..

31. Since $\dfrac{2^k}{3^k+4^k} \le \dfrac{2^k}{3^k} = \left(\dfrac{2}{3}\right)^k$ for $k \ge 0$, the series converges by comparison with $\displaystyle\sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k$..

32. Since $a_n = \dfrac{n}{n+2}$ and $\lim\limits_{n\to\infty} a_n = 1$, we see that the terms do not converge to zero. Therefore, the series diverges.

33. Applying the ratio test, we have $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \dfrac{(n+1)p^{n+1}}{np^n} = \lim\limits_{n\to\infty} \left(1+\dfrac{1}{n}\right) p = p$. Thus, the series converges for $0 \le p < 1$ and diverges for $p > 1$. For $p = 1$, the series is $\displaystyle\sum_{k=1}^{\infty} k$, which diverges.

34. Applying the ratio test, we have $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \dfrac{(n+1)^2 2^{n+1}/p^{n+1}}{n^2 2^n/p^n} = \lim\limits_{n\to\infty} \dfrac{2}{p}\left(1+\dfrac{1}{n}\right)^2 = \dfrac{2}{p}$. For $2/p < 1$ or $p > 2$, the series converges, and for $0 < p < 2$, it diverges. For $p = 2$, the series is $\displaystyle\sum_{k=1}^{\infty} k^2$, which diverges.

35. Applying the ratio test, we have $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \dfrac{(n+1)^p/(n+1)!}{n^p/n!} = \lim\limits_{n\to\infty} \left(\dfrac{n+1}{n}\right)^p \dfrac{1}{n+1} = 0$. The series converges for all real values of p.

36. For $p \le 1$ and $k \ge 3$, the general term of the series $\dfrac{\ln k}{k^p} \ge \dfrac{1}{k}$ and the series diverges by comparison with the harmonic series. For $p > 1$, we note that $f(x) = \dfrac{\ln x}{x^p}$ has derivative $f'(x) = \dfrac{1 - p\ln x}{x^{p+1}}$ and that $f'(x) < 0$ for $x \ge 3$. Thus, $f(x)$ is continuous and decreasing on

$[3, \infty)$ and we may apply the integral test:

$$\begin{aligned}
\int_3^\infty \frac{\ln x}{x^p}\,dx \qquad & \boxed{t = \ln x,\ x = e^t,\ dx = e^t\,dt} \\
&= \int_{\ln 3}^\infty \frac{t}{e^{pt}} e^t\,dt = \int_{\ln 3}^\infty t e^{(1-p)t}\,dt \\
& \boxed{u = t,\ du = dt; \quad dv = e^{(1-p)t}\,dt,\ v = \frac{1}{1-p} e^{(1-p)t}} \\
&= \frac{1}{1-p} t e^{(1-p)t}\Big]_{\ln 3}^\infty - \int_{\ln 3}^\infty \frac{1}{1-p} e^{(1-p)t}\,dt \\
&= \frac{1}{1-p} t e^{(1-p)t}\Big]_{\ln 3}^\infty - \frac{1}{(1-p)^2} e^{(1-p)t}\Big]_{\ln 3}^\infty \\
&= \frac{1}{1-p}\left[\lim_{t\to\infty} \frac{t}{e^{(p-1)t}} - (\ln 3)3^{1-p}\right] - \frac{1}{(1-p)^2}\left[\lim_{t\to\infty} e^{(1-p)t} - 3^{1-p}\right] \\
&\overset{h}{=} \frac{1}{1-p}\left[\lim_{t\to\infty} \frac{1}{(p-1)e^{(p-1)t}} - 3^{1-p}\ln 3\right] - \frac{1}{(1-p)^2}(-3^{1-p}) \\
&= \frac{3^{1-p}}{(1-p)^2} - \frac{3^{1-p}\ln 3}{1-p}
\end{aligned}$$

Thus, the integral converges and $\displaystyle\sum_{k=2}^\infty \frac{\ln k}{k^p}$ converges for $p > 1$.

37. (a)
$$\begin{aligned}
F_n + F_{n-1} &= \frac{1}{\sqrt5}\left(\frac{1+\sqrt5}{2}\right)^n - \frac{1}{\sqrt5}\left(\frac{1-\sqrt5}{2}\right)^n + \frac{1}{\sqrt5}\left(\frac{1+\sqrt5}{2}\right)^{n-1} - \frac{1}{\sqrt5}\left(\frac{1-\sqrt5}{2}\right)^{n-1} \\
&= \frac{1}{\sqrt5}\left(\frac{1+\sqrt5}{2}\right)^{n-1}\left(\frac{1+\sqrt5}{2}+1\right) - \frac{1}{\sqrt5}\left(\frac{1-\sqrt5}{2}\right)^{n-1}\left(\frac{1-\sqrt5}{2}+1\right) \\
&= \frac{1}{\sqrt5}\left(\frac{1+\sqrt5}{2}\right)^{n-1}\left(\frac{1+\sqrt5}{2}\right)^2 - \frac{1}{\sqrt5}\left(\frac{1-\sqrt5}{2}\right)^{n-1}\left(\frac{1-\sqrt5}{2}\right)^2 \\
&= \frac{1}{\sqrt5}\left(\frac{1+\sqrt5}{2}\right)^{n-1} - \frac{1}{\sqrt5}\left(\frac{1-\sqrt5}{2}\right)^{n-1} = F_{n+1}
\end{aligned}$$

(b)
$$\begin{aligned}
F_1 &= \frac{1}{\sqrt5}\left(\frac{1+\sqrt5}{2}\right) - \frac{1}{\sqrt5}\left(\frac{1-\sqrt5}{2}\right) = 1 \\
F_2 &= \frac{1}{\sqrt5}\left(\frac{1+\sqrt5}{2}\right)^2 - \frac{1}{\sqrt5}\left(\frac{1-\sqrt5}{2}\right)^2 = 1 \\
F_3 &= \frac{1}{\sqrt5}\left(\frac{1+\sqrt5}{2}\right)^3 - \frac{1}{\sqrt5}\left(\frac{1-\sqrt5}{2}\right)^3 = 2 \\
F_4 &= \frac{1}{\sqrt5}\left(\frac{1+\sqrt5}{2}\right)^4 - \frac{1}{\sqrt5}\left(\frac{1-\sqrt5}{2}\right)^4 = 3
\end{aligned}$$

$$F_5 = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^5 - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^5 = 5$$

38. $\displaystyle\lim_{n\to\infty}\frac{F_{n+1}}{F_n} = \lim_{n\to\infty}\frac{\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n}} = \lim_{n\to\infty}\frac{1}{2}\cdot\frac{(1+\sqrt{5})^{n+1}-(1-\sqrt{5})^{n+1}}{(1+\sqrt{5})^n-(1-\sqrt{5})^n}$

Dividing numerator and denominator by $(1+\sqrt{5})^n$ we have

$$\lim_{n\to\infty}\frac{F_{n+1}}{F_n} = \frac{1}{2}\lim_{n\to\infty}\frac{1+\sqrt{5}-(1-\sqrt{5})\left(\dfrac{1-\sqrt{5}}{1+\sqrt{5}}\right)^n}{1-\left(\dfrac{1-\sqrt{5}}{1+\sqrt{5}}\right)^n}.$$

Now $\dfrac{1-\sqrt{5}}{1+\sqrt{5}} = \dfrac{1-\sqrt{5}}{1+\sqrt{5}}\cdot\dfrac{1-\sqrt{5}}{1+\sqrt{5}} = \dfrac{1-2\sqrt{5}+5}{1-5} = \dfrac{\sqrt{5}-3}{2}$. Since $|(\sqrt{5}-3)/2|<1$, $\displaystyle\lim_{n\to\infty}\left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^n = \lim_{n\to\infty}\left(\frac{\sqrt{5}-3}{2}\right)^n = 0$, and $\displaystyle\lim_{n\to\infty}\frac{F_{n+1}}{F_n} = \frac{1}{2}(1+\sqrt{5})$.

39. Applying the ration test to $\displaystyle\sum_{n-1}^{\infty}\frac{1}{F_n}$, we have

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{1/F_{n+1}}{1/F_n} = \lim_{n\to\infty}\frac{F_n}{F_{n+1}} = \frac{2}{1+\sqrt{5}} < 1.$$

Thus, the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{F_n}$ converges.

40. (a) Using the ratio test, we have

$$\begin{aligned}\lim_{n\to\infty}\frac{a_{n+1}}{a_n} &= \lim_{n\to\infty}\frac{1103+36390(n+1)}{1103+26390n}\,\frac{(4n+4)!}{[(n+1)!]^4[4.99]^{n+1}}\,\frac{(n!)^4(4.99)^n}{(4n)!}\\ &= \lim_{n\to\infty}\frac{1103+36390(n+1)}{1103+26390n}\,\frac{(4n+4)(4n+3)(4n+2)(4n+1)}{(n+1)^4(4)(99)}\\ &= \lim_{n\to\infty}\frac{4\cdot4\cdot4\cdot4}{4\cdot99} = \frac{64}{99} < 1.\end{aligned}$$

Thus, the series converges.

(b) The first term of the series is $1/\pi \approx 2\sqrt{2}(1103)/9801$, which gives $\pi \approx 9801/2\sqrt{2}(1103) \approx 3.14159273$. This is accurate to 6 decimal places.

(c) The first two terms of the series give

$$1/\pi \approx \frac{2\sqrt{2}}{9801}\left[1103 + 27,493\left(\frac{4!}{(4\cdot99)^4}\right)\right] = \frac{2\sqrt{2}[1103\cdot99^4+27493\cdot6]}{9801\cdot99^4}.$$

Then $\pi \approx 9801(99)^4/2\sqrt{2}[1103\cdot99^4+27493\cdot6] \approx 3.14159265358979388$, which is accurate to 14 decimal places.

9.7 Alternating Series

1. Since $a_{k+1} = \dfrac{1}{k+3} < \dfrac{1}{k+2} = a_k$ and $\lim\limits_{k\to\infty} \dfrac{1}{k+2} = 0$, the series converges.

2. Since $a_{k+1} = \dfrac{1}{\sqrt{k+1}} < \dfrac{1}{\sqrt{k}} = a_k$ and $\lim\limits_{k\to\infty} \dfrac{1}{\sqrt{k}} = 0$, the series converges.

3. Since $\lim\limits_{k\to\infty} \dfrac{k}{k+1} = 1 \neq 0$, the series diverges.

4. Let $f(x) = \dfrac{x}{x^2+1}$. Then $f'(x) = \dfrac{1-x^2}{(x^2+1)^2} < 0$ for $x > 1$ and $a_{k+1} < a_k$. Since $\lim\limits_{k\to\infty} \dfrac{k}{k^2+1} = \lim\limits_{k\to\infty} \dfrac{1}{k+1/k} = 0$, the series converges.

5. Let $f(x) = \dfrac{x^2+2}{x^3}$. Then $f'(x) = -\dfrac{x^2+6}{x^4} < 0$ for $x \geq 1$ and $a_{k+1} < a_k$. Since $\lim\limits_{k\to\infty} \dfrac{k^2+2}{k^3} = \lim\limits_{k\to\infty} \dfrac{1/k+2/k^3}{1} = 0$, the series converges.

6. Since $\lim\limits_{k\to\infty} \dfrac{3k-1}{k+5} = 3 \neq 0$, the series diverges.

7. Since $a_{k+1} = \dfrac{1}{k+1} + \dfrac{1}{3^{k+1}} < \dfrac{1}{k} + \dfrac{1}{3^k} = a_k$ and $\lim\limits_{k\to\infty} \left(\dfrac{1}{k} + \dfrac{1}{3^k}\right) = 0$, the series converges.

8. Since $a_{k+1} = \dfrac{k+2}{4^{k+1}} = \dfrac{k/4+1/2}{4^k} < \dfrac{k+1}{4^k} = a_k$ and using L'Hôpital's Rule, $\lim\limits_{k\to\infty} \dfrac{k+1}{4^k} \stackrel{h}{=} \lim\limits_{k\to\infty} \dfrac{1}{4^k \ln 4} = 0$, the series converges.

9. Let $f(x) = \dfrac{4\sqrt{x}}{2x+1}$. Then $f'(x) = \dfrac{2-4x}{\sqrt{x}(2x+1)^2} < 0$ for $x \geq 1$ and $a_{n+1} < a_n$. Since $\lim\limits_{n\to\infty} \dfrac{4\sqrt{n}}{2n+1} = \lim\limits_{n\to\infty} \dfrac{4}{2\sqrt{n}+1/\sqrt{n}} = 0$, the series converges.

10. Let $f(x) = \dfrac{x^{1/3}}{x+1}$. Then $f'(x) = \dfrac{1-2x}{3x^{2/3}(x+1)^2} < 0$ for $x \geq 1$ and $a_{n+1} < a_n$. Since $\lim\limits_{n\to\infty} \dfrac{\sqrt[3]{n}}{n+1} = \lim\limits_{n\to\infty} \dfrac{1}{n^{2/3}+n^{-1/3}} = 0$, the series converges.

11. Note that $\cos n\pi = (-1)^n$. Let $f(x) = \dfrac{\sqrt{x+1}}{x+2}$. Then $f'(x) = \dfrac{-2x}{2\sqrt{x+1}(x+2)^2} < 0$ for $x \geq 2$ and $a_{n+1} < a_n$. Since $\lim\limits_{n\to\infty} \dfrac{\sqrt{n+1}}{n+2} = \lim\limits_{n\to\infty} \dfrac{\sqrt{1+1/n}}{\sqrt{n}+2/\sqrt{n}} = 0$, the series converges.

12. Let $f(x) = \dfrac{\sqrt{x^2+1}}{x^3}$. Then $f'(x) = \dfrac{-2x^2-3}{x^4\sqrt{x^2+1}} < 0$ for $x \geq 2$ and $a_{k+1} < a_k$. Since $\lim\limits_{n\to\infty} \dfrac{\sqrt{k^2+1}}{k^3} = \lim\limits_{n\to\infty} \dfrac{\sqrt{1+1/k^2}}{k^2} = 0$, the series converges.

13. Using L'Hôpital's Rule, $\lim_{x\to\infty} \frac{x}{\ln x} = \lim_{x\to\infty} \frac{1}{1/x} = \lim_{x\to\infty} x \neq 0$, and the series diverges.

14. Since $a_{k+1} = \frac{1}{\ln(k+1)} < \frac{1}{\ln k} = a_k$ and $\lim_{k\to\infty} \frac{1}{\ln k} = 0$, the series converges.

15. Apply the limit comparison test to $\sum_{k=1}^{\infty} \frac{1}{2k+1}$ with $a_k = \frac{1}{2k+1}$ and $b_k = \frac{1}{k}$:

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{1/(2k+1)}{1/k} = \lim_{k\to\infty} \frac{k}{2k+1} = \frac{1}{2}.$$

Since $\sum_{k=1}^{\infty} b_k$ diverges, the given series is not absolutely convergent. Since $a_{k+1} = \frac{1}{2k+3} < \frac{1}{2k+1} = a_k$ and $\lim_{k\to\infty} \frac{1}{2k+1} = 0$, the series is conditionally convergent.

16. Apply the limit comparison test to $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k+5}}$ with $a_k = \frac{1}{\sqrt{k+5}}$ and $b_k = \frac{1}{\sqrt{k}}$:

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{1/\sqrt{k+5}}{1/\sqrt{k}} = \lim_{k\to\infty} \frac{1}{\sqrt{1+5/k}} = 1.$$

Since $\sum_{k=1}^{\infty} b_k$ diverges, the given series is not absolutely convergent. Since $a_{k+1} = \frac{1}{\sqrt{k+6}} < \frac{1}{\sqrt{k+5}} = a_k$ and $\lim_{k\to\infty} \frac{1}{\sqrt{k+5}} = 0$, the series is conditionally convergent.

17. Since $\sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^k$ is a geometric series with $r = \frac{2}{3} < 1$, the series is absolutely convergent.

18. Since $\frac{2^{2k}}{3^k} = \left(\frac{4}{3}\right)^k$, then $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \left(\frac{4}{3}\right)^k = \infty$, and the series diverges by the n-th term test.

19. Since $$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{(k+1)/5^{k+1}}{k/5^k} = \lim_{k\to\infty} \frac{k+1}{5k} = \frac{1}{5} < 1,$$ the series is absolutely convergent by the ratio test.

20. Since $$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{(k+1)^2[2^{-(k+1)}]^2}{k^2(2^{-k})^2} = \lim_{k\to\infty} \frac{(k+1)^2}{k^2 4} = \frac{1}{4} < 1,$$ the series is absolutely convergent by the ratio test.

21. Since $$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{1/(k+1)!}{1/k!} = \lim_{k\to\infty} \frac{1}{k+1} = 0 < 1,$$ the series is absolutely convergent by the ratio test.

22. Since $$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{[(k+1)!]^2/(2k+2)!}{(k!)^2/(2k)!} = \lim_{k\to\infty}\frac{(k+1)^2}{(2k+2)(2k+1)} = \lim_{k\to\infty}\frac{k^2+2k+1}{4k^2+6k+2} = \frac{1}{4} < 1,$$
the series is absolutely convergent by the ratio test.

23. Since $$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{(k+1)!/100^{k+1}}{k!/100^k} = \lim_{k\to\infty}\frac{k+1}{100} = \infty,$$
the series is divergent by the ratio test.

24. Since
$$\lim_{k\to\infty} a_k = \lim_{k\to\infty}\frac{5^{2k-3}}{10^{k+2}} = \lim_{k\to\infty}\frac{5^{-3}}{10^2}\left(\frac{5^{2k}}{10^k}\right) = \frac{1}{12500}\lim_{k\to\infty}\frac{25^k}{10^k} = \frac{1}{12500}\lim_{k\to\infty}\left(\frac{5}{2}\right)^k = \infty,$$
the series diverges by the n-th term test.

25. Apply the limit comparison test to $\sum_{k=1}^{\infty}\frac{k}{1+k^2}$ with $a_k = \frac{k}{1+k^2}$ and $b_k = \frac{1}{k}$: $\lim_{k\to\infty}\frac{a_k}{b_k} = \lim_{k\to\infty}\frac{k/(1+k^2)}{1/k} = \lim_{k\to\infty}\frac{k^2}{1+k^2} = 1$. Since $\sum_{k=1}^{\infty} b_k$ diverges, the given series is not absolutely convergent. Let $f(x) = \frac{x}{1+x^2}$. Then $f'(x) = \frac{1-x^2}{(1+x^2)^2} < 0$ for $x > 1$ and $a_{k+1} < a_k$. Also, $\lim_{k\to\infty} a_k = \lim_{k\to\infty}\frac{k}{1+k^2} = 0$, so the series is conditionally convergent.

26. Since $\frac{k}{1+k^4} = \frac{1}{k^3+1/k} < \frac{1}{k^3}$, the series is absolutely convergent by comparison with the p-series $\sum_{k=1}^{\infty}\frac{1}{k^3}$.

27. Since $\cos k\pi = (-1)^k$ and $\lim_{k\to\infty}(-1)^k$ is not 0, the series diverges by the n-th term test.

28. Since $\sin\left(\frac{2k+1}{2}\right)\pi = \sin\left(k\pi + \frac{\pi}{2}\right) = (-1)^k$, the series is $\sum_{k=1}^{\infty}\frac{(-1)^k}{\sqrt{k+1}}$. Apply the limit comparison test to $\sum_{k=1}^{\infty}\frac{1}{\sqrt{k+1}}$ with $a_k = \frac{1}{\sqrt{k+1}}$ and $b_k = \frac{1}{\sqrt{k}}$: $\lim_{k\to\infty}\frac{a_k}{b_k} = \lim_{k\to\infty}\frac{1/\sqrt{k+1}}{1/\sqrt{k}} = \lim_{k\to\infty}\frac{\sqrt{k}}{\sqrt{k+1}} = 1$. Since $\sum_{k=1}^{\infty} b_k$ is a p-series with $p = \frac{1}{2} < 1$, it is divergent and the given series is not absolutely convergent. Now, $a_{k+1} = \frac{1}{\sqrt{k+2}} < \frac{1}{\sqrt{k+1}} = a_k$ and $\lim_{k\to\infty} a_k = \lim_{k\to\infty}\frac{1}{\sqrt{k+1}} = 0$, so the original series is conditionally convergent.

29. Apply the limit comparison test to $\sum_{k=1}^{\infty} \sin\left(\frac{1}{k}\right)$ with $a_k = \sin\left(\frac{1}{k}\right)$ and $b_k = \frac{1}{k}$:

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\sin(1/k)}{1/k} \overset{h}{=} \lim_{k\to\infty} \frac{(-1/k^2)\cos(1/k)}{-1/k^2} = \lim_{k\to\infty} \cos(1/k) = 1.$$

Since $\sum_{k=1}^{\infty} b_k$ diverges, the given series is not absolutely convergent. Let $f(x) = \sin\left(\frac{1}{x}\right)$. Then $f'(x) = \left(-\frac{1}{x^2}\right)\cos\left(\frac{1}{x}\right) < 0$ for $x \geq 1$ and $a_{k+1} < a_k$. Since $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \sin\left(\frac{1}{k}\right) = 0$, the original series is conditionally convergent.

30. Apply the limit comparison test to $\sum_{k=1}^{\infty} \frac{1}{k^2} \sin\left(\frac{1}{k}\right)$ with $a_k = \frac{1}{k^2}\sin\left(\frac{1}{k}\right)$ and $b_k = \frac{1}{k^3}$:

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{[\sin(1/k)]/k^2}{1/k^3} = \lim_{k\to\infty} \frac{\sin(1/k)}{1/k} \overset{h}{=} \lim_{k\to\infty} \frac{(-1/k^2)\cos(1/k)}{-1/k^2} \lim_{k\to\infty} \cos\left(\frac{1}{k}\right) = 1.$$

Since $\sum_{k=1}^{\infty} b_k$ is a p-series with $p = 3 > 1$, it converges and the original series is absolutely convergent.

31. Since $\frac{1}{k+1} - \frac{1}{k} = -\frac{1}{k^2+k}$, the series can be written as $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k^2+k}$. Now, $\frac{1}{k^2+k} < \frac{1}{k^2}$, so the series is absolutely convergent by comparison with the p-series $\sum_{k=1}^{\infty} \frac{1}{k^2}$.

32. Since $\quad \sqrt{k+1} - \sqrt{k} = (\sqrt{k+1} - \sqrt{k})\left(\frac{\sqrt{k+1}+\sqrt{k}}{\sqrt{k+1}+\sqrt{k}}\right) = \frac{1}{\sqrt{k+1}+\sqrt{k}}$,

the series can be written as $\sum_{k=1}^{\infty} (-1)^k \frac{1}{\sqrt{k+1}+\sqrt{k}}$. Apply the limit comparison test to $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k+1}+\sqrt{k}}$ with $a_k = \frac{1}{\sqrt{k+1}+\sqrt{k}}$ and $b_k = \frac{1}{\sqrt{k}}$:

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{1/(\sqrt{k+1}+\sqrt{k})}{1/\sqrt{k}} = \lim_{k\to\infty} \frac{\sqrt{k}}{\sqrt{k+1}+\sqrt{k}} = \lim_{k\to\infty} \frac{1}{\sqrt{1+1/k}+1} = \frac{1}{2} > 0.$$

Since $\sum_{k=1}^{\infty} b_k$ is a p-series with $p = \frac{1}{2} < 1$, it diverges and the given series is not absolutely convergent. Now let $f(x) = \sqrt{x+1} - \sqrt{x}$. Then $f'(x) = \frac{1}{2\sqrt{x+1}} - \frac{1}{2\sqrt{x}} < 0$ for $x \geq 1$ and $a_{k+1} < a_k$. Since $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{1}{\sqrt{k+1}+\sqrt{k}} = 0$, the original series is conditionally convergent.

33. Since $\lim_{k\to\infty}\left(\frac{2k}{k+50}\right)^k = \lim_{k\to\infty}\left(\frac{2}{1+50/k}\right)^k = \infty$, the series diverges by the n-th term test.

34. Since $\lim_{k\to\infty}|a_k|^{1/k} = \lim_{k\to\infty}\left(\frac{6^{3k}}{k^k}\right)^{1/k} = \lim_{k\to\infty}\frac{216}{k} = 0 < 1$, the series is absolutely convergent by the root test.

35. We must have $a_{n+1} = \frac{1}{(2n+1)!} < 0.000005$. Taking $n = 4$ we have $a_5 = \frac{1}{9!} \approx 0.000003 < 0.000005$. Thus, $S_4 = \frac{1}{1!} - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} \approx 0.84147$ has the desired accuracy.

36. We must have $a_{n+1} = \frac{1}{(n+1)!} < 0.0005$. Taking $n = 6$ we have $a_7 = \frac{1}{7!} \approx 0.0002 < 0.0005$. Thus, $S_6 = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!} \approx 0.632$ has the desired accuracy.

37. We must have $a_{n+1} = \frac{1}{(n+1)^3} < 0.005$. Taking $n = 5$ we have $a_6 = \frac{1}{6^3} \approx 0.0046 < 0.005$. Thus, S_5 has the desired accuracy.

38. We must have $a_{n+1} = \frac{1}{\sqrt{n+1}} < 0.0005 = \frac{1}{2000}$. Taking $n+1 = (2000)^2 + 1 = 4,000,001$ we have $a_{4,000,001} < 0.005$. Thus, $S_{4,000,000}$ has the desired accuracy.

39. We must have $a_{n+1} = \frac{1}{4^{n+1}} < 0.001$. Taking $n = 4$ we have $a_5 = \frac{1}{4^5} \approx 0.00098 < 0.001$. Thus, $S_4 = 1 - \frac{1}{4^2} + \frac{1}{4^3} - \frac{1}{4^4} \approx 0.9492$ has the desired accuracy.

40. We must have $a_{n+1} = \frac{n+1}{5^{n+1}} < 0.0001$. Taking $n = 6$ we have $a_7 = \frac{7}{5^7} \approx 0.00009 < 0.0001$. Thus, $S_6 = 1 - \frac{2}{5^2} + \frac{3}{5^3} - \frac{4}{5^4} + \frac{5}{5^5} - \frac{6}{5^6} \approx 0.93882$ has the desired accuracy.

41. The error will be less than $a_{101} = \frac{1}{101} \approx 0.009901$.

42. The error will be less than $a_7 = \frac{1}{7(2^7)} \approx 0.00112$.

43. This is not an alternating series since, for $k = 1$ to $k = 5$, the terms are positive, while for $k = 7$ to $k = 11$, the terms are negative. Since $|a_{k+1}| \le \frac{1}{k^2}$, the series is absolutely convergent by comparison with the p-series $\sum_{k=1}^{\infty}\frac{1}{k^2}$. Hence, the series is convergent.

44. This is not an alternating series since for $k \le 6$, the terms are positive. For $k \ge 7$, the terms alternate but do not satisfy $|a_{k+1}| \le |a_k|$ since $a_7 = \frac{100 - 2^7}{3^7} \approx -0.013$ and $a_8 = \frac{100 + 2^8}{3^8} \approx 0.054$. Write $\sum_{k=1}^{\infty}\frac{100 + (-1)^k 2^k}{3^k} = \sum_{k=1}^{\infty}\left[\frac{100}{3^k} + (-1)^k\left(\frac{2}{3}\right)^k\right]$ and apply Theorem 9.3.5 in the

text. Since $\sum_{k=1}^{\infty} \frac{100}{3^k}$ and $\sum_{k=1}^{\infty} (-1)^k \left(\frac{2}{3}\right)^k$ are geometric series with $|r| < 1$, they both converge and the original series is convergent.

45. This is not an alternating series. Since $|a_k| = \frac{1}{2^{k-1}}$, $\sum_{k=1}^{\infty} |a_k|$ is a geometric series with $r = \frac{1}{2} < 1$, and the original series is absolutely convergent.

46. This is not an alternating series. Since $|a_k| = \frac{1}{k^2}$, $\sum_{k=1}^{\infty} |a_k|$ is a p-series with $p = 2 > 1$, and the original series is absolutely convergent.

47. The terms of the series do not satisfy $|a_{n+1}| \leq |a_n|$ since $|a_5| = \frac{2}{3} > \frac{1}{2} = |a_4|$. Grouping pairs of terms, we obtain the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \cdots$. Thus, the sequence of partial sums $\{S_{2n}\}$ is the same as the sequence of partial sums for the harmonic series. Since the latter sequence diverges, so does $\{S_{2n}\}$. Finally, if $\{S_{2n}\}$ diverges, so must $\{S_n\}$. Thus, the original series diverges.

48. This is not an alternating series. The sequence of partial sums S_2, S_5, S_9, S_{14}, S_{20}, $\ldots$ is 1, 0, 1, 0, 1, $\ldots$. Since the sequence diverges, so must $\{S_n\}$. Thus, the original series diverges.

49. The terms do not approach 0, so the series diverges.

50. The terms of the series are all 0, so the series converges.

51. All terms of the series after the first are 0, so the series converges.

52. All odd terms of the series are 1 or -1, so the terms do not approach 0 and the series diverges.

53. The statement is true because a positive-term series $\sum a_k$ is the same as the series of its terms' absolute values $\sum |a_k|$. If this series is convergent, then it is also absolutely convergent, and so, as stated in the discussion, its terms can be rearranged in any manner and the resulting series will converge to the same number as the original series.

54. S can be written as $S = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$. When rearranged as shown in the exercise, the resulting series is $\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \frac{1}{14} - \cdots$, which can be written as $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{2k}$. Factoring, we get

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{2k} = \sum_{k=1}^{\infty} (-1)^{k+1} \left(\frac{1}{2}\right)\left(\frac{1}{k}\right) = \frac{1}{2} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} = \frac{1}{2} S.$$

55. Let $\sum_{n=1}^{\infty} a_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$, and from Problem 54 let $\sum_{n=1}^{\infty} b_n = 0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + \cdots$. Then by Theorem 9.3.5 in the text,

$$\begin{aligned}\frac{3}{2}S = S + \frac{1}{2}S = \sum_{n=1}^{\infty}(a_n + b_n) &= 1 + 0 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + 0 + \frac{1}{7} - \frac{1}{4} + \cdots \\ &= 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots .\end{aligned}$$

56. Write the series in the form $\sum_{k=0}^{\infty}(-1)^{k+1}\frac{1}{3^{k+(-1)^k}}$. Applying the ratio test, we have

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{1/3^{k+1+(-1)^{k+1}}}{1/3^{k+(-1)^k}} = \lim_{k\to\infty}\frac{1}{3^{1+(-1)^{k+1}-(-1)^k}} = \lim_{k\to\infty}\frac{1}{3^{1+2(-1)^{k+1}}}.$$

This limit does not exist, so the ratio test is inconclusive.

Applying the root test, we have

$$\lim_{k\to\infty}|a_k|^{1/k} = \lim_{k\to\infty}\left[\frac{1}{3^{k+(-1)^k}}\right]^{1/k} = \lim_{k\to\infty}\frac{1}{3^{1+(-1)^k/k}} = \frac{1}{3} < 1.$$

Thus, the series is absolutely convergent by the root test.

57. Since $\sum_{k=1}^{\infty} a_k$ is absolutely convergent, $\lim_{n\to\infty} a_k = 0$. Thus, for n sufficiently large, $|a_n| < 1$ and $a_n^2 < |a_n|$. Therefore, $\sum_{k=1}^{\infty} a_k^2$ converges by the comparison test.

58. $\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{\sqrt{k}}$ converges (see Exercise 2), but $\sum_{k=1}^{\infty}\left[\frac{(-1)^{k-1}}{\sqrt{k}}\right]^2 = \sum_{k=1}^{\infty}\frac{1}{k}$ is the harmonic series, which diverges.

59. The alternating harmonic series converges (see Example 2 in this section) and the series consisting of its terms' squares, $\sum_{k=1}^{\infty}\frac{1}{x^2}$, is a p-series with $p = 2 > 1$, which also converges.

60. The harmonic series $\sum_{k=1}^{\infty}\frac{1}{k}$ is divergent while the p-series with $p = 2 > 1$ $\sum_{k=1}^{\infty}\frac{1}{k^2}$ converges.

61. $e^{-x}\sin x + e^{-2x}\sin 2x + e^{-3x}\sin 3x + \cdots$ can be written as $\sum_{k=1}^{\infty} e^{-kx}\sin kx$, and for $x > 0$, $|e^{-kx}\sin kx| \leq |e^{-kx}| = (e^{-x})^k$. Now, $\sum_{k=1}^{\infty}(e^{-x})^k$ is a convergent geometric series since

$e^{-x} < 1$ for $x > 0$. Thus, $\sum_{k=1}^{\infty} |e^{-kx} \sin kx|$ converges by the direct comparison test for all positive x, and so $\sum_{k=1}^{\infty} e^{-kx} \sin kx$ must also converge for all positive x.

9.8 Power Series

1. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}/(n+1)}{x^n/n}\right| = \lim_{n\to\infty} \frac{n}{n+1}|x| = |x|$

   The series is absolutely convergent on $(-1, 1)$. At $x = -1$, the series $\sum_{k=1}^{\infty} \frac{1}{k}$ is the harmonic series which diverges. At $x = 1$, the series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ converges by the alternating series test. Thus, the given series converges on $(-1, 1]$.

2. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}/(n+1)^2}{x^n/n^2}\right| = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 |x| = |x|$

   The series is absolutely convergent on $(-1, 1)$. At $x = -1$, the series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$ converges by the alternating series test. At $x = 1$, the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is a convergent p-series.. Thus, the given series converges on $[-1, 1]$.

3. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{2^{n+1}x^{n+1}/(n+1)}{2^n x^n/n}\right| = \lim_{n\to\infty} \frac{2n}{n+1}|x| = 2|x|$

   The series is absolutely convergent for $2|x| < 1$ or $|x| < 1/2$. At $x = -1/2$, the series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ converges by the alternating series test. At $x = 1/2$, the series $\sum_{k=1}^{\infty} \frac{1}{k}$ is the harmonic series which diverges. Thus, the given series converges on $[-1/2, 1/2)$.

4. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{5^{n+1}x^{n+1}/(n+1)!}{5^n x^n/n!}\right| = \lim_{n\to\infty} \frac{5}{n+1} = 0$

   The series is absolutely convergent on $(-\infty, \infty)$.

5. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-3)^{n+1}/(n+1)^3}{(x-3)^n/n^3}\right| = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^3 |x-3| = |x-3|$

   The series is absolutely convergent for $|x-3| < 1$ or on $(2, 4)$. At $x = 2$, the series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^3}$ converges by the alternating series test. At $x = 4$, the series $\sum_{k=1}^{\infty} \frac{1}{k^3}$ is a convergent p-series. Thus, the given series converges on $[2, 4]$.

6. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-3)^{n+1}/\sqrt{n+1}}{(x+7)^n/\sqrt{n}}\right| = \lim_{n\to\infty}\sqrt{\frac{n}{n+1}}|x+7| = |x+7|$
The series is absolutely convergent for $|x+7|<1$ or on $(-8,6)$. At $x=-8$, the series $\sum_{k=1}^{\infty}\frac{(-1)^k}{\sqrt{k}}$ converges by the alternating series test. At $x=-6$, the series $\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}$ is a divergent p-series. Thus, the given series converges on $[-8,-6)$.

7. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-5)^{n+1}/10^{n+1}}{(x-5)^n/10^n}\right| = \lim_{n\to\infty}\frac{1}{10}|x-5| = \frac{1}{10}|x-5|$
The series is absolutely convergent for $\frac{1}{10}|x-5|<1$, $|x-5|<10$, or on $(-5,15)$. At $x=-5$, the series $\sum_{k=1}^{\infty}\frac{(-1)^k(-10)^k}{10^k} = \sum_{k=1}^{\infty}1$ diverges by the n-th term test. At $x=15$, the series $\sum_{k=1}^{\infty}\frac{(-1)^k 10^k}{10^k} = \sum_{k=1}^{\infty}(-1)^k$ diverges by the n-th term test. Thus, the series converges on $(-5,15)$.

8. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(x-4)^{n+1}/(n+3)^2}{n(x-4)^n/(n+2)^2}\right| = \lim_{n\to\infty}\frac{(n+1)(n+2)^2}{n(n+3)^2}|x-4| = |x-4|$
The series is absolutely convergent for $|x-4|<1$ or on $(3,5)$. At $x=3$, the series $\sum_{k=1}^{\infty}\frac{(-1)^k k}{(k+2)^2}$ converges by the alternating series test. At $x=5$, the series $\sum_{k=1}^{\infty}\frac{k}{(k+2)^2}$ diverges by the limit comparison test with $\sum_{k=1}^{\infty}\frac{1k}{k}$. Thus, the series converges on $[3,5)$..

9. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!2^{n+1}x^{n+1}}{n!2^n x^n}\right| = \lim_{n\to\infty}2(n+1)|x| = \infty, \quad x\neq 0$
The series converges only at $x=0$.

10. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{nx^{n+1}/(n+1)^{2(n+1)}}{(n-1)x^n/n^{2n}}\right| = \lim_{n\to\infty}\frac{nn^{2n}}{(n-1)(n+1)2^{2n+2}}|x|$

$$= \lim_{n\to\infty}\frac{n}{(n+1)(n+1)^2}\left(\frac{n}{n+1}\right)^{2n}|x| = \lim_{n\to\infty}\frac{n}{(n-1)(n+1)^2}\cdot\frac{1}{\left[\left(\frac{n+1}{n}\right)^n\right]^2}|x|$$

$$= \lim_{n\to\infty}\frac{n}{(n-1)(n+1)^2}\cdot\frac{1}{[(1+1/n)^n]^2}|x| = 0\cdot\frac{1}{e^2}|x| = 0$$

The series is convergent on $(-\infty,\infty)$.

11. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(3x-1)^{n+1}/[(n+1)^2+(n+1)]}{(3x-1)^n/(n^2+n)}\right| = \lim_{n\to\infty}\frac{n^2+n}{n^2+3n+2}|3x-1| = |3x-1|$
The series is absolutely convergent for $|3x-1|<1$ or on $(0,2/3)$. At $x=0$, the series

$\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2+k}$ converges by the alternating series test. At $x = 2/3$, the series $\sum_{k=1}^{\infty}\frac{1}{k^2+k}$ converges by comparison with the p-series $\sum_{k=1}^{\infty}\frac{1}{k^2}$. Thus, the given series converges on $[0, 2/3]$.

12. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(4x-5)^{n+1}/3^{n+1}}{(4x-5)^n/3^n}\right| = \lim_{n\to\infty}\frac{1}{3}|4x-5| = \frac{1}{3}|4x-5|$

The series is absolutely convergent for $\frac{1}{3}|4x-5| < 1$, $|4x-5| < 3$, or on $(1/2, 2)$. At $x = 1/2$, the series $\sum_{k=1}^{\infty}\frac{(-3)^k}{3^k} = \sum_{k=0}^{\infty}(-1)^k$ diverges by the n-th term test. At $x = 2$, the series $\sum_{k=0}^{\infty}\frac{3^k}{3^k} = \sum_{k=0}^{\infty}1$ diverges by the n-th term test. Thus, the given series converges on $(1/2, 2)$.

13. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}/\ln(n+1)}{x^n/\ln n}\right| = \lim_{n\to\infty}\frac{\ln n}{\ln(n+1)}|x|$

By L'Hôpital's Rule, $\lim_{n\to\infty}\frac{\ln n}{\ln(n+1)} \overset{h}{=} \lim_{n\to\infty}\frac{1/n}{1/(n+1)} = \lim_{n\to\infty}\frac{n+1}{n} = 1$. Thus, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x|$. The series is absolutely convergent on $(-1, 1)$. At $x = -1$, the series $\sum_{k=2}^{\infty}\frac{(-1)^k}{\ln k}$ converges by the alternating series test. At $x = 1$, the series $\sum_{k=2}^{\infty}\frac{1}{\ln k}$ diverges by comparison with $\sum_{k=2}^{\infty}\frac{1}{k}$. Thus, the given series converges on $[-1, 1)$.

14. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}/(n+1)\ln(n+1)}{x^n/n\ln n}\right| = \lim_{n\to\infty}\frac{n}{n+1}\cdot\frac{\ln n}{\ln(n+1)}|x|$

By L'Hôpital's Rule, $\lim_{n\to\infty}\frac{\ln n}{\ln(n+1)} \overset{h}{=} \lim_{n\to\infty}\frac{1/n}{1/(n+1)} = \lim_{n\to\infty}\frac{n+1}{n} = 1$. Thus, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x|$. The series is absolutely convergent on $(-1, 1)$. At $x = -1$, the series $\sum_{k=2}^{\infty}\frac{1}{k\ln k}$ diverges by the integral test. At $x = 1$, the series $\sum_{k=2}^{\infty}\frac{(-1)^k}{k\ln k}$ converges by the alternating series test. Thus, the given series converges on $(-1, 1]$.

15. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2(x+7)^{n+1}/3^{2n+2}}{n^2(x+7)^n/3^{2n}}\right| = \lim_{n\to\infty}\frac{1}{9}\left(\frac{n+1}{n}\right)^2|3x+7| = \frac{1}{9}|x+7|$

The series is absolutely convergent for $\frac{1}{9}|x+7| < 1$, $|x+7| < 9$, or on $(-16, 2)$. At $x = -16$, the series $\sum_{k=1}^{\infty}\frac{(-9)^k k^2}{9^k} = \sum_{k=1}^{\infty}(-1)^k k^2$ diverges by the n-th term test. At $x = 2$, the series

$\sum_{k=1}^{\infty} \frac{9^k k^2}{9^k} = \sum_{k=1}^{\infty} k^2$ diverges by the n-th term test. Thus, the given series converges on $(-16, 2)$.

16. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)^3 2^{4(n+1)}(x-1)^{n+1}}{n^3 2^{4n}(x-1)^n}\right| = \lim_{n\to\infty} 16\left(\frac{n+1}{n}\right)^3 |x-1| = 16|x-1|$
The series is absolutely convergent for $16|x-1| < 1$, $|x-1| < 1/16$, or on $(15/16, 17/16)$. At $x = 15/16$, the series $\sum_{k=1}^{\infty} k^3 2^{4k}\left(-\frac{1}{16}\right)^k = \sum_{k=1}^{\infty}(-1)^k k^3$ diverges by the n-th term test. At $x = 17/16$, the series $\sum_{k=1}^{\infty} k^3 2^{4k}\left(\frac{1}{16}\right)^k = \sum_{k=1}^{\infty} k^3$ diverges by the n-th term test. Thus, the given series converges on $(15/16, 17/16)$.

17. Write the series as $\sum_{k=1}^{\infty}\left(\frac{32}{75}\right)^k x^k$. Then
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(35/75)^{n+1}x^{n+1}}{(32/75)^n x^n}\right| = \lim_{n\to\infty}\left|\frac{32}{75}x\right| = \frac{32}{75}|x|.$$
The series is absolutely convergent for $\frac{32}{75}|x| < 1$, or on $(-75/32, 75/32)$. At $x = -75/32$ the series $\sum_{k=1}^{\infty}(-1)^k$ diverges by the n-th term test. At $x = 75/32$ the series $\sum_{k=1}^{\infty} 1$ diverges by the n-th term test. Thus, the given series converges on $(-75/32, 75/32)$

18. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1000^{n+1}x^{n+1}/(n+1)^{n+1}}{1000^n x^n/n^n}\right| = \lim_{n\to\infty}\frac{1000}{n+1}\left(\frac{n}{n+1}\right)^n |x|$
$= \lim_{n\to\infty} 1000n + 1\left(\frac{1}{e}\right)|x| = 0$
The series is absolutely convergent on $(-\infty, \infty)$.

19. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}(x-1)^{n+1}/(n+2)(n+3)}{3^n(x-1)^n/(n+1)(n+2)}\right| = \lim_{n\to\infty} 3\left(\frac{n+1}{n+3}\right)|x-1| = 3|x-1|$
The series is absolutely convergent for $3|x-1| < 1$, $|x-1| < 1/3$, or on $(2/3, 4/3)$. At $x = 2/3$, the series $\sum_{k=0}^{\infty}\frac{(-3)^k}{(k+1)(k+2)}\left(-\frac{1}{3}\right)^k = \sum_{k=0}^{\infty}\frac{1}{(k+1)(k+2)}$ converges by comparison with the p-series $\sum_{k=1}^{\infty}\frac{1}{k^2}$. At $x = 4/3$, the series $\sum_{k=0}^{\infty}\frac{(-3)^k}{(k+1)(k+2)}\left(\frac{1}{3}\right)^k = \sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)(k+2)}$ converges by the alternating series test. Thus, the given series converges on $[2/3, 4/3]$.

20. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}(x+5)^{n+1}/2^{n+1}(n+1)(n+2)}{3^n(x+5)^n/2^n(n+1)(n+2)}\right| = \lim_{n\to\infty}\frac{3}{2}\left(\frac{n}{n+2}\right)|x+5| = \frac{3}{2}|x+5|$
The series is absolutely convergent for $\frac{3}{2}|x+5| < 1$, $|x+5| < 2/3$, or on $(-17/3, -13/3)$. At $x = -17/3$, the series $\sum_{k=1}^{\infty}\frac{3^k}{(-2)^k k(k+1)}\left(-\frac{2}{3}\right)^k = \sum_{k=1}^{\infty}\frac{1}{k(k+1)}$ converges by comparison

with the p-series $\sum_{k=0}^{\infty}\frac{1}{k^2}$. At $x=-13/3$, the series $\sum_{k=1}^{\infty}\frac{3^k}{(-2)^k k(k+1)}\left(\frac{2}{3}\right)^k=\sum_{k=1}^{\infty}\frac{(-1)^k}{k(k+1)}$ converges by the alternating series test. Thus, the given series converges on $[-17/3,-13/3]$.

21. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(x-2)^{n+1}/(n+1)!(n+1)!3^{n+1}}{(x-2)^n/n!n!3^n}\right|=\lim_{n\to\infty}\frac{1}{3(n+1)^2}|x-2|=0$
The series converges on $(-\infty,\infty)$.

22. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(6-x)^{n+2}/\sqrt{2n+3}}{(6-x)^{n+1}/\sqrt{2n+1}}\right|=\lim_{n\to\infty}\sqrt{\frac{2n+1}{2n+3}}|6-x|=|6-x|$
The series is absolutely convergent for $|6-x|<1$ or on $(5,7)$. At $x=5$, the series $\sum_{k=0}^{\infty}\frac{1}{\sqrt{2k+1}}$ diverges by the limit comparison test with the p-series $\sum_{k=0}^{\infty}\frac{1}{\sqrt{k}}$. At $x=7$, the series $\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{\sqrt{2k+1}}$ converges by the alternating series test. Thus, the given series converges on $(5,7]$.

23. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{x^{2(n+1)+1}/9^{n+1}}{x^{2n+1}/9^n}\right|=\lim_{n\to\infty}\frac{1}{9}x^2=\frac{1}{9}x^2$
The series is absolutely convergent for $\frac{1}{9}x^2<1$ or on $(-3,3)$. At $x=-3$ the series $\sum_{k=0}^{\infty}(-1)^k(-3)$ diverges by the n-th term test. At $x=3$ the series $\sum_{k=0}^{\infty}(-1)^k 3$ diverges by the n-th term test. Thus, the given series converges on $(-3,3)$.

24. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{5^{n+1}x^{2(n+1)}/(2n+1)!}{5^n x^{2n}/(2n)!}\right|=\lim_{n\to\infty}\frac{5x^2}{(2n+2)(2n+1)}=0$
The series is absolutely convergent on $(-\infty,\infty)$.

25. $\lim_{n\to\infty}|a_n|^{1/n}=\lim_{n\to\infty}\left|\frac{x^n}{(\ln n)^n}\right|^{1/n}=\lim_{n\to\infty}\frac{|x|}{\ln n}=0$
The series is absolutely convergent on $(-\infty,\infty)$.

26. $\lim_{n\to\infty}|a_n|^{1/n}=\lim_{n\to\infty}|(n+1)^n(x+1)^n|^{1/n}=\lim_{n\to\infty}(n+1)|x+1|=\infty, x\neq -1$
The series converges only at $x=-1$.

27. $\lim_{n\to\infty}|a_n|^{1/n}=\lim_{n\to\infty}\left|\left(\frac{4}{3}^n(x+3)^n\right)\right|^{1/n}=\lim_{n\to\infty}\frac{4}{3}|x+3|=\frac{4}{3}|x+3|$
The series is absolutely convergent for $\frac{4}{3}|x+3|<1$, $|x+3|<\frac{3}{4}$, or on $(-15/4,-9/4)$. At $x=-15/4$, the series $\sum_{k=1}^{\infty}\left(\frac{4}{3}\right)^k\left(-\frac{3}{4}\right)^k=\sum_{k=1}^{\infty}(-1)^k$ is divergent by the n-th term test. At

$x = -9/4$, the series $\displaystyle\sum_{k=1}^{\infty}\left(\frac{4}{3}\right)^k\left(\frac{3}{4}\right)^k = \sum_{k=1}^{\infty} 1^k$ is divergent by the n-th term test. Thus, the given series converges on $(-15/4, -9/4)$.

28. $$\lim_{n\to\infty}|a_n|^{1/n} = \lim_{n\to\infty}\left|\left(\frac{n}{n+1}\right)^{n^2}(x-e)^n\right|^{1/n} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n|x-e|$$
$$= \lim_{n\to\infty}\frac{1}{(1+1/n)^n}|x-e| = \frac{1}{e}|x-e|$$
The series is absolutely convergent for $\frac{1}{e}|x-e| < 1$, $|x-e| < e$, or on $(0, 2e)$. At $x = 0$, we have the series $\displaystyle\sum_{k=1}^{\infty}\left(\frac{k}{k+1}\right)^{k^2}(-e)^k = \sum_{k=1}^{\infty}(-1)^k\left[\frac{e}{(1+1/k)^k}\right]^k$. We will now show that $(1+1/k)^k < e$. Using the definition of $\ln x$, we have

$$\ln(1+1/k) = \int_1^{1+1/k}\frac{1}{t}dt \le \int_1^{1+1/k} dt \quad \boxed{\text{since } 1/2 \le 1 \text{ for } t \ge 1}$$
$$= \frac{1}{k}.$$

Thus, $\ln 1{+}1/k) \le 1/k$ and since e^x is an increasing function, $(1+1/k) \le e^{1/k}$ and $(1+1/k)^k \le e$. Hence, $\dfrac{e}{(1+1/k)^k} \ge 1$ and the series $\displaystyle\sum_{k=1}^{\infty}(-1)^k\left[\frac{e}{(1+1/k)^k}\right]^k$ diverges by the n-th term test. Similarly, at $x = 2e$ the series $\displaystyle\sum_{k=1}^{\infty}\left(\frac{k}{k+1}\right)^{k^2} = \sum_{k=1}^{\infty}\left[\frac{e}{(1+1/k)^k}\right]^k$ diverges by the n-th term test. Thus, the given series converges on $(0, 2e)$.

29. $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{(n+1)!(x/2)^{n+1}}{1\cdot 3\cdot 5\cdots(2n-1)(2n+1)}}{\dfrac{n!(x/2)^n}{1\cdot 3\cdot 5\cdots(2n-1)}}\right| = \lim_{n\to\infty}\frac{n+1}{2n+1}\left|\frac{x}{2}\right| = \frac{1}{4}|x|$$
The series has radius of convergence 4.

30. $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{1\cdot 3\cdot 5\cdots(2n-1)(2n+1)(x-1)^{n+1}}{3^{n+1}(x+1)!}}{\dfrac{1\cdot 3\cdot 5\cdots(2n-3)(2n-1)(x-1)^n}{3^n n!}}\right|$$
$$= \lim_{n\to\infty}\frac{2n-1}{3(n+1)}|x-1| = \frac{2}{3}|x-1|$$
The series has radius of convergence $3/2$.

31. $\displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{x}\right)^k$ is a geometric series with common ratio $r = 1/x$. It converges for $|1/x| < 1$ or $x > 1$.

32. Applying the ratio test, $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\dfrac{7^{n+1}/x^{2n+2}}{7^n/x^{2n}}\right| = \lim_{n\to\infty}\dfrac{7}{x^2} = \dfrac{7}{x^2}$. The series converges for $7/x^2 < 1$, $x^2 > 7$, or $(-\infty, -\sqrt{7})\bigcup(\sqrt{7}, \infty)$. At $x = \pm\sqrt{7}$, the series $\displaystyle\sum_{k=1}^{\infty}\frac{7^k}{7^k} = \sum_{k=1}^{\infty} 1$ diverges by the n-th term test. Thus, the given series converges on $(-\infty, -\sqrt{7})\bigcup(\sqrt{7}, \infty)$.

33. This is a geometric series with common ratio $r = \dfrac{x+1}{x}$ and will converge for $\left|\dfrac{x+1}{x}\right| < 1$ or $|x+1| < |x|$. If $x \geq 0$, this means that $x+1 < x$ which has no solution. For $x < 0$, $|x| = -x$ and the inequality can be written as $|x+1| < -x$ or $x < x+1 < -x$. Since $x < x+1$ is valid for all x, we have $x+1 < -x$, $2x < -1$, or $x < -1/2$. Thus, the given series converges on $(-\infty, -1/2)$.

34. This is a geometric series with common ratio $r = \dfrac{x}{2x+4}$ and will converge for $\left|\dfrac{x}{2x+4}\right| < 1$ or $|x| < |2x+4|$. If $x \geq 0$, this means $x < 2x+4$ which is true for all $x \geq 0$. Thus, the series converges on $[0, \infty)$. If $x < 0$, $|x| = -x$ and the inequality can be written as $-x < |2x+4|$. This is equivalent to $2x+4 < -x(-x)$ or $2x+4 > -x$. Solving for x, we obtain $x < -4$ or $x > -4/3$. Therefore, the given series converges on $(-\infty, -4)\bigcup(-4/3, \infty)$.

35. Applying the root test, we have $\lim_{n\to\infty}|a_n|^{1/n} = \lim_{n\to\infty}\left|\left(\dfrac{x^2+2}{6}\right)^{n^2}\right|^{1/n} = \lim_{n\to\infty}\left(\dfrac{x^2+2}{6}\right)^n < 1$. Setting $\left(\dfrac{x^2+2}{6}\right)^n < 1$, we obtain $\dfrac{x^2+2}{6} < 1$ or $x^2 < 4$. Thus, the series converges on $(-2, 2)$. At $x = \pm 2$, the series $\displaystyle\sum_{k=0}^{\infty} 1^{k^2}$ diverges by the n-th term test. Therefore, the given series converges on $(-2, 2)$.

36. Applying the ratio test, we have

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(n+1)!/[(n+1)x]^{n+1}}{n!/(nx)^n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)n^n}{(n+1)^{n+1}x}\right| \\ &= \lim_{n\to\infty}\left|\left(\frac{n}{n+1}\right)^n\frac{1}{x}\right| = \lim_{n\to\infty}\frac{1}{(1+1/n)^n}\cdot\frac{1}{|x|} = \frac{1}{e|x|}.\end{aligned}$$

Setting $1/e|x| < 1$, we obtain $|x| > 1/e$. Thus, the series converges on $(\infty, -1/e)\bigcup(1/e, \infty)$. At $x = -1/e$ and $1/e$ we obtain the series $\displaystyle\sum_{k=1}^{\infty}\frac{k!}{k^k(-1/e)^k} = \sum_{k=1}^{\infty}(-1)^k\frac{k!e^k}{k^k}$ and $\displaystyle\sum_{k=1}^{\infty}\frac{k!e^k}{k^k}$. Letting $a_k = \dfrac{k!e^k}{k^k}$, we will show that $a_{k+1} \geq a_k$. First, compute

$$\frac{a_{k+1}}{a_k} = \frac{(k+1)!e^{k+1}/(k+1)^{k+1}}{k!e^k/k^k} = \frac{(k+1)k^ke}{(k+1)^{k+1}} = \frac{e}{\left(\dfrac{k+1}{k}\right)^k}.$$

Then $a_{k+1} = \dfrac{e}{(1+1/k)^k} a_k$. We will now show that $(1+1/k)^k < e$. Using the definition of $\ln x$, we have

$$\begin{aligned}\ln(1+1/k) &= \int_1^{1+1/k} \frac{1}{t}dt \le \int_1^{1+1/k} dt \quad \boxed{\text{since } 1/t \le 1 \text{ for } t \le 1}\\ &= \frac{1}{k}.\end{aligned}$$

Thus, $\ln(1+1/k) \le 1/k$ and since e^x is an increasing function, $(1+1/k) \le e^{1/k}$ and $(1+1/k)^k \le e$. Hence, $\dfrac{e}{(1+1/k)^k} \ge 1$ and $a_{k+1} = \dfrac{e}{(1+1/k)^k} a_k \ge a_k$. Therefore, the two series diverge because their terms do not approach 0. The given series converges on $(\infty, -1/e) \bigcup (1/e, \infty)$.

37. This is a geometric series with common ratio $r = x^x$ and will converge for $|e^x| = e^x < 1$ or $x < 0$. Thus, the series converges on $(-\infty, 0)$.

38. Applying the ratio test, we have $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!e^{-(n+1)x^2}}{n!e^{-nx^2}}\right| = \lim_{n\to\infty}(n+1)e^{-x^2} = \infty$. Thus, the series diverges for all x.

39. This is a geometric series with $r = \dfrac{2}{\sqrt{3}}\sin x$ and will converge for $\left|\dfrac{2}{\sqrt{3}}\sin x\right| < 1$ or $|\sin x| < \sqrt{3}/2$. On $[0, 2\pi]$ this will be on $[0, \pi/3) \bigcup (2\pi/3, 4\pi/3) \bigcup (5\pi/3, 2\pi]$.

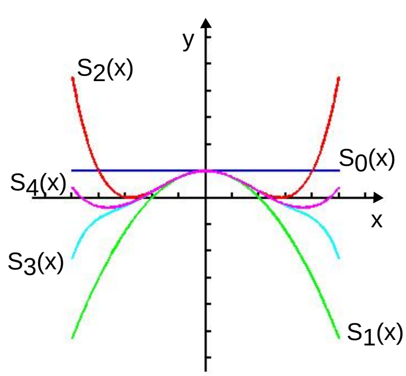
40. Since $|\dfrac{\sin kx}{k^2}| \le \dfrac{1}{k^2}$, we see that $\displaystyle\sum_{k=1}^{\infty}\frac{\sin kx}{k^2}$ converges for all x by comparison with the p-series $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2}$.

41. (a) $\displaystyle\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{x^{2(n+1)}}{x^{2n}} \cdot \frac{2^{2n}(n!)^2}{2^{2(n+1)[(n+1)!]^2}}$ for all x values. Therefore, the interval of

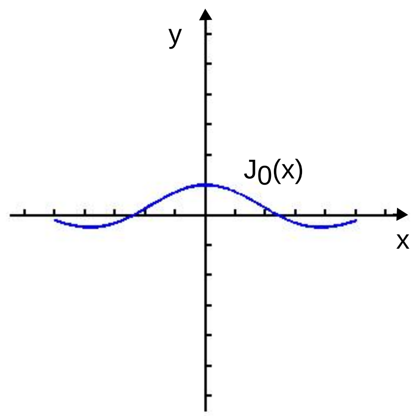
$$= \lim_{n\to\infty} x^2 \cdot \frac{1}{4(n+1)^2} = 0 < 1$$

convergence is $(-\infty, \infty)$ and hence the domain is $(-\infty, \infty)$.

(b)



(c)



9.9 Representing Functions by Power Series

In this section we will use the fact that the geometric series $\sum_{k=0}^{\infty} ar^k$ converges to $\frac{a}{1-r}$ for $r < 1$.

1. $\frac{1}{3-x} = \frac{1}{3} \cdot \frac{1}{1-\frac{x}{3}} = \frac{1}{3}\sum_{k=0}^{\infty}\left(\frac{x}{3}\right)^k$ for $|\frac{x}{3}| < 1$ or x in $(-3, 3)$. At $x = -3$ and $x = 3$, the series diverges by the n-th term test. Thus, the interval of convergence is $(-3, 3)$.

2. $\frac{1}{4+x} = \frac{1}{4} \cdot \frac{1}{1+\frac{x}{4}} = \frac{1}{4}\sum_{k=0}^{\infty}(-1)^k\left(\frac{x}{4}\right)^k$ for $|\frac{x}{4}| < 1$ or x in $(-4, 4)$. At $x = -4$ and $x = 4$, the series diverges by the n-th term test. Thus, the interval of convergence is $(-4, 4)$.

3. $\frac{1}{2+x} = \sum_{k=0}^{\infty}(-1)^k (2x)^k$ for $|2x| < 1$ or x in $\left(-\frac{1}{2}, \frac{1}{2}\right)$. At $x = -\frac{1}{2}$ and $x = \frac{1}{2}$, the series diverges by the n-th term test. Thus, the interval of convergence is $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

4. $\frac{1}{5+2x} = \frac{1}{5} \cdot \frac{1}{1+\frac{2x}{5}} = \frac{1}{5}\sum_{k=0}^{\infty}(-1)^k\left(\frac{2x}{5}\right)^k$ for $|\frac{2x}{5}| < 1$ or x in $\left(-\frac{5}{2}, \frac{5}{2}\right)$. At $x = -\frac{5}{2}$ and $x = \frac{5}{2}$, the series diverges by the n-th term test. Thus, the interval of convergence is $\left(-\frac{5}{2}, \frac{5}{2}\right)$.

5. Identify $a = 1$ and $r = -x^2$. Then, $\frac{1}{1+x^2} = \sum_{k=0}^{\infty}(-x^2)^k = \sum_{k=0}^{\infty}(-1)^k x^{2k}$ for $x^2 < 1$ or x in $(-1, 1)$.
At $x = -1$ and $x = 1$, the series diverges by the n-th term test. Thus, the interval of convergence is $(-1, 1)$.

6. Using the result from Problem 5, we have $\dfrac{x}{1+x^2} = x \cdot \dfrac{1}{1+x^2} = x \cdot \displaystyle\sum_{k=0}^{\infty} (-1)^k x^{2k} = \sum_{k=0}^{\infty} (-1)^k x^{2k+1}$ for x in $(-1,1)$. At $x=-1$ and $x=1$, the series diverges by the n-th term test. Thus, the interval of convergence is $(-1,1)$.

7. $\dfrac{1}{4+x^2} = \dfrac{1}{4} \cdot \dfrac{1}{1+\left(\frac{x}{2}\right)^2} = \dfrac{1}{4} \displaystyle\sum_{k=0}^{\infty} (-1)^k \left(\frac{x}{2}\right)^{2k}$ for $\left|\left(\dfrac{x}{2}\right)^2\right| < 1$ or x in $(-2,2)$. At $x=-2$ and $x=2$, the series diverges by the n-th term test. Thus, the interval of convergence is $(-2,2)$.

8. $\dfrac{4}{4-x^2} = \dfrac{1}{1-\left(\frac{x}{2}\right)^2} = \displaystyle\sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^{2k}$ for $\left|\left(\dfrac{x}{2}\right)^2\right| < 1$ or x in $(-2,2)$. At $x=-2$ and $x=2$, the series diverges by the n-th term test. Thus, the interval of convergence is $(-2,2)$.

9. $\dfrac{1}{(3-x)^2} = -\dfrac{d}{dx}\left(\dfrac{1}{3-x}\right) = -\dfrac{d}{dx}\left(\dfrac{1}{3}\displaystyle\sum_{k=0}^{\infty}\left(\frac{x}{3}\right)^k\right) = -\dfrac{1}{3}\sum_{k=1}^{\infty} k\left(\frac{x}{3}\right)^{k-1}$ The integral of convergence remains $(-3,3)$.

10. $\dfrac{1}{(1+2x)^2} = -\dfrac{d}{dx}\left(\dfrac{1}{1+2x}\right) = -\dfrac{d}{dx}\left(\displaystyle\sum_{k=0}^{\infty}(-1)^k(2x)^k\right) = -\sum_{k=1}^{\infty} k(-1)^k(2x)^{k-1}$. The integral of convergence remains $\left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$.

11.
$$\begin{aligned}
\frac{1}{(5+2x)^3} &= \frac{1}{8}\frac{d^2}{dx^2}\left(\frac{1}{5+2x}\right) = \frac{1}{8}\frac{d^2}{dx^2}\left(\frac{1}{5}\sum_{k=0}^{\infty}(-1)^k\left(\frac{2x}{5}\right)^k\right) \\
&= \frac{1}{250}\sum_{k=2}^{\infty} k(k-1)(-1)^k \frac{2x}{5}^{k-2}.
\end{aligned}$$
The integral of convergence remains $\left(-\dfrac{5}{2}, \dfrac{5}{2}\right)$.

12.
$$\begin{aligned}
\frac{1}{(4+x)^3} &= \frac{1}{2}\frac{d^2}{dx^2}\left(\frac{1}{4+x}\right) = \frac{1}{2}\frac{d^2}{dx^2}\left(\frac{1}{4}\sum_{k=0}^{\infty}(-1)^k\left(\frac{x}{4}\right)^k\right) \\
&= \frac{1}{128}\sum_{k=2}^{\infty} k(k-1)(-1)^k \frac{x}{4}^{k-2}.
\end{aligned}$$
The integral of convergence remains $(-4,4)$.

13.
$$\begin{aligned}
\frac{x}{(1+x^2)^2} &= -\frac{1}{2}\frac{d}{dx}\left(\frac{1}{1+x^2}\right) = -\frac{1}{2}\frac{d}{dx}\left(\sum_{k=0}^{\infty}(-1)^k x^{2k}\right) \\
&= -\frac{1}{2}\sum_{k=1}^{\infty} 2k(-1)^k x^{2k-1}.
\end{aligned}$$
The integral of convergence remains $(-1,1)$.

14. $$\frac{1-x^2}{(1+x^2)^2} = \frac{d}{dx}\left(\frac{x}{1+x^2}\right) = \frac{d}{dx}\left(\sum_{k=0}^{\infty}(-1)^k x^{2k+1}\right)$$
$$= \sum_{k=0}^{\infty}(2k+1)(-1)^k x^{2k}.$$
The integral of convergence remains $(-1, 1)$.

15. Using Problem 5,
$$\tan^{-1} x = \int_0^x \frac{1}{1+t^2} = \int_0^x \sum_{k=0}^{\infty}(-1)^k t^{2k} dt = \sum_{k=0}^{\infty}(-1)^k \int_0^x t^{2k} dt = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{2k+1}$$
for x in $(-1, 1)$. At $x = -1$ and $x = 1$, the series converges by the alternating series test. Thus, the interval of convergence is $[-1, 1]$.

16. $$\tan^{-1}\left(\frac{x}{2}\right) = 2\int \frac{1}{4+x^2} dx = 2\int \left(\sum_{k=0}^{\infty}(-1)^k \left(\frac{x}{2}\right)^{2k}\right) dx$$
$$= 2\sum_{k=0}^{\infty} \frac{2(-1)^k \left(\frac{x}{2}\right)^{2k+1}}{2k+1}$$
which converges for x in $(-2, 2)$. At $x = 2$, we have the series $2\sum_{k=0}^{\infty} \frac{2(-1)^k}{2k+1}$ which is convergent. At $x = -2$, we have the series $2\sum_{k=0}^{\infty} \frac{2(-1)^k(-1)^{2k+1}}{2k+1} = 2\sum_{k=0}^{\infty} \frac{2(-1)^{3k+1}}{2k+1}$ which is also convergent. Thus, the interval of convergence is $[-2, 2]$.

17. Using Problem 5, $\frac{1}{1+x^2} = \sum_{k=0}^{\infty}(-1)^k x^{2k}$ and
$$\frac{2x}{1+x^2} = 2x\sum_{k=0}^{\infty}(-1)^k x^{2k} = \sum_{k=0}^{\infty} 2(-1)^k x^{2k+1}.$$
Then
$$\ln(1+x^2) = \int_0^x \frac{2t}{1+t^2} dt = \int_0^x \sum_{k=0}^{\infty} 2(-1)^k t^{2k+1} dt = 2\sum_{k=0}^{\infty}(-1)^k \int_0^x t^{2k+1} dt$$
$$= 2\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+2}}{2k+2} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} x^{2k+2}$$
for $x^2 < 1$ or x in $(-1, 1)$. At $x = -1$ and $x = 1$, the series converges by the alternating series test. Thus, the interval of convergence is $[-1, 1]$.

18. $\ln(5+2x) = 2\int \frac{1}{5+2x}dx = 2\int \left(\sum_{k=0}^{\infty}(-1)^k\left(\frac{2x}{5}\right)^k\right)dx$

$$= 2\sum_{k=0}^{\infty}\frac{\frac{5}{2}(-1)^k\left(\frac{2x}{5}\right)^{k+1}}{k+1}$$

which converges for x in $\left(-\frac{5}{2},\frac{5}{2}\right)$. At $x=-\frac{5}{2}$, we have the series $2\sum_{k=0}^{\infty}\frac{\frac{5}{2}(-1)^k(-1)^{k+1}}{k+1} = 2\sum_{k=0}^{\infty}\frac{\frac{5}{2}(-1)^{2k+1}}{k+1} = 5\sum_{k=0}^{\infty}\frac{-1}{k+1}$ which diverges. At At $x=\frac{5}{2}$, we have the series $2\sum_{k=0}^{\infty}\frac{\frac{5}{2}(-1)^k}{k+1} = 5\sum_{k=0}^{\infty}\frac{-1}{k+1}$ which converges. Thus, the interval of convergence is $\left(-\frac{5}{2},\frac{5}{2}\right]$.

19. Writing $\frac{1}{4+x} = \frac{1}{4}\cdot\frac{1}{1+x/4}$, we have

$$\ln(4+x)-\ln 4 = \int_0^x \frac{dt}{4+t} = \int_0^x \frac{1}{4}\cdot\frac{dt}{1+t/4} = \frac{1}{4}\int_0^x\sum_{k=0}^{\infty}(-t/4)^k dt$$
$$= \frac{1}{4}\sum_{k=0}^{\infty}(-1)^k\int_0^x \frac{t^k}{4^k}dt = \frac{1}{4}\sum_{k=0}^{\infty}\frac{(-1)^k}{k+1}\cdot\frac{x^{k+1}}{4^k}.$$

Thus $\ln(4+x) = \ln 4 + \sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)4^{k+1}}x^{k+1}$. This series converges for $|x/4|<1$, $|x|<4$, or on $(-4,4)$. At $x=-4$, the series diverges since it is the negative harmonic series. At $x=4$, the series converges by the alternating series test. Thus, the interval of convergence is $(-4,4]$.

20. $\ln\left(\frac{3+x}{3-x}\right) = \ln(3+x)-\ln(3-x)$

$$= \int\frac{1}{3+x}dx + \int\frac{1}{3-x}dx$$
$$= \int\left(\frac{1}{3}\sum_{k=0}\infty(-1)^k\left(\frac{x}{3}\right)^k\right)dx + \int\left(\frac{1}{3}\sum_{k=0}\infty\left(\frac{x}{3}\right)^k\right)dx$$
$$= \sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{x}{3}\right)^{k+1}}{k+1} + \sum_{k=0}^{\infty}\frac{\left(\frac{x}{3}\right)^{k+1}}{k+1}$$
$$= \sum_{k=0}^{\infty}\frac{(1+(-1)^k)\left(\frac{x}{3}\right)^{k+1}}{k+1}$$

which converges for x in $(-3,3)$. At $x=3$, we have the series $\sum_{k=0}^{\infty}\frac{(1+(-1)^k)}{k+1}$ which diverges.

At $x=-3$, we have the series $\sum_{k=0}^{\infty}\frac{(1+(-1)^k)(-1)^{k+1}}{k+1}$ which also diverges. Thus, the interval of convergence is $(-3,3)$.

21. $$\begin{aligned}\frac{1-x}{1+2x} &= (1-x)\cdot\left(\frac{1}{1+2x}\right)(1-x)\cdot\sum_{k=0}^{\infty}(-1)^k(2x)^k\\ &= \sum_{k=0}^{\infty}(-1)x^k(2x)^k-\frac{1}{2}(-1)^k(2x)^{k+1}\\ &= 1+\frac{3}{2}\sum_{k=1}^{\infty}(-1)x^k(2x)^k\end{aligned}$$
which converges for x in $\left(-\frac{1}{2},\frac{1}{2}\right)$ and diverges at $x=-\frac{1}{2}$ and $x=\frac{1}{2}$. Thus, the interval of convergence is $\left(-\frac{1}{2},\frac{1}{2}\right)$.

22. $$\begin{aligned}\frac{3-x}{1-x} &= (3-x)\cdot\left(\frac{1}{1-x}\right)(3-x)\cdot\sum_{k=0}^{\infty}x^k\\ &= \sum_{k=0}^{\infty}3x^k-x^{k+1}=3+\sum_{k=1}^{\infty}3x^k-x^k\\ &= 3+\sum_{k=1}^{\infty}2x^k\end{aligned}$$
which converges for x in $(-1,1)$ and diverges at $x=-1$ and $x=1$. Thus, the interval of convergence is $(-1,1)$.

23. $$\begin{aligned}\frac{x^2}{(1-x)^3} &= x^2\cdot\frac{1}{(1+x)^3}=x^2\cdot\frac{1}{2}\frac{d^2}{dx^2}\left[\frac{1}{(1+x)}\right]\\ &= x^2\cdot\frac{1}{2}\cdot\sum_{k=2}^{\infty}k(k-1)(-1)^kx^{k-2}\\ &= \frac{1}{2}\cdot\sum_{k=2}^{\infty}k(k-1)(-1)^kx^k\end{aligned}$$
which converges for x in $(-1,1)$ and diverges at $x=-1$ and $x=1$. Thus, the interval of convergence is $(-1,1)$.

24. $$\begin{aligned}\frac{x^3}{8+2x} &= x^3\cdot\frac{1}{8}\cdot\frac{1}{1+\frac{x}{4}}=\frac{x^3}{8}\cdot\sum_{k=0}^{\infty}(-1)^k\left(\frac{x}{4}\right)^k\\ &= \frac{1}{8}\cdot\sum_{k=0}^{\infty}\frac{x^{k+3}}{4^k}\end{aligned}$$
which converges for x in $(-4,4)$ and diverges at $x=-4$ and $x=4$. Thus, the interval of convergence is $(-4,4)$.

25. $$\begin{aligned} s\ln(1+x^2) &= x\cdot 2\int \frac{x}{1+x^2}dx \\ &= x\cdot 2\int\left(\sum_{k=0}^{\infty}(-1)^k x^{2k+1}\right)dx \\ &= x\cdot 2\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+2}}{2k+2} = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+3}}{k+1} \end{aligned}$$

which converges for x in $(-1,1)$. At $x=-1$, we have the series $\sum_{k=0}^{\infty}\frac{(-1)^{3k+3}}{k+1}$ which converges. At $x=1$, we have the series $\sum_{k=0}^{\infty}\frac{(-1)^k}{k+1}$ which converges. Thus, the interval of convergence is $[-1,1]$.

26. $$\begin{aligned} x^2\tan^{-1}(x) &= x^2\cdot\int\left[\frac{1}{1+x^2}\right]dx \\ &= x^2\cdot\int\left[\sum_{k=0}^{\infty}(-1)^k x^{2k}\right]dx \\ &= x^2\cdot\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{2k+1} = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+3}}{2k+1} \end{aligned}$$

which converges for x in $(-1,1)$. At $x=-1$, we have the series $\sum_{k=0}^{\infty}\frac{(-1)^{3k+3}}{2k+1}$ which converges. At $x=1$, we have the series $\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}$ which converges. Thus, the interval of convergence is $[-1,1]$.

27. Since $\tan^{-1} t = \sum_{k=0}^{\infty}\frac{(-1)^k t^{2k+1}}{2k+1}$, we have

$$\begin{aligned} \int_0^x \tan^{-1} t\,dt &= \int_0^x\left(\sum_{k=0}^{\infty}\frac{(-1)^k t^{2k+1}}{2k+1}\right)dt \\ &= \sum_{k=0}^{\infty}\left(\int_0^x\frac{(-1)^k t^{2k+1}}{2k+1}dt\right) \\ &= \sum_{k=0}^{\infty}\left(\frac{(-1)^k t^{2k+2}}{(2k+1)(2k+2)}\right)\Bigg|_0^x \\ &= \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+2}}{(2k+1)(2k+2)} \end{aligned}$$

which converges for x in $(-1,1)$. At $x=-1$, we have the series $\sum_{k=0}^{\infty}\frac{(-1)^{3k+2}}{(2k+1)(2k+2)}$ which

converges. At $x = 1$, we have the series $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)(2k+2)}$ which converges. Thus, the interval of convergence is $[-1, 1]$.

28. Since $\ln(1+t^2) = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+2}}{k+1}$, we have

$$\begin{aligned}
\int_0^x \ln(1+t^2)dt &= \int_0^x \left(\sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+2}}{k+1}\right) dt \\
&= \sum_{k=0}^{\infty} \left(\int_0^x \frac{(-1)^k t^{2k+2}}{k+1} dt\right) \\
&= \sum_{k=0}^{\infty} \left(\frac{(-1)^k t^{2k+3}}{(k+1)(2k+3)}\right)\Bigg|_0^x \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{(k+1)(2k+3)}
\end{aligned}$$

which converges for x in $(-1, 1)$. At $x = -1$, we have the series $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^{3k+3}}{(2k+1)(2k+3)}$ which converges. At $x = 1$, we have the series $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)(2k+3)}$ which converges. Thus, the interval of convergence is $[-1, 1]$.

29. $$\begin{aligned}
\frac{1}{1-x} &= \frac{1}{-5-(x-6)} = -\frac{1}{5} \cdot \frac{1}{1+\left(\frac{x+6}{5}\right)} \\
&= -\frac{1}{5} \sum_{k=0}^{\infty} (-1)^k \left(\frac{x-6}{5}\right)^k
\end{aligned}$$

which converges for $\left|\dfrac{x-6}{5}\right| < 1$ or $|x-6| < 5$. Since the series diverges at $x = 11$ and $x = 1$, the interval of convergence is $(1, 11)$.

30. $$\begin{aligned}
\frac{1}{x} &= \frac{1}{-2+(x+2)} = -\frac{1}{2} \cdot \frac{1}{1-\left(\frac{x+2}{2}\right)} \\
&= -\frac{1}{2} \sum_{k=0}^{\infty} (\left(\frac{x+2}{2}\right)^k
\end{aligned}$$

which converges for $\left|\dfrac{x+2}{2}\right| < 1$ or $|x+2| < 2$. Since the series diverges at $x = -4$ and $x = 0$, the interval of convergence is $(-4, 0)$.

31. $$\begin{aligned}\frac{x}{2+x} &= \frac{(x+2)-2}{x+2} = 1 - \frac{2}{x+2} \\ &= 1 - \frac{2}{1+(x+1)} = 1 - 2\sum_{k=0}^{\infty}(-1)^k(x+1)^k \\ &= -1 - 2\sum_{k=1}^{\infty}(-1)^k(x+1)^k = -1 + 2\sum_{k=0}^{\infty}(-1)^k(x+1)^{k+1}\end{aligned}$$
which converges for $|x+1| < 1$. Since the series diverges at $x = -2$ and $x = 0$, the interval of convergence is $(-2, 0)$.

32. $$\begin{aligned}\frac{x-2}{x-1} &= \frac{(x-1)-1}{x-1} = 1 - \frac{1}{x-1} \\ &= 1 - \frac{1}{1+(x-2)} = 1 - \sum_{k=0}^{\infty}(-1)^k(x-2)^k\end{aligned}$$
which converges for $|x-2| < 1$. Since the series diverges at $x = 1$ and $x = 3$, the interval of convergence is $(1, 3)$.

33. $$\begin{aligned}\frac{7x}{x^2+x-12} &= \frac{4}{x+4} + \frac{3}{x-3} \\ &= \frac{1}{1+\left(\frac{x}{4}\right)} - \frac{1}{1-\left(\frac{x}{3}\right)} \\ &= \sum_{k=0}^{\infty}(-1)^k\left(\frac{x}{4}\right)^k - \sum_{k=0}^{\infty}\left(\frac{x}{3}\right)^k \\ &= \sum_{k=0}^{\infty}\left[\frac{(-1)^k}{4^k} - \frac{1}{3^k}\right]x^k\end{aligned}$$
which converges for $(-3, 3)$. Since the series diverges at $x = -3$ and $x = 3$, the interval of convergence is $(-3, 3)$.

34. $$\begin{aligned}\frac{3}{x^2-x-2} &= \frac{1}{x-2} - \frac{1}{x+2} \\ &= -\frac{1}{2}\cdot\frac{1}{1-\left(\frac{x}{2}\right)} - \frac{1}{1+x} \\ &= -\frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{x}{2}\right)^k - \sum_{k=0}^{\infty}(-1)^k x^k \\ &= \sum_{k=0}^{\infty}\left[\frac{-1}{2^{k+1}} + (-1)^k\right]x^k\end{aligned}$$
which converges for $(-1, 1)$. Since the series diverges at $x = -1$ and $x = 1$, the interval of convergence is $(-1, 1)$.

35. $$\frac{1}{(2-x)(1-x)} = \frac{1}{2-x}\cdot\frac{1}{1-x} = \left[\frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{x}{2}\right)^k\right]\cdot\left[\sum_{k=0}^{\infty}x^k\right]$$
$$= \frac{1}{2}\left[1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{8}+\cdots\right]\cdot[1+x+x^2+x^3+\cdots]$$
$$= \frac{1}{2}+\frac{3}{4}x+\frac{7}{8}x^2+\frac{15}{16}x^3+\cdots$$

36. $$\frac{x}{(1+2x)(1+x^2)} = \frac{1}{1+2x}\cdot\frac{x}{1+x^2} = \left[\sum_{k=0}^{\infty}(-1)^k(2x)^k\right]\cdot\left[\sum_{k=0}^{\infty}(-1)^k x^{2k+1}\right]$$
$$= \left[1-2x+4x^2-8x^3+\cdots\right]\cdot[x-x^3+x^5-\cdots]$$
$$= x-2x^2+3x^3-10x^4+\cdots$$

37. Writing $f(x) = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k3^k}$ and applying the ratio test, we have
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}/(n+1)3^{n+1}}{x^n/n3^n}\right| = \lim_{n\to\infty}\frac{1}{3}\left(\frac{n}{n+1}\right)|x| = \frac{1}{3}|x|.$$
The series is absolutely convergent for $\frac{1}{3}|x| < 1$, $|x| < 3$, or on $(-3,3)$. At $x=-3$, the series is divergent since it is the negative harmonic series. At $x=3$, the series converges by the alternating series test. Thus, the domain of the function is $(-3,3]$.

38. Writing $f(x) = \sum_{k=0}^{\infty}\frac{2^k}{k!}x^k$ and applying the ratio test, we have
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}x^{n+1}/(n+1)!}{2^n x^n/n!}\right| = \lim_{n\to\infty}\frac{2}{n+1}|x| = 0.$$
Thus, the series is absolutely convergent for all x and the domain of the function is $(-\infty,\infty)$.

39. Using Example 7 in the text with $x = 0.1$,
$$\ln 1.1 = 0.1 - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4} + \frac{(0.1)^5}{5} - \cdots$$
$$\approx 0.1 - 0.005 + 0.00033 - 0.00003 + \cdots \approx 0.0953.$$

40. Using Problem 15,
$$\tan^{-1}(.2) = \sum_{k=0}^{\infty}(-1)^k\frac{(0.2)^{2k+1}}{2k+1} = 0.2 - \frac{(0.2)^3}{3} + \frac{(0.2)^5}{5} - \frac{(0.2)^7}{7} + \cdots$$
$$\approx 0.2 - 0.00267 + 0.00006 - 0.000002 + \cdots \approx 0.1974.$$

41. Identify $a = 1$ and $r = -x^3$. Then $\dfrac{1}{1+x^3} = \sum_{k=0}^{\infty}(-x^3)^k = \sum_{k=0}^{\infty}(-1)^k x^{3k}$ and

$$\int_0^x \frac{dt}{1+t^3} = \int_0^x \sum_{k=0}^{\infty}(-1)^k t^{3k}dt = \sum_{k=0}^{\infty}(-1)^k \int_0^x t^{3k}dt == \sum_{k=0}^{\infty}(-1)^k \frac{x^{3k+1}}{3k+1}.$$

Letting $x = 1/2$, we have

$$\begin{aligned}\int_0^{1/2} \frac{dx}{1+x^3} &= \sum_{k=0}^{\infty}(-1)^k \frac{(1/2)^{3k+1}}{3k+1} = \sum_{k=0}^{\infty}(-1)^k \frac{1}{2^{3k+1}(3k+1)}\\ &= \frac{1}{2} - \frac{1}{2^4(4)} + \frac{1}{2^7(7)} - \frac{1}{2^{11}(11)} + \cdots\\ &\approx 0.5 - 0.01563 + 0.00112 - 0.00004 + \cdots \approx 0.4854.\end{aligned}$$

42. Identify $a = 1$ and $r = -x^4$. Then $\dfrac{1}{1+x^4} = \sum_{k=0}^{\infty}(-x^4)^k = \sum_{k=0}^{\infty}(-1)^k x^{4k}$ and
$\dfrac{x}{1+x^4} = x\sum_{k=0}^{\infty}(-1)^k x^{4k} = \sum_{k=0}^{\infty}(-1)^k x^{4k+1}$ and

$$\int_0^x \frac{dt}{1+t^4} = \int_0^x \sum_{k=0}^{\infty}(-1)^k t^{4k}dt = \sum_{k=0}^{\infty}(-1)^k \int_0^x t^{4k}dt == \sum_{k=0}^{\infty}(-1)^k \frac{x^{4k+2}}{4k+2}.$$

Letting $x = 1/3$, we have

$$\begin{aligned}\int_0^{1/3} \frac{dx}{1+x^4} &= \sum_{k=0}^{\infty}(-1)^k \frac{(1/3)^{4k+2}}{4k+2} = \sum_{k=0}^{\infty}(-1)^k \frac{1}{3^{4k+2}(4k+2)}\\ &= \frac{1}{3^2(2)} - \frac{1}{3^6(7)} + \frac{1}{3^{10}(10)} - \cdots\\ &\approx 0.05556 - 0.00023 + 0000002 - \cdots \approx 0.0553.\end{aligned}$$

43. Using Problem 15, $x\tan^{-1}x = x\sum_{k=0}^{\infty}(-1)^k \dfrac{x^{2k+1}}{2k+1} = \sum_{k=0}^{\infty}(-1)^k \dfrac{x^{2k+2}}{2k+1}$, so

$$\int_0^x t\tan^{-1}t dt = \int_0^x \sum_0^{\infty}(-1)^k \frac{t^{2k+2}}{2k+1}dt = \sum_0^{\infty} \frac{(-1)^k}{2k+1}\int_0^x t^{2k+2}dt = \sum_0^{\infty} \frac{(-1)^k}{2k+1}\frac{x^{2k+3}}{2k+3}.$$

Letting $x = 0.3$, we have

$$\begin{aligned}\int_0^{0.3} x\tan^{-1}x dx &= \sum_0^{\infty} \frac{(-1)^k}{2k+1}\frac{(0.3)^{2k+3}}{2k+3} = \frac{(0.3)^3}{1\cdot 3} - \frac{(0.3)^5}{3\cdot 5} + \frac{(0.3)^7}{5\cdot 7} - \cdot\\ &\approx 0.0090 - 0.0016 + 0.00001 - \cdot \approx 0.008.\end{aligned}$$

44. Using Problem 15, $\tan^{-1} x^2 = \sum_{k=0}^{\infty}(-1)^k \frac{(x^2)^{2k+1}}{2k+1} = \sum_{k=0}^{\infty}(-1)^k \frac{x^{4k+2}}{2k+1}$, so

$$\int_0^x \tan^{-1} t^2 dt = \int_0^x \sum_0^{\infty}(-1)^k \frac{t^{4k+2}}{2k+1} dt = \sum_0^{\infty} \frac{(-1)^k}{2k+1} \int_0^x t^{4k+2} dt = \sum_0^{\infty} \frac{(-1)^k}{2k+1} \frac{x^{4k+3}}{4k+3}.$$

Letting $x = 1/2$, we have

$$\begin{aligned}\int_0^{1/2} \tan^{-1} x^2 dx &= \sum_0^{\infty} \frac{(-1)^k}{2k+1} \frac{(1/2)^{4k+3}}{4k+3} = \sum_0^{\infty}(-1)^k \frac{1}{2^{4k+3}(2k+1)(4k+3)} \\ &= \frac{1}{2^3 \cdot 1 \cdot 3} - \frac{1}{2^7 \cdot 3 \cdot 7} + \frac{1}{2^1 1 \cdot 5 \cdot 11} - \cdot \\ &\approx 0.04167 - 0.00037 + 0.00001 - \cdot \approx 0.0413.\end{aligned}$$

45. Using $\tan^{-1} x = \sum_0^{\infty}(-1)^k \frac{x^{2k+1}}{2k+1}$, we have $\frac{\pi}{4} = \tan^{-1} 1 = \sum_0^{\infty} \frac{(-1)^k}{2k+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

46. Using Theorem 9.7.2 in the text, we must have $a_{n+1} = \frac{1}{2(n+1)+1} = \frac{1}{2n+3} < 0.00005$ or $2n+3 > 20,000$. This means $n > 9998.5$. Thus, it will require S_{9999} to approximate $\pi/4$ to four decimal places.

47. $y' = \sum_{k=1}^{\infty}(-1)^{k+1} x^{k-1} = -\sum_{k=1}^{\infty}(-1)^k x^{k-1} \quad y'' = \sum_{k=2}^{\infty}(k-1)(-1)^{k+1} x^{k-2}$

$$\begin{aligned}(x+1)y'' &= \sum_{k=2}^{\infty}(k-1)(-1)^{k+1} x^{k-1} + \sum_{k=2}^{\infty}(k-1)(-1)^{k+1} x^{k-2} \\ &= \sum_{k=1}^{\infty} k(-1)^{k+2} x^k + \sum_{k=1}^{\infty} k(-1)^{k+2} x^{k-1} \\ &= \sum_{k=1}^{\infty} k(-1)^k x^k + \sum_{k=1}^{\infty}(-1)^k x^{k-1}\end{aligned}$$

$$\begin{aligned}(x+1)y'' + y' &= \sum_{k=1}^{\infty} k(-1)^k x^k + \sum_{k=1}^{\infty} k(-1)^k x^{k-1} - \sum_{k=1}^{\infty}(-1)^k x^{k-1} \\ &= \sum_{k=1}^{\infty} k(-1)^k x^k + \sum_{k=1}^{\infty}(k-1)(-1)^k x^{k-1} \\ &= \sum_{k=1}^{\infty} k(-1)^k x^k + \sum_{k=2}^{\infty}(k-1)(-1)^k x^{k-1} \\ &= \sum_{k=1}^{\infty} k(-1)^k x^k - \sum_{k=1}^{\infty} k(-1)^k x^k = 0\end{aligned}$$

48. Letting $y = J_0(x)$, we have

$$y' = \sum_{k=1}^{\infty} \frac{2k(-1)^k}{2^{2k}(k!)^2} x^{2k-1}, \quad y'' = \sum_{k=1}^{\infty} \frac{2k(2k-1)(-1)^k}{2^{2k}(k!)^2} x^{2k-2}.$$

$$\begin{aligned}
xy'' + y' + xy &= \sum_{k=1}^{\infty} \frac{(2k)(2k-1)(-1)^k x^{2k-1}}{2^{2k}(k!)^2} + \sum_{k=1}^{\infty} \frac{(2k)(-1)^k}{2^{2k}(k!)^2} x^{2k-1} + \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k}(k!)^2} x^{2k+1} \\
&= \sum_{k=1}^{\infty} \frac{(2k)^2(-1)^k x^{2k-1}}{2^{2k}(k!)^2} + \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k}(k!)^2} x^{2k+1} \\
&= \sum_{k=1}^{\infty} \frac{(2k)^2(-1)^k x^{2k-1}}{(2k)^2 2^{2(k-1)}[(k-1)!]^2} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2^{2(k-1)}[(k-1)!]^2} x^{2k-1} \\
&= \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k-1}}{2^{2(k-1)}[(k-1)!]^2} - \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k-1}}{2^{2(k-1)}[(k-1)!]^2} = 0
\end{aligned}$$

49. (a) $f'(x) = \sum_{k=1}^{\infty} \frac{x^{k-1}}{(k-1)!} = \sum_{k=0}^{\infty} \frac{x^k}{k!} = f(x)$

(b) e^x

50. (a) $f'(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k},$

$$\begin{aligned}
f''(x) &= \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k-1)!} x^{2k-1} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k+1)!} x^{2k+1} \\
&= -\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = -f(x)
\end{aligned}$$

(b) $f(x) = \sin x$ and $f(x) = \cos x$ both satisfy $f''(x) = -f(x)$. Since $\sin x$ is an odd function while $\cos x$ is an even function, the function represented is $f(x) = \sin x$.

9.10 Taylor Series

1.
$$\begin{aligned}
f(x) &= \frac{1}{2-x}, & f(0) &= \frac{1}{2}\\
f'(x) &= \frac{1}{(2-x)^2}, & f'(0) &= \frac{1}{2^2}\\
f''(x) &= \frac{2}{(2-x)^3}, & f''(0) &= \frac{2}{2^3}\\
f'''(x) &= \frac{2\cdot 3}{(2-x)^4}, & f'''(0) &= \frac{3!}{2^4}\\
&\vdots\\
f^{(k)}(x) &= \frac{k!}{(2-x)^{k+1}}, & f^{k}(0) &= \frac{k!}{2^{k+1}}
\end{aligned}$$

The Maclaurin series is $\displaystyle\sum_{k=0}^{\infty}\frac{k!/2^{k+1}}{k!}x^k = \sum_{k=0}^{\infty}\frac{1}{2^{k+1}}x^k$.

2.
$$\begin{aligned}
f(x) &= \frac{1}{1+5x}, & f(0) &= 1\\
f'(x) &= \frac{-5}{(1+5x)^2}, & f'(0) &= -5\\
f''(x) &= \frac{5^2\cdot 2}{(1+5x)^3}, & f''(0) &= 5^2\cdot 2\\
f'''(x) &= \frac{-5^3\cdot 2\cdot 3}{(1+5x)^4}, & f'''(0) &= -5^3\cdot 3!\\
&\vdots\\
f^{(k)}(x) &= \frac{(-1)^k 5^k k!}{(1+5x)^{k+1}}, & f^{k}(0) &= (-1)^k 5^k k!
\end{aligned}$$

The Maclaurin series is $\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k 5^k k!}{k!}x^k = \sum_{k=0}^{\infty}(-1)^k 5^k x^k$.

3.

$$
\begin{aligned}
&f(x) = \ln(1+x), && f(0) = 0\\
&f'(x) = \frac{1}{1+x}, && f'(0) = 1\\
&f''(x) = -\frac{1}{(1+x)^2}, && f''(0) = -1\\
&f'''(x) = \frac{2}{(1+x)^3}, && f'''(0) = 2\\
&f^{(4)}(x) = -\frac{2\cdot 3}{(1+x)^4}, && f^{(4)}(0) = -3!\\
&\vdots && \\
&f^{(k)}(x) = (-1)^{k-1}\frac{(k-1)!}{(1+x)^k}, && f^{(k)}(0) = (-1)^{k-1}(k-1)!
\end{aligned}
$$

The Maclaurin series is $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(k-1)!}{k!}x^k = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}(k-1)!}{k!}x^k = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}x^k.$

4.

$$
\begin{aligned}
&f(x) = \ln(1+2x), && f(0) = 0\\
&f'(x) = \frac{2}{1+2x}, && f'(0) = 2\\
&f''(x) = -\frac{2^2}{(1+2x)^2}, && f''(0) = -2^2\\
&f'''(x) = \frac{2^3\cdot 2}{(1+2x)^3}, && f'''(0) = 2^3\cdot 2\\
&f^{(4)}(x) = -\frac{2^4\cdot 2\cdot 3}{(1+2x)^4}, && f^{(4)}(0) = -2^4\cdot 3!\\
&\vdots && \\
&f^{(k)}(x) = (-1)^{k-1}\frac{2^k(k-1)!}{(1+2x)^k}, && f^{(k)}(0) = (-1)^{k-1}2^k(k-1)!
\end{aligned}
$$

The Maclaurin series is $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k-1}2^k(k-1)!}{k!}x^k = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}2^k}{k}x^k.$

5.
$$\begin{aligned}
&f(x)=\sin x, && f(0)=0\\
&f'(x)=\cos x, && f'(0)=1\\
&f''(x)=-\sin x, && f''(0)=0\\
&f'''(x)=-\cos x, && f'''(0)=-1\\
&\vdots\\
&f^{(2k+1)}(x)=(-1)^k\cos x, && f^{(2k+1)}(0)=(-1)^k
\end{aligned}$$

The Maclaurin series is $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k+1)!}x^{2k+1}$.

6.
$$\begin{aligned}
&f(x)=\cos 2x, && f(0)=1\\
&f'(x)=-2\sin 2x, && f'(0)=0\\
&f''(x)=-2^2\cos 2x, && f''(0)=-2^2\\
&f'''(x)=2^3\sin 2x, && f'''(0)=0\\
&\vdots\\
&f^{(2k)}(x)=(-1)^k2^{2k}\cos 2x, && f^{(2k)}(0)=(-1)^k2^{2k}
\end{aligned}$$

The Maclaurin series is $\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k2^{2k}}{(2k)!}x^{2k}$.

7.
$$\begin{aligned}
&f(x)=e^x, && f(0)=1\\
&f'(x)=e^x, && f'(0)=1\\
&\vdots\\
&f^{(2k)}(x)=e^x, && f^{(k)}(0)=1
\end{aligned}$$

The Maclaurin series is $\displaystyle\sum_{k=0}^{\infty}\frac{1}{k!}x^k$.

8.
$$\begin{aligned}
&f(x)=e^{-x}, && f(0)=-1\\
&f'(x)=-e^{-x}, && f'(0)=1\\
&\vdots\\
&f^{(k)}(x)=(-1)^ke^{-x}, && f^{(k)}(0)=(-1)^k
\end{aligned}$$

The Maclaurin series is $\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}x^k$.

9.

$$\begin{aligned}
&f(x) = \sinh x, && f(0) = 0\\
&f'(x) = \cosh x, && f'(0) = 1\\
&f''(x) = \sinh x, && f''(0) = 0\\
&\vdots\\
&f^{(2k+1)}(x) = \cosh x, && f^{(2k+1)}(0) = 1
\end{aligned}$$

The Maclaurin series is $\displaystyle\sum_{k=0}^{\infty} \frac{1}{(2k+1)!} x^{2k+1}$.

10.

$$\begin{aligned}
&f(x) = \cosh x, && f(0) = 1\\
&f'(x) = \sinh x, && f'(0) = 0\\
&f''(x) = \cosh x, && f''(0) = 1\\
&\vdots\\
&f^{(2k)}(x) = \cosh x, && f^{(2k)}(0) = 1
\end{aligned}$$

The Maclaurin series is $\displaystyle\sum_{k=0}^{\infty} \frac{1}{(2k)!} x^{2k}$.

11.

$$\begin{aligned}
&f(x) = \tan x, && f(0) = 0\\
&f'(x) = \sec^2 x = 1 + \tan^2 x, && f'(0) = 1\\
&f''(x) = 2\tan x(1+\tan^2 x) = 2\tan x + 2\tan^3 x, && f''(0) = 0\\
&f'''(x) = 2 + 8\tan^2 x + 6\tan^4 x, && f'''(0) = 2\\
&f^{(4)}(x) = 16\tan x + 40\tan^3 x + 24\tan^5 x, && f^{(4)}(0) = 0\\
&f^{(5)}(x) = 16 + 136\tan^2 x + 240\tan^4 x + 120\tan^6 x, && f^{(5)}(0) = 16\\
&f^{(6)}(x) = 276\tan x + 1246\tan^3 x + 1680\tan^5 x + 720\tan^7 x, && f^{(6)}(0) = 0\\
&f^{(7)}(x) = 276 + 4014\tan^2 x + 12,138\tan^4 x + 13,440\tan^6 x + 5040\tan^8 x, && f^{(7)}(0) = 276
\end{aligned}$$

The Maclaurin series is $x + \frac{2}{3!}x^3 + \frac{16}{5!}x^5 + \frac{272}{7!}x^7 + \cdots = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots$

12.

$$f(x) = \sin^{-1} x, \qquad f(0) = 0$$

$$f'(x) = (1-x^2)^{-1/2}, \qquad f'(0) = 1$$

$$f''(x) = x(1-x^2)^{-3/2}, \qquad f''(0) = 0$$

$$f'''(x) = 3x^2(1-x^2)^{-5/2} + (1-x^2)^{-3/2}, \qquad f'''(0) = 1$$

$$f^{(4)}(x) = 15x^3(1-x^2)^{-7/2} + 9x(1-x^2)^{-5/2}, \qquad f^{(4)}(0) = 0$$

$$f^{(5)}(x) = 105x^4(1-x^2)^{-9/2} + 90x^2(1-x^2)^{-7/2} + 9(1-x^2)^{-5/2}, \qquad f^{(5)}(0) = 9$$

$$f^{(6)}(x) = 945x^5(1-x^2)^{-11/2} + 1050x^3(1-x^2)^{-9/2} + 225x(1-x^2)^{-7/2}, \qquad f^{(6)}(0) = 0$$

$$f^{(7)}(x) = 10,395x^6(1-x^2)^{-13/2} + 14,175x^4(1-x^2)^{-11/2} + 4,725x^2(1-x^2)^{-9/2} + 225(1-x^2)^{-7/2},$$

$$f^7(0) = 225$$

The Maclaurin series is $x + \frac{1}{3!}x^3 + \frac{9}{5!}x^5 + \frac{225}{7!}x^7 + \cdots$.

13.

$$f(x) = \frac{1}{1+x}, \qquad f(4) = \frac{1}{5}$$

$$f'(x) = -\frac{1}{(1+x)^2}, \qquad f'(4) = -\frac{1}{5^2}$$

$$f''(x) = \frac{2}{(1+x)^3}, \qquad f''(4) = \frac{3}{5^3}$$

$$f'''(x) = -\frac{2\cdot 3}{(1+x)^4}, \qquad f'''(4) = -\frac{3!}{5^4}$$

$$\vdots$$

$$f^{(2k+1)}(x) = \frac{(-1)^k k!}{(1+x)^{k+1}}, \qquad f^{(2k+1)}(4) = \frac{(-1)^k k!}{5^{k+1}}$$

The Taylor series is $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k k!/5^{k+1}}{k!}(x-4)^k = \sum_{k=0}^{\infty} \frac{(-1)^k}{5^{k+1}}(x-4)^k$.

14.

$$f(x) = x^{1/2}, \qquad f(1) = 1$$

$$f'(x) = \frac{1}{2}x^{-3/2}, \qquad f'(1) = \frac{1}{2}$$

$$f''(x) = -\frac{3}{2^2}x^{-5/2}, \qquad f''(1) = -\frac{3}{2^2}$$

$$f'''(x) = \frac{3 \cdot 5}{2^3}x^{-7/2}, \qquad f'''(1) = \frac{3 \cdot 5}{2^3}$$

$$\vdots$$

$$f^k(x) = \frac{(-1)^{k+1} 1 \cdot 3 \cdot 5 \cdots (2k+1)}{2^k} x^{-(2k+2)/2} \qquad f^{(k)}(1) = \frac{(-1)^{k+1} 1 \cdot 3 \cdot 5 \cdots (2k+1)}{2^k}$$

The Taylor series is

$$1 + \sum_{k=0}^{\infty} \frac{(-1)^{k+1} 1 \cdot 3 \cdot 5 \cdots (2k+1)/2^k}{k!}(x-1)^k = 1 + \sum_{k=0}^{\infty} \frac{(-1)^{k+1} 1 \cdot 3 \cdot 5 \cdots (2k-1)}{2^k k!}(x-1)^k.$$

15.

$$f(x) = \frac{1}{x}, \qquad f(1) = 1$$

$$f'(x) = -\frac{1}{x^2}, \qquad f'(1) = -1$$

$$f''(x) = \frac{2}{x^3}, \qquad f''(1) = 2$$

$$f'''(x) = -\frac{6}{x^4}, \qquad f'''(1) = -6$$

$$\vdots$$

$$f^n(x) = \frac{(-1)^n n!}{x^{n+1}} \qquad f^{(n)}(1) = (-1)^n n!$$

The Taylor series is

$$\sum_{k=0}^{\infty} \frac{f^(k)(1)}{k!}(x-1)^k = \sum_{k=0}^{\infty} \frac{(-1)^k k!}{k!}(x-1)^k = \sum_{k=0}^{\infty} (-1)^k (x-1)^k$$

16.

$$\begin{aligned}
f(x) &= \frac{1}{x}, & f(-5) &= -\frac{1}{5} \\
f'(x) &= -\frac{1}{x^2}, & f'(-5) &= -\frac{1}{25} \\
f''(x) &= \frac{2}{x^3}, & f''(-5) &= -\frac{2}{125} \\
f'''(x) &= -\frac{6}{x^4}, & f'''(-5) &= -\frac{6}{125} \\
&\vdots \\
f^n(x) &= \frac{(-1)^n n!}{x^{n+1}} & f^{(n)}(-5) &= -\frac{n!}{5^{n+1}}
\end{aligned}$$

The Taylor series is

$$\sum_{k=0}^{\infty} \frac{f^(k)(-5)}{k!}(x+5)^k = \sum_{k=0}^{\infty} \frac{-1}{5^{k+1}}(x+5)^k$$

17.

$$\begin{aligned}
f(x) &= \sin x, & f(\pi/4) &= \sqrt{2}/2 \\
f'(x) &= \cos x, & f'(\pi/4) &= \sqrt{2}/2 \\
f''(x) &= -\sin x, & f''(\pi/4) &= -\sqrt{2}/2 \\
f'''(x) &= -\cos x, & f'''(\pi/4) &= -\sqrt{2}/2
\end{aligned}$$

The Taylor series is $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{2 \cdot 2!}\left(x - \frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{2 \cdot 3!}\left(x - \frac{\pi}{4}\right)^3 + \cdots$

18.

$$\begin{aligned}
f(x) &= \sin x, & f(\pi/2) &= 1 \\
f'(x) &= \cos x, & f'(\pi/2) &= 0 \\
f''(x) &= -\sin x, & f''(\pi/2) &= -1 \\
f'''(x) &= -\cos x, & f'''(\pi/2) &= 0 \\
&\vdots \\
f^{(2k)}(x) &= (-1)^k \sin x, & f^{(2k)}(\pi/2) &= (-1)^k
\end{aligned}$$

The Taylor series is $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\left(x - \frac{\pi}{2}\right)^{2k}$.

19.
$$\begin{aligned}
f(x) &= \cos x, & f(\pi/3) &= 1/2 \\
f'(x) &= -\sin x, & f'(\pi/3) &= -\sqrt{3}/2 \\
f''(x) &= -\cos x, & f''(\pi/3) &= -1/2 \\
f'''(x) &= \sin x, & f'''(\pi/3) &= \sqrt{3}/2
\end{aligned}$$

The Taylor series is $\frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{2\cdot 2!}\left(x - \frac{\pi}{3}\right)^2 + \frac{\sqrt{3}}{2\cdot 3!}\left(x - \frac{\pi}{3}\right)^3 + \cdots$.

20.
$$\begin{aligned}
f(x) &= \cos x, & f(\pi/6) &= \sqrt{3}/2 \\
f'(x) &= -\sin x, & f'(\pi/6) &= -1/2 \\
f''(x) &= -\cos x, & f''(\pi/6) &= -\sqrt{3}/2 \\
f'''(x) &= \sin x, & f'''(\pi/6) &= 1/2
\end{aligned}$$

The Taylor series is $\frac{\sqrt{3}}{2} - \frac{1}{2}\left(x - \frac{\pi}{6}\right) - \frac{\sqrt{3}}{2\cdot 2!}\left(x - \frac{\pi}{6}\right)^2 + \frac{1}{2\cdot 3!}\left(x - \frac{\pi}{6}\right)^3 + \cdots$.

21.
$$\begin{aligned}
f(x) &= e^x, & f(1) &= e \\
f'(x) &= e^x, & f'(1) &= e \\
&\vdots \\
f^{(k)}(x) &= e^x, & f^{(k)}(1) &= e
\end{aligned}$$

The Taylor series is $\sum_{k=0}^{\infty} \frac{e}{k!}(x-1)^k$.

22.
$$\begin{aligned}
f(x) &= e^{-2x}, & f(1) &= e^{-1} \\
f'(x) &= -2e^{-2x}, & f'(1) &= -2e^{-1} \\
f''(x) &= 2^2e^{-2x}, & f''(1) &= 2^2e^{-1} \\
f'''(x) &= -2^3e^{-2x}, & f'''(1) &= -2^3e^{-1} \\
&\vdots \\
f^{(k)}(x) &= (-1)^k2^ke^{-2x}, & f^{(k)}(1) &= (-1)^k2^ke^{-1}
\end{aligned}$$

The Taylor series is $\sum_{k=0}^{\infty} \frac{(-1^k 2^k e^{-1})}{k!}\left(x - \frac{1}{2}\right)^k$.

23.

$$\begin{aligned}
f(x) &= \ln x, & f(2) &= 0\\
f'(x) &= \frac{1}{x}, & f'(2) &= \frac{1}{2}\\
f''(x) &= -\frac{1}{x^2}, & f''(2) &= -\frac{1}{2^2}\\
f'''(x) &= \frac{2}{x^3}, & f'''(2) &= \frac{2}{2^3}\\
f^{(4)}(x) &= -\frac{2\cdot 3}{x^4}, & f^{(4)}(2) &= -\frac{3!}{2^4}\\
&\vdots\\
f^{(k)}(x) &= (-1)^{k-1}\frac{(k-1)!}{x^k}, & f^{(k)}(2) &= (-1)^{k-1}\frac{(k-1)!}{2^k}
\end{aligned}$$

The Taylor series is

$$\ln 2 + \sum_{k=1}^{\infty}\frac{(-1)^{k-1}(k-1)!/2^k}{k!}(x-2)^k = \ln 2 + \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k2^k}(x-2)^k.$$

24.

$$\begin{aligned}
f(x) &= \ln(x+1), & f(2) &= \ln 3\\
f'(x) &= \frac{1}{x+1}, & f'(2) &= \frac{1}{3}\\
f''(x) &= -\frac{1}{(x+1)^2}, & f''(2) &= -\frac{1}{3^2}\\
f'''(x) &= \frac{2}{(x+1)^3}, & f'''(2) &= \frac{2}{3^3}\\
f^{(4)}(x) &= -\frac{2\cdot 3}{(x+1)^4}, & f^{(4)}(2) &= -\frac{3!}{3^4}\\
&\vdots\\
f^{(k)}(x) &= (-1)^{k-1}\frac{(k-1)!}{(x+1)^k}, & f^{(k)}(2) &= (-1)^{k-1}\frac{(k-1)!}{3^k}
\end{aligned}$$

The Taylor series is $\ln 3 + \sum_{k=1}^{\infty}\frac{(-1)^{k-1}(k-1)!/3^k}{k!}(x-2)^k = \ln 3 + \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k3^k}(x-2)^k$.

25. Substituting x^2 for x in Problem 8, we have $\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}(x^2)^k = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}x^{2k}$.

26. Substituting $3x$ for x and mutiplying by x^2 in Problem 8, we have $x^2\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}(3x)^k = \sum_{k=0}^{\infty}\frac{(-1)^k 3^k}{k!}x^{k+2}$.

27. Multiplying by x in Example 3 in the text, we have $x\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}x^{2k}=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}x^{2k+1}$.

28. Substituting x^3 for x in Problem 5, we have $\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}(x^3)^{6k+3}=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}x^{6k+3}$.

29. Substituting $-x$ for x in Problem 3, we have $\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}(-x)^k=\sum_{k=1}^{\infty}\frac{-1}{k}x^k$.

30. Using Problems 3 and 20 and the fact the $\ln\left(\frac{1+x}{1-x}\right)=\ln(1+x)-\ln(1-x)$, we have

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}x^k-\sum_{k=1}^{\infty}\frac{-1}{k}x^k=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}+1}{k}x^k=2x+\frac{2}{3}x^3+\frac{2}{5}x^5+\cdots=\sum_{k=1}^{\infty}\frac{2}{2k-1}x^{2k-1}.$$

31. Using Problem 11 and the fact that $\sec^2 x=\frac{d}{dx}\tan x$, we have

$$\frac{d}{dx}\left(x+\frac{1}{3}x^3+\frac{2}{15}x^5+\frac{17}{315}x^7+\cdots\right)=1+x^2+\frac{2}{3}x^4+\frac{17}{45}x^6+\cdots.$$

32. Using Problem 11 and the fact that $\ln(\cos x)=-\int_0^x\tan t\,dt$, we have

$$-\int_0^x\left(t+\frac{2}{3!}t^3+\frac{16}{5!}t^5+\frac{276}{7!}t^7+\cdots\right)dt=-\frac{1}{2}x^2-\frac{2}{4!}x^4-\frac{16}{6!}x^6-\frac{276}{8!}x^8-\cdots.$$

33.
$$\begin{aligned}\lim_{x\to0}\frac{x^3}{x-\sin x}&=\lim_{x\to0}\frac{x^3}{x-\left(x-\dfrac{x^3}{6}+\dfrac{x^5}{120}-\cdots\right)}\\&=\lim_{x\to0}\frac{x^3}{\frac{x^3}{6}-\frac{x^5}{120}+\cdots}\\&=\lim_{x\to0}\frac{1}{\frac{1}{6}-\frac{x^2}{120}+\cdots}\\&=\frac{1}{\frac{1}{6}}=6\end{aligned}$$

34.
$$\begin{aligned}\lim_{x\to0}\frac{1+x-e^x}{1-\cos x}&=\lim_{x\to0}\frac{1+x-\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\cdots\right)}{1-\left(1-\frac{x^2}{2}+\frac{x^4}{24}-\cdots\right)}\\&=\lim_{x\to0}\frac{-\frac{x^2}{2}-\frac{x^3}{6}-\frac{x^4}{24}-\cdots}{\frac{x^2}{2}-\frac{x^4}{24}+\cdots}\\&=\lim_{x\to0}\frac{-1-\frac{x}{3}-\frac{x^2}{12}-\cdots}{1-\frac{x^2}{12}}=\frac{-1}{1}=-1\end{aligned}$$

35. $$\begin{aligned}\cosh x &= \frac{e^x + e^{-x}}{2}\\ &= \frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\right) + \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!}\right)}{2}\\ &= \frac{2\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)}{2}\\ &= \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}\end{aligned}$$

36. $$\begin{aligned}\sinh x &= \frac{e^x - e^{-x}}{2}\\ &= \frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!}\right)}{2}\\ &= \frac{2\left(x + \frac{x^3}{4!} + \frac{x^5}{5!} + \cdots\right)}{2}\\ &= \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}\end{aligned}$$

37. $$\begin{aligned}\frac{e^x}{1-x} &= e^x \cdot \left(\frac{1}{1-x}\right)\\ &= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) \cdot \left(1 + x + x^2 + x^3 + x^4 + \cdots\right)\\ &= 1 + 2x + \frac{5}{2}x^2 + \frac{8}{3}x^3 + \frac{65}{24}x^4 + \cdots\end{aligned}$$

38. $$\begin{aligned}e^x \sin x &= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}\right) \cdots \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)\\ &= x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} - \cdots\end{aligned}$$

39. $$\begin{aligned}\frac{e^x}{\cos x} &= \frac{1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots}{1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots}\\ &= 1 + x + x^2 + \frac{2x^3}{3} + \frac{x^4}{2} + \cdots\end{aligned}$$

40. $$\begin{aligned}\sec x &= \frac{1}{\cos x} = \frac{1}{1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots}\\ &= 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \frac{277x^8}{8064} + \cdots\end{aligned}$$

41. $$\int_0^{-1} e^{-x^3}\,dx = \int_0^{-1} 1 + (-x^2) + \frac{(-x^2)^2}{2} + \frac{(-x^2)^3}{3!} + \cdots dx$$
$$= \int_0^{-1} \left(1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \cdots\right) dx$$
$$= \left(x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42}\right)\Big|_0^1$$
$$= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \cdots$$

42. $$\int_0^1 \frac{\sin x}{x} dx = \int_0^1 \frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) dx$$
$$= \int_0^1 \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots\right) dx$$
$$= \left(x - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!} - \frac{x^7}{7\cdot 7!}\right)\Big|_0^1$$
$$= 1 - \frac{1}{3\cdot 3!} + \frac{1}{5\cdot 5!} - \frac{1}{7\cdot 7!} + \cdots$$

43. Using $\tan^{-1} x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$, we have

$$\frac{\pi}{4} = \tan^{-1} 1 = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

44. Using $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$, we have

$$e^{-1} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots$$
$$= \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots$$

45. Using $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$, we have

$$\cos\pi = 1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \cdots$$

46. Using $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$, we have

$$\sin\pi = \pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!}$$

47. Using Problem 17 and the fact that $46° \approx 0.802851456$ radians, we have

$$\sin 46° \approx= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(0.802851456 - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(0.802851456 - \frac{\pi}{4}\right)^2 \approx 0.719340424.$$

Now, for $f(x) = \sin x$, $f'''(x) = -\cos x$ and $|R_2(x)| = \frac{|\cos c|}{3!}|x - \pi/4|^3 < \frac{|x - \pi/4|^3}{3!}$. Thus, $|R_2(0.802851456)| = \frac{|0.802851456 - \pi/4|^3}{6} < 0.000001$.

48. Using Problem 20 and the fact that $29° \approx 0.506145483$ radians, we have

$$\cos 29° \approx \frac{\sqrt{3}}{2} - \frac{1}{2}\left(0.506145483 - \frac{\pi}{6}\right) - \frac{\sqrt{3}}{4}\left(0.506145483 - \frac{\pi}{6}\right)^2 \approx 0.874620147.$$

Now, for $f(x) = \cos x$, $f'''(x) = \sin x$ and $|R_2(x)| = \frac{|\sin c|}{3!}|x - \pi/6|^3 < \frac{|x - \pi/6|^3}{3!}$. Thus, $|R_2(0.506145483)| < \frac{|0.506145483 - \pi/6|^3}{6} < 0.000001$.

49. Using Problem 7, we have $e^{0.3} \approx 1 + 0.3 + \frac{(0.3)^2}{2} + \frac{(0.3)^3}{3!} + \frac{(0.3)^4}{4!} \approx 1.349837500$. Now, $|R_4(x)| = \frac{e^c}{5!}|x|^5 < \frac{3|x|^5}{5!}$ since $0 < c < 1$ and $e^c < e < 3$. Thus, $|R_4(0.3)| < \frac{3|0.3|^5}{5!} < 0.0001$.

50. Using Problem 9, we have $\sinh(0.1) \approx 0.1 + \frac{(0.1)^3}{3!} \approx 0.100166667$. Now, for $f(x) = \sin hx$, $f^{(4)}(x) = \sinh x$ and $|R_3(x)| = \frac{|\sinh c|}{4!}|x|^4 < \frac{(\sinh 1)|x|^4}{4!}$ since $0 < c < 0.1$ and $\sinh c < \sinh 1 = \frac{e - e^{-1}}{2} = \frac{e^2 - 1}{2e} < \frac{9 - 1}{4} = 2$. Thus, $|R_3(0.1)| < \frac{2(0.1)^4}{24} < 0.00001$.

51. We use $|f^{(n+1)}(x)| = \begin{cases} |\cos x|, & n \text{ even} \\ |\sin x|, & n \text{ odd} \end{cases}$. Since $|\cos x| \leq 1$ and $|\sin x| \leq 1$, $|R_n(x)| = \frac{|f^{(n+1)}(c)|}{(n+1)!}|x|^{n+1} \leq \frac{|x|^{n+1}}{(n+1)!}$ and $\lim_{n\to\infty} |R_n(x)| = 0$. Thus, the series represents $\sin x$ for all x.

52. We use $R_n(x) = \frac{e^c}{(n+1)!}|x|^{n+1}$, where c is between 0 and x. If $x < 0$, then $e^c < 1$ and $|R_n(x)| = \frac{e^c}{(n+1)!}|x|^{n+1} < \frac{|x|^{n+1}}{(n+1)!}$. If $x > 0$, then $e^c < e^x$ and $|R_n(x)| = \frac{e^c}{(n+1)!}|x|^{n+1} < \frac{e^x|x|^{n+1}}{(n+1)!}$. In either case, for x fixed, $\lim_{n\to\infty} |R_n(x)| = 0$ and the series represents e^x for all x.

53. We use $R_n(x) = \begin{cases} \frac{\sinh c}{(n+1)!}x^{n+1}, & n \text{ odd} \\ \frac{\cosh c}{(n+1)!}x^{n+1}, & n \text{ even} \end{cases}$ for c between 0 and x. Now, $|\sinh c| < |\sinh x|$ and $\cosh c < \cosh x$ for c between 0 and x, so for n odd, $|R_n(x)| = \frac{|\sinh c|}{(n+1)!}|x|^{n+1} < \frac{|\sinh x||x|^{n+1}}{(n+1)!}$,

and for n even, $|R_n(x)| = \dfrac{\cosh c}{(n+1)!}|x|^{n+1} < \dfrac{(\cosh x)|x|^{n+1}}{(n+1)!}$. Thus, $\lim_{n\to\infty} |R_n(x)| = 0$ and the series represents $\sinh x$ for all x.

54. We use $R_n(x) = \begin{cases} \dfrac{\cosh c}{(n+1)!}x^{n+1}, & n \text{ odd} \\ \dfrac{\sinh c}{(n+1)!}x^{n+1}, & n \text{ even} \end{cases}$ for c between 0 and x. Now, $|\sinh c| < |\sinh x|$ and $\cosh c < \cosh x$ for c between 0 and x, so for n even, $|R_n(x)| = \dfrac{|\sinh c|}{(n+1)!}|x|^{n+1} < \dfrac{|\sinh x||x|^{n+1}}{(n+1)!}$, and for n odd, $|R_n(x)| = \dfrac{\cosh c}{(n+1)!}|x|^{n+1} < \dfrac{(\cosh x)|x|^{n+1}}{(n+1)!}$. Thus, $\lim_{n\to\infty} |R_n(x)| = 0$ and the series represents $\cosh x$ for all x.

55. (a) From Figure 9.10.3, $L = Rx$ and $\sec x = \dfrac{R+y}{R}$. This gives $y = R\sec x - R = R\sec\left(\dfrac{L}{R}\right) - R$

(b) We need to find the Maclaurin series for $f(x) = \sec x$. We compute

$$\begin{aligned}
&f(x) = \sec x, && f(0) = 1 \\
&f'(x) = \sec x \tan x, && f'(0) = 0 \\
&f''(x) = 2\sec^3 x - \sec x, && f''(0) = 1 \\
&f'''(x) = 6\sec^3 x \tan x - \sec x \tan x, && f'''(0) = 0 \\
&f^{(4)} = 6\sec^5 x + 18\sec^3 \tan^2 x - \sec^3 x - \sec x \tan x, && f^{(4)}(0) = 5
\end{aligned}$$

Therefore, $\sec x = 1 + \dfrac{x^2}{2} + \cdots$.

Approximating $\sec x$ by $1 + \dfrac{x^2}{2}$, we have

$$y = R\left(\sec\left(\frac{L}{R}\right) - 1\right) = R\left(1 + \frac{(L/R)^2}{2} - 1\right) = R\left(\frac{L^2}{2r^2}\right) = \frac{L^2}{2R}$$

(c) Using $y \approx \dfrac{L^2}{2R}$, we have

$$y = \frac{(5280)^2}{2(4000)(5280)} \text{ ft} \approx 0.66 \text{ ft} = 7.92 \text{ in}$$

(d) Approximating $\sec x$ by $1 + \dfrac{x^2}{2} + \dfrac{5}{24}x^4$, we have

$$\begin{aligned}
y &= R\left(1 + \frac{(L/R)^2}{2} + \frac{5}{24}(L/R)^4 - 1\right) \\
&= \frac{L^2}{2R} + \frac{5L^4}{24R^3}.
\end{aligned}$$

56. (a) Since $\lim_{d\to\infty} \tan h\dfrac{2\pi d}{L} = 1$, for d large, $\nu \approx \sqrt{gL/2\pi}$.

(b) $f(x) = \tanh x, \quad f(0) = 0$
$f'(x) = \operatorname{sech}^2 x = 1 - \tanh^2 x, \quad f'(0) = 1$
$f''(x) = -2\tanh x(1-\tanh^2 x) = 2\tanh^3 x - 2\tanh x, \quad f''(0) = 0$
$f'''(x) = 6\tanh^2 x(1-\tanh^2 x) - 2(1-\tanh^2 x) = -6\tanh^4 x + 8\tanh^2 x - 2, \quad f'''(0) = -2$
$f^{(4)}(x) = -24\tanh^3 x(1-\tanh^2 x) + 16\tanh x(1-\tanh^2 x), \quad f^{(4)}(0) = 0$
$\qquad = 24\tanh^5 x - 40\tanh^3 x + 16\tanh x,$
$f^{(5)}(x) = 120\tanh^4 x(1-\tanh^2 x) - 120\tanh^2 x(1-\tanh^2 x) + 16(1-\tanh^2 x), \quad f^{(5)}(0) =$ 16 The Maclaurin series is $\tanh x = x - \dfrac{2}{6}x^3 + \dfrac{16}{120}x^5 - \cdots = x - \dfrac{1}{3}x^3 + \dfrac{2}{15}x^5 - \cdots$.
Now, for small d/L, $\tanh(2\pi d/L) \approx 2\pi d/L$ and $\nu \approx \sqrt{(gL/2\pi)(2\pi d/L)} = \sqrt{gd}$.

57. $$\begin{aligned}\sin^2 x &= (\sin x)(\sin x)\\ &= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)\\ &= x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \cdots\end{aligned}$$
Also,
$$\begin{aligned}\sin^2 x &= 1 - \cos^2 x - 1 - (\cos x)(\cos x)\\ &= 1 - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)\\ &= 1 - \left(1 - x^2 + \frac{x^4}{3} - \frac{2x^6}{45} + \cdots\right)\\ &= x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \cdots\end{aligned}$$

58. $$\begin{aligned}\sin x\cos x &= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)\\ &= x - \frac{2x^3}{3} + \frac{2x^5}{15} - \cdots\end{aligned}$$
Also, using the result from Problem 57, we have
$$\begin{aligned}\sin x\cos x &= \frac{1}{2}\frac{d}{dx}[\sin^2 x] = \frac{1}{2}\frac{d}{dx}\left[x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \cdots\right]\\ &= \frac{1}{2}\left[2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \cdots\right]\\ &= x - \frac{2}{3}x^3 + \frac{2}{15}x^5 - \cdots\end{aligned}$$

59. Using $e^x = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{3!} + \cdots$, we have

$$\begin{aligned}(x+1)^2e^x &= (x+1)^2e^{x+1-1} = e^{-1}(x+1)^2e^{x+1}\\ &= e^{-1}(x+1)^2\left[1+(x+1)+\frac{(x+1)^2}{2}+\frac{(x+1)^3}{3!}+\cdots\right]\\ &= e^{-1}\left[(x+1)^2+(x+1)^3+\frac{(x+1)^4}{2}+\frac{(x+1)^5}{3!}+\cdots\right]\\ &= \sum_{k=0}^{\infty}\frac{e^{-1}(x+1)^{k+2}}{k!}\end{aligned}$$

60. No, the function $f(x) = \cot x$ is undefined at $x = 0$.

61. $\cos x$ is an even function while $\sin x$ is an odd function. From (18), (19), and (20), we see that $\tan^{-1} x$ is an odd function, $\cosh x$ is an even function, and $\sinh x$ is an odd function.

62. We have

$$\begin{aligned}f(x) &= x^4\sin x^2 = x^4\left((x^2)-\frac{(x^2)^3}{3!}+\cdots\right)\\ &= x^6-\frac{x^{10}}{3!}+\cdots\end{aligned}$$

The coefficient of x^{10} should be $\dfrac{f^{(10)}(0)}{10!}$. Therefore, $f^{(10)}(0) = \dfrac{-10!}{3!} = -604,800$.

63.

9.11 Binomial Series

1. With $r = 1/3$, for $|x| < 1$,

$$\begin{aligned}\sqrt[3]{1+x} &= 1+\frac{1}{3}x+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}x^2+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}x^3+\cdots\\ &= 1+\frac{1}{3}x-\frac{2}{3^2\cdot 2!}x^2+\frac{2\cdot 5}{3^3\cdot 3!}x^3-\cdots\end{aligned}$$

.

2. With $r = 1/2$, for $|x| < 1$,

$$\begin{aligned}\sqrt{1+x} &= 1+\frac{1}{2}(-x)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(-x)^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(-x)^3+\cdots\\ &= 1-\frac{1}{2}x-\frac{1}{2^2\cdot 2!}x^2-\frac{3}{2^2\cdot 3!}x^3-\cdots.\end{aligned}$$

3. With $r = 1/2$, for $|x/9| < 1$ or $|x| < 9$,

$$\begin{aligned}\sqrt{9-x} &= 3\sqrt{1-x/9} = 3\left[1+\frac{1}{2}\left(-\frac{x}{9}\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(-\frac{x}{9}\right)^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\left(-\frac{x}{9}\right)^3+\cdots\right]\\ &= 3-\frac{3}{2\cdot 9}x-\frac{3\cdot 1}{2^2\cdot 2!\cdot 9^2}x^2-\frac{3\cdot 1\cdot 3}{2^3\cdot 3!\cdot 9^3}x^3-\cdots.\end{aligned}$$

4. With $r = -1/2$, for $|5x| < 1$ or $|x| < 1/5$,

$$\frac{1}{\sqrt{1+5x}} = 1 + \left(-\frac{1}{2}\right)(5x) + \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)}{2!}(5x)^2 + \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{3!}(5x)^3 + \cdots$$
$$= 1 - \frac{5}{2}x - \frac{3\cdot 5^2}{2^2\cdot 2!}x^2 - \frac{3\cdot 5\cdot 5^3}{2^3\cdot 3!}x^3 - \cdots.$$

5. With $r = -1/2$, for $|x| < 1$,

$$\frac{1}{\sqrt{1+x^2}} = 1 + \left(-\frac{1}{2}\right)(x^2) + \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)}{2!}(x^2)^2 + \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{3!}(x^2)^3 + \cdots$$
$$= 1 - \frac{1}{2}x - \frac{3}{2^2\cdot 2!}x^4 - \frac{3\cdot 5}{2^3\cdot 3!}x^6 - \cdots.$$

6. With $r = -1/3$, for $|x| < 1$,

$$\frac{x}{\sqrt[3]{1-x^2}} = x\left[1 - \frac{1}{3}(-x^2) + \frac{-\frac{1}{3}\left(-\frac{1}{3}-1\right)}{2!}(-x^2)^2 + \frac{-\frac{1}{3}\left(-\frac{1}{3}-1\right)\left(-\frac{1}{3}-2\right)}{3!}(-x^2)^3 + \cdots\right]$$
$$= x + \frac{1}{3}x^3 + \frac{4}{3^2\cdot 2!}x^5 + \frac{4\cdot 7}{3^3\cdot 3!}x^6 + \cdots.$$

7. With $r = 3/2$, for $|x/4| < 1$ or $|x| < 4$,

$$(4+x)^{3/2} = 8(1+x/4)^{3/2} = 8\left[1 + \frac{3}{2}\left(\frac{x}{4}\right) + \frac{\frac{3}{2}\left(\frac{3}{2}-1\right)}{2!}\left(\frac{x}{4}\right)^2 + \frac{\frac{3}{2}\left(\frac{3}{2}-1\right)\left(\frac{3}{2}-2\right)}{3!}\left(\frac{x}{4}\right)^3 + \cdots\right]$$
$$= 8 + \frac{8\cdot 3}{2\cdot 4}x + \frac{8\cdot 3\cdot 1}{3^2\cdot 2!\cdot 4^2}x^2 + \frac{8\cdot 3\cdot 1\cdot(-1)}{2^3\cdot 3!\cdot 4^3}x^3 + \cdots.$$

8. With $r = -5/2$, for $|x| < 1$,

$$\frac{x}{\sqrt{(1+x)^5}} = x\left[1 - \frac{5}{2}x + \frac{-\frac{5}{2}\left(-\frac{5}{2}-1\right)}{2!}x^2 + \frac{-\dfrac{5}{2}\left(-\frac{5}{2}-1\right)\left(-\frac{5}{2}-2\right)}{3!}x^3 + \cdots\right]$$
$$= x - \frac{5}{2}x^2 + \frac{5\cdot 7}{2^2\cdot 2!}x^3 - \frac{5\cdot 7\cdot 9}{2^3\cdot 3!}x^4 + \cdots$$

.

9. With $r = -2$, for $|x/2| < 1$ or $|x| < 2$,

$$\frac{x}{(2+x)^2} = \frac{x}{4}(1+x/2)^{-2}$$
$$= \frac{x}{4}\left[1 - 2\left(\frac{x}{2}\right) + \frac{-2(-2-1)}{2!}\left(\frac{x}{2}\right)^2 + \frac{-2(-2-1)(-2-2)}{3!}\left(\frac{x}{2}\right)^3 + \cdots\right]$$
$$= \frac{1}{4}x - \frac{1}{4}x^2 + \frac{2\cdot 3}{4\cdot 2!\cdot 2^2}x^3 - \frac{2\cdot 3\cdot 4}{4\cdot 3!\cdot 2^3}x^4 + \cdots$$

.

10. With $r=-3$, for $|x|<1$,

$$\begin{aligned}x^2(1-x^2)^{-3} &= x^2\left[1-3(-x)^2+\frac{-3(-3-1)}{2!}((-x)^2)^2+\frac{-3(-3-1)(-3-2)}{3!}((-x)^2)^3+\cdots\right]\\ &= x^2+3x^4+\frac{3\cdot 4}{2!}x^6+\frac{3\cdot 4\cdot 5}{3!}x^8+\cdots\end{aligned}$$

.

11. See Problem 1. Since the series is alternating on $(0,1)$, by Theorem 9.7.2 the approximation to the sum using $S_2=1+\dfrac{x}{3}$ is accurate within $a_3=\dfrac{1}{9}x^2$.

12. See Problem 5. Since the series is alternating on $(-1,1)$, by Theorem 9.7.2 the approximation to the sum using $S_3=1-\dfrac{x^2}{2}+\dfrac{3}{8}x^4$ is accurate within $a_4=\dfrac{3\cdot 5}{2^3\cdot 3!}x^6=\dfrac{5}{16}x^6$.

13. With $r=-1/2$,

$$\begin{aligned}\sin^{-1}x &= \int_0^x (1-t^2)^{(-1/2)}dt\\ &= \int_0^x\left[1-\frac{1}{2}(-t^2)+\frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)}{2!}(-t^2)^2+\frac{-\dfrac{1}{2}\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{2!}(-t^2)^3+\cdots\right]dt\\ &= \int_0^x\left[1+\frac{1}{2}t^2+\frac{3}{2^2\cdot 2!}t^4+\frac{3\cdot 5}{2^3\cdot 3!}+\cdots\right]dt\\ &= x+\frac{1}{2\cdot 3}x^3+\frac{3}{2^2\cdot 2!\cdot 5}x^5+\frac{3\cdot 5}{2^3\cdot 3!\cdot 7}x^7+\cdots\\ &= x+\sum_{k=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2k-1)}{2^k k!(2k+1)}x^{2k+1}.\end{aligned}$$

14. (a) The equation of the ellipse can be written as $y=\dfrac{b}{a}\sqrt{a^2-x^2}$. Then $y'=-\dfrac{bx}{a\sqrt{a^2-x^2}}$ and

$$L=\int_0^a\sqrt{1+\frac{b^2x^2}{a^2(a^2-x^2)}}dx=\frac{1}{a}\int_0^a\sqrt{\frac{a^4-x^2(a^2-b^2)}{a^2-x^2}}dx.$$

Let $k^2a^2=a^2-b^2$ and $x=a\sin\theta$. The $dx=a\cos\theta d\theta$ and

$$\begin{aligned}L &= \frac{1}{a}\int_0^a\sqrt{\frac{a^4-a^2k^2x^2}{a^2-x^2}}dx=\int_0^a\sqrt{\frac{a^2-k^2x^2}{a^2-x^2}}dx=\int_0^{\pi/2}\sqrt{\frac{a^2-k^2a^2\sin^2\theta}{a^2-a^2\sin^2\theta}}a\cos\theta d\theta\\ &= a\int_0^{\pi/2}\sqrt{\frac{1-k^2\sin^2\theta}{1-\sin^2\theta}}\cos\theta d\theta=a\int_0^{\pi/2}\sqrt{1-k^2\sin^2\theta}d\theta.\end{aligned}$$

(b) With $r = 1/2$,

$$a\int_0^{\pi/2}\sqrt{1-k^2\sin^2\theta}d\theta = a\int_0^{\pi/2}\left[1+\frac{1}{2}(-k^2\sin^2\theta)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(-k^2\sin^2\theta)^2+\cdots\right]d\theta$$

$$= a\int_0^{\pi/2}\left[1-\frac{1}{2}k^2\sin^2\theta-\frac{1}{8}k^4\sin^4\theta-\cdots\right]d\theta$$

$$= a\left[\theta-\frac{1}{2}k^2\left(\frac{1}{2}\theta-\frac{1}{4}\sin 2\theta\right)\right.$$

$$\left.\left.-\frac{1}{8}k^4\left(\frac{3}{8}\theta-\frac{1}{4}\sin 2\theta+\frac{1}{32}\sin 4\theta\right)-\cdots\right]\right|_0^{\pi/2}$$

$$= a\left[\frac{\pi}{2}-\frac{1}{2}k^2\left(\frac{\pi}{4}\right)-\frac{1}{8}k^4\left(\frac{3\pi}{16}\right)-\cdots\right]$$

$$= a\frac{\pi}{2}-\frac{a}{2}\cdot\frac{\pi}{4}-\frac{a}{8}\cdot\frac{3\pi}{16}k^4-\cdots.$$

15. $y' = \dfrac{8d}{l^2}x$. Using the formula for arc length and $r = 1/2$, we have

$$s = 2\int_0^{l/2}\sqrt{1+\frac{64d^2}{l^4}x^2}dx = 2\int_0^{l/2}\left[1+\frac{1}{2}\cdot\frac{64d^2}{l^4}x^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(\frac{64d^2}{l^4}x^2\right)^2+\cdots\right]dx$$

$$= 2\int_0^{l/2}\left[1+\frac{32d^2}{l^4}x^2-\frac{64^2d^4}{8l^8}{}^4+\cdots\right]dx = 2\left.\left[x+\frac{32d^2}{3l^4}x^3-\frac{64^2d^4}{5\cdot 8l^8}x^5+\cdots\right]\right|_0^{l/2}$$

$$= l+\frac{8d^2}{3l}-\frac{32d^4}{5l^3}+\cdots.$$

16. (a) $$\int_0^{0.2}\sqrt{1+x^3}dx = \int_0^{0.2}\left[1+\frac{1}{2}x^3+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(x^3)^2+\cdots\right]dx$$

$$= \int_0^{0.2}\left[1+\frac{1}{2}x^3-\frac{1}{8}x^6+\cdots\right]dx = \left.\left[x+\frac{1}{8}x^4-\frac{1}{56}x^7+\cdots\right]\right|_0^{0.2}$$

$$= 0.2+\frac{(0.2)^4}{8}-\frac{(0.2)^7}{56}+\cdots$$

After the first term this is an alternating series. Since $\dfrac{(0.2)^7}{56} < 0.0005$, we have will three decimal place accuracy $\int_0^{0.2}\sqrt{1+x^3}dx \approx 0.2+\dfrac{(0.2)^4}{8} = 0.2002$.

(b) $$\int_0^{1/2}\sqrt[3]{1+x^4}dx = \int_0^{1/2}\left[1+\frac{1}{3}x^4+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}(x^4)^2+\cdots\right]dx$$

$$= \int_0^{1/2}\left(1+\frac{1}{3}x^4-\frac{2}{18}x^8+\cdots\right)dx = \left.\left(x+\frac{1}{15}x^5-\frac{2}{18\cdot 9}x^9+\cdots\right)\right|_0^{1/2}$$

$$= \frac{1}{2}+\frac{1}{15\cdot 32}-\frac{2}{18\cdot 9\cdot 2^9}+\cdots$$

After the first term this is an alternating series. Since $\dfrac{2}{18 \cdot 9 \cdot 2^9} < 0.0005$, we have with three decimal place accuracy $\int_0^{1/2} \sqrt[3]{x + x^4} \approx \dfrac{1}{2} + \dfrac{1}{15 \cdot 32} \approx 0.502$.

17. From Theorem 9.11.1 in the text with $r = -1/2$, we have

$$(1 - 2xr + r^2)^{-1/2} = [1 + r(r - 2x)]^{-1/2} = 1 - \frac{1}{2}r(r - 2x) + \frac{-\frac{1}{2}\left(-\frac{1}{2} - 1\right)}{2!}r^2(r - 2x)^2 + \cdots$$

$$= 1 - \frac{1}{2}r^2 + xr + \frac{3}{8}(r^4 - 4xr^3 + 4x^2r^2) + \cdots = 1 + xr + \left(\frac{3}{2}x^2 - \frac{1}{2}\right)r^2 + \cdots$$

Thus, $P_0 = 1$ $P_1 = x$, and $P_2 = \dfrac{3}{2}x^2 - \dfrac{1}{2}$.

18. (a) $f'(x) = r + r(r-1)x + \dfrac{r(r-1)(r-2)}{2!}x^2 + \cdots + \dfrac{r(r-1)\cdots(r-n+1)}{(n-1)!}x^{n-1} + \cdots$

$$xf'(x) = rx + r(r-1)x^2 + \frac{r(r-1)(r-2)}{2!}x^3 + \cdots + \frac{r(r-1)\cdots(r-n+1)}{(n-1)!}x^n + \cdots$$

(b) $(n+1)\dfrac{r(r-1)\cdots(r-n)}{(n+1)!} + n\dfrac{r(r-1)\cdots(r-n+1)}{n!}$

$$= \frac{r(r-1)\cdots(r-n)}{n!} + \frac{nr(r-1)\cdots(r-n+1)}{n!}$$

$$= \frac{r(r-1)\cdots(r-n+1)}{n!}(r - n + n) = r\frac{r(r-1)\cdots(r-n+1)}{n!}$$

(c) $f'(x) + xf'(x) = r + r(r-1)x + \cdots + \dfrac{r(r-1)\cdots(r-n+1)}{(n-1)!}x^{n-1} + \cdots$

$$rx + r(r-1)x^2 + \cdots + \frac{r(r-1)\cdots(r-n+1)}{(n-1)}x^n + \cdots$$

$$= \sum_{k=1}^{\infty} \frac{r(r-1)\cdots(r-k+1)}{(k-1)!}x^{k-1} + \sum_{j=1}^{\infty} \frac{r(r-1)\cdots(r-j+1)}{(j-1)!}x^j$$

$$\boxed{\text{Let } k = j+1}$$

$$= \sum_{j=0}^{\infty} \frac{r(r-1)\cdots(r-j)}{j!}x^j + \sum_{j=1}^{\infty} \frac{r(r-1)\cdots(r-j+1)}{(j-1)!}x^j$$

$$= r + \sum_{j=1}^{\infty} \left[\frac{r(r-1)\cdots(r-j)}{j!} + \frac{r(r-1)\cdots(r-j+1)}{(j-1)!}\right]x^j$$

$$= r + \sum_{j=1}^{\infty} \left[(j+1)\frac{r(r-1)\cdots(r-j)}{(j+1)!} + j\frac{r(r-1)\cdots(r-j+1)}{j!}\right]x^j$$

$$\boxed{\text{By part } \textbf{(b)}}$$

$$= r + \sum_{j=1}^{\infty} r\frac{r(r-1)\cdots(r-j+1)}{j!}x^j$$

$$= r\left[1 + r + \frac{r(r-1)}{2}x^2 + \cdots \frac{r(r-1)\cdots(r-n+1)}{n!}x^n + \cdots\right] = rf(x)$$

(d) Write the equation in the form $(1+x)\dfrac{dy}{dx} = ry$. Then, separating variables, we have

$$\frac{dy}{y} = \frac{rdx}{1+x}; \quad \ln y = r\ln(1+x) + c; \quad y = e^{r\ln(1+x)+c} = e^c e^{\ln(1+x)^r} = c_1(1+x)^r$$

or $f(x) = c_1(1+x)^r$. Now, $1 + f(0) = c_1$ so $f(x) = (1+x)^r$.

19. $$(1+x)^{1/2} = [2 + (x-1)]^{1/2} = \sqrt{2}\left(1 + \frac{x-1}{2}\right)^{1/2}$$
$$= \sqrt{2}\left[1 + \left(\frac{1}{2}\right)\frac{x-1}{2} + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}\frac{(x-1)^2}{2^2} + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\frac{(x-1)^4}{2^4} + \cdots\right]$$
$$= \sqrt{2} + \frac{\sqrt{2}}{2^2}(x-1) - \frac{\sqrt{2}}{2^2 2!}(x-1)^2 + \frac{\sqrt{2}\cdot 1\cdot 3}{2^6\cdot 3!}(x-1)^3 + \cdots$$

20. $$(1+x)^{-2} = [2 + (x+1)]^{-2} = 2^{-2}\left(1 + \frac{x-1}{2}\right)^{-2}$$
$$= \frac{1}{4}\left[1 - 2\frac{x-1}{2} + \frac{-2(-2-1)}{2!}\frac{(x-1)^2}{2^2} + \frac{-2(-2-1)(-2-2)}{3!}\frac{(x-1)^3}{2^3}\right]$$
$$= \frac{1}{4} - \frac{1}{4}(x-1) + \frac{3}{2^2\cdot 2!}(x-1)^2 - \frac{3\cdot 4}{2^4 3!}(x-1)^3 + \cdots$$

Chapter 9 in Review

A. True/False

1. False; since $|a_n| = \dfrac{n}{2n+1} \to \dfrac{1}{2}$, the series diverges by the n-th term test.
2. False; $\{(-1)^n\}$ is bounded and divergent.
3. False; $\{(-1)^n/n\}$ is convergent and not monotonic.
4. True; the first three terms are $1/2,\ 100/6$, and $1000/512$.
5. True; since $a_{n+1}/a_n \geq 1$, $a_{n+1} \geq a_n$ and $\{a_n\}$ is a bounded monotonic sequence.
6. True
7. False; $\{1/n\}$ converges, but $\displaystyle\sum_{k=1}^{\infty} 1/k$ is the divergent harmonic series.
8. True; $$0.999\ldots = \frac{9}{10}\left(1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots\right))$$
$$= \frac{9}{10}\left(\frac{1}{1-\frac{1}{10}}\right) = \frac{9}{10}\left(\frac{1}{\frac{9}{10}}\right) = 1$$
9. True; if this were false, the series would diverge by the n-th term test.

10. False; consider the harmonic series
11. False; let $a_k = 1/k$.
12. True
13. True
14. True
15. False; let $a_k = (-1)^k/k$.
16. True
17. True
18. False; let $b_k = 1/k^2$ and $a_k = a/k$.
19. False; the ratio test is inconclusive in this case.
20. False; $\sum_{k=0}^{\infty} k!x^k$ converges only at $x = 0$.
21. False; $\sum_{k=1}^{\infty} \frac{1}{k}x^k$ converges at $x = -1$ but is not absolutely convergent there.
22. True
23. False; let $c_k = -1/k$.
24. False; at $x = r$ the series diverges
25. False; the integral test simply indicates that the series converges.
26. True; the series is absolutely convergent
27. True; $\ln x$ is not differentiable at $x = 0$.
28. True
29. True
30. True; $f^{(4)} = 4!c_4 = -$ since $c_4 = 0$.

B. Fill in the Blanks

1. 20, 9, 4/5, 16
2. $\pi/2$
3. 4, since $a_5 = 1/10^5 = 0.00001 < 0.00005$.
4. 12

5. $\dfrac{n}{9}$; $\dfrac{22}{9}$

$$0.nnnn\ldots = \frac{n}{10}\left(1 + \frac{1}{10} + \frac{1}{10^2} + \ldots\right)$$
$$= \frac{n}{10}\left(\frac{1}{1-\frac{1}{10}}\right) = \frac{n}{10}\left(\frac{1}{\frac{9}{10}}\right) = \frac{n}{9}$$
$$2.444\ldots = 2 + .044\ldots = 2 + \frac{4}{9} = \frac{22}{9}$$

6. $-\pi/4$

7. e^x

8. 4

9. $x < -5$ and $x > 5$.

10. $\displaystyle\sum_{k=0}^{\infty} \frac{(-x^3)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{3k}}{k!}$

11. The series converges absolutely for x in $(-1, 1)$. At $x = -1$, the series diverges. At $x = 1$, the series converges conditionally. Thus, the interval of convergence is $(-1, 1]$.

12. 5

C. Exercises

1. Since $\dfrac{k}{(k^2+1)^2} < \dfrac{1}{k^3}$, the series converges by comparison with the p-series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^p}$.

2. Since $\displaystyle\lim_{n\to\infty} \frac{1}{1+e^{-k}} = 1$, the series diverges by the n-th term test.

3. This is a geometric series with $r = \dfrac{1}{\pi} < 1$. Thus, the series converges.

4. This is a geometric series with $r = \dfrac{1}{\ln(2.5)} > 1$. Thus, the series diverges.

5. Since $\ln k < k$, $\dfrac{\sqrt{k}\ln k}{k^4+4} < \dfrac{1}{k^{5/2}}$ and the series converges by comparison with the p-series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{5/2}}$

6. Since $\left|\dfrac{\sin k}{k^{3/2}}\right| < \dfrac{1}{k^{3/2}}$, the series is absolutely convergent, and thus convergent, by comparison with the p-series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$.

7. Since $\dfrac{k}{\sqrt[3]{k^6-4k}} > \dfrac{k}{k^2} = \dfrac{1}{k}$, the series diverges by comparison with the harmonic series.

8. The function $f(x) = 1/x\sqrt{\ln x}$ is continuous and decreasing on $[2,\infty)$. Since

$$\begin{aligned}\int_2^\infty \frac{dx}{x\sqrt{\ln x}} &= \lim_{t\to\infty}\int_2^t \frac{dx}{x\sqrt{\ln x}} \quad \boxed{u=\ln x,\quad du=\frac{1}{x}dx}\\ &= \lim_{t\to\infty}\int_{\ln 2}^{\ln t}\frac{du}{\sqrt{u}} = \lim_{t\to\infty} 2\sqrt{u}\Big|_{\ln 2}^{\ln t} = \lim_{t\to\infty} 2\left(\sqrt{\ln t}-\sqrt{\ln 2}\right) = \infty,\end{aligned}$$

the series diverges by the integral test.

9. All of the odd-numbered terms of this series are 0. We may thus express the series as $\displaystyle\sum_{k=1}^\infty \frac{2}{\sqrt{2k}}$ or $\displaystyle\sum_{k=1}^\infty \frac{1}{\sqrt{k}}$. This is a divergent p-series.

10. Since

$$\begin{aligned}\lim_{n\to\infty}\frac{a_{n+1}}{a_n} &= \lim_{n\to\infty}\frac{[(n+1)^2!]/[(n+1)!]}{(n^2)!/(n!)^2} = \lim_{n\to\infty}\frac{(n^2+2n+1)!}{(n^2)!}\left[\frac{n!}{(n+1)!}\right]^2\\ &= \lim_{n\to\infty}(n^2+2n+1)(n^2+2n)\cdots(n^2+1)\left(\frac{1}{n+1}\right)^2\\ &= \lim_{n\to\infty}\frac{(n^2+2n+1)(n^2+2n)\cdots(n^2+1)}{(n^2+2n+1)}\\ &= \lim_{n\to\infty}(n^2+2n)\cdots(n^2+1) = \infty,\end{aligned}$$

the series diverges by the ratio test.

11. Since $\dfrac{1}{3k^2+4k+6} < \dfrac{1}{k^2}$, the series converges by comparison with the p-series $\displaystyle\sum_{k=1}^\infty \frac{1}{k^2}$.

12. $a_n = \ln\left(\dfrac{3n}{n+1}\right) \to \ln(3) \neq 0$. Thus, the series diverges by the n-th term test.

13. $$\begin{aligned}\sum_{k=1}^\infty \frac{(-1)^{k-1}+3}{(1.01)^{k-1}} &= \sum_{k=1}^\infty\left(-\frac{1}{1.01}\right)^{k-1} + 3\sum_{k=1}^\infty\left(\frac{1}{1.01}\right)^{k-1} = \frac{1}{1-(-1/1.01)} + \frac{3}{1-(1/1.01)}\\ &= \frac{1.01}{1.01+1} + \frac{3.03}{1.01-1} = \frac{101}{201} + 303 = \frac{61,004}{201}\end{aligned}$$

14. Write $a_k = \dfrac{1}{k+1} - \dfrac{1}{k+6}$. Then $S_n = \left(\dfrac{1}{6}-\dfrac{1}{7}\right) + \left(\dfrac{1}{7}-\dfrac{1}{8}\right) + \cdots + \left(\dfrac{1}{n+5}-\dfrac{1}{n+6}\right) = \dfrac{1}{6} - \dfrac{1}{n+6}$

and $\displaystyle\lim_{n\to\infty}\sum_{k=1}^n \frac{1}{k^2+11k+30} = \lim_{n\to\infty} S_n = \frac{1}{6}$.

15. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \lim_{n\to\infty}\left|\frac{3^{n+1}x^{n+1}/(n+1)^3}{3^n x^n/n^3}\right| = \lim_{n\to\infty} 3\left(\frac{n}{n+1}\right)^3 |x| = 3|x|$
The series is absolutely convergent for $3|x| < 1$ or on $(-1/3, 1/3)$. At $x = -1/3$, the series $\sum_{k=1}^{\infty}\frac{(-1)^k}{k^3}$ converges by the alternating series test. At $x = 1/3$, the series $\sum_{k=1}^{\infty}\frac{1}{k^3}$ is a convergent p-series. Thus, the given series converges on $[-1/3, 1/3]$.

16. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \lim_{n\to\infty}\left|\frac{(n+1)(2x-1)^{n+1}/4^{n+1}}{n(2x-1)^n/4^n}\right| = \lim_{n\to\infty}\frac{1}{4}\left(\frac{n+1}{n}\right)|2x-1| = \frac{1}{4}|2x-1|$
The series is absolutely convergent for $\frac{1}{4}|2x-1| < 1$, $|2x-1| < 4$ or on $(-3/2, 5/2)$. At $x = -3/2$, the series $\sum_{k=1}^{\infty}(-1)^k k$ diverges by the n-th term test. At $x = 5/2$, the series $\sum_{k=1}^{\infty} k$ diverges by the n-th term test. Thus, the given series converges on $(-3/2, 5/2)$.

17. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \lim_{n\to\infty}\left|\frac{(n+1)!(x+5)^{n+1}}{n!(x+5)^n}\right| = \lim_{n\to\infty}(n+1)|x+5| = \infty$ for $x \neq -5$. Thus, the series converges only for $x = -5$.

18.
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(2x)^{n+1}/\ln(n+1)}{(2x)^n/\ln n}\right|$$
$$= \lim_{n\to\infty}\frac{\ln n}{\ln(n+1)}|2x| \stackrel{h}{=} \lim_{n\to\infty}\frac{1/n}{1/(n+1)}|2x|$$
$$= \lim_{n\to\infty}\frac{n+1}{n}|2x| = |2x|$$
The series is absolutely convergent for $|2x| < 1$ or on $(-1/2, 1/2)$. At $x = -1/2$, the series $\sum_{k=2}^{\infty}\frac{1}{\ln 2}$ diverges by comparison with the harmonic series. Thus, the given series converges on $[-1/2, 1/2)$.

19. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{2\cdot 5\cdots(3n+2)x^{n+1}}{3\cdot 7\cdots(4n+3)}}{\dfrac{2\cdot 5\cdots(3n-1)x^n}{3\cdot 7\cdots(4n-1)}}\right| = \lim_{n\to\infty}\frac{3n+2}{4n+3}|x| = \frac{3}{4}|x$
The series converges for $\frac{3}{4}|x| < 1$ or $|x| < \frac{4}{3}$. Thus, the radius of convergence is $4/3$.

20. Applying the ratio test, we have $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(\cos x)^{n+1}}{(\cos x)^n}\right| = \lim_{n\to\infty}|\cos x| = |\cos x|$.
Since $|\cos x| < 1$ for $x \neq k\pi$ where k is an integer, the series converges for all $x \neq k\pi$. When $x = k\pi$, the series is either $\sum_{k=1}^{\infty}(-1)^k$ or $\sum_{k=1}^{\infty}1$, both of which diverge by the n-th term test. Thus, the series converges on all intervals of the form $(k\pi, k\pi + \pi)$, where k is an integer.

21. $$\frac{1}{\alpha}+\frac{1}{\alpha^2}+\frac{1}{\alpha^3}+\cdots=\left(1+\frac{1}{\alpha}+\frac{1}{\alpha^2}+\cdots\right)-1$$
$$=\frac{1}{1-\left(\frac{1}{\alpha}\right)}-1=\frac{\alpha}{\alpha-1}-1=\frac{1}{\alpha-1}$$

22. The argument is invalid since both series diverge. This implies $S=\infty$ and hence S cannot be subtracted from both sides of $2S=S-1$.

23. Using a binomial series expansion,
$$\frac{1}{\sqrt[3]{1+x^5}}=(1+x^5)^{-1/3}=1-\frac{1}{3}x^5+\frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)}{2!}(x^5)^2+\cdots=1-\frac{1}{3}x^5+\frac{2}{9}x^{10}-\cdots.$$

24. Using a binomial series expansion,
$$\frac{x}{2-x}=\frac{2-x-2}{2-x}=-1+\frac{2}{2-x}=-1+\left(1-\frac{x}{2}\right)^{-1}$$
$$=-1+\left[1+(-1)\left(-\frac{x}{2}\right)+\frac{(-1)(-1)}{2!}\left(-\frac{x}{2}\right)^2+\frac{(-1)(-1-1)(-1-2)}{3!}\left(-\frac{x}{2}\right)^3+\cdots\right]$$
$$=\frac{1}{2}x+\frac{1}{4}x^2+\frac{1}{8}x^3+\cdots.$$

25. Using the Maclaurin series for $\sin x$,
$$\sin x\cos x=\frac{1}{2}\sin 2x=\frac{1}{2}\left[2x-\frac{(2x)^3}{3!}+\frac{(2x)^5}{5!}-\cdots\right]=x-\frac{2}{3}x^3+\frac{2}{15}x^5-\cdots.$$

26. Using the Maclaurin series for e^t,
$$\int_0^x e^{t^2}\,dt=\int_0^x\left[1+t^2+\frac{1}{2}(t^2)^2+\cdots\right]dt=\int_0^x\left(1+t^2+\frac{1}{2}t^4+\cdots\right)dt$$
$$=\left(1+\frac{1}{3}t^3+\frac{1}{10}t^5+\cdots\right)\Big|_0^x=x+\frac{1}{3}x^3+\frac{1}{10}x^5+\cdots.$$

27.
$$\begin{array}{ll}
f(x)=\cos x, & f(\pi/2)=0\\
f'(x)=-\sin x, & f'(\pi/2)=-1\\
f''(x)=-\cos x, & f''(\pi/2)=0\\
f'''(x)=\sin x, & f'''(\pi/2)=1\\
\vdots & \\
f^{(2k+1)}(x)=(-1)^{k+1}\sin x, & f^{(2k+1)}(\pi/2)=(-1)^{k+1}
\end{array}$$
The Taylor series is $\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{(2k+1)!}\left(x-\frac{\pi}{2}\right)^{2k+1}$.

28. We use $|f^{(n+1)}(x)| = \begin{cases} 2^{n-1}|\sin(2x)|, & n \text{ even} \\ 2^{n-1}|\cos(2x)|, & n \text{ odd} \end{cases}$ Since $|\cos(2x)| \le 1$ and $|\sin(2x)| \le 1$,

$$|R_n(x)| = \frac{|f^{(n+1)}(c)|}{(n+1)!}|x|^{n+1} \le \frac{2^n|x|^{n+1}}{(n+1)!}$$

Using the ratio test, $\begin{aligned} \frac{|a_{n+1}|}{a_n} &= \frac{2^{n+1}|x|^{n+2}}{2^n|x|^{n+1}} \cdot \frac{(n+1)!}{(n+2)!} \\ &= \frac{2|x|}{n+2}. \end{aligned}$

which converges to 0 as $n \to \infty$. Thus, the series found in Problem 55 represents $\sin x \cos x$ for all x.

29. $$3\left(\frac{2}{3}\right) + 2\left(\frac{2}{3}\right) + \frac{4}{3}\left(\frac{2}{3}\right) + \frac{8}{9}\left(\frac{2}{3}\right) + \frac{16}{27}\left(\frac{2}{3}\right) + \cdots$$
$$= 3\left(\frac{2}{3}\right) + 3\left(\frac{2}{3}\right)^2 + 3\left(\frac{2}{3}\right)^3 + 3\left(\frac{2}{3}\right)^4 + 3\left(\frac{2}{3}\right)^5 + \cdots = \frac{2}{1-2/3} = 6 \text{ million dollars.}$$

30. (a) Solving $2P = P(1+r)^n$ for n, we obtain the doubling time $n = \ln 2/\ln(1+r)$.

(b) Since $0 < r < 1$, $\ln(1+r) = r - r^2/2 + r^3/3 - \cdots \approx r$. Then $(\ln 2)/\ln(1+r) \approx (\ln 2)/r \approx 0.69/r \approx 70/100r$.

(c) We want to solve $70/100r = \ln 2/\ln(1+r)$ or $7\ln(1+r) = 10r\ln 2$ for r. Using $\ln(1+r) \approx r - r^2/2 + r^3/3$, this equation can be written as

$$\frac{7}{3}r^3 - \frac{7}{2}r^2 + (7 - 10\ln 2)r = r\left(\frac{7}{3}r^2 - \frac{7}{2}r + 7 - 10\ln 2\right) = 0.$$

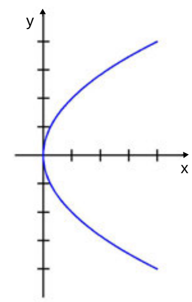
The quadratic formula gives $r \approx 1.4802$ and $r \approx 0.0198$. Since $r < 1$ we see that the Rule of 70 gives the true doubling time for $r \approx 1.98\%$.

Chapter 10

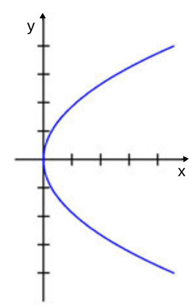
Conics and Polar Coordinates

10.1 Conic Sections

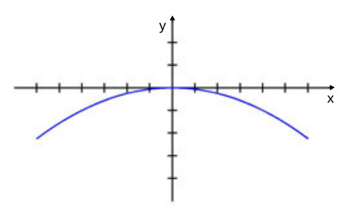
1. vertex: $(0, 0)$
   focus: $(1, 0)$
   directrix: $x = -1$
   axis: $y = 0$



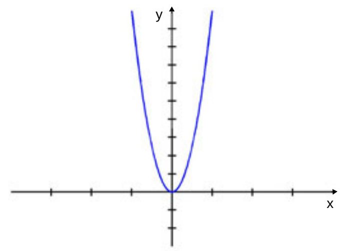
2. vertex: $(0, 0)$
   focus: $(7/8, 0)$
   directrix: $x = -\frac{7}{8}$
   axis: $y = 0$



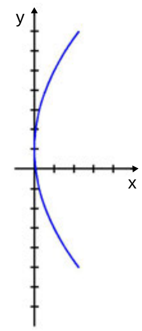
3. vertex: $(0, 0)$
   focus: $(0, -4)$
   directrix: $y = 4$
   axis: $x = 0$



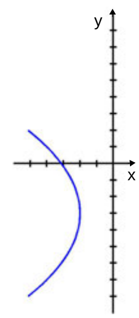
4. vertex: $(0,0)$
   focus: $\left(0,\frac{1}{40}\right)$
   directrix: $y=-\frac{1}{40}$
   axis: $x=0$



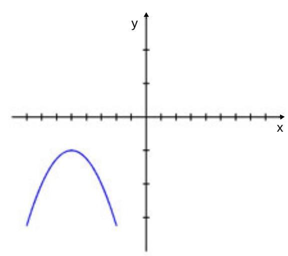
5. vertex: $(0,1)$
   focus: $(4,1)$
   directrix: $x=-4$
   axis: $y=1$



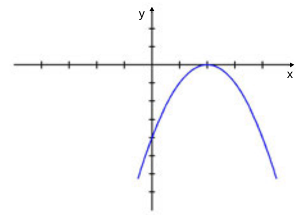
6. vertex: $(-2,-3)$
   focus: $(-4,-3)$
   directrix: $x=0$
   axis: $y=-3$



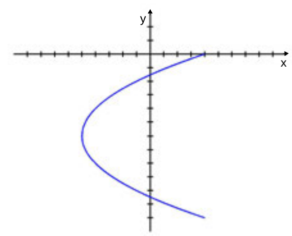
7. vertex: $(-5,-1)$
   focus: $(-5,2)$
   directrix: $y=0$
   axis: $x=-5$



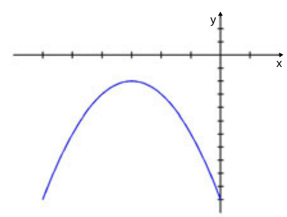
8. vertex: $(2,0)$
focus: $\left(2,-\frac{1}{4}\right)$
directrix: $y=\frac{1}{4}$
axis: $x=2$



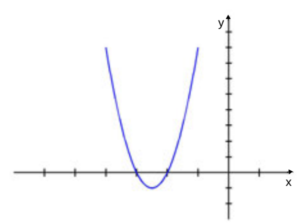
9. $(y+6)^2=4(x+5)$
vertex: $(-5,-6)$
focus: $(-4,-6)$
directrix: $x=-6$
axis: $y=-6$



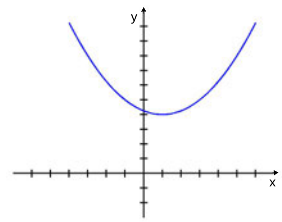
10. $(x+3)^2=-(y+2)$
vertex: $(-3,-2)$
focus: $(-3,-9/4)$
directrix: $y=-7/4$
axis: $x=-3$



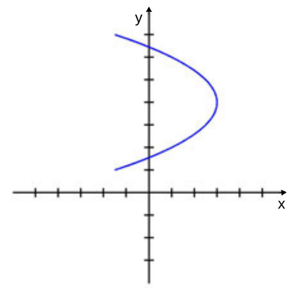
11. $\left(x+\frac{5}{2}\right)^2=\frac{1}{4}(y-1)$
vertex: $(-5/2,-1)$
focus: $(-5/2,-15/16)$
directrix: $y=-17/16$
axis: $x=--5/2$



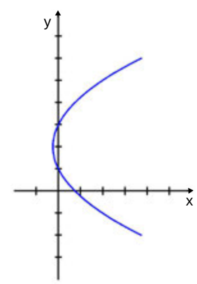
12. $(x-1)^2 = 4(y-4)$
vertex: $(1,4)$
focus: $(1,5)$
directrix: $y=3$
axis: $x=1$



13. $(y-4)^2 = -2(x+3)$
vertex: $(3,4)$
focus: $(5/2,4)$
directrix: $x=7/2$
axis: $y=4$



14. $(y-2)^2 = 4\left(x+\frac{1}{4}\right)$
vertex: $(-1/4,2)$
focus: $(3/4,2)$
directrix: $x=-5/4$
axis: $y=2$



15. $x^2 = 28$

16. $y^2 = -16x$

17. $y^2 = 10x$

18. $x^2 = -40y$

19. The parabola is of the form $(y-k)^2 = 4p(x-h)$ with $(h,k) = (-2,-7)$ and $p=3$. Thus the equation is $(y+7)^2 = 12(x+2)$.

20. The parabola is of the form $(x-h)^2 = 4p(y-5)$ with $(h,k) = (2,0)$ and $p=3$, so the equation of the parabola is $(x-2)^2 = 12y$.

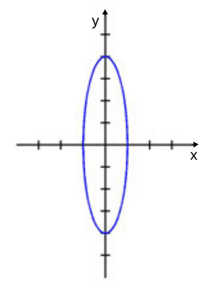
21. The parabola is of the form $x^2 = 4py$ with $(-2)^2 = 4p(8)$. Thus $p = \frac{1}{8}$ and the equation is $x^2 = \frac{1}{2}y$.

22. The parabola is of the form $y^2 = 4px$ with $\left(\frac{1}{4}\right)^2 = 4p(1)$ so $p = \frac{1}{64}$. Thus the equation is $y^2 = \frac{1}{16}x$.

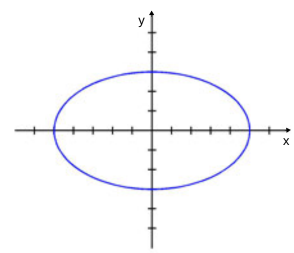
23. To find the x-intercept set $y = 0$. Solving $4^2 = 4(x+1)$ gives $x = 3$. The x-intercept is $(3, 0)$. To find the y-intercept set $x = 0$. Solving $(y+4)^2 = 4$ gives $y = -4 \pm 2$. The y-intercepts are $(0, -2)$ and $(0, -6)$.

24. To find the x-intercept set $y = 0$. Solving $(x-1)^2 = 2$ gives $x = 1 \pm \sqrt{2}$. The x-intercepts are $(1+\sqrt{2}, 0)$ and $(1-\sqrt{2}, 0)$. To find the y-intercept set $x = 0$. Solving $1 = -2(y-1)$ gives $y = \frac{1}{2}$. The y-intercept is $(0, 1/2)$.

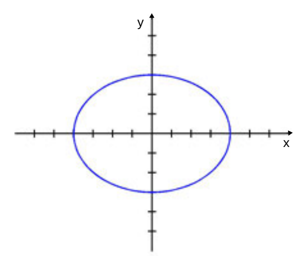
25. center: $(0, 0)$
foci: $(0, \pm\sqrt{15})$
vertices: $(0, \pm 4)$
endpoints of the minor axis: $(\pm 1, 0)$
eccentricity: $\dfrac{\sqrt{15}}{4}$



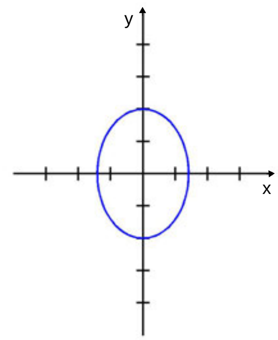
26. center: $(0, 0)$
foci: $(\pm 4, 0)$
vertices: $(\pm 5, 0)$
endpoints of the minor axis: $(0, \pm 3)$
eccentricity: $\dfrac{4}{5}$



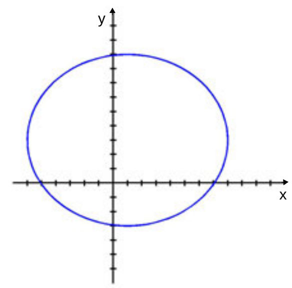
27. $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$ center: $(0, 0)$
foci: $(\pm\sqrt{7}, 0)$
vertices: $(\pm 4, 0)$
endpoints of the minor axis: $(0, \pm 3)$
eccentricity: $\dfrac{\sqrt{7}}{4}$



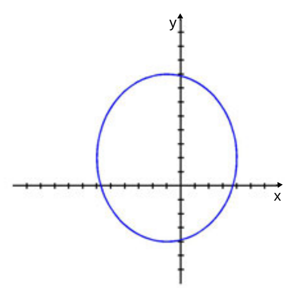
28. $\dfrac{x^2}{2} + \dfrac{y^2}{4} = 1$ center: $(0,0)$
foci: $(0, \pm\sqrt{2})$
vertices: $(0, \pm 2)$
endpoints of the minor axis: $(\pm\sqrt{2}, 0)$
eccentricity: $\dfrac{\sqrt{2}}{2}$



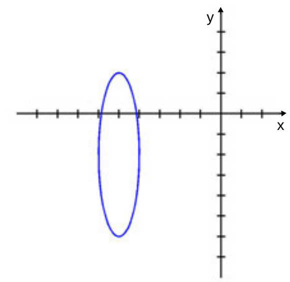
29. center: $(1,3)$
foci: $(1 \pm \sqrt{13}, 3)$
vertices: $(-6,3)$, $(8,3)$
endpoints of the minor axis: $(1,-3)$, $(1,9)$
eccentricity: $\dfrac{\sqrt{13}}{7}$



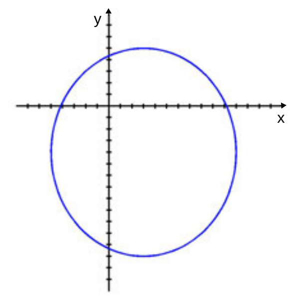
30. center: $(-1,2)$
foci: $(-1, 2 \pm \sqrt{11})$
vertices: $(-1,-4)$, $(-1,8)$
endpoints of the minor axis: $(-6,2)$, $(4,2)$
eccentricity: $\dfrac{\sqrt{11}}{6}$



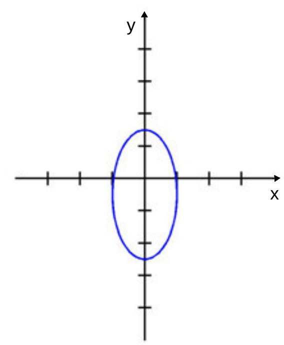
31. center: $(-5,-2)$
foci: $(-5,-2\pm\sqrt{15})$
vertices: $(-5,-6)$, $(-5,2)$
endpoints of the minor axis: $(-6,-2)$, $(-4,-2)$
eccentricity: $\dfrac{\sqrt{15}}{4}$



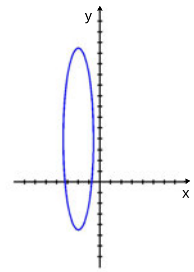
32. center: $(3,-4)$
foci: $(3,-4\pm\sqrt{17})$
vertices: $(3,-13)$, $(3,5)$
endpoints of the minor axis: $(-5,-4)$, $(11,-4)$
eccentricity: $\dfrac{\sqrt{17}}{9}$



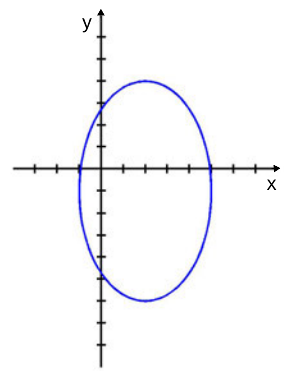
33. $x^2+\dfrac{\left(y+\frac{1}{2}\right)^2}{4}=1$
center: $(0,-1/2)$
foci: $(0,-1/2\pm\sqrt{3})$
vertices: $(0,-5/2)$, $(0,3/2)$
endpoints of the minor axis: $(-1,-1/2)$, $(1,-1/2)$
eccentricity: $\dfrac{\sqrt{3}}{2}$



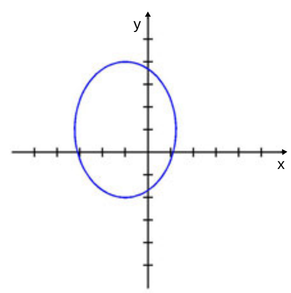
34. $\dfrac{(x+2)^2}{2} + \dfrac{(y-4)^2}{72} = 1$
center: $(-2, 4)$
foci: $(-2, 4 \pm \sqrt{70})$
vertices: $(-2, 4 \pm 6\sqrt{2})$
endpoints of the minor axis: $(-2 \pm \sqrt{2}, 4)$
eccentricity: $\dfrac{\sqrt{35}}{6}$



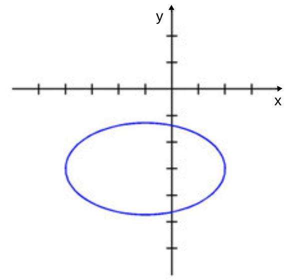
35. $\dfrac{(x-7)^2}{9} + \dfrac{(y+1)^2}{25} = 1$
center: $(2, -1)$
foci: $(2, -5)$, $(2, 3)$
vertices: $(2, -6)$, $(2, 4)$
endpoints of the minor axis: $(-1, -1)$, $(5, -1)$
eccentricity: $\dfrac{4}{5}$



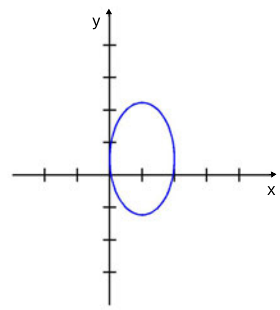
36. $\dfrac{(x+1)^2}{5} + \dfrac{(y-1)^2}{9} = 1$
center: $(-1, 1)$
foci: $(-1, -1)$, $(-1, 3)$
vertices: $(-1, -2)$, $(-1, 4)$
endpoints of the minor axis: $(-1 \pm \sqrt{5}, 1)$
eccentricity: $\dfrac{2}{3}$



37. $\dfrac{x^2}{9} + \dfrac{(y+3)^2}{3} = 1$
center: $(0, -3)$
foci: $(\pm\sqrt{6}, -3)$
vertices: $(\pm 3, -3)$
endpoints of the minor axis: $(0, -3 \pm \sqrt{3})$
eccentricity: $\dfrac{\sqrt{6}}{3}$



38. $(x-1)^2 + \dfrac{(y-1/2)^2}{3} = 1$
center: $(1, 1/2)$
foci: $(1, 1/2 \pm \sqrt{2})$
vertices: $(1, 1/2 \pm \sqrt{3})$
endpoints of the minor axis: $(0, 1/2)$, $(2, 1/2)$
eccentricity: $\dfrac{\sqrt{2}}{3}$



39. The center is $(0,0)$ with the x-axis as the major axis. $a = 5$ and $c = 3$, so $b = 4$. Thus the equation is $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$.

40. The center is $(0,0)$ with the x-axis as the major axis. $a = 9$ and $c = 2$, so $b = \sqrt{77}$. Thus the equation is $\dfrac{x^2}{81} + \dfrac{y^2}{77} = 1$.

41. The center is $(1,-3)$ with the x-axis as the major axis. $a = 4$ and $b = 2$. Thus the equation is $\dfrac{(x-1)^2}{16} + \dfrac{(y+3)^2}{4} = 1$.

42. The center is $(1,-2)$ with the y-axis as the major axis. $a = 4$ and $b = 3$. Thus the equation is $\dfrac{(x-1)^2}{9} + \dfrac{(y+2)^2}{16} = 1$.

43. The center is $(0,0)$ with the x-axis as the major axis. $c = \sqrt{2}$ and $b = 3$, so $a = \sqrt{11}$. Thus the equation is $\dfrac{x^2}{11} + \dfrac{y^2}{9} = 1$.

44. The center is $(0,0)$ with the y-axis as the major axis. $c=\sqrt{5}$ and $a=8$, so $b=\sqrt{59}$. Thus the equation is $\dfrac{x^2}{59}+\dfrac{y^2}{64}=1$.

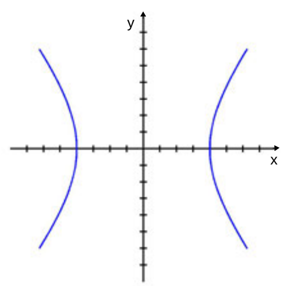
45. The center is $(0,0)$ with the y-axis as the major axis. $c=3$ thus $9=a^2-b^2$ and $a=\sqrt{9+b^2}$. Thus the equation is of the form $\dfrac{x^2}{b^2}+\dfrac{y^2}{9+b^2}=1$. The ellipse passes through the point $(-1,2\sqrt{2})$, thus $\dfrac{(-1)^2}{b^2}+\dfrac{(2\sqrt{2})^2}{9+b^2}=1$. Solving this for b, we obtain $b=\sqrt{3}$. Thus $a^2=12$ and the equation is $\dfrac{x^2}{3}+\dfrac{y^2}{12}=1$.

46. The center is $(0,0)$ with the x-axis as the major axis and $a=5$. The equation is of the form $\dfrac{x^2}{25}+\dfrac{y^2}{b^2}=1$. The ellipse passes through the point $(\sqrt{5},4)$ so $\dfrac{5}{25}+\dfrac{19}{b^2}=1$. Solving for b^2, we obtain $b^2=20$. Thus the equation of the ellipse is $\dfrac{x^2}{25}+\dfrac{y^2}{20}=1$.

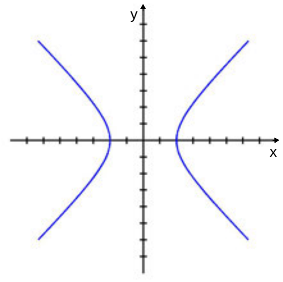
47. The y-axis as the major axis with $c=3$ and $a=4$. Thus $b=\sqrt{7}$ and the equation of the ellipse is $\dfrac{(x-1)^2}{7}+\dfrac{(y-3)^2}{16}=1$

48. The center is $(15/2,4)$ with the x-axis as the major axis. $a=1/2$ and $c=7/2$, thus $b=\sqrt{18}$. Thus the equation is $\dfrac{(x-15/2)^2}{(11/2)^2}+\dfrac{(y-4)^2}{18}=1$.

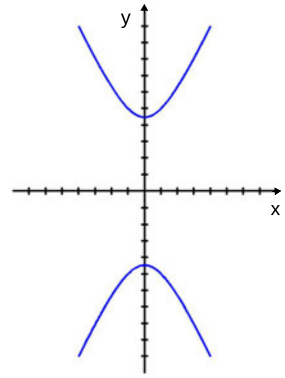
49. center: $(0,0)$
foci: $(\pm\sqrt{41},0)$
vertices: $(\pm 4,0)$
asymptotes: $y=\pm\frac{5}{4}x$
eccentricity: $\dfrac{\sqrt{41}}{4}$



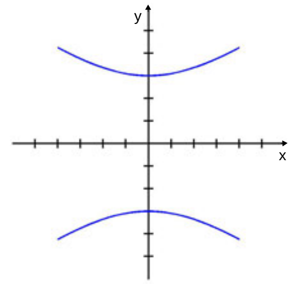
50. center: $(0,0)$
foci: $(\pm\sqrt{8},0)$
vertices: $(\pm 2,0)$
asymptotes: $y=\pm x$
eccentricity: $\sqrt{2}$



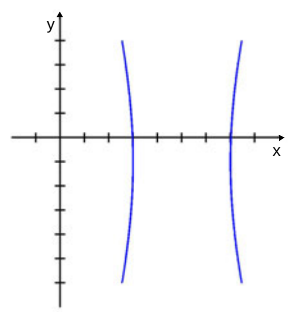
51. $\dfrac{y^2}{20} - \dfrac{x^2}{4} = 1$ center: $(0,0)$
foci: $(0, \pm 2\sqrt{6})$
vertices: $(0, \pm 2\sqrt{5})$
asymptotes: $y = \pm\sqrt{5}x$
eccentricity: $\sqrt{\dfrac{6}{5}}$



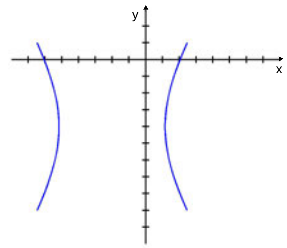
52. $\dfrac{y^2}{9} - \dfrac{x^2}{16} = 1$ center: $(0,0)$
foci: $(0, \pm 5)$
vertices: $(0, \pm 3)$
asymptotes: $y = \pm\frac{3}{4}x$
eccentricity: $\dfrac{5}{3}$



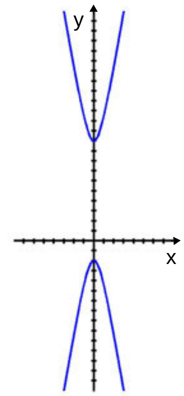
53. center: $(5, -1)$
foci: $(5 \pm \sqrt{53}, -1)$
vertices: $(3, -1)$, $(7, -1)$
asymptotes: $y = -1 \pm \frac{7}{2}(x-5)$
eccentricity: $\dfrac{\sqrt{53}}{2}$



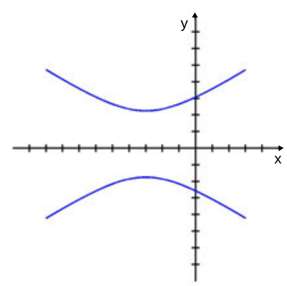
54. center: $(-2, -4)$
foci: $(-2 \pm \sqrt{35}, -4)$
vertices: $(-2 \pm \sqrt{10}, -4)$
asymptotes: $y = -4 \pm \dfrac{5x + 10}{\sqrt{10}}$
eccentricity: $\sqrt{\dfrac{7}{2}}$



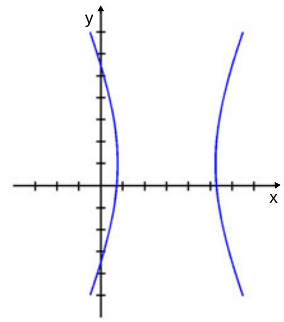
55. center: $(0, 4)$
foci: $(0, 4 \pm \sqrt{37})$
vertices: $(0, -2)$, $(0, 10)$
asymptotes: $y = 4 \pm 6x$
eccentricity: $\dfrac{\sqrt{37}}{6}$



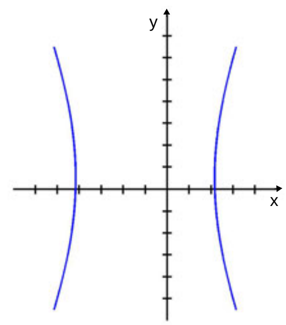
56. center: $(-3, 1/4)$
foci: $(-3, 1/4 \pm \sqrt{13})$
vertices: $(-3, 9/4)$, $(-3, -7/4)$
asymptotes: $y = \dfrac{1}{2} \pm \frac{2}{3}(x + 3)$
eccentricity: $\dfrac{\sqrt{13}}{2}$



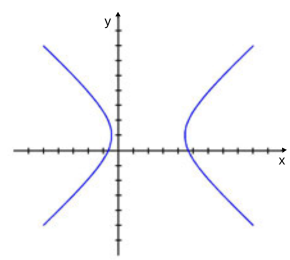
57. $\dfrac{(x-3)^2}{5} - \dfrac{(y-1)^2}{25} = 1$ center: $(3,1)$
foci: $(3 \pm \sqrt{30}, 1)$
vertices: $(3 \pm \sqrt{5}, 1)$
asymptotes: $y = 1 \pm \sqrt{5}(x-3)$
eccentricity: $\sqrt{6}$



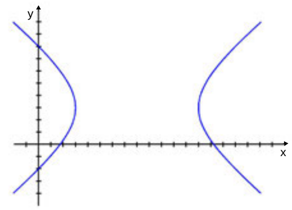
58. $\dfrac{(x+1)^2}{10} - \dfrac{(y-1/2)^2}{50} = 1$ center: $(-1, 1/2)$
foci: $(-1 \pm \sqrt{60}, 1/2)$
vertices: $(-1 \pm \sqrt{10}, 1/2)$
asymptotes: $y = 1/2 \pm \sqrt{5}(x+1)$
eccentricity: $\sqrt{6}$



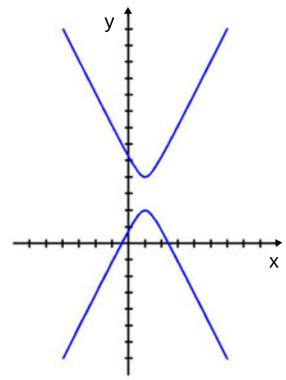
59. $\dfrac{(x-2)^2}{6} - \dfrac{(y-1)^2}{5} = 1$ center: $(2,1)$
foci: $(2 \pm \sqrt{11}, 1)$
vertices: $(2 \pm \sqrt{6}, 1)$
asymptotes: $y = 1 \pm \sqrt{\dfrac{5}{6}}(x-2)$
eccentricity: $\sqrt{\dfrac{11}{6}}$



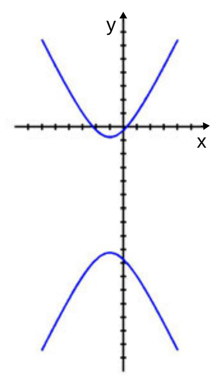
60. $\dfrac{(x-8)^2}{25} - \dfrac{(y-3)^2}{16} = 1$ center: $(8,3)$
foci: $(8 \pm \sqrt{41}, 3)$
vertices: $(13,3)$, $(3,3)$
asymptotes: $y = 3 \pm 4/5(x-8)$
eccentricity: $\dfrac{\sqrt{41}}{5}$



61. $(y-3)^2 - \dfrac{(x-1)^2}{1/4} = 1$ center: $(1,3)$
foci: $(1, 3 \pm \sqrt{5}/2)$
vertices: $(1,2)$, $(1,4)$
asymptotes: $y = 3 \pm 2(x-1)$
eccentricity: $\dfrac{\sqrt{5}}{2}$



62. $\dfrac{(y+5)^2}{18} - \dfrac{(x+1)^2}{4} = 1$ center: $(-1,-5)$
foci: $(-1, -5 \pm \sqrt{22})$
vertices: $(-1, -5 \pm \sqrt{18})$
asymptotes: $y = -5 \pm \frac{\sqrt{18}}{2}(x+1)$
eccentricity: $\dfrac{\sqrt{11}}{3}$



63. The center is $(0,0)$ with the y-axis as the transverse axis. $c = 4$ and $a = 2$, thus $b = \sqrt{12}$. The equation is $\dfrac{y^2}{4} - \dfrac{x^2}{12} = 1$

64. The center is $(0,0)$ with the y-axis as the transverse axis. $c = 3$ and $a = 3/2$, thus $b = \dfrac{3\sqrt{3}}{2}$.

The equation is $\dfrac{y^2}{9/4} - \dfrac{x^2}{27/4} = 1$

65. The center is $(1, -3)$ with the y-axis as the transverse axis. $c = 3$ and $a = 2$, thus $b = \sqrt{5}$. The equation is $\dfrac{(y+3)^2}{4} - \dfrac{(x-1)^2}{5} = 1$

66. The center is $(2, 2)$ with the y-axis as the transverse axis. $c = 5$ and $a = 32$, thus $b = 4$. The equation is $\dfrac{(y-2)^2}{9} - \dfrac{(x-2)^2}{16} = 1$.

67. The center is $(-1, 3)$ with the y-axis as the transverse axis. $a = 1$ and the equation is of the form $\dfrac{(y-3)^2}{1} - \dfrac{(x+1)^2}{b^2} = 1$. The hyperbola passes through the point $(-5, 3+\sqrt{5})$ thus $(3+\sqrt{5}-3)^2 - \dfrac{(-5+1)^2}{b^2} = 1$. Thus $b^2 = 4$ and the equation is $(y-3)^2 - \dfrac{(x+1)^2}{4} = 1$.

68. The center is $(3, -5)$ with the y-axis as the transverse axis. $a = 3$ and the equation is of the form $\dfrac{(y-3)^2}{9} - \dfrac{(x+5)^2}{b^2} = 1$. The hyperbola passes through the point $(1, -1)$ thus $\dfrac{(-4)^2}{9} - \dfrac{(6)^2}{b^2} = 1$. Thus $b^2 = \dfrac{324}{7}$ and the equation is $\dfrac{(y-3)^2}{9} - \dfrac{(x+5)^2}{324/7} = 1$.

69. The center is $(2, 4)$ with the y-axis as the transverse axis and $a = 1$. After solving the asymptote given in the problem for y, we obtain $y = \dfrac{x+6}{2} = \dfrac{x}{2} + 3$. The equation of the hyperbola is of the form $(y-4)^2 - \dfrac{(x-2)^2}{b^2} = 1$. The asymptote equations for this hyperbola are $y - 4 = \dfrac{x-2}{b}$ and $y - 4 = \dfrac{-x+2}{b}$ (these are also equivalent to $y = \dfrac{x}{b} + \left(4 - \dfrac{2}{b}\right)$ and $y = -\dfrac{x}{b} + \left(4 + \dfrac{2}{b}\right)$). Letting b equal 2 or -2 will yield one asymptote with the equation $y = \dfrac{x}{2} + 3$. In either case, the equation of the hyperbola is $(y-4)^2 - \dfrac{(x-2)^2}{4} = 1$.

70. The y-axis is the conjuate axis. The center is $(-5, 7)$ with $b = 3$ and $\dfrac{c}{a} = \sqrt{10}$. Thus $c = \sqrt{10}a$. Since $c^2 = b^2 + a^2$ then $10a^2 = 9 + a^2$ and $a^2 = 1$. Thus the equation is $(x+5)^2 - \dfrac{(y+7)^2}{9} = 1$.

71. We place the coordinate axes so that the origin is at the vertex of the parabola. The point $(2, 2)$ lies on the parabola. Thus the equation is $x^2 = 2y$ with $p = 1/2$. The focus of this parabola occurs at the point $(0, 1/2)$. Thus the light source is 6 inches from the vertex.

72. We place the coordinate axes so that the origin is at the vertex of the parabola. The point $(10, 4)$ lies on the parabola. Thus the equation is $x^2 = 25y$ with $p = 25/4$. The focus is located at $(0, 25/4)$. The eyepiece should be located 6.25 ft from the vertex.

73. We place the coordinate axes so that the origin is at the vertex of the parabola. The parabola is of the form $x^2 = 4py$ and contains the point $(20, 1)$. Thus the equation of the parabola is $x^2 = 400y$. The towers are located at $x = 175$ and $x = -175$. Hence the height of the towers

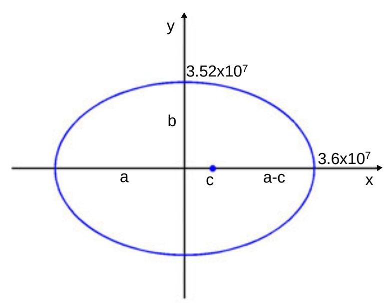
is found by solving $(175)^2 = 400y$. Solving this equation yields $y = 76.5625$. Therefore the towers are 76.5625 ft above the road.

74. We place the coordinate axes so that the origin is at the vertex of the parabola. The parabola is of the form $x^2 = 4py$ and contains the point $(125, 75)$. Thus the equation of the parabola is $x^2 = \frac{625}{12}$. We need to find the y-value of the point on the parabola when we are 50 ft from the tower or when $x = 75ft$. Hence this y-value is found by solving the equation $75^2 = \frac{625}{3}y$ which yield the solution $y = 27$ ft. The height of the cable above the roadway at a point 50 ft from one of the towers is 27 ft.

75. We place the coordinate axes so that the origin is at end of the pipe with the parabola in Quadrants 3 and 4. The equation is of the form $x^2 = 4py$ and the point $(4, -2)$ lies on the parabola. Therefore the equation is $x^2 = -8y$. The water hits the ground at $y = -20$. The point on the parabola with y-value -20 is found by solving $x^2 = -8(-20)$. This point is $x = 12.65$. Thus the water hits the ground 12.65 m from the point on the ground directly beneath the end of the pipe.

76. We place the coordinate axes with the x-axis along the ground and the y-axis to be through the dart thrower. Thus the dart was released at the point $(0, 5)$ and hits the ground at the point $(10\sqrt{10}, 0)$. The parabola is of the form $x^2 = 4p(y-5)$ and contains the point $(10\sqrt{10}, 0)$. Therefore the equation of the parabola is $x^2 = -200y$. To find the height of the dart 10 ft from the thrower, we need to find the y-value of the point on the parabola corresponding to the x-value of 10. Hence, we need to solve the equation $10^2 = -200y$ which yields $y = -1/2ft$. So 10 feet from the thrower the dart will be 0.5 ft below the thrower or it will be 4.5 ft from the ground.

77. Taking the center of the ellipse to be at the origin, we have $a = 3.6 \times 10^7$ and $b = 3.52 \times 10^7$. Since $c^2 = a^2 - b^2$, $c^2 = 12.96 \times 10^{14} - 12.3904 \times 10^{14} = 0.5696 \times 10^{14}$ and $c \approx 0.75 \times 10^7$. The perihelion or least distance is $a - c \approx 2.85 \times 10^7$ miles or 28.5 million miles. And the aphelion or greatest distance is $a + c \approx 4.35 \times 10^7$ miles or 43.5 million miles.



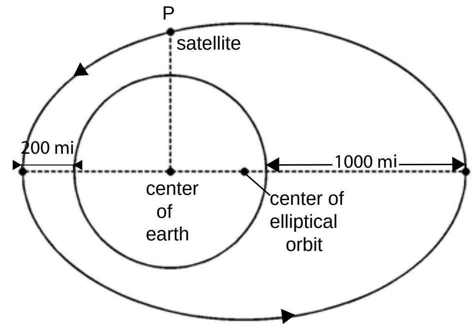
78. Using $a = 3.6 \times 10^7$ and $c = 0.75 \times 10^7$, we compute the eccentricity $e = \dfrac{0.75 \times 10^7}{3.6 \times 10^7} \approx 0.20833$.

79. From $a = 1.67 \times 10^9$ and 4.25×10^8 we obtain

$$\begin{aligned} c^2 = a^2 - b^2 &= 2.7889 \times 10^{18} - 18.0625 \times 10^6 = 2.78.89 \times 10^{16} - 18.0625 \times 10^{16} \\ &= 260.8275 \times 10^{16} = 2.608275 \times 10^{18}. \end{aligned}$$

Then $c \approx 1.615 \times 10^9$ and the eccentricity is $\dfrac{c}{a} \approx \dfrac{1.615 \times 10^9}{1.67 \times 10^9} \approx 0.967$.

80. We place the coordinate axes so that the origin is at the center of the ellipse. The length of the major axis is $200+2(4000)+1000 = 9200$ so that $a = 4600$. Therefore $c = 4600 - (200 + 4000) = 400$. Then $b^2 = a^2 - c^2 = 4600^2 - 400^2 = (1000\sqrt{21})^2$, and the equation is $\dfrac{x^2}{4600^2} + \dfrac{y^2}{(1000\sqrt{12})^2} = 1$



81. We place the coordinate axes so that the origin is at the point midway across the base. Thus $a = 5$ and $b = 15$ so the equation of the doorway is $y = \sqrt{15^2\left(1 - \dfrac{x^2}{25}\right)}$. The height of the doorway at a point on the base 4 ft from the center is $y = \sqrt{15^2\left(1 - \dfrac{3^2}{25}\right)} = 12$ ft.

82. The base of the room is an ellipse with $a = 20$ and $b = 16$. We place the coordinate axes so that the origin is at the point in the center of the room and let the major axes be in the x-direction. $c^2 = a^2 - b^2$, so $c^2 = 20^2 - 16^2$, which gives $c = 12$. Thus the foci occur at the points $(-12, 0)$ and $(12, 0)$. Therefore the listening and whispering posts occur along the center line on the longer part of the base 12 ft in each direction from the center.

83.

	ellipse	shifted ellipse
center	$(0,1)$	$(4,1)$
vertices	$(-2,1), (2,1), (0,-2), (0,4)$	$(2.1), (6.1), (4.-2), (4.4)$
foci	$(0, 1-\sqrt{5}), (0, 1+\sqrt{5})$	$(4, 1-\sqrt{5}), (4, 1+\sqrt{5})$

84.

	ellipse	shifted ellipse
center	$(1,4)$	$(-4,7)$
vertices	$(-2,4), (4,4), (1,3), (1,5)$	$(-7,7), (-1,7), (-4,6), (-4.8)$
foci	$(1-2\sqrt{2}, 4), (1+2\sqrt{2}, 4)$	$(-4-2\sqrt{2}, 7), (-4+2\sqrt{2}, 7)$

85. (a) $\dfrac{y^2}{144} - \dfrac{x^2}{25} = 1$

(b) Conjugate hyperbolas have the same asymptotes and do not intersect.

86. (a) The equation of the hyperbola can be written as $\dfrac{(y+5/2)^2}{5} - \dfrac{(x-3/2)^2}{5} = 1$, The equations of the asymptotes are $y = 7/2 - x$ and $x = 1 + x$. These lines are perpendicular, thus the hyperbola is a rectangular hyperbola.

(b) The hyperbola in Problem 50 is a rectangular hyperbola.

87. Since $a = 4$ and $b = \sqrt{20}$, we have $c^2 = a^2 + b^2 = 16 + 20 = 36$ and hence $c = 6$. Thus the foci occur at $F_1 = (-6, 0)$ and $F_2 = (6, 0)$. The line joining $(-6, -5)$ and F_2 is given by $y = \dfrac{5}{12}x - \dfrac{5}{2}$. The ray of light travels southwest along this line.

88. (a) The distance from $(0, b)$ to $(a, 0)$ is $\sqrt{a^2 + b^2}$. Thus $R = \sqrt{a^2 + b^2} = r$.

(b) From $A = a + r$ and $B = R - b = \sqrt{a^2 + b^2} + r - b = \sqrt{a^2 + b^2} + (A - a) - b = A - (a + b) + \sqrt{a^2 + b^2}$ we see that

$$A - B = a + b - \sqrt{a^2 + b^2} = \sqrt{(a + b)^2} - \sqrt{a^2 + b^2} = \sqrt{a^2 + 2ab + b^2} - \sqrt{a^2 + b^2} > 0.$$

Thus, $A > B$.

10.2 Parametric Equations

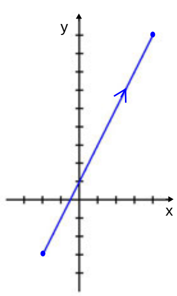
1.

t	-3	-2	-1	0	1	2	3
x	-5	-3	-1	1	3	5	7
y	6	2	0	0	2	6	12

2.

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{5\pi}{6}$	$\frac{7\pi}{4}$
x	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$
y	0	1/4	1/2	3/4	1	1/4	1/2

3.



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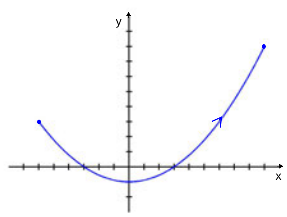

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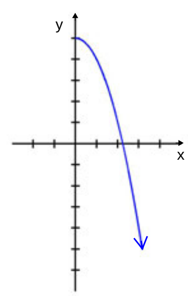

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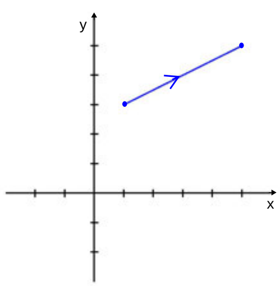

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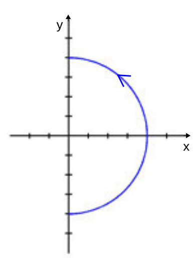

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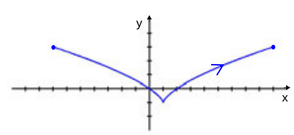

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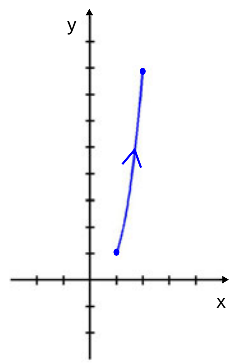

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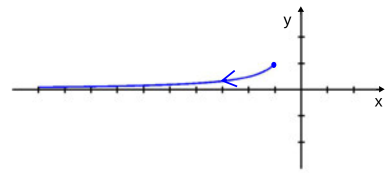


11. $y = (t^2)^2 + 3t^2 - 1 = x^2 + 3x - 1; \quad y = x^2 + 3x - 1, \quad x \geq 0$

12. $-\frac{1}{2}y = t^3 + t; \quad x = -\frac{1}{2}y + 4; \quad 2x + y = 4$

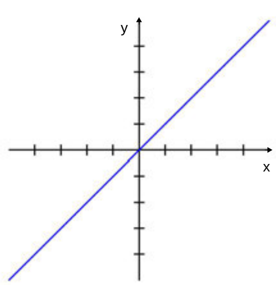
13. $x = \cos 2t = \cos^2 t - \sin^2 t = 1 - 2\sin^2 t = 1 - 2y^2; \quad y = 1 - 2y^2, \quad -1 \leq y \leq 1$

14. $\ln x = t; \quad y = \ln(\ln x), \quad x > 1$. Alternatively, $e^y = t; \quad x = e^{e^y}, \ x > 1$

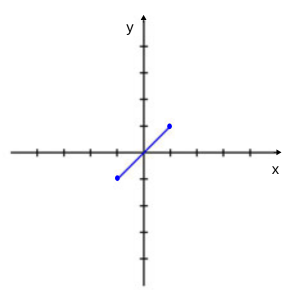
15. $t = x^{1/3}; \ \ y = 3\ln x^{1/3}; \ \ y = \ln x, \ \ x > 0$

16. $x^2 \tan^2 x, \ \ y^2 = \sec^2 t; \ \ x^2 + 1 = \tan^2 t + 1 = \sec^2 t = y^2; \ \ y^2 - x^2 = 1. \ \ y \geq 1$

17.

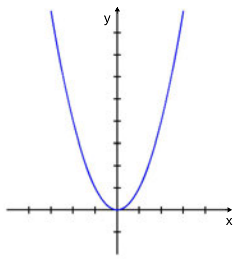


$y = x$

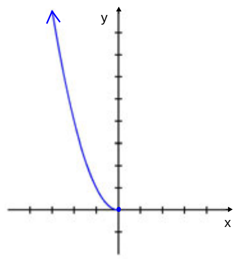


$x = \sin t$
$y = \sin t$

18.

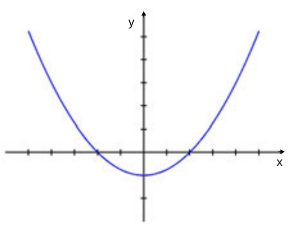


$y = x^2$

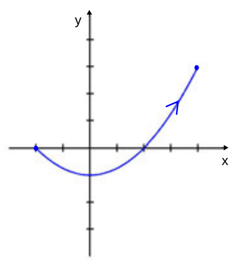


$x = -\sqrt{t}$
$y = t$
$t \geq 0$

19.

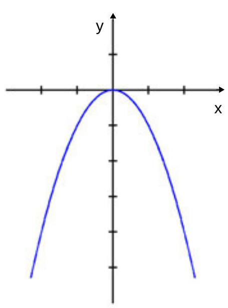


$y = \dfrac{x^2}{4} - 1$

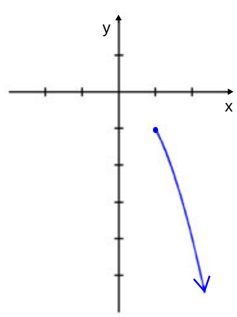


$x = 2t$
$y = t^2 - 1$
$-1 \leq t \leq 2$

20.

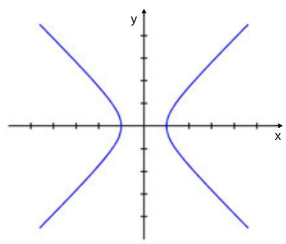


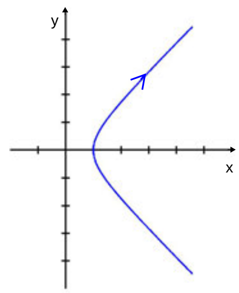
$y = -x^2$



$x = a^t$
$y = -e^{2t}$
$t \geq 0$

21.

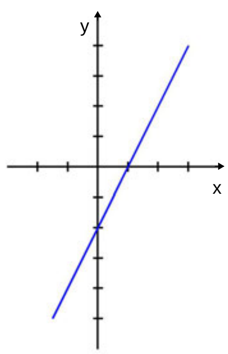


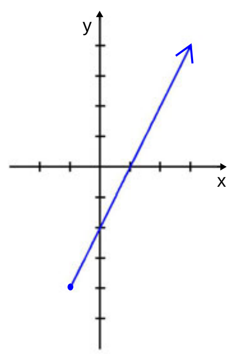


$x^2 - y^2 = 1$

$x = \cosh t$
$y = \sinh t$

22.

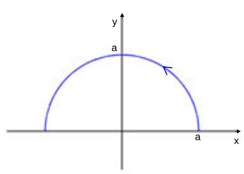


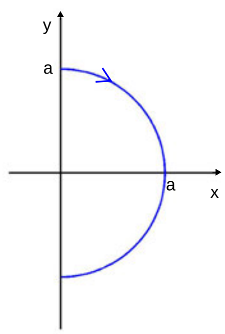


$y = 2x - 2$

$x = x = t^2 - 1$
$y = 2t^2 - 4$

23.

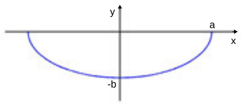


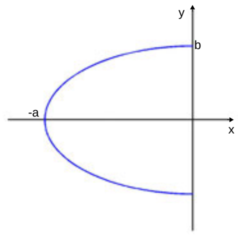


$x = a\cos t$
$y = a\sin t$
$a > 0$
$0 \leq t \leq \pi$

$x = a\sin t$
$y = a\cos t$
$a > 0$
$0 \leq t \leq \pi$

24.

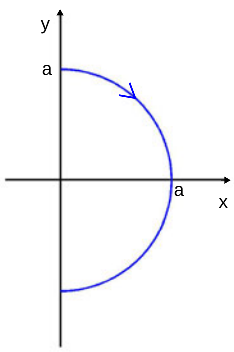


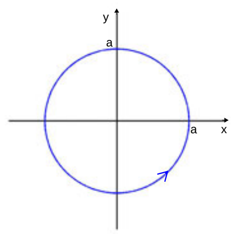


$x = a \cos t$
$y = b \sin t$
$a > b > 0$
$\pi \leq t \leq 2\pi$

$x = a \sin t$
$y = b \cos t$
$a > b > 0$
$\pi \leq t \leq 2\pi$

25.

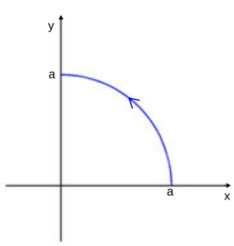


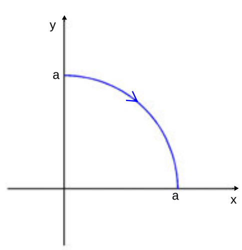


$x = a \cos t$
$y = a \sin t$
$a > 0$
$-\pi/2 \leq t \leq \pi/2$

$x = a \cos 2t$
$y = a \sin 2t$
$a > 0$
$-\pi/2 \leq t \leq \pi/2$

26.

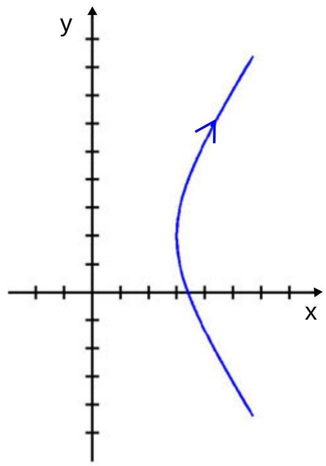




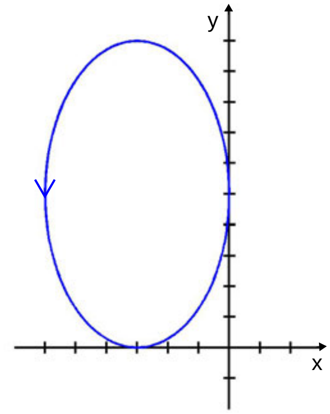
$x = a \cos t/2$
$y = a \sin t/2$
$a > 0$
$0 \leq t \leq \pi/2$

$x = a \cos(-t/2)$
$y = a \sin(-t/2)$
$a > 0$
$-\pi \leq t \leq 0$

27.



28.



29. This is the same as $x = 1/y$ or $xy = 1$. The graphs are the same.

30. Since $x = t^{1/2} \geq 0$ for all t, x can never be -1. But $(-1, -1)$ is on the $xy = 1$, so the graphs are not the same.

31. Since $|\cos t| \geq 1$, x can never be 2. But $(2, 1/2)$ is on $xy = 1$, so the graphs are not the same.

32. Since $t^2 + 1 \geq 1$ for all t, x can never be -1. But $(-1, -1)$ is on $xy = 1$, so the graphs are not the same.

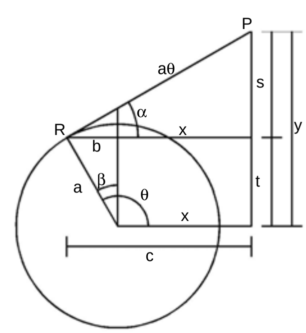
33. Since $e^{-2t} > 0$ for all t, x can never be -1. But $(-1, -1)$ is on $xy = 1$, so the graphs are not the same.

34. This is the same as $x = 1/y$ or $xy = 1$. The graphs are the same.

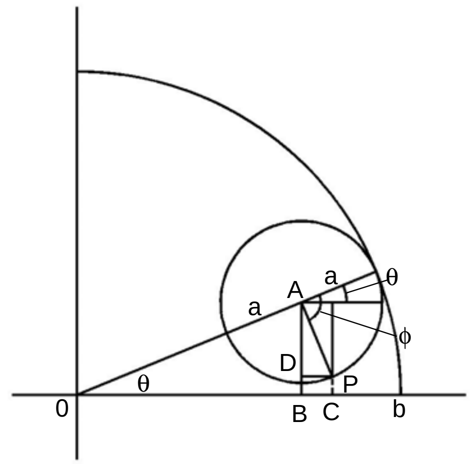
35. From $\sin\phi = \frac{y}{L}$ we have $t = L\sin\phi$. Since (x, y) is on the circle $x^2 + y^2 = r^2$, $x = \pm\sqrt{r^2 - y^2} = \pm\sqrt{r^2 - l^2\sin^2\phi}$.

36. From the figure in the text, we see that $x = r\cos 3\theta + R\cos\theta$ and $y = r\sin 3\theta + r\sin\theta$. (The actual curve generated by these equations will have the general appearance of the curve in Figure 10.2.12 in the text only when $R > 3r$.)

37. From the figure we see that $\beta = \theta - \pi/2$ and $\alpha = \beta = \theta - \pi/2$. The length of the line segment from R to P is equal to the arc of the circle subtended by θ; that is $a\theta$. Now, $x = a\theta\sin\alpha = a\theta\sin(\theta - \pi/2) = -a\theta\cos\theta$, $t = a\cos\beta = a\cos(\theta - \pi/2) = a\sin\theta$, $b = a\sin\beta = a\sin(\theta - \pi/2) = -a\cos\theta$, and $c = a\theta\cos\alpha = a\theta\cos(\theta - \pi/2) = a\theta\sin\theta$. Thus



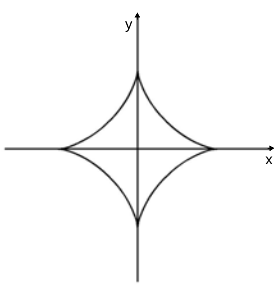
$$x = c - b = a\theta\sin\theta - (-a\cos\theta) = a(\cos\theta + \theta\sin\theta)$$
$$y = s + t = -a\theta\cos\theta = a(\sin\theta - \theta\cos\theta).$$



38. The hypotenuse of the triangle OAB is $b-a$ and $\overline{OB} = (b-a)\cos\theta$. The actue angle at A in the right triangle with hypotenuse AP is $\phi - \theta$. Thus, $\overline{BC} = a\cos(\phi-\theta)$ and $x = \overline{OB} + \overline{BC} = (b-a)\cos\theta + a\cos(\phi-\theta)$. Similarly, $y = \overline{AB} - \overline{AD} = (b-a)\sin\theta - a\sin(\phi-\theta)$. Now, the arc on the smaller circle subtended by the angle ϕ has length $a\phi$ and the arc on the larger circle subended by the angle θ has length $b\theta$. From the definition of the hypocycloid, $a\phi = b\theta$. Then $\phi = \frac{b\theta}{a}$ and $\phi - \theta = \left(\frac{b\theta}{a}\right) - \theta$. Thus, the parametric equations of the hypocycloid are $x = (b-a)\cos\theta + a\cos[(b-a)/a]\theta$, $y = (b-a)\sin\theta - a\sin[(b-a)/a]\theta$.

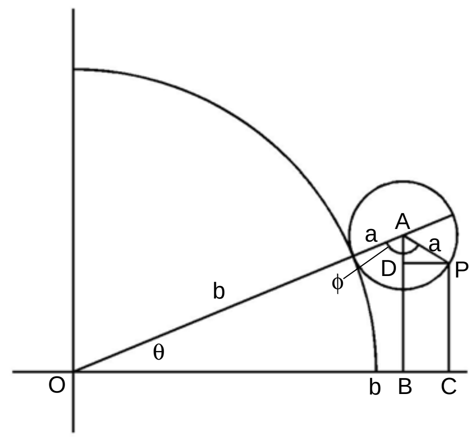
39. (a) When $b = 4a$, the equations become $x = 3a\cos\theta + a\cos 3\theta$, $y = 3a\sin\theta - a\sin 3\theta$. Using the identities $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ and $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$, the parametric equations of the hypocycloid of four cusps become $x = 4a\cos^3\theta = b\cos^3\theta$, $y = 4a\sin^3\theta = b\sin^3\theta$.

(b)



(c) Writing $x^{2/3} = b^{2/3}\cos^2\theta$, $y^{2/3} = b^{2/3}\sin^2\theta$ we obtain $x^{2/3} + y^{2/3} = b^{2/3}$.

40. The hypotenuse of triangle OAB is $a+b$ and $\overline{OB}=(a+b)\cos\theta$. The acute angle at A in triangle ADP is $\phi-\left(\frac{1}{2}-\theta\right)=(\phi+\theta)-\frac{\pi}{2}$. Thus, $BC=DP=a\sin(\phi+\theta-\pi/2)=-a\cos(\phi+\theta)$ and $x=\overline{OB}+\overline{BC}=(a+b)\cos\theta-a\cos(\phi+\theta)$. Similarly,

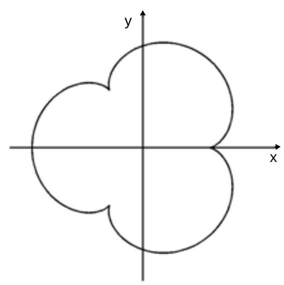


$$y=\overline{AB}-\overline{AD}=(a+b)\sin\theta-a\cos\left(\phi+\theta-\frac{\pi}{2}\right)$$
$$=(a+b)\sin\theta-a\sin(\phi+\theta).$$

Now, the arc on the smaller circle subtended by the angle ϕ has length $a\phi$ and the arc on the large circle subtended by the angle θ has length $b\theta$. From the definition of the epicycloid, $a\phi=b\theta$. Then $\phi=\frac{b\theta}{a}$ and $\phi+\theta=\frac{b\theta}{a}+\theta=\left[\frac{a+b}{a}\right]\theta$. Thus, the parametric equations of the epicycloid are $x=(a+b)\cos\theta-a\cos\left[\frac{a+b}{a}\right]\theta$, $y=(a+b)\sin\theta-a\sin\left[\frac{a+b}{a}\right]\theta$.

41. (a) When $b=3a$, the equations become $x=4a\cos\theta-a\cos 4\theta$, $y=4a\sin\theta-a\sin 4\theta$.

(b)



42. (a) The Q be the point $(0,2a)$ at the top of the circle and let (x,y) be the coordinates of point P. Then the measure of angle $\angle OBQ$ is equal to θ. The gives

$$\tan\theta=\frac{2a}{x}\quad\text{or}\quad x=\frac{2a}{\tan\theta}.$$

Also, the measure of angle $\angle AQO$ is equal to θ. Letting r represent the length of the segment $\overline{AO}$, we have

$$\sin\theta=\frac{r}{2a}\quad\text{or}\quad r=2a\sin\theta.$$

Since $y=r\sin\theta$, we have $y=(2a\sin\theta)\sin\theta=2a\sin^2\theta$.

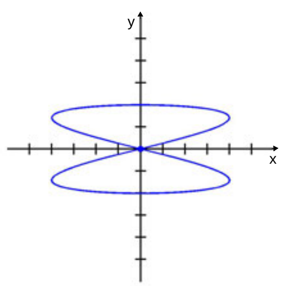
(b) Rewrite x as $x = \dfrac{2a\cos\theta}{\sin\theta}$. Then

$$x^2y = \frac{4a^2\cos^2\theta}{\sin^2\theta}(2a\sin^2\theta) = 8a^3\cos^2\theta$$

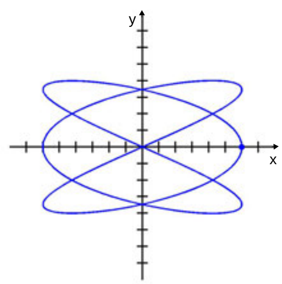
and $4a^2y = 8a^3\sin^2\theta$. This gives

$$\begin{aligned} x^2y + 4a^2y &= 8a^3\cos^2\theta + 8a^3\sin^2\theta \\ y(x^2+4a^2) &= 8a^3 \\ y &= \frac{8a^3}{x^2+4a^2} \end{aligned}$$

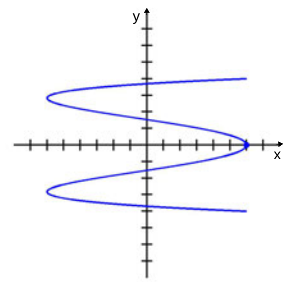
43.



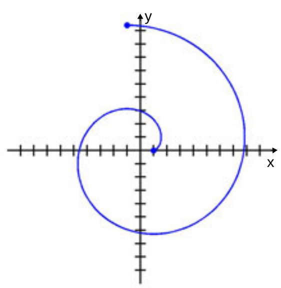
44.



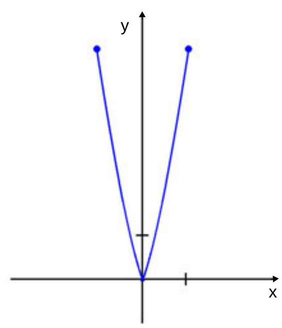
45.



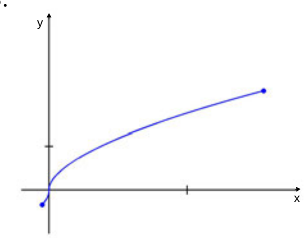
46.



47.



48.



49. Using the equation from x to solve for t, we have $t = \dfrac{x-x_1}{x_2-x_1}$. Plugging this into the equation for y yields

$$\begin{aligned} y &= y_1 + (y_2-y_1)\left(\frac{x-x_1}{x_2-x_1}\right) \\ &= y_1 + \left(\frac{y_2-y_1}{x_2-x_1}\right)(x-x_1) \end{aligned}$$

which is the equation of a line joining (x_1, y_1) and (x_2, y_2). When $0 \le t \le 1$, we get the line segment with endpoints (x_1, y_1) and (x_2, y_2).

50. (a) $x = -2 + (4 - (-2))t = -2 + 6t$
$y = 5 + (8 - 5) = 5 + 3t$
(b) $y = \frac{1}{2}x + 6$
(c) $x = -2 + 6t$, $y = 5 + 3t$; $0 \leq t \leq 1$

51. If the launch point is designated as the origin, then the equations describing the skier's motion from launch until landing are given by

$$x = 75t \quad \text{and} \quad y = -16t^2$$

where $t = 0$ at the moment of launch. At the moment of impact, we have

$$\tan 33^\circ = \frac{|y|}{|x|} = \frac{16}{15}t.$$

Thus, $t = \dfrac{75}{16}\tan 33^\circ \approx 3.044$, and therefore $x \approx 228.3$ ft, $y \approx -148.25$ ft.

10.3 Calculus and Parametric Equations

1. $\dfrac{dx}{dt} = 3t^2 - 2t; \quad \dfrac{dy}{dt} = 2t + 5; \quad \dfrac{dy}{dx} = \dfrac{2t+5}{3t^2-2t}; \quad \left.\dfrac{dy}{dx}\right|_{t=-1} = \dfrac{3}{5}$

2. $\dfrac{dx}{dt} = -\dfrac{4}{t^2}; \quad \dfrac{dy}{dt} = 6t^2 - 1; \quad \dfrac{dy}{dx} = \dfrac{6t^2-1}{-4/t^2} = -\dfrac{6t^4-t^2}{4}; \quad \left.\dfrac{dy}{dx}\right|_{t=2} = -\dfrac{92}{4} = -23$

3. $\dfrac{dx}{dt} = \dfrac{t}{\sqrt{t^2+1}}; \quad \dfrac{dy}{dt} = 4t^3; \quad \dfrac{dy}{dx} = \dfrac{4t^3}{t/\sqrt{t^2+1}} = 4t^2\sqrt{t^2+1}; \quad \left.\dfrac{dy}{dx}\right|_{t=\sqrt{3}} = 4(3)\sqrt{4} = 24$

4. $\dfrac{dx}{dt} = 2e^{2t}; \quad \dfrac{dy}{dt} = -4e^{-4t}; \quad \dfrac{dy}{dx} = -\dfrac{4e^{-4t}}{2e^{2t}} = -2e^{-6t}$
$\left.\dfrac{dy}{dx}\right|_{t=\ln 2} = -2e^{-6\ln 2} = -2e^{\ln 2^{-6}} = -2(2^{-6}) = -\dfrac{1}{32}$

5. $\dfrac{dx}{d\theta} = 2\cos\theta(-\sin\theta); \quad \dfrac{dy}{d\theta}\cos\theta; \quad \dfrac{dy}{dx} = \dfrac{\cos\theta}{-2\sin\theta\cos\theta} = -\dfrac{1}{2\sin\theta}$
$\left.\dfrac{dy}{dx}\right|_{\theta=\pi/6} = -\dfrac{1}{2(1/2)} = -1$

6. $\dfrac{dx}{d\theta} = 2 - 2\cos\theta; \quad \dfrac{dy}{d\theta} = 2\sin\theta; \quad \dfrac{dy}{dx} = \dfrac{2\sin\theta}{2-2\cos\theta} = \dfrac{\sin\theta}{1-\cos\theta}$
$\left.\dfrac{dy}{dx}\right|_{\theta=\pi/4} = \dfrac{\sqrt{2}/2}{1-\sqrt{2}/2} = \dfrac{\sqrt{2}}{2-\sqrt{2}} = \sqrt{2}+1$

7. $\dfrac{dy}{dx} = \dfrac{12t}{3t^2+3} = \dfrac{4t}{t^2+1}$.
At $t = -1$ we observe $x = -4$, $y = 7$, and $m = dy/dx = -2$. The tangent line is $y = -2x - 1$.

8. $\dfrac{dy}{dx} = \dfrac{2t+1/t}{2} = t + \dfrac{1}{2t}$.
At $t = 1$ we observe $x = 6$, $y = 1$, and $m = dy/dx = 3/2$. The tangent line is $y = \dfrac{3}{2}x - 8$.

9. $\dfrac{dy}{dt} = \dfrac{2t}{(2t+1)}$. At $(2,4)$, $t = -2$ and $m = dy/dx = 4/3$. The tangent line is $y = \dfrac{4}{3}x + \dfrac{4}{3}$.

10. $\dfrac{dy}{dx} = \dfrac{(4t^3 - 2t)}{4t^3} = 1 - \dfrac{1}{2t^2}$. At $(0,6)$, $t = \sqrt{3}$ or $-\sqrt{3}$ and $m = 5/6$. The tangent line is $y = \dfrac{5}{6}x + 6$.

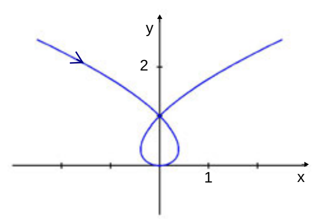
11. $\dfrac{dy}{dx} = \dfrac{-2\sin t}{8\cos 2t}$. When $y = 1$, $\cos t = 1/2$ and $t = \pi/3$ or $5\pi/3$. For $t = \pi/3$, $x = 4\sin(2\pi/3) = 4(\sqrt{3}/2) = 2\sqrt{3}$, and $m = \dfrac{dy}{dx} = \dfrac{-2\sin(\pi/3)}{8\cos(2\pi/3)} = \dfrac{\sqrt{3}}{4}$.

12. $\dfrac{dy}{dx} = \dfrac{3t^2}{2t} = \dfrac{3t}{2}$. The slope of the tangent line is -3. Solving $\dfrac{3t}{2} = -3$, we obtain $t = -2$. At $t = -2$ we observe $x = 4$ and $y = -7$. The point on the graph is $(4,7)$.

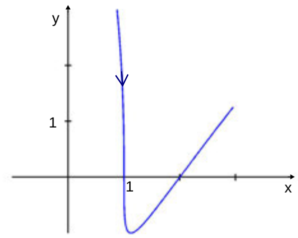
13. $\dfrac{dx}{dt} = 2$; $\dfrac{dy}{dt} = 2t - 4$; $\dfrac{dy}{dx} = \dfrac{2t-4}{2} = t - 2$
We want $t - 2 = 3$. Then $t = 5$ and the point of tangency is $(5,8)$. The equation of the tangent line is $y - 8 = 3(x - 5)$ or $y = 3x - 7$.

14. For $\theta = \pi/2$ we observe $x = -2/\pi$ and $y = 1 - 1 = 0$. For $\theta = -\pi/2$ we observe $x = -2/\pi$ and $y = -1 + 1 = 0$. Thus, the curve intersects itself when $\theta = \pi/2$ and $\theta = -\pi/2$.
$\dfrac{dx}{d\theta} = -\sin\theta$; $\dfrac{dy}{d\theta} = \cos\theta - \dfrac{2}{\pi}$; $\dfrac{dy}{dx} = \dfrac{\cos\theta - 2/\pi}{-\sin\theta} = \dfrac{2}{\pi}\csc\theta - \cot\theta$
When $\theta = \pi/2$, the slope of the tangent line is $\left.\dfrac{dy}{dx}\right|_{\theta=\pi/2} = \dfrac{2}{\pi}$ and its equation is $y - 0 = \dfrac{2}{\pi}\left(x + \dfrac{2}{\pi}\right)$ or $y = \dfrac{2}{\pi}x + \dfrac{4}{\pi^2}$. When $\theta = -\pi/2$, the slope of the tangent line is $\left.\dfrac{dy}{dx}\right|_{\theta=-\pi/2} = -\dfrac{2}{\pi}$ and its equation is $y - 0 = -\dfrac{2}{\pi}\left(x + \dfrac{2}{\pi}\right)$ or $y = -\dfrac{2}{\pi}x - \dfrac{4}{\pi^2}$.

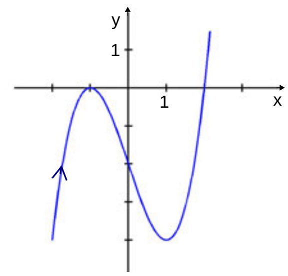
15. $\dfrac{dx}{dt} = 3t^2 - 1$; $\dfrac{dy}{dt} = 2t$; $\dfrac{dy}{dt} = \dfrac{2t}{3t^2 - 1}$. The tangent line is horizontal when $2t = 0$ or $t = 0$, and vertical when $3t^2 - 1 = 0$ or $t = \pm 1/\sqrt{3}$. Thus, there is a horizontal tangent at $(0,0)$ and vertical at $(-2/3\sqrt{3}, 1/3)$ and $(2/3\sqrt{3}, 1/3)$.



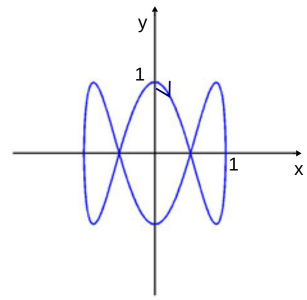
16. $\dfrac{dx}{dt} = \dfrac{3}{8}t^2$; $\dfrac{dy}{dt} = 2y - 2$; $\dfrac{dy}{dx} = \dfrac{2t-2}{3t^2/8} = \dfrac{16t-16}{3t^2}$. The tangent line is horizontal when $16t - 16 = 0$ or $t = 1$, and vertical when $3t^2 = 0$ or $t = 0$. Thus, there is a horizontal tangent at $(9/8, -1)$ and a vertical tangent at $(1, 0)$.



17. $\dfrac{dx}{dt} = 1$; $\dfrac{dy}{dt} = 3t^2 - 6t$; $\dfrac{dy}{dx} = 3t^2 - 6t$. The tangent line is horizontal when $3t^2 - 6t = 3t(t-2) = 0$ or $t = 0, 2$. Thus, there are horizontal tangents at $(-1, 0)$ and $(1, -4)$. There are no vertical tangents.



18. $\dfrac{dx}{dt} = \cos t$; $\dfrac{dy}{dt} = -3\sin 3t$; $\dfrac{dy}{dx} = \dfrac{-3\sin 3t}{\cos t}$. The tangent line is horizontal when $\sin 3t = 0$ or $t = 0,\ \pi/3,\ 2\pi/3,\ \pi,\ 4\pi/3,\ 5\pi/3,\ 2\pi$ and vertical when $\cos t = 0$ or $t = \pi/2,\ 3\pi/2$. Thus, there are horizontal tangents at $(0, 1)$, $(\sqrt{3}/2, -1)$, $(\sqrt{3}/2, 1)$, $(0, -1)$, $(-\sqrt{3}/2, 1)$, $(\sqrt{3}/2, 1)$ and vertical tangents at $(1, 0)$ and $(-1, 0)$.



19. $\dfrac{dy}{dx} = \dfrac{18t^2}{6t} = 3t$; $\dfrac{d^2y}{dx^2} = \dfrac{3}{6t} = \dfrac{1}{2t}$; $\dfrac{d^3y}{dx^3} = \dfrac{-1/2t^2}{6t} = -\dfrac{1}{12t^3}$

20. $\dfrac{dy}{dx} = \dfrac{\cos t}{-\sin t} = -\cot t$; $\dfrac{d^2y}{dx^2} = \dfrac{\csc^2 t}{-\sin t} = -\csc^3 t$; $\dfrac{d^3y}{dx^3} = \dfrac{3\csc^3 t\cot t}{-\sin t} = -3\csc^4 t\cot t$

21. $\dfrac{dy}{dx} = \dfrac{2e^{2t} + 3e^{3t}}{-e^{-t}} = -2e^{3t} - 3e^{4t}$; $\dfrac{d^2y}{dx^2} = \dfrac{-6e^{3t} - 12e^{4t}}{-e^{-t}} = 6e^{4t} + 12e^{5t}$
$\dfrac{d^3y}{dx^3} = \dfrac{24e^{4t} + 60e^{5t}}{-e^{-t}} = -24e^{5t} - 60e^{6t}$

22. $\dfrac{dy}{dx} = \dfrac{t-1}{t+1}$; $\dfrac{d^2y}{dx^2} = \dfrac{2/(t+1)^2}{t+1} = \dfrac{2}{(t+1)^3}$; $\dfrac{d^3y}{dx^3} = \dfrac{-6/(t+1)^4}{t+1} = -\dfrac{6}{(t+1)^5}$

23. Using Problem 16, $\dfrac{d^2y}{dx^2} = \dfrac{dy'/dt}{dx/dt} = \dfrac{(32-16t)/3t^3}{3t^2/8} = \dfrac{256-128t}{9t^5} = \dfrac{128}{9}\left(\dfrac{2-t}{t^5}\right)$. Then d^2y/dx^2 is 0 when $t = 2$ and undefined when $t = 0$. The graph is concave downward on $(-\infty, 0)$ and $(2, \infty)$ and concave upward on $(0, 2)$.

t		0		2	
y''	-	und	+	0	-

24. $\dfrac{dx}{dt} = 2;\quad \dfrac{dy}{dt} = 6t^2 + 12t + 4;\quad \dfrac{dy}{dx} = 3t^2 + 6t + 2;\quad \dfrac{d^2y}{dx^2} = \dfrac{dy'/dt}{dx/dt} = \dfrac{6t+6}{2} = 3t + 3$
Solving $3t + 3 = 0$ we obtain $t = -1$. Since $d^2y/dx^2 < 0$ for $t < -1$ and $d^2y/dx^2 > 0$ for $t > -1$, the graph has a point of inflection when $t = -1$ or at $(3, 0)$.

25. $x'(t) = 5t^2,\quad y'(t) = 12t^2;\quad s = \displaystyle\int_0^2 \sqrt{25t^4 + 144t^4}dt = 13\int_0^2 t^2 dt = \frac{13}{3}t^3\Big|_0^2 = \frac{104}{3}$

26. $x'(t) = t^2,\quad y'(t) = t$
$s = \displaystyle\int_0^{\sqrt{3}} \sqrt{t^4 + t^2}dt = \int_0^{\sqrt{3}} t\sqrt{t^2+1}dt \quad \boxed{u = t^2 + 1,\ \ du = 2tdt}$
$\displaystyle\int_1^4 \frac{1}{2}u^{1/2}du = \frac{1}{3}u^{3/2}\Big|_1^4 = \frac{1}{3}(8-1) = \frac{7}{3}$

27. $x'(t) = e^t\cos t + e^t\sin t;\quad y'(t) = -e^t\sin t + e^t\cos t$
$s = \displaystyle\int_0^{\pi} [e^{2t}(\cos^2 t + 2\sin t\cos t + \sin^2 t) + e^{2t}(\sin^2 t - 2\sin t\cos t + \cos^2 t)]^{1/2}dt$
$= \displaystyle\int_0^{\pi} e^t(2)^{1/2}dt = \sqrt{2}e^t\Big|_0^{\pi} = \sqrt{2}(e^{\pi} - 1)$

28. $x'(\theta) = a(1-\cos\theta),\quad y'(\theta) = a\sin\theta$. Using symmetry,

$$s = 2\int_0^{\pi}\sqrt{a^2(1-\cos\theta)^2 + a^2\sin^2\theta}d\theta = 2a\int_0^{\pi}\sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}d\theta$$
$$= 2\sqrt{2}a\int_0^{\pi}\sqrt{1-\cos\theta}d\theta = 2\sqrt{2}a\int_0^{\pi}\sqrt{1-\cos\theta}\left(\frac{\sqrt{1+\cos\theta}}{\sqrt{1+\cos\theta}}\right)d\theta$$
$$= 2\sqrt{2}a\int_0^{\pi}\frac{\sqrt{1-\cos^2\theta}}{\sqrt{1+\cos^2\theta}}d\theta = 2\sqrt{2}a\int_0^{\pi}\frac{\sin\theta}{\sqrt{1+\cos\theta}}d\theta \quad \boxed{u = 1+\cos\theta,\ \ du = -\sin\theta d\theta}$$
$$= 2\sqrt{2}a\int_2^0\frac{-du}{\sqrt{u}} = 2\sqrt{2}a\int_2^0 u^{-1/2}du = 2\sqrt{2}a\lim_{b\to0^+}\int_b^2 u^{1/2}du$$
$$= 2\sqrt{2}a\lim_{b\to0^+}\left(2\sqrt{u}\Big|_b^2\right) = 2\sqrt{2}a\lim_{b\to0^+}(2\sqrt{2} - 2\sqrt{b}) = 8a.$$

29. $x'(\theta) = -3b\cos^2\theta\sin\theta;\quad y'(\theta) = 3b\sin^2\theta\cos\theta$
$s = \displaystyle\int_0^{\pi/2}(9b^2\cos^4\theta\sin^2\theta + 9b^2\sin^4\theta\cos^2\theta)^{1/2}d\theta = 3|b|\int_0^{\pi/2}\sin\theta\cos\theta\sqrt{\cos^2\theta + \sin^2\theta}d\theta$
$= 3|b|\displaystyle\int_0^{\pi/2}\frac{1}{2}\sin 2\theta d\theta = -\frac{3}{4}|b|\cos 2\theta\Big|_0^{\pi/2} = -\frac{3}{4}|b|(-1-1) = \frac{3}{2}|b|$

30. $x'(\theta) = -4a\sin\theta + 4a\sin 4\theta;\quad y'(\theta) = 4a\cos\theta - 4a\cos 4\theta$

$$\begin{aligned}
s &= \int_0^{2\pi/3} [(-4a\sin\theta + 4a\sin 4\theta)^2 + (4a\cos\theta - 4a\cos 4\theta)^2]^{1/2} d\theta \\
&= 4|a| \int_0^{2\pi/3} (\sin^2\theta - 2\sin\theta\sin 4\theta + \sin^2 4\theta + \cos^2\theta - 2\cos\theta\cos 4\theta + \cos^2 4\theta)^{1/2} d\theta \\
&= 4|a| \int_0^{2\pi/3} (2 - 2\sin\theta\sin 4\theta - 2\cos\theta\cos 4\theta)^{1/2} d\theta \\
&= 4\sqrt{2}|a| \int_0^{2\pi/3} [1 - (\cos 4\theta\cos\theta + \sin 4\theta\sin\theta)]^{1/2} d\theta \\
&= 4\sqrt{2}|a| \int_0^{2\pi/3} \sqrt{1 - \cos(4\theta - \theta)} d\theta = 4\sqrt{2}|a| \int_0^{2\pi/3} \sqrt{1 - \cos 3\theta} d\theta \\
&= 4\sqrt{2}|a| \int_0^{2\pi/3} \sqrt{2\sin^2 3\theta/2} d\theta = 8|a| \int_0^{2\pi/3} \sin\frac{3\theta}{2} d\theta \\
&= 8|a| \left(-\frac{2}{3}\cos\frac{3\theta}{2}\right)\Bigg|_0^{2\pi/3} = -\frac{16}{3}|a|(\cos\pi - \cos 0) = \frac{32}{3}|a|
\end{aligned}$$

31. (a) Setting $x = 0$ we have $t^2 - 4t - 2 = 0$ which implies $t = 2 \pm \sqrt{6}$. When $t = 2 - \sqrt{6}$, $y \approx -0.6551$. When $t = 2 + \sqrt{6}$, $y \approx 1390.66$.

(b) Using Newton's Method to solve $t^5 - 4t^3 - 1 = 0$ we obtain $t \approx -1.96687, \ -0.654175, \ 2.02968$ with corresponding x values $9.73606, \ 1.04465, \ -5.99912$.

32. If $y = F(x)$ and $x = f(t)$, then $g(y) = y = F(f(t))$. Thus,

$$\begin{aligned}
A &= \int_{x_1}^{x_2} F(x)dx \boxed{x = f(t), \quad dx = f'(t)dt} \\
&= \int_a^b F(f(t))f'(t)dt = \int_a^b g(t)f'(t)dt.
\end{aligned}$$

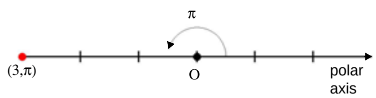
33. From Example 7 in Section 10.2, $f(\theta) = a(\theta - \sin\theta)$ and $g(\theta) = a(1 - \cos\theta)$ for $0 \le \theta \le 2\pi$. Then $f'(\theta) = a(1 - \cos\theta)$, and using symmetry,

$$\begin{aligned}
A &= 2\int_0^{\pi} a^2(1 - \cos\theta)^2 d\theta = 2a^2 \int_0^{\pi} (1 - 2\cos\theta + \cos^2\theta) d\theta \\
&= 2a^2 \int_0^{\pi} (1 - 2\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta) d\theta = 2a^2 \left(\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right)\Bigg|_0^{\pi} \\
&= 2a^2\left(\frac{3}{2}\pi\right) = 3(\pi a^2).
\end{aligned}$$

Thus, the area under an arch of the cycloid is three times the area of the circle.

10.4 Polar Coordinate System

1.



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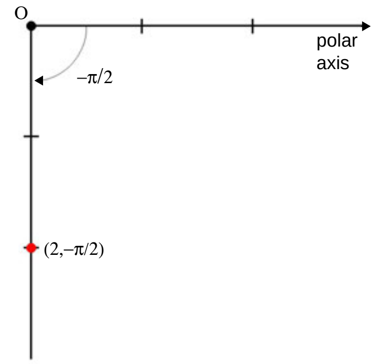

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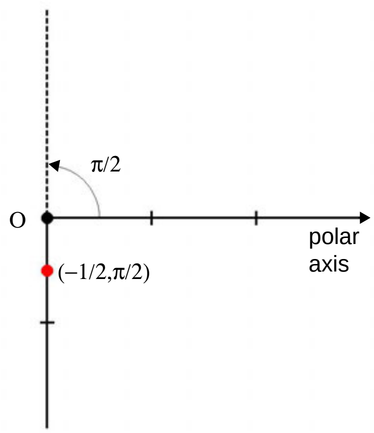

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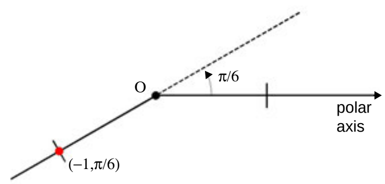

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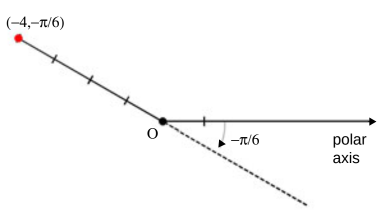
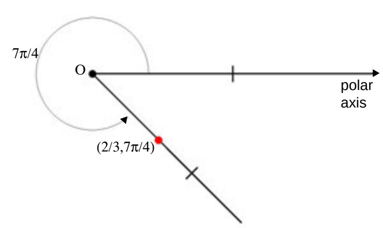


6.



7. (a) $\left(2, -\frac{5\pi}{4}\right)$ (b) $\left(2, \frac{11\pi}{4}\right)$ (c) $\left(-2, \frac{7\pi}{4}\right)$ (d) $\left(-2, -\frac{\pi}{4}\right)$

8. (a) $\left(5, -\frac{3\pi}{2}\right)$ (b) $\left(5, \frac{5\pi}{2}\right)$ (c) $\left(-5, \frac{3\pi}{2}\right)$ (d) $\left(-5, -\frac{\pi}{2}\right)$

9. (a) $\left(4, -\frac{5\pi}{3}\right)$ (b) $\left(4, \frac{7\pi}{3}\right)$ (c) $\left(-4, \frac{4\pi}{3}\right)$ (d) $\left(-4, -\frac{2\pi}{3}\right)$

10. (a) $\left(3, -\frac{7\pi}{4}\right)$ (b) $\left(3, \frac{9\pi}{4}\right)$ (c) $\left(-3, \frac{5\pi}{4}\right)$ (d) $\left(-3, -\frac{3\pi}{4}\right)$

11. (a) $\left(1, -\frac{11\pi}{6}\right)$ (b) $\left(1, \frac{13\pi}{6}\right)$ (c) $\left(-1, \frac{7\pi}{6}\right)$ (d) $\left(-1, -\frac{5\pi}{6}\right)$

12. (a) $\left(3, -\frac{5\pi}{6}\right)$ (b) $\left(3, \frac{19\pi}{6}\right)$ (c) $\left(-3, \frac{\pi}{6}\right)$ (d) $\left(-3, -\frac{11\pi}{6}\right)$

13. With $r = 1/2$ and $\theta = 2\pi/3$ we have $x = 1/2\cos 2\pi/3 = 1/2(-1/2) = -1/4$, $y = 1/2\sin 2\pi/3 = 1/2(\sqrt{3}/2) = \sqrt{3}/4$. The point is $\left(-\dfrac{1}{4}, \dfrac{\sqrt{3}}{4}\right)$ in rectangular coordinates.

14. With $r = -1$ and $\theta = 7\pi/4$ we have $x = -1\cos 7\pi/4 = -1(\sqrt{2}/2) = -\sqrt{2}/2$, $y = -1\sin 7\pi/4 = -1(-\sqrt{2}/2) = \sqrt{2}/2$. The point is $\left(-\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$ in rectangular coordinates.

15. With $r = -6$ and $\theta = -\pi/3$ we have $x = -6\cos(-\pi/3) = -6(1/2) = -3$, $y = -6\sin(-\pi/3) = -6(-\sqrt{3}/2) = 3\sqrt{3}$. The point is $\left(-3, 3\sqrt{3}\right)$ in rectangular coordinates.

16. With $r = \sqrt{2}$ and $\theta = 11\pi/6$ we have $x = \sqrt{2}\cos 11\pi/6 = \sqrt{2}(\sqrt{3}/2) = \sqrt{6}/2$, $y = \sqrt{2}\sin 11\pi/6 = \sqrt{2}(-1/2) = -\sqrt{2}/2$. The point is $\left(\dfrac{\sqrt{6}}{2}, -\dfrac{\sqrt{2}}{2}\right)$ in rectangular coordinates.

17. With $r = 4$ and $\theta = 5\pi/4$ we have $x = 4\cos 5\pi/4 = 4(-\sqrt{2}/2) = -2\sqrt{2}$, $y = 4\sin 5\pi/4 = 4(-\sqrt{2}/2) = -2\sqrt{2}$. The point is $\left(-2\sqrt{2}, -2\sqrt{2}\right)$ in rectangular coordinates.

18. With $r = -5$ and $\theta = \pi/2$ we have $x = -5\cos\pi/2 = 0$, $y = -5\sin\pi/2 = -5$. The point is $(0, -5)$ in rectangular coordinates.

19. With $x = -2$ and $y = -2$ we have $r^2 = 8$ and $\tan\theta = 1$.
(a) $\left(2\sqrt{2}, -\frac{3\pi}{4}\right)$ **(b)** $\left(-2\sqrt{2}, \frac{\pi}{4}\right)$

20. With $x = 0$ and $y = -4$ we have $r^2 = 16$ and $\tan\theta$ undefined.
(a) $\left(4, -\frac{\pi}{2}\right)$ **(b)** $\left(-4, \frac{\pi}{2}\right)$

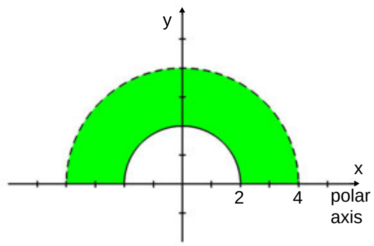
21. With $x = 1$ and $y = -\sqrt{3}$ we have $r^2 = 4$ and $\tan\theta = -\sqrt{3}$.
(a) $\left(2, -\frac{\pi}{3}\right)$ **(b)** $\left(-2, \frac{3\pi}{3}\right)$

22. With $x = \sqrt{6}$ and $y = \sqrt{2}$ we have $r^2 = 8$ and $\tan\theta = 1/\sqrt{3}$.
(a) $\left(2\sqrt{2}, \frac{\pi}{6}\right)$ **(b)** $\left(-2\sqrt{2}, -\frac{5\pi}{6}\right)$

23. With $x = 7$ and $y = 0$ we have $r^2 = 49$ and $\tan\theta = 0$.
(a) $(7, 0)$ **(b)** $(-7, \pi)$ or $(-7, -\pi)$

24. With $x = 1$ and $y = 2$ we have $r^2 = 5$ and $\tan\theta = 2$.
(a) $\left(\sqrt{5}, \tan^{-1} 2\right)$ or $\left(\sqrt{5}, 1.1071\right)$ **(b)** $\left(-\sqrt{5}, -\pi + \tan^{-1} 2\right)$ or $\left(-\sqrt{5}, -2.0344\right)$

25.



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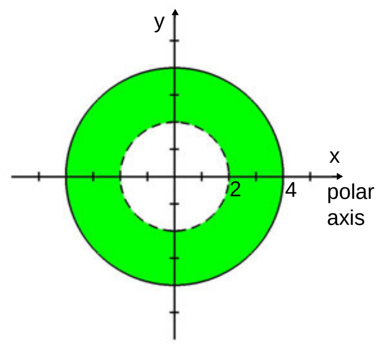

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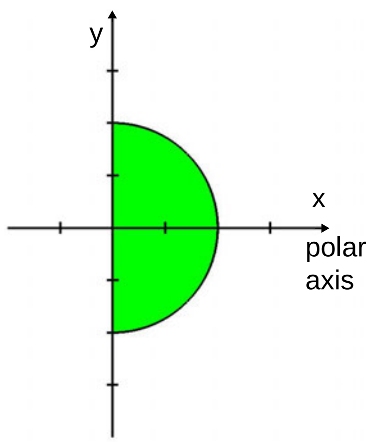

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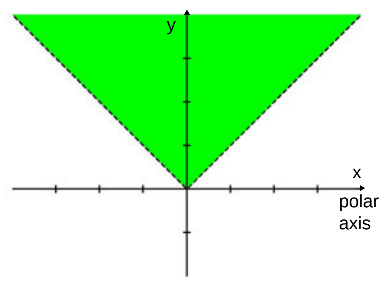

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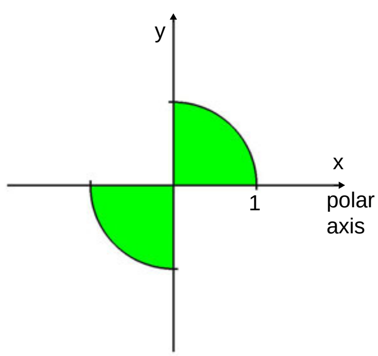
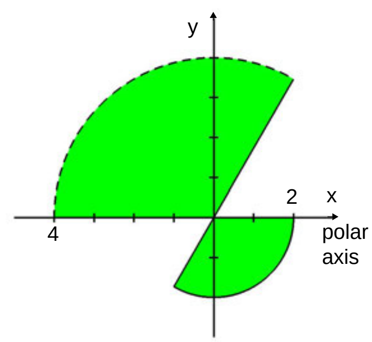


30.



In Problems 31-40, we use $x = r\cos\theta$ and $y = r\sin\theta$.

31. $r\sin\theta = 5$; $\quad r = 5\csc\theta$

32. $r\cos\theta + 1 = 0$; $\quad r = -\sec\theta$

33. $r\sin\theta = 7r\cos\theta$; $\quad \tan\theta = 7$; $\quad \theta = \tan^{-1} 7$

34. $3r\cos\theta + 8r\sin\theta + 6 = 0$; $\quad r(3\cos\theta + 8\sin\theta) = -6$; $\quad r = -\dfrac{6}{(3\cos\theta + 8\sin\theta)}$

35. $r^2\sin^2\theta = -4r\cos\theta$; $r^2(1-\cos^2\theta) + 4r\cos\theta = 0$; $r^2 - (r^2\cos^2\theta - 4r\cos\theta + 4) = 0$; $r^2 - (r\cos\theta - 2)^2 = 0$; $[r - (r\cos\theta - 2)][r + (r\cos\theta) - 2] = 0$
Solving for r, we obtain $r = \dfrac{-2}{(1-\cos\theta)}$ or $r = \dfrac{2}{(1+\cos\theta)}$. Since replacement of (r,θ) by $(-r, \theta+\pi)$ in the first equation gives the second equation, we take the polar equation to be $r = \dfrac{2}{1+\cos\theta}$.

36. $r^2\cos^2\theta - 12r\sin\theta - 36 = 0$; $r^2(1-\sin^2\theta) - 12r\sin\theta - 36 = 0$; $r^2 - (r^2\sin^2\theta + 12r\sin\theta + 36) = 0$; $r^2 - (r\sin\theta + 6)^2 = 0$; $[r - (r\sin\theta + 6)][r + (r\sin\theta + 6)] = 0$
Solving for r, we obtain $r = \dfrac{6}{(1-\sin\theta)}$ or $r = \dfrac{-6}{(1+\sin\theta)}$. Since replacement of (r,θ) by $(-r, \theta+\pi)$ in the second equation gives the first equation, we take the polar equation to be $r = \dfrac{6}{(1-\sin\theta)}$.

37. $r^2 = 36$. Since $r = -6$ has the same graph as $r = 6$, we take the equation to be $r = 6$.

38.
$$\begin{aligned}
(r\cos\theta)^2 - (r\sin\theta)^2 &= 1 \\
r^2\cos^2\theta - r^2\sin^2\theta &= 1 \\
r^2\left(\cos^2\theta - \sin^2\theta\right) &= 1 \\
r^2\left(1 - 2\sin^2\theta\right) &= 1
\end{aligned}$$

39. $r^2 + r\cos\theta = \sqrt{r^2} = \pm r$; $r(r + \cos\theta \mp 1) = 0$. Solving for r, we obtain $r = 0$ or $r = \pm 1 - \cos\theta$. Since replacement of (r,θ) in $r = -1-\cos\theta$ by $(-r, \theta+\pi)$ gives $r = 1-\cos\theta$, and since $\theta = 0$ gives $r = 0$, we take the polar equation to be $r = 1-\cos\theta$.

40. $r^3\cos^3\theta + r^3\sin^3\theta - r^2\sin\theta\cos\theta = 0$; $r^2[r(\cos^3\theta + \sin^3\theta) - \frac{1}{2}\sin 2\theta] = 0$
Solving for r gives $r = 0$ or $r = \dfrac{\sin 2\theta}{2(\cos^3\theta + \sin^3\theta)}$. Since $\pi = 0$ gives $r = 0$ in the second equation, we take the polar equation to be $r = \dfrac{\sin 2\theta}{2(\cos^3\theta + \sin^3\theta)}$.

In Problems 41-52, we use $r^2 = x^2 + y^2$, $r\cos\theta = x$, $r\sin\theta = y$, and $\tan\theta = y/x$.

41. $r\cos\theta = 2$; $\quad x = 2$

42. $x = -4$

43. $r = 12\sin\theta\cos\theta$; $\quad r^3 = 12r\sin\theta r\cos\theta$; $\quad (x^2+y^2)^{3/2} = 12xy$; $\quad (x^2+y^2)^3 = 144x^2y^2$

44. $2(x^2+y^2)^{1/2} = y/x$; $\quad 4x^2(x^2+y^2) = y^2$

45. $r^2 = 8\sin\theta\cos\theta$; $\quad r^4 = 8r\sin\theta r\cos\theta$; $\quad (x^2+y^2)^2 = 8xy$

46. $r^2(\cos^2\theta - r^2\sin^2\theta) = 16;\quad r^2\cos^2\theta - r^2\sin^2\theta = 16;\quad x^2 - y^2 = 16$

47. $r^2 + 5r\sin\theta = 0;\quad x^2 + y^2 + 5y = 0$

48. $r^2 = 2r + r\cos\theta$; $x^2+y^2 = 2(x^2+y^2)^{1/2}+x$; $x^2+y^2-x = 2(x^2+y^2)^{1/2}$; $(x^2+y^2-x)^2 = 4(x^2+y^2)$

49. $r + 3r\cos\theta = 2$; $(x^2+y^2)^{1/2} + 3x = 2$; $(x^2+y^2)^{1/2} = 2 - 3x$; $x^2 + y^2 = 4 - 12x + 9x^2$; $8x^2 - y^2 - 12x + 4 = 0$

50. $4r - r\sin\theta = 10$; $4\sqrt{x^2+y^2} - y = 10$; $4\sqrt{x^2+y^2} = y + 10$; $16(x^2+y^2) = y^2 + 20y + 100$; $16x^2 + 15y^2 - 20y - 100 = 0$

51. $3r\cos\theta + 8r\sin\theta = 5$; $3x + 8y = 5$

52. $r\cos\theta = 3\cos\theta + 3$; $r^2\cos\theta = 3r\cos\theta + 3r$; $r(r\cos\theta - 3) = 3r\cos\theta$; $(x^2+y^2)^{1/2}(x-3) = 3x$; $(x^2+y^2)(x-3)^2 = 9x^2$

53.
$$\begin{aligned}
\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} &= \sqrt{(r_2\cos\theta_2 - r_1\cos\theta_1)^2 + (r_2\sin\theta_2 - r_1\sin\theta_1)^2} \\
&= \sqrt{r_2^2\cos^2\theta_2 - 2r_2r_1\cos\theta_2\cos\theta_1 + r_1^2\cos^2\theta_1 + r_2^2\sin^2\theta_2 - 2r_2r_2\sin\theta_2\sin\theta_1 + r_1^2\sin^2\theta_1} \\
&= \sqrt{r_2^2 + r_1^2 - 2r_1r_2(\cos\theta_2\cos\theta_1 + \sin\theta_2\sin\theta_1)} \\
&= \sqrt{r_2^2 + r_1^2 - 2r_1r_2\cos(\theta_2 - \theta_1)}
\end{aligned}$$

54. Consider the general linear function $y = ax + b$. Transforming this function into polar coordinates, we have $r\sin\theta = ar\cos\theta + b$. To find a line passing through (r_1, θ_1) and (r_2, θ_2), we would need to solve the following system for a and b:

$$r_1\sin\theta_1 = ar_1\cos\theta_1 + br_2\sin\theta_2 = ar_2\cos\theta_2 + b$$

To find the line passing through $\left(3, \frac{3\pi}{4}\right)$ and $\left(1, \frac{\pi}{4}\right)$, we solve the system

$$\begin{aligned}
\frac{3\sqrt{2}}{2} &= -3a\frac{\sqrt{2}}{2} + b \\
\frac{\sqrt{2}}{2} &= a\frac{\sqrt{2}}{2} + b
\end{aligned}$$

for a and b. This yields $a = -\frac{1}{2}$, $b = \frac{3\sqrt{2}}{4}$. Thus, the equation of the line is

$$r\sin\theta = -\frac{1}{2}r\cos\theta + \frac{3\sqrt{2}}{4}$$

or

$$r = \frac{3\frac{\sqrt{2}}{4}}{\sin\theta + \frac{1}{2}\cos\theta}$$

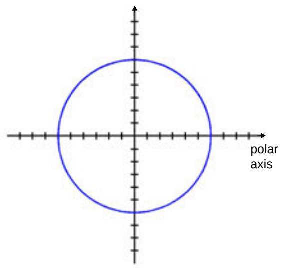
The y-intercept occurs when $r\cos\theta = 0$ and hence $r\sin\theta = \dfrac{3\sqrt{2}}{4}$. Together, these yield $\theta = \frac{\pi}{2}$ and $r = \dfrac{3\sqrt{2}}{4}$. The y-intercept is thus $\left(\dfrac{3\sqrt{2}}{4}, \dfrac{\pi}{2}\right)$ in polar coordinates. The x-intercept

occurs when $r\sin\theta = 0$ and hence $r\cos\theta = \dfrac{3\sqrt{2}}{2}$. Together, these yield $\theta = 0$ and $r = \dfrac{3\sqrt{2}}{2}$. The x-intercept is thus $\left(\dfrac{3\sqrt{2}}{2}, 0\right)$.

55. Solutions of $f(\theta) = 0$ are θ values at which the graph of $r = f(\theta)$ passes through the origin.

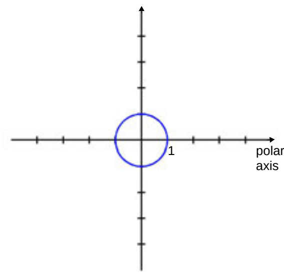
10.5 Graphs of Polar Equations

1.



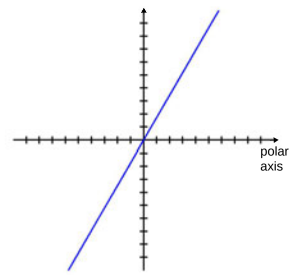
circle

2.



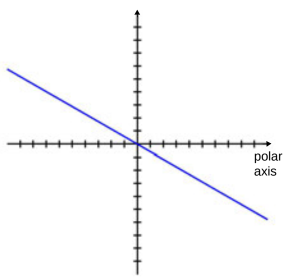
circle

3.



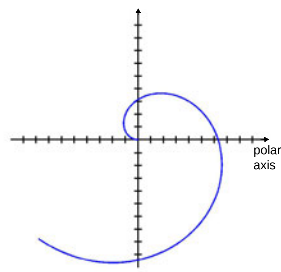
line through origin

4.



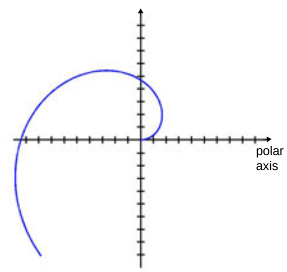
line through origin

5.



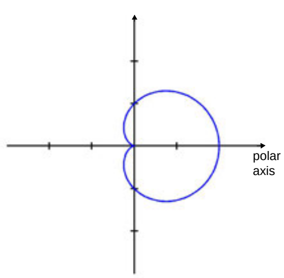
spiral

6.



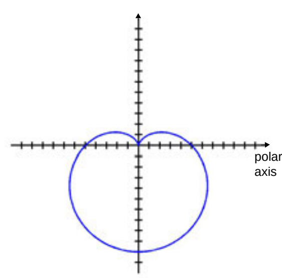
spiral

7.



cardioid

8.



cardioid

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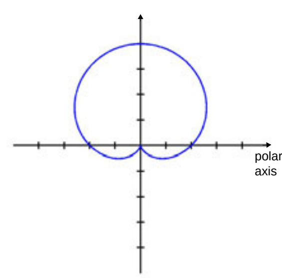
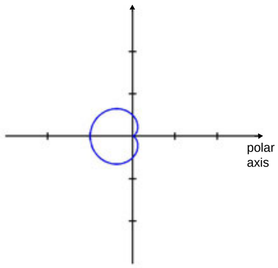


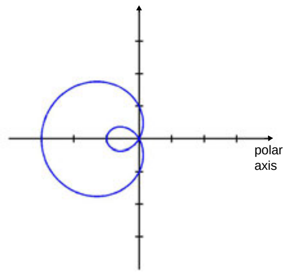
cardioid

10.



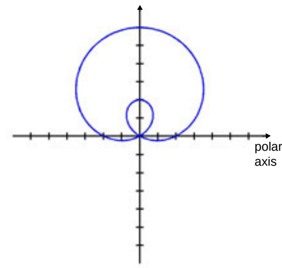
cardioid

11.



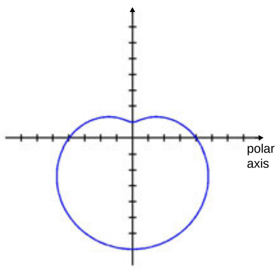
limacon with an interior loop

12.



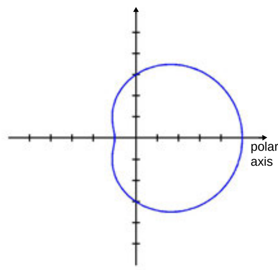
limacon with an interior loop

13.



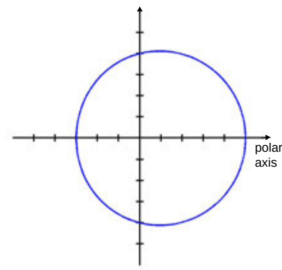
dimpled limacon

14.



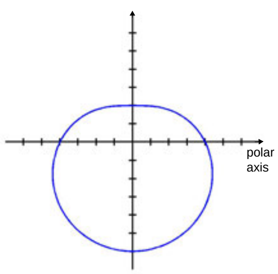
dimpled limacon

15.



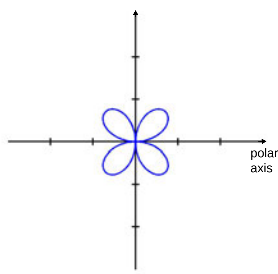
convex limacon

16.



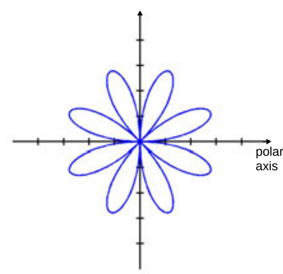
convex limacon

17.



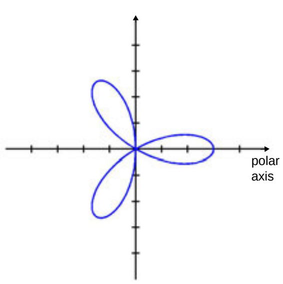
rose curve

18.



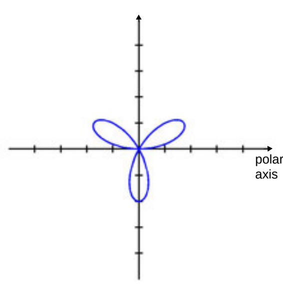
rose curve

19.



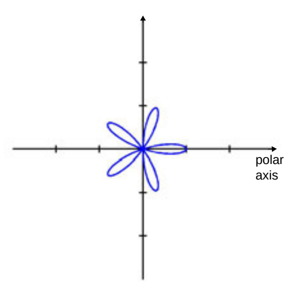
rose curve

20.



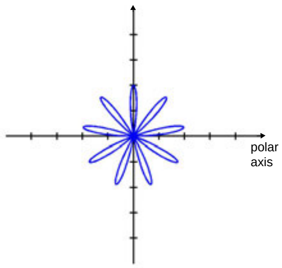
rose curve

21.



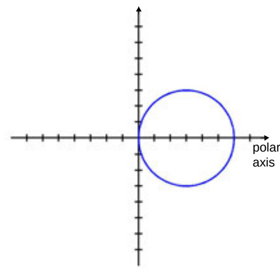
rose curve

22.



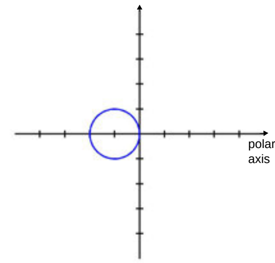
rose curve

23.



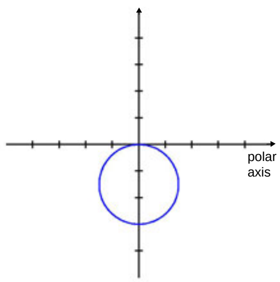
circle with center on x-axis

24.



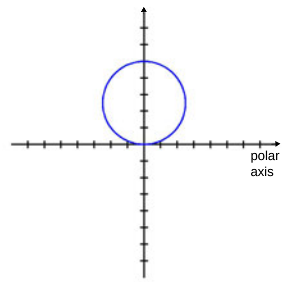
circle

25.



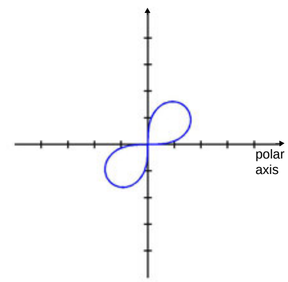
circle with center on y-axis

26.



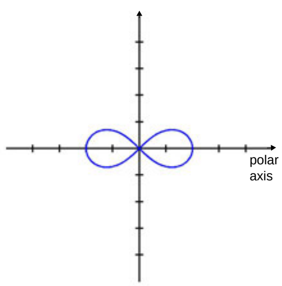
circle with center on y-axis

27.



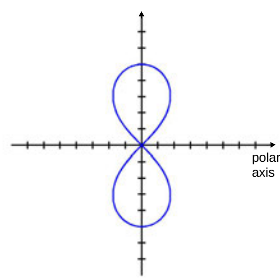
lemniscate

28.



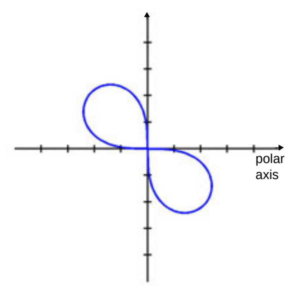
lemniscate

29.



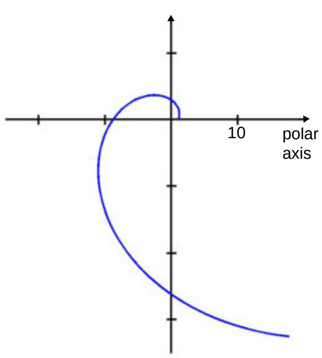
lemniscate

30.

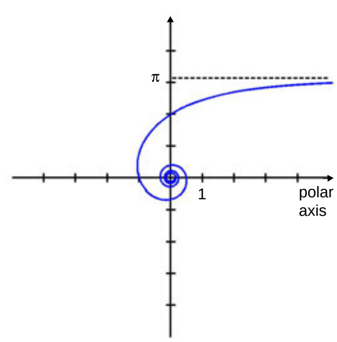


lemniscate

31.



32.



33. $r = 2.5$

34. $r = -4\cos\theta$

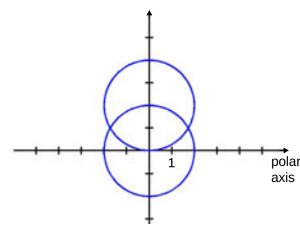
35. $r = 4 - 3\cos\theta$

36. $r = 2 + 3\sin\theta$

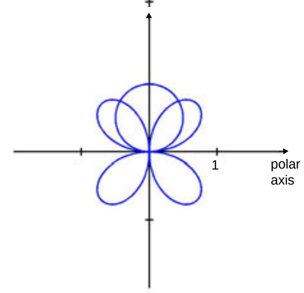
37. $r = 2\cos 4\theta$

38. $r = 5\cos 2\theta$

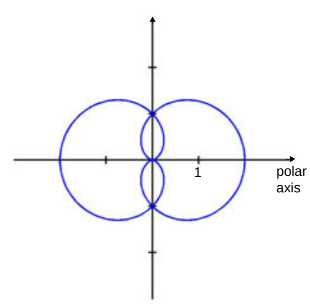
39. Solving $4\sin\theta = 2$ we have $\sin\theta = 1/2$ and $\theta = \pi/6$ and $5\pi/6$. The points of intersection are $(2, \pi/6)$ and $(2, 5\pi/6)$.



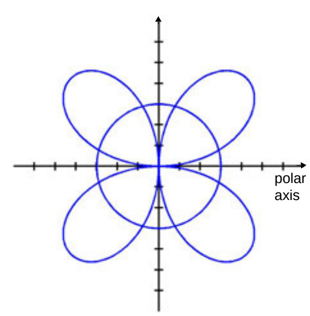
40. Writing $\sin 2\theta = 2\sin\theta\cos\theta$ and setting $\sin\theta = 2\sin\theta\cos\theta$, we obtain $\sin\theta(2\cos\theta - 1) = 0$. This gives $\theta = 0,\ \pi,\ \pi/3$, and $5\pi/3$. For $\theta = 0$ and $\theta = \pi$ we obtain the pole $(0,0)$. The other two points of intersection are $(\sqrt{3}/2, \pi/3)$ and $(-\sqrt{3}/2, 5\pi/3)$.



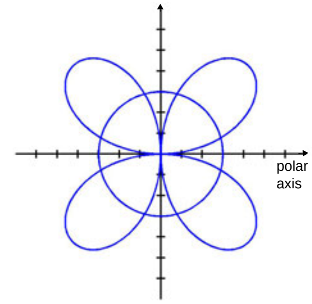
41. Setting $1 - \cos\theta = 1 + \cos\theta$, we obtain $2\cos\theta = 0$. This gives $\theta = \pm\pi/2$. Two points of intersection are $(1, \pi/2)$ and $(1, -\pi/2)$. From the figure we see that the pole ($(0,0)$ on $r = 1 - \cos\theta$ and $(0, \pi)$ on $r = 1 + \cos\theta$) is also a point of intersection.



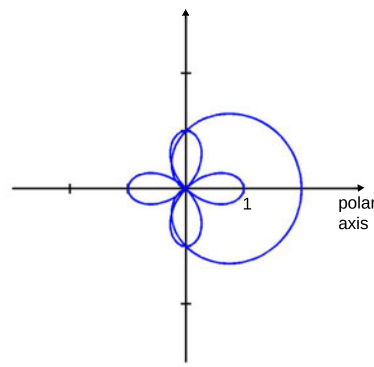
42. Setting $3 - 3\cos\theta - 3\cos\theta$, we obtain $3 = 6\cos\theta$ or $\frac{1}{2} = \cos\theta$. This yields $\theta = \pm\frac{\pi}{3}$. Two points of intersection are $\left(\frac{3}{2}, \frac{\pi}{3}\right)$ and $\left(\frac{3}{2}, -\frac{\pi}{3}\right)$. From the figure we see that the pole ($(0,0)$ on $r = 3 - 3\cos\theta$ and $(0, \pi/2)$ on $r = 3\cos\theta$) is also a point of intersection.



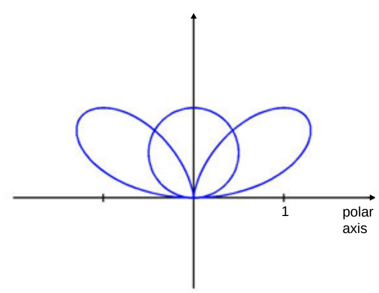
43. Setting $6 \sin 2\theta = 3$, we obtain $\sin 2\theta = 1/2$. Then $2\theta = \pi/6$, $5\pi/6$, $13\pi/6$, and $17\pi/6$. This gives the points of intersection $(3, \pi/12)$, $(3, 5\pi/12)$, $(3, 13\pi/12)$, and $(3, 17\pi/12)$. Writing the second equation in the form $-r = 6 \sin 2(\theta + \pi)$, we obtain $r = -6 \sin 2\theta$. Setting $-6 \sin 2\theta = 3$, we obtain $\sin 2\theta = -1/2$. Then $2\theta = -\pi/6$, $-5\pi/6$, $-13\pi/6$, and $-17\pi/6$. This gives the points of intersection $(3, -\pi/12)$, $(3, -5\pi/12)$, $(3, -13\pi/12)$, and $(3, -17\pi/12)$.



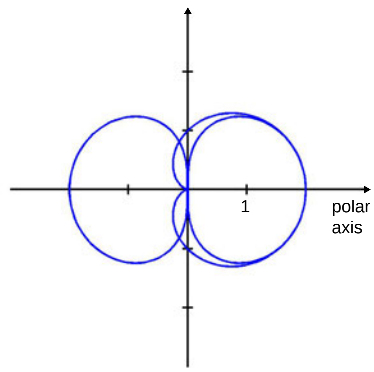
44. Using $\cos 2\theta = 2\cos^2\theta - 1$, we have $2\cos^2\theta - 1 = 1 + \cos\theta$ or $2\cos^2\theta - \cos\theta - 2 = 0$. From the quadratic formula, $\cos\theta = \dfrac{1 \pm \sqrt{17}}{4}$. Since $\dfrac{1 + \sqrt{17}}{4} > 1$, we solve only $\cos\theta = \dfrac{1 - \sqrt{17}}{4}$. This gives $\theta \approx 2.4667$ and $\theta \approx 3.8165$. For both of these values $r \approx 0.219$. Writing $-r = \cos 2(\theta + \pi)$, we have $r = -\cos 2\theta = -2\cos^2\theta + 1$. Solving this equation with $r = 1 + \cos\theta$, we have $-2\cos^2\theta + 1 = 1 + \cos\theta$ or $\cos\theta(2\cos\theta + 1) = 0$. For $\cos\theta = 0$ we obtain $\theta = \pi/2$ and $\theta = 3\pi/2$. For both of these values $r = 1$. From $\cos\theta = -1/2$ we obtain $\theta = 2\pi/3$ and $\theta = 4\pi/3$. For both of these values $r = 1/2$. Thus, $(0.219, 2.47)$, $(0.219, 3.82)$, $(1, \pi/2)$, $(1, 3\pi/2)$, $(1/2, 2\pi/3)$, and $(1/2, 4\pi/3)$ are points of intersection. From the graph we see that the pole $((0, \pi/4)$ on $r = \cos 2\theta$ and $(0, \pi)$ on $r = 1 + \cos\theta)$ is also a point of intersection.



45. Setting $4 \sin\theta \cos^2\theta = \sin\theta$ we obtain $\sin\theta(4\cos^2\theta - 1) = 0$. This gives $\theta = 0$, $\theta = \pi/3$, and $\theta = 2\pi/3$. The points of intersection are $(0, 0)$, $(\sqrt{3}/2, \pi/3)$, and $(\sqrt{3}/2, 2\pi/3)$.



46. From the figure we see that the graphs intersect at the pole (which occurs for $\theta = \pi$ on the cardioid and $\theta = \pi/2$ on the lemniscate) and at $(2, 0)$. Setting $(1 + \cos\theta)^2 = 4\cos\theta$ we have



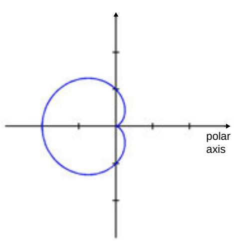
$$\cos^2\theta - 2\cos\theta + 1 = 0 \Longrightarrow (\cos\theta - 1) = 0 \Longrightarrow \cos\theta = 1$$

which yields only the point (2,0). The points of intersection in the second and third quadrants occur when $\pi/2 < \theta < 3\pi/2$ on the cardioid and when $-\pi/2 < \theta < \pi/2$ and $r < 0$ on the lemniscate. If (r_2, θ_2) represents a point in the second or third quadrants on the cardioid, we have $r_2 = 1 + \cos\theta_2$. If (r_1, θ_1) is a point in the second or third quadrants on the lemniscate, we have $r_1^2 = 4\cos\theta$. At points of intersection then, we have $r_2 = 1 + \cos\theta$, $r_1^2 = 4\cos\theta$, $r_2 = -r_1$, and $\theta_2 = \theta_1 + \pi$. Substituting the last two equations into the first equation, we obtain $-r_1 = 1 + \cos(\theta_1 + \pi) = 1 - \cos\theta_1$ or $r_1^2 = 1 - 2\cos\theta_1 + \cos^2\theta_1$. Combining with $r_1^2 = 4\cos\theta_1$ we have

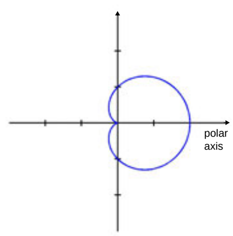
$$1 - 2\cos\theta_1 + \cos^2\theta_1 = 4\cos\theta_1 \Longrightarrow \cos^2\theta_1 - 6\cos\theta_1 + 1 = 0 \Longrightarrow \cos\theta_1 = \frac{6 \pm \sqrt{36-4}}{2} = 3 \pm 2\sqrt{2}.$$

Since $\cos\theta_1 \leq 1$, $\cos\theta_1 = 3 - 2\sqrt{2}$ and $\theta_1 \approx 1.40$ or $\theta_1 \approx -1.40$. In either case $r_1 \approx -0.83$. The point of intersection in the second quadrant occurs when $\theta \approx -1.40$ and $r \approx -0.83$ on the lemniscate and when $\theta \approx -1.40 + \pi \approx 1.74$ and $r \approx 0.83$ on the cardioid. In the third quadrant the point of intersection occurs when $\theta \approx 1.40$ and $r \approx -0.83$ on the lemniscate and when $\theta \approx 1.40 + \pi \approx 4.54$ and $r \approx 0.83$ on the cardioid.

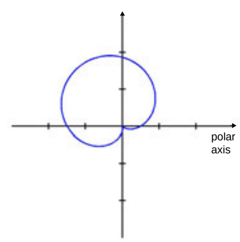
47. **(a)**



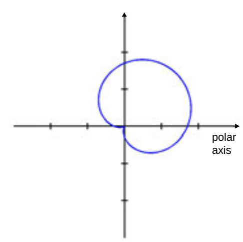
(b)



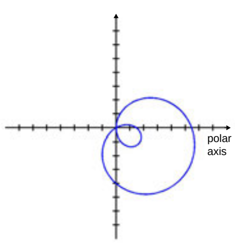
(c)



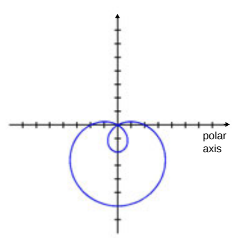
(d)



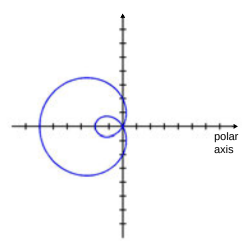
48. **(a)**



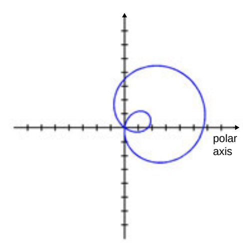
(b)



(c)



(d)



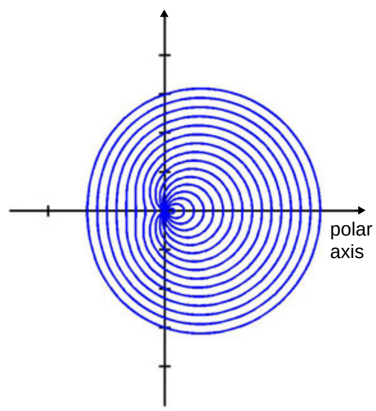
49. **(d)**

50. **(c)**

51. **(b)**

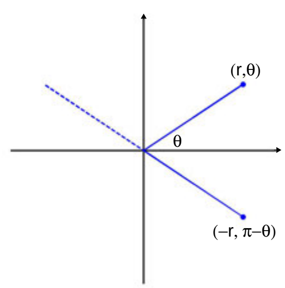
52. **(a)**

53.



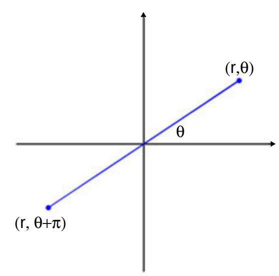
54. For $a = 0$, the graph is a circle.
For $a = \frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$, the graph is a limacon with an interior loop.
For $a = 1$, the graph is a cardiod.
For $a = \frac{5}{4}$, $\frac{3}{2}$, and $\frac{7}{4}$, the graph is a dimpled limacon
For $a = 2$, $\frac{9}{4}$, $\frac{5}{2}$, $\frac{11}{4}$, and 3, the graph is a convex limacon.
As $a \to \infty$, the graphs more closely approximate a circle.

55.



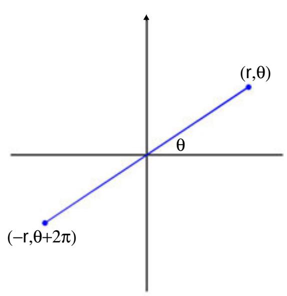
Symmetric with respect to the x-axis.

56.



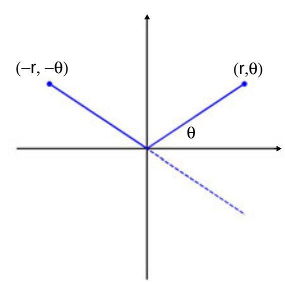
Symmetric with respect to the origin.

57.



Symmetric with respect to the origin.

58.



Symmetric with respect to the y-axis .

59. Symmetric with respect to the x-axis.

60. The graph is symmetric with respect to the y-axis.

61. (a) The graphs are identical.

    (b) The graphs are identical.

62. From the statement in the text preceding Problem 33 in Section 4.1 we have that the component of acceleration in the direction of the ramp is $-g\sin\theta$, where g is the acceleration due to gravity and $-\pi \leq \theta \leq 0$. Thus the distance traveled in time t along the ramp at angle θ is $r = -\frac{1}{2}gt^2\sin\theta$. But this is the equation of a circle of radius $gt^2/4$ centered at $(0, -gt^2/4)$, whose topmost point is $(0,0)$ which is taken at the point of release.

10.6 Calculus in Polar Coordinates

1. $$\frac{dy}{d\theta} = \theta\cos\theta + \sin\theta$$
$$\frac{dx}{d\theta} = -\theta\sin\theta + \cos\theta$$
$$\frac{dy}{dx} = \frac{\theta\cos\theta + \sin\theta}{-\theta\sin\theta + \cos\theta} = -\frac{2}{\pi} \text{ at } \theta = \tfrac{\pi}{2}.$$

2. At $\theta = 3$,
$$\frac{dy}{d\theta} = \frac{\cos\theta}{\theta} - \frac{\sin\theta}{\theta^2} = \frac{\cos 3}{3} - \frac{\sin 3}{9}$$
$$\frac{dx}{d\theta} = \frac{-\sin\theta}{\theta} - \frac{\cos\theta}{\theta^2} = -\frac{\sin 3}{3} - \frac{\cos 3}{9}$$
$$\frac{dy}{dx} = \frac{3\cos 3 - \sin 3}{-3\sin 3 - \cos 3}$$

3. At $\theta = \frac{\pi}{3}$,
$$\begin{aligned}\frac{dy}{d\theta} &= (4 - 2\sin\theta)\cos\theta + (-2\cos\theta)\sin\theta \\ &= 4 - 4\sin\theta\cos\theta = 4 - 4\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) = 4 - \sqrt{3}\end{aligned}$$
$$\begin{aligned}\frac{dx}{d\theta} &= -(4 - 2\sin\theta)\sin\theta + (-2\cos\theta)\cos\theta \\ &= -4 + 2\sin^2\theta - 2\cos^2\theta = -4 + 2\left(\frac{3}{4}\right) - 2\left(\frac{1}{4}\right) \\ &= -4 + \frac{3}{2} - \frac{1}{2} = -3\end{aligned}$$
$$\frac{dy}{dx} = \frac{4 - \sqrt{3}}{-3}$$

4. At $\theta = \dfrac{3\pi}{4}$,

$$\begin{aligned}
\frac{dy}{d\theta} &= (2 - \cos\theta)\cos\theta + (\sin\theta)\sin\theta \\
&= 1 - \cos^2\theta + \sin^2\theta = 1 - \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) = 1 \\
\frac{dx}{\theta} &= -(1 - \cos\theta)\sin\theta + (\sin\theta)\cos\theta \\
&= -\sin\theta + 2\cos\theta\sin\theta \\
&= -\frac{\sqrt{2}}{2} + 2\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) \\
&= 1 - \frac{\sqrt{2}}{2} \\
\frac{dy}{dx} &= \frac{1}{1 - \frac{\sqrt{2}}{2}}
\end{aligned}$$

5. At $\theta = \pi/6$,

$$\begin{aligned}
\frac{dy}{d\theta} &= \sin\theta\cos\theta + \cos\theta\sin\theta \\
&= 2\cos\theta\sin\theta = 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{2} \\
\frac{dx}{d\theta} &= -\sin^2\theta + \cos^2\theta = -\left(\frac{1}{4}\right) + \left(\frac{3}{4}\right) = \frac{1}{2} \\
\frac{dy}{dx} &= \sqrt{3}
\end{aligned}$$

6. At $\theta = \pi/4$,

$$\begin{aligned}
\frac{dy}{d\theta} &= (10\cos\theta)\cos\theta + (-10\sin\theta)\sin\theta \\
&= 10\cos^2\theta = 1 - \sin^2\theta = 10\left(\frac{1}{2}\right) - 10\left(\frac{1}{2}\right) \\
&= 0 \\
\frac{dx}{d\theta} &= -(10\cos\theta)\sin\theta + (-10\sin\theta)\cos\theta \\
&= -10\cos\theta\sin\theta = -20\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) \\
&= -10 \\
\frac{dy}{dx} &= \frac{0}{-10} = 0
\end{aligned}$$

7. $$\begin{aligned}\frac{dy}{d\theta} &= (2+2\cos\theta)\cos\theta + (-2\sin\theta)\sin\theta \\ &= 2(1+\cos^2\theta - \sin^2\theta) \\ \frac{dx}{d\theta} &= -(2+2\cos\theta)\sin\theta + (-2\sin\theta)\cos\theta \\ &= -2-4\cos\theta\sin\theta \\ \frac{dy}{dx} &= \frac{2(1+\cos^2\theta-\sin^2\theta)}{2(-2-2\cos\theta\sin\theta)} = \frac{1+\cos^2\theta-\sin^2\theta}{-1-2\cos\theta\sin\theta}\end{aligned}$$
If the tangent line is horizontal, we must have
$$1+\cos^2\theta-\sin^2\theta = 0$$
which requires $\sin\theta = \pm 1$ and thus $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$. Hence, the polar coordinates of points on the graph with horizontal tangents are $(2, \pi/2)$ and $(2, 3\pi/2)$. If the tangent line is vertical, we must have
$$-1-2\cos\theta\sin\theta = 0 \quad \text{or} \quad \cos\theta\sin\theta = -\frac{1}{2}$$
which occurs at $\theta = \frac{3\pi}{4}$ or $\theta = \frac{7\pi}{4}$. Hence the polar coordinates of points on the graph with vertical tangents are $(2-\sqrt{3}, 3\pi/4)$ and $(2+\sqrt{3}, 7\pi/4)$.

8. $$\begin{aligned}\frac{dy}{d\theta} &= (1-\sin\theta)\cos\theta + (-\cos\theta)\sin\theta \\ &= 1-2\sin\theta\cos\theta \\ \frac{dx}{d\theta} &= -(1-\sin\theta)\sin\theta + (-\cos\theta)\cos\theta \\ &= -\sin\theta + \sin^2\theta - \cos^2\theta \\ \frac{dy}{dx} &= \frac{1-2\sin\theta\cos\theta}{-\sin\theta+\sin^2\theta-\cos^2\theta}\end{aligned}$$
If the tangent line is horizontal, we must have
$$1-2\sin\theta\cos\theta = 0 \quad \text{or} \quad \sin\theta\cos\theta = \frac{1}{2}$$
which occurs at $\theta = \frac{\pi}{4}$ and $\theta = \frac{5\pi}{4}$. Hence, the polar coordinates of points on the graph with horizontal tangents are $\left(1-\frac{\sqrt{2}}{2}, \frac{\pi}{4}\right)$ and $\left(1+\frac{\sqrt{2}}{2}, \frac{5\pi}{4}\right)$. If the tangent line is vertical, we must have
$$\begin{aligned}-\sin\theta+\sin^2\theta-\cos^2\theta &= 0 \\ -\sin\theta+\sin^2\theta-(1-\sin^2\theta) &= 0 \\ 2\sin^2\theta-\sin\theta-1 &= 0\end{aligned}$$
which gives $\sin\theta = -\frac{1}{2}$ or $\sin\theta = 1$. This occurs at $\theta = \frac{\pi}{2}$, $\theta = \frac{7\pi}{6}$, and $\theta = \frac{11\pi}{6}$. Hence, the polar coordinates of points on the graph with vertical tangents are $\left(0, \frac{\pi}{2}\right)$, $\left(\frac{3}{2}, \frac{7\pi}{6}\right)$, and $\left(\frac{3}{2}, \frac{11\pi}{6}\right)$.

9. $$\begin{aligned}\frac{dy}{d\theta} &= (4\cos 3\theta)(\cos\theta) + (-12\sin 3\theta)(\sin\theta) \\ \frac{dx}{d\theta} &= -(4\cos 3\theta)(\sin\theta) + (-12\sin 3\theta)(\cos\theta)\end{aligned}$$

The points on the graph correspond to $\theta = \frac{\pi}{3}$ and $\theta = \frac{2\pi}{3}$. At $\theta = \frac{\pi}{3}$, we have

$$\frac{dy}{dx} = \frac{(-4)\left(\frac{1}{2}\right) + (0)\left(\frac{\sqrt{3}}{2}\right)}{-(-4)\left(\frac{\sqrt{3}}{2}\right) + (0)\left(\frac{1}{2}\right)} = \frac{-2}{2\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

and the rectangular coordinates of the point are $(-2, -2\sqrt{3})$. Hence, the equation of the tangent line is $y = -2\sqrt{3} - \dfrac{\sqrt{3}}{3}(x+2)$. At $\theta = \frac{2\pi}{3}$, we have

$$\frac{dy}{dx} = \frac{4\left(-\frac{1}{2}\right) + (0)\left(\frac{\sqrt{3}}{2}\right)}{-(4)\left(\frac{\sqrt{3}}{2}\right) + (0)\left(-\frac{1}{2}\right)} = \frac{2}{2\sqrt{3}} = \frac{\sqrt{3}}{3}$$

and the rectangular coordinates of the point are $(-2, 2\sqrt{3})$. Hence, the equation of the tangent line is $y = 2\sqrt{3} + \dfrac{\sqrt{3}}{3}(x+2)$.

10. $$\frac{dy}{d\theta} = (1 + 2\cos\theta)\cos\theta + (-2\sin\theta)\sin\theta = 1 + 2\cos^2\theta - 2\sin^2\theta$$
$$\frac{dx}{d\theta} = -(1 + 2\cos\theta)\sin\theta + (-2\sin\theta)\cos\theta = -\sin\theta - 4\cos\theta\sin\theta$$
At $\theta = \frac{\pi}{3}$, we have

$$\frac{dy}{dx} = \frac{1 + 2\left(\frac{1}{2}\right)^2 - 2\left(\frac{\sqrt{3}}{2}\right)^2}{-\frac{\sqrt{3}}{2} - 4\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)} = 0.$$

Also, $r = 2$ so the rectangular coordinates of the point are $(1, \sqrt{3})$. Hence, the equation of the tangent line is $y = \sqrt{3}$.
At $\theta = \dfrac{5\pi}{3}$, we have

$$\frac{dy}{dx} = \frac{1 + 2\left(\frac{1}{2}\right)^2 - 2\left(-\frac{\sqrt{3}}{2}\right)^2}{\frac{\sqrt{3}}{2} - 4\left(\frac{1}{2}\right)\left(-\frac{\sqrt{3}}{2}\right)} = 0.$$

Also, $r = 2$ so the rectangular coordinates of the point are $(1, -\sqrt{3})$. Hence, the equation of the tangent line is $y = -\sqrt{3}$.

11. $r = 0$ when $\sin\theta = 0$ which occurs at $\theta = 0$ and $\theta = \pi$. $\dfrac{dr}{d\theta} = -2\cos\theta \neq 0$ at either $\theta = 0$ or $\theta = \pi$. Therefore, $\theta = 0$ and $\theta = \pi$ define tangent lines to the graph at the origin.

12. $r = 0$ when $\cos\theta = 0$ which occurs at $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$. $\dfrac{dr}{d\theta} = -3\sin\theta \neq 0$ at either $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$. Therefore, $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$ define tangent lines to the graph at the origin.

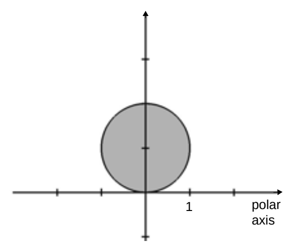
13. $r = 0$ when $\sin\theta = -\dfrac{1}{\sqrt{2}} = -\dfrac{\sqrt{2}}{2}$ which occurs at $\theta = \dfrac{5\pi}{4}$ and $\theta = \dfrac{7\pi}{4}$. $\dfrac{dr}{d\theta} = \sqrt{2}\cos\theta \neq 0$ at $\theta = \dfrac{5\pi}{4}$ or $\theta = \dfrac{7\pi}{4}$. Therefore, $\theta = \dfrac{5\pi}{4}$ and $\theta = \dfrac{7\pi}{4}$ define tangent lines to the graph at the origin.

14. $r = 0$ when $\sin\theta = \dfrac{1}{2}$ which occurs at $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$. $\dfrac{dr}{d\theta} = -2\cos\theta \neq 0$ at $\theta = \frac{\pi}{6}$ or $\theta = \frac{5\pi}{6}$. Therefore, $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$ define tangent lines to the graph at the origin.

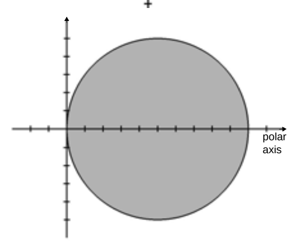
15. $r = 0$ when $\cos 5\theta = 0$ which occurs at $\theta = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{5\pi}{10}, \frac{7\pi}{10}, \frac{9\pi}{10}, \frac{11\pi}{10}, \frac{13\pi}{10}, \frac{15\pi}{10}, \frac{17\pi}{10}$, and $\frac{19\pi}{10}$. $\dfrac{dr}{d\theta} = -5\sin 5\theta \neq 0$ at any of these θ values. Therefore, $\theta = \dfrac{n\pi}{10}$ defines a tangent line to the graph at the origin for $n = 1, 3, 5, \ldots, 19$.

16. $r = 0$ when $\sin 2\theta = 0$ which occurs at $\theta = 0$, $\frac{\pi}{2}$, π, and $\frac{3\pi}{2}$. $\dfrac{dr}{d\theta} = 4\cos 2\theta \neq 0$ at any of these θ values. Therefore, $\theta = 0$, $\theta = \frac{\pi}{2}$, $\theta = \pi$ and $\theta = \frac{3\pi}{2}$ define tangent lines to the graph at the origin.

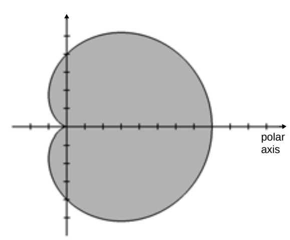
17. $A = \dfrac{1}{2}\displaystyle\int_0^\pi 4\sin^2\theta d\theta = \left(\theta - \dfrac{1}{2}\sin 2\theta\right)\Big|_0^\pi = \pi$



18. $A = \dfrac{1}{2}\displaystyle\int_0^\pi 100\cos^2\theta d\theta = \left(25\theta + \dfrac{25}{2}\sin 2\theta\right)\Big|_0^\pi = 25\pi$

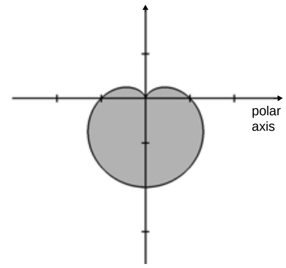


19. $A = \dfrac{1}{2}\displaystyle\int_0^{2\pi} (4 + 4\cos\theta)^2 d\theta = 8\int_0^{2\pi} (1 + 2\cos\theta + \cos^2\theta) d\theta$
$= (8\theta + 16\sin\theta + 4\theta + 2\sin 2\theta)\big|_0^{2\pi} = 24\pi$



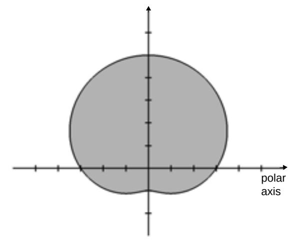
20. $A = \frac{1}{2}\int_0^{2\pi}(1-\sin\theta)^2 d\theta = \frac{1}{2}\int_0^{2\pi}(1 - 2\sin\theta + \sin^2\theta)d\theta$

$= \left(\frac{1}{2}\theta + \cos\theta + \frac{1}{4}\theta - \frac{1}{8}\sin 2\theta\right)\Big|_0^{2\pi} = \frac{3}{2}\pi$



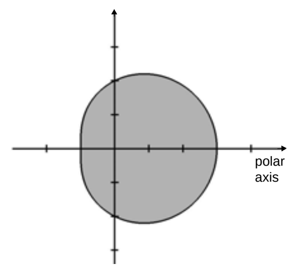
21. $A = \frac{1}{2}\int_0^{2\pi}(3+2\sin\theta)^2 d\theta = \frac{1}{2}\int_0^{2\pi}(9 + 12\sin\theta + 4\sin^2\theta)d\theta$

$= \left(\frac{9}{2}\theta - 6\cos\theta + \theta - \frac{1}{2}\sin 2\theta\right)\Big|_0^{2\pi} = 11\pi$



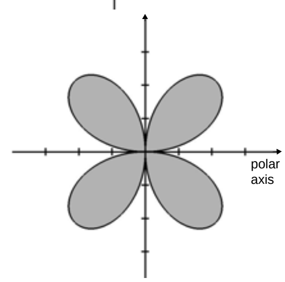
22. $A = \frac{1}{2}\int_0^{2\pi}(2+\cos\theta)^2 d\theta = \frac{1}{2}\int_0^{2\pi}(4 + 4\cos\theta + \cos^2\theta)d\theta$

$= \left(2\theta + 2\sin\theta + \frac{1}{4}\theta + \frac{1}{8}\sin 2\theta\right)\Big|_0^{2\pi} = \frac{9}{2}\pi$



23. $A = \frac{1}{2}\int_0^{2\pi} 9\sin^2 2\theta d\theta \quad \boxed{u = 2\theta \;\; du = 2d\theta}$

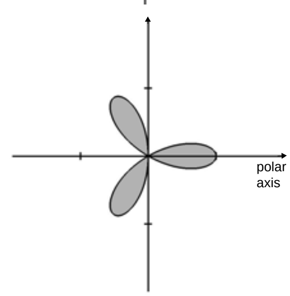
$= \frac{9}{4}\int_0^{4\pi}\sin^2 u du = \left(\frac{9}{8}u - \frac{9}{16}\sin 2u\right)\Big|_0^{4\pi} = \frac{9}{2}\pi$



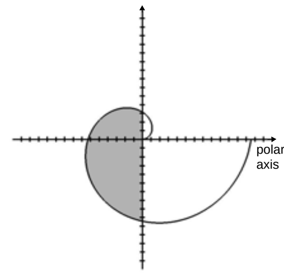
24. $A = \frac{1}{2}\int_0^{\pi}\cos^2 3\theta d\theta \quad \boxed{u = 3\theta, \;\; du = 3d\theta}$

$= \frac{1}{6}\int_0^{3\pi}\cos^2 u du$

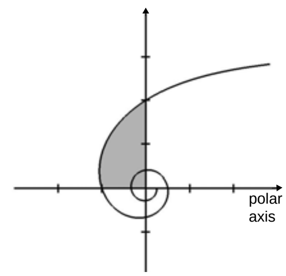
$= \frac{1}{6}\left[\frac{1}{2}u + \frac{1}{4}\sin 2u\right]\Big|_0^{3\pi} = \frac{\pi}{4}$



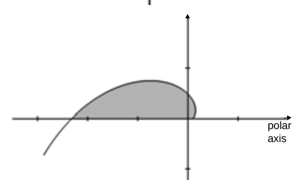
25. $A = \frac{1}{2}\int_0^{3\pi/2} 4\theta^2 d\theta = \left.\frac{2}{3}\theta^3\right|_0^{3\pi/2} = \frac{29}{4}\pi^3$



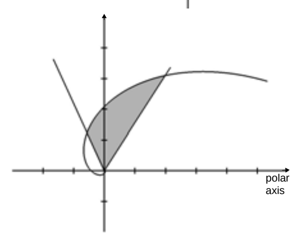
26. $A = \frac{1}{2}\int_{\pi/2}^{\pi} \left(\frac{\pi}{\theta}\right) d\theta = \left.-\frac{\pi^2}{2\theta}\right|_{\pi/2}^{\pi} = -\left(\frac{\pi}{2} - \pi\right) = \frac{\pi}{2}$



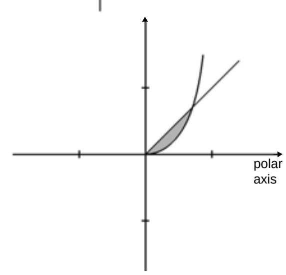
27. $A = \frac{1}{2}\int_0^{\pi} e^{2\theta} d\theta = \left.\frac{1}{4}e^{2\theta}\right|_0^{\pi} = \frac{1}{4}e^{2\theta} - \frac{1}{4}$



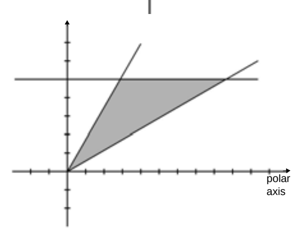
28. $A = \frac{1}{2}\int_1^2 100e^{-2\theta} d\theta$
$= -25e^{-2\theta}\Big|_1^2 = -25(e^{-4} - e^{-2}) = 25e^{-2} - 25e^{-4}$



29. $A = \frac{1}{2}\int_0^{\pi/4} \tan^2\theta d\theta = \frac{1}{2}\int_0^{\pi/4} (\sec^2\theta - 1) d\theta$
$= \left.\left(\frac{1}{2}\tan\theta - \frac{1}{2}\theta\right)\right|_0^{\pi/4} = \left(\frac{1}{2} - \frac{\pi}{8}\right) - 0 = \frac{4-\pi}{8}$



30. $A = \frac{1}{2}\int_{\pi/6}^{\pi/3} 25\csc^2\theta d\theta = \left.-\frac{25}{2}\cot\theta\right|_{\pi/6}^{\pi/3}$
$= -\frac{25}{2}\left(\frac{\sqrt{3}}{3} - \sqrt{3}\right) = \frac{25\sqrt{3}}{3}$

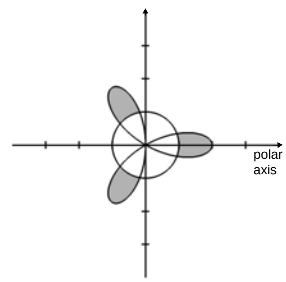


31. $$\begin{aligned} A &= \frac{1}{2}\int_{2\pi/3} 4\pi/3(1+2\cos\theta)^2 d\theta \\ &= \frac{1}{2}\int_{2\pi/3} 4\pi/3(1+4\cos\theta+4\cos^2\theta)d\theta \\ &= \frac{1}{2}\left[\theta+4\sin\theta+4\left(\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta\right)\right]\Bigg|_{2\pi/3}^{4\pi/3} \\ &= \frac{2\pi-3\sqrt{3}}{2} \end{aligned}$$

32. $$\begin{aligned} A &= \frac{1}{2}\int_{-2\pi/3}^{2\pi/3}(1+2\cos\theta)^2 d\theta - \frac{1}{2}\int_{2\pi/3}^{4\pi/3}(1+2\cos\theta)^2 d\theta \\ &= \frac{1}{2}\int_{-2\pi/3}^{2\pi/3}(1+4\cos\theta+4\cos^2\theta)d\theta - \frac{1}{2}\int_{2\pi/3}^{4\pi/3}(1+4\cos\theta+4\cos^2\theta)d\theta \\ &= \frac{1}{2}\left(\theta+4\sin\theta+4\left(\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta\right)\right)\Bigg|_{-2\pi/3}^{2\pi/3} - \frac{1}{2}\left(\theta+4\sin\theta+4\left(\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta\right)\right)\Bigg|_{2\pi/3}^{4\pi/3} \\ &= \frac{1}{2}(3\sqrt{3}+4\pi) - \frac{1}{2}(2\pi-3\sqrt{3}) \\ &= 3\sqrt{3}+\pi \end{aligned}$$

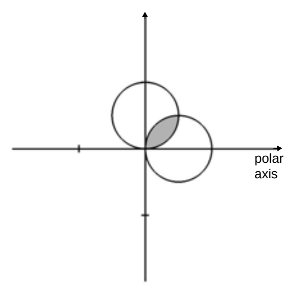
33. Solving $2\cos 3\theta = 1$ in the first quadrant, we obtain $\cos 3\theta = 1/2$, $3\theta = \pi/3$, and $\theta = \pi/9$. Using symmetry,

$$\begin{aligned} A &= 6\left[\int_0^{\pi/9}(4\cos^2 3\theta - 1)d\theta\right] \qquad \boxed{u = 3\theta,\ \ du = 3d\theta} \\ &= 4\int_0^{\pi/3}(\cos^2 u - 1)du = (2u + \sin 2u - u)\big|_0^{\pi/3} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}. \end{aligned}$$

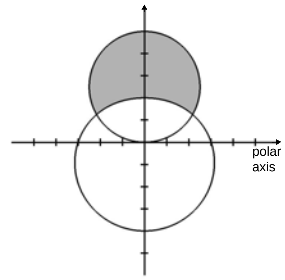


34. The circles intersect at $\theta = \pi/4$. Using symmetry,

$$A = 2\left[\frac{1}{2}\int_0^{\pi/4}\sin^2\theta d\theta\right] = \left(\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right)\Bigg|_0^{\pi/4} = \frac{\pi}{8} - \frac{1}{4} = \frac{\pi-2}{8}.$$

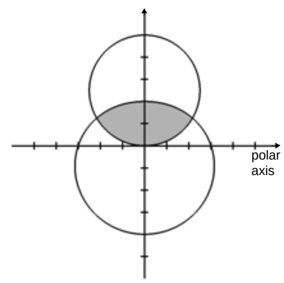


35. Solving $5\sin\theta = 3 - \sin\theta$ in the first quadrant, we obtain $\sin\theta = 1/2$ and $\theta = \pi/6$. Using symmetry,



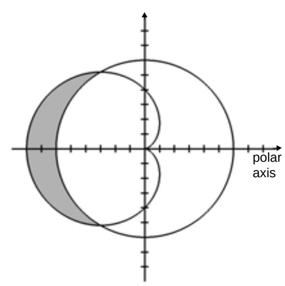
$$\begin{aligned}
A &= 2\left[\frac{1}{2}\int_{\pi/6}^{\pi/2}(25\sin^2\theta - (3-\sin\theta)^2)d\theta\right]\\
&= \int_{\pi/6}^{\pi/2}(24\sin^2\theta + 6\sin\theta - 9)d\theta\\
&= (12\theta - 6\sin 2\theta - 6\cos\theta - 9\theta)|_{\pi/6}^{\pi/2}\\
&= \frac{3\pi}{2} - \left(\frac{\pi}{2} - 3\sqrt{3} - 3\sqrt{3}\right) = \pi + 6\sqrt{3}.
\end{aligned}$$

36. From Problem 35, the point of intersection in the first quadrant is at $\theta = \pi/6$. Using symmetry,



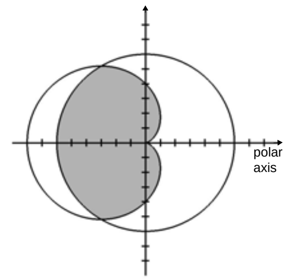
$$\begin{aligned}
A &= 2\left[\frac{1}{2}\int_{0}^{\pi/6}25\sin^2\theta d\theta + \frac{1}{2}\int_{\pi/6}^{\pi/3}(3-\sin\theta)^2 d\theta\right]\\
&= 25\int_{0}^{\pi/6}\sin^2\theta d\theta + \int_{\pi/6}^{\pi/3}(9 - 6\sin\theta + \sin^2\theta)d\theta\\
&= \left(\frac{25}{2}\theta - \frac{25}{4}\sin 2\theta\right)\Bigg|_{0}^{\pi/6} + \left(9\theta + 6\sin\theta + \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right)\Bigg|_{\pi/6}^{\pi/2}\\
&= \left(\frac{25\pi}{12} - \frac{25\sqrt{3}}{8}\right) + \frac{19\pi}{4} - \left(\frac{19\pi}{12} + 3\sqrt{3} - \frac{\sqrt{3}}{8}\right) = \frac{21\pi}{4} - 6\sqrt{3}.
\end{aligned}$$

37. Solving $4 - 4\cos\theta = 6$ in the second quadrant, we obtain $\cos\theta = -1/2$ and $\theta = 2\pi/3$. Using symmetry,



$$\begin{aligned}
A &= 2\left[\frac{1}{2}\int_{2\pi/3}^{\pi}\left((4-4\cos\theta)^2 - 36\right)\right]d\theta\\
&= \int_{2\pi/3}^{\pi}(16\cos^2\theta - 32\cos\theta - 20)d\theta\\
&= (8\theta + 4\sin 2\theta - 32\sin\theta - 20\theta)|_{2\pi/3}^{\pi}\\
&= -12\pi - (8\pi - 2\sqrt{3} - 16\sqrt{3}) = 18\sqrt{3} - 4\pi.
\end{aligned}$$

38. From Problem 37, the point of intersection in the second quadrant is at $\theta = 2\pi/3$. Using symmetry,



$$\begin{aligned} A &= 2\left[\frac{1}{2}\int_0^{2\pi/3}(4-4\cos\theta)^2 d\theta + \frac{1}{2}\int_{2\pi/3}^{\pi} 36 d\theta\right] \\ &= 16\int_0^{2\pi/3}(1-2\cos\theta+\cos^2\theta)d\theta + 36\left(\pi - \frac{2\pi}{3}\right) \\ &= (16\theta - 32\sin\theta + 8\theta + 4\sin 2\theta)\big|_0^{2\pi/3} + 12\pi \\ &= 16\pi - 16\sqrt{3} - 2\sqrt{3} + 12\pi = 28\pi - 18\sqrt{3}. \end{aligned}$$

39. $\dfrac{dr}{d\theta} = 0$ so we have $L = \int_0^{2\pi}\sqrt{3^2}d\theta = \int_0^{2\pi} 3d\theta = 6\pi$

40. $\dfrac{dr}{d\theta} = -6\sin\theta$ so we have

$$\begin{aligned} L &= \int_0^{\pi}\sqrt{36\cos^2\theta + 36\sin^2\theta}d\theta \\ &= \int_0^{\pi} 6d\theta = 6\pi \end{aligned}$$

41. $\dfrac{dr}{d\theta} = \dfrac{1}{2}e^{\theta/2}$; $\left(\dfrac{dr}{d\theta}\right)^2 + r^2 = \dfrac{1}{4}e^{\theta} + e^{\theta} = \dfrac{5}{4}e^{\theta}$; $s = \dfrac{\sqrt{5}}{2}\int_0^4 e^{\theta/2}d\theta = \sqrt{5}e^{\theta/2}\big|_0^4 = \sqrt{4}(e^2-1)$

42. $\dfrac{dr}{d\theta} = -2e^{-\theta}$; $\left(\dfrac{dr}{d\theta}\right)^2 + r^2 = 4e^{-2\theta} + 4e^{-2\theta} = 8e^{-2\theta}$;
$s = 2\sqrt{2}\int_0^{\pi} e^{-\theta}d\theta = -2\sqrt{2}e^{-\theta}\big|_0^{\pi} = -2\sqrt{2}(e^{-\pi-1}) = 2\sqrt{2}(1-e^{-\pi})$

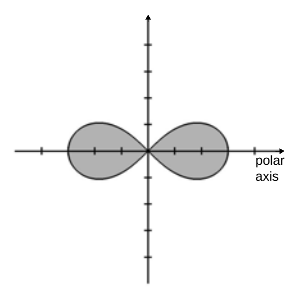
43. $\dfrac{dr}{d\theta} = 3\sin\theta$ so we have

$$\begin{aligned}
L &= \int_0^{2\pi} \sqrt{(3-3\cos\theta)^2+(3\sin\theta)^2}d\theta \\
&= \int_0^{2\pi} \sqrt{9-18\cos\theta+9\cos^2\theta+9\sin^2\theta}d\theta \\
&= \int_0^{2\pi} \sqrt{18-18\cos\theta}d\theta \\
&= \sqrt{18}\int_0^{2\pi} \sqrt{1-\cos\theta}d\theta \\
&\boxed{\cos^2\left(\frac{\theta}{2}\right)=\frac{1}{2}(1+\cos\theta) \longrightarrow 1-\cos\theta = 2\sin^2\left(\frac{\theta}{2}\right)} \\
&= \sqrt{18}\int_0^{2\pi} \sqrt{2\sin^2\left(\frac{\theta}{2}\right)}d\theta \\
&= 6\int_0^{2\pi} \sin\left(\frac{\theta}{2}\right)d\theta \quad \boxed{u=\frac{\theta}{2},\quad du = 2d\theta} \\
&= 12\int_0^{\pi} \sin u du \\
&= 12\left(-\cos u\right)\Big|_0^{\pi} \\
&= 24
\end{aligned}$$

44. $\dfrac{dr}{d\theta} = \sin^2\left(\frac{\theta}{3}\right)\cos\left(\frac{\theta}{3}\right)$ so we have

$$\begin{aligned}
L &= \int_0^{\pi} \sqrt{\sin^6\left(\frac{\theta}{3}\right)+\sin^4\left(\frac{\theta}{3}\right)\cos^2\left(\frac{\theta}{3}\right)}d\theta \\
&= \int_0^{\pi} \sqrt{\sin^4\left(\frac{\theta}{3}\right)}d\theta \qquad = \int_0^{\pi} \sin^2\left(\frac{\theta}{3}\right)d\theta \quad \boxed{u=\frac{\theta}{3},\quad du=\frac{1}{2}d\theta} \\
&= 3\int_0^{\frac{\pi}{3}} \sin^2 u du = 3\left(\frac{1}{2}u-\frac{1}{4}\sin 2u\right)\Big|_0^{\pi/3} \\
&= \frac{4\pi-3\sqrt{3}}{8}
\end{aligned}$$

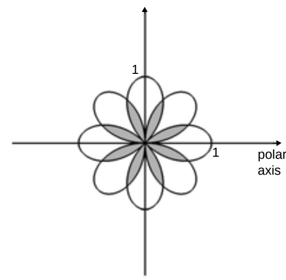
45. (a) The lemniscate $r^2 = 9\cos 2\theta$ is only defined for $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$ and $\frac{3\pi}{4} \leq \theta \leq \frac{5\pi}{4}$.



(b) $A = \dfrac{1}{2}\displaystyle\int_{-\pi/4}^{\pi/4} 9\cos 2\theta d\theta + \dfrac{1}{2}\displaystyle\int_{3\pi/4}^{5\pi/4} 9\cos 2\theta d\theta$

$$\begin{aligned}
&\boxed{u=2\theta,\quad du=2d\theta} \\
&= \frac{9}{4}\left(\sin u\right)\Big|_{-\pi/2}^{\pi/2} + \frac{9}{4}\left(\sin u\right)\Big|_{3\pi/2}^{5\pi/2} \\
&= \frac{9}{2}+\frac{9}{2} = 9
\end{aligned}$$

46. Example 8 uses the fact that $\sqrt{\cos^2\left(\frac{\theta}{2}\right)} = \cos\left(\frac{\theta}{2}\right)$ on the interval $0 \le \theta \le \pi$. This same equality does not hold on the interval $\pi < \theta < 2\pi$ since $\cos\left(\frac{\theta}{2}\right)$ is negative on this interval. In Problem 43, the equality $\sqrt{\sin^2\left(\frac{\theta}{2}\right)} = \sin\left(\frac{\theta}{2}\right)$ is used which holds for all θ in the interval $0 \le \theta \le 2\pi$.

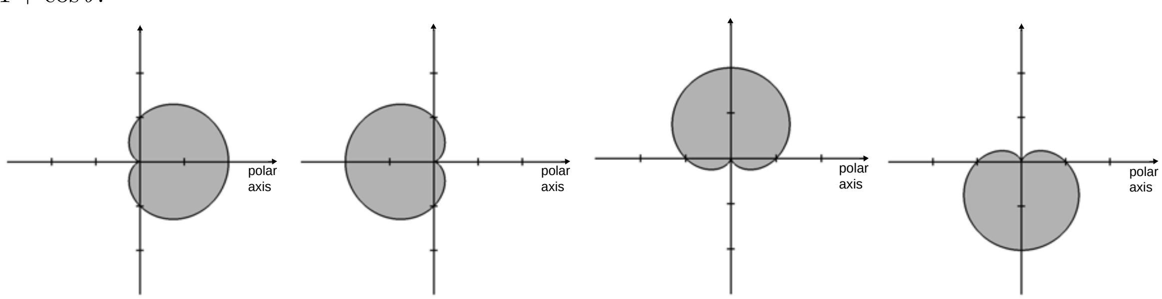


47. To obtain the area, we can compute the area of half of one of the petals and then use symmetry, multiplying by 16.

$$\begin{aligned}\text{area of a half-petal} &= \frac{1}{2}\int_0^{\pi/8} \sin^2 2\theta d\theta \quad \boxed{u = 2\theta,\ du = 2d\theta} \\ &= \frac{1}{4}\int_0^{\pi/4} \sin^2 u du \\ &= \frac{1}{4}\left(\frac{1}{2}u - \frac{1}{4}\sin 2u\right)\Big|_0^{\pi/4} \\ &= \frac{\pi - 2}{32}\end{aligned}$$

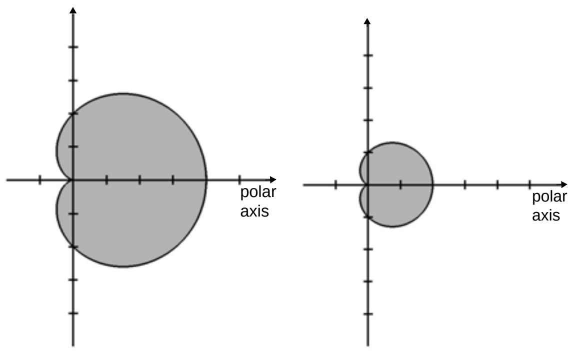
$$\text{Total area} = 16 \cdot \left(\frac{\pi - 2}{32}\right) = \frac{\pi - 2}{2}$$

48. Each of the areas will equal $\frac{3\pi}{2}$ since the graphs are simply rotations of the graph of $r = 1 + \cos\theta$.

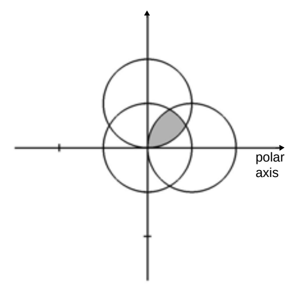


49. No; Let A_1 denote the area of the graph of $r = 2(1 + \cos\theta)$ and let A_2 denote the area of the graph of $r = 1 + \cos\theta$. Then

$$\begin{aligned}A &= \frac{1}{2}\int_0^{2\pi} 4(1 + \cos\theta)^2 d\theta = 4\left[\frac{1}{2}\int_0^{2\pi} (1 + \cos\theta)^2 d\theta\right] \\ &= 4A_1\end{aligned}$$



50. The equations of the circles are $r = 1$, $r = 2\sin\theta$, and $r = 2\cos\theta$. We split the area into three regions to get



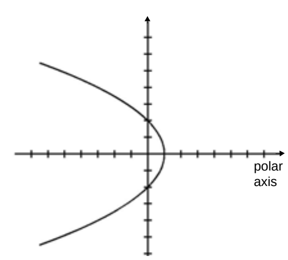
$$A = \frac{1}{2}\int_0^{\pi/6} 4\sin^2\theta d\theta + \frac{1}{2}\int_{\pi/6}^{\pi/3} 1 d\theta + \frac{1}{2}\int_{\pi/3}^{\pi} 4\cos^2\theta d\theta$$
$$= 2\left(\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right)\Big|_0^{\pi/6} + \frac{1}{2}\,(\theta)|_{\pi/6}^{\pi/3} + 2\left(\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right)\Big|_{\pi/3}^{\pi}$$
$$= \frac{\pi}{6} - \frac{\sqrt{3}}{4} + \frac{\pi}{12} + \frac{2\pi}{3} - \frac{\sqrt{3}}{4}$$

51. From $L = mr^3 d\theta = dt$ we obtain $r^2 d\theta = L dt/m$. Then

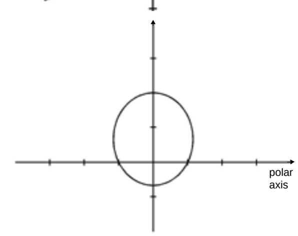
$$A = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2 d\theta = \frac{1}{2}\int_a^b \frac{L}{m} dt = \frac{L}{2m}(b-a).$$

10.7 Conic Sections in Polar Coordinates

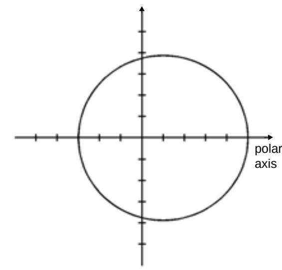
1. Identifying $e = 1$, the graph is a parabola.



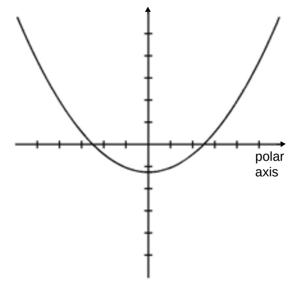
2. Writing $r = \dfrac{1}{1-(1/2)\sin\theta}$, we identify $e = 1/2$. The graph is an ellipse.



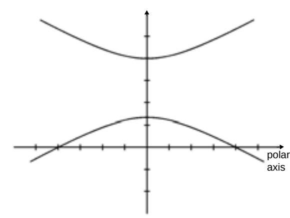
3. Writing $r = \dfrac{15/4}{1-(1/4)\cos\theta}$, we identify $e = 1/4$. The graph is an ellipse.



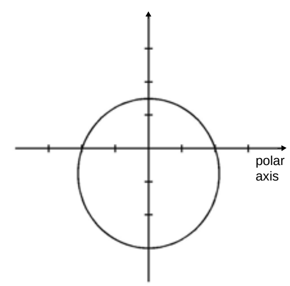
4. Writing $r = \dfrac{5/2}{1-\sin\theta}$, we identify $e = 1$. The graph is a parabola.



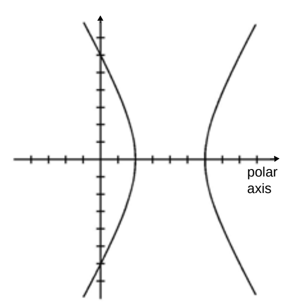
5. Identifying $e = 2$, the graph is a hyperbola.



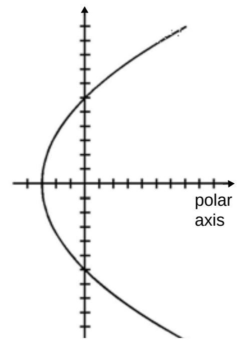
6. Writing $r = \dfrac{2}{1+(1/3)\sin\theta}$, we identify $e = 1/3$. The graph is an ellipse.



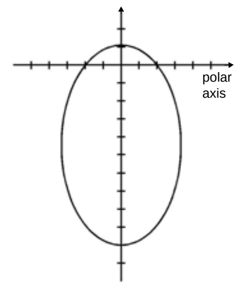
7. Writing $r = \dfrac{6}{1+2\cos\theta}$, we identify $e = 2$. The graph is a hyperbola.



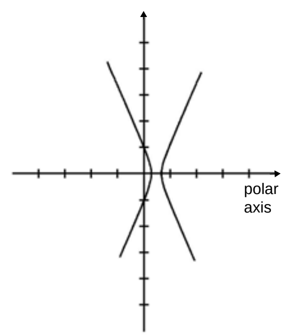
8. Writing $r = \dfrac{6}{1-\cos\theta}$, we identify $e = 1$. The graph is a parabola.



9. Writing $r = \dfrac{2}{1+(4/5)\sin\theta}$, we identify $e = 4/5$. The graph is an ellipse.



10. Writing $r = \dfrac{1}{1+(5/2)\cos\theta}$, we identify $e = 5/2$. The graph is an hyperbola.



11. From $r = \dfrac{6}{1+2\sin\theta}$, we have $e = 2$. Converting to a rectangular equation, we get

$$\begin{aligned} r &= \frac{6}{1+2\sin\theta} \\ r + 2r\sin\theta &= 6 \\ r &= 6 - 2r\sin\theta \\ \sqrt{x^2+y^2} &= 6 - 2y \\ x^2 + y^2 &= 36 - 24y + 4y^2 \\ \frac{(y-4)^2}{4} - \frac{x^2}{12} &= 1 \end{aligned}$$

with $a = 2$ and $b = \sqrt{12}$ so $c^2 = a^2 + b^2 = 4 + 12 = 16$ so $c = 4$. Thus $e = \dfrac{c}{a} = \dfrac{4}{2} = 2$.

12. From $r = \dfrac{10}{2 - 3\cos\theta} = \dfrac{5}{1 - \frac{3}{2}\cos\theta}$, we have $e = \frac{3}{2}$. Converting to a rectangular equation, we get

$$\begin{aligned} r &= \frac{10}{2 - 3\cos\theta} \\ 2r &= 10 + 3r\cos\theta \\ 2\sqrt{x^2 + y^2} &= 10 + 3x \\ 4(x^2 + y^2) &= 100 + 60x + 9x^2 \\ \frac{(x+6)^2}{16} - \frac{y^2}{20} &= 1 \end{aligned}$$

with $a = 4$ and $b = \sqrt{20}$ so $c^2 = a^2 + b^2 = 16 + 20$. Thus $c = 6$ and $e = \frac{c}{a} = \frac{6}{4} = \frac{3}{2}$.

13. From $r = \dfrac{12}{3 - 2\cos\theta} = \dfrac{4}{1 - \frac{2}{3}\cos\theta}$, we have $e = \frac{2}{3}$. Converting to a rectangular equation, we get

$$\begin{aligned} r &= \frac{12}{3 - 2\cos\theta} \\ 3r &= 12 + 2r\cos\theta \\ 3\sqrt{x^2 + y^2} &= 12 + 2x \\ \frac{\left(x - \frac{24}{5}\right)^2}{\frac{1296}{25}} + \frac{y^2}{\frac{144}{5}} &= 1 \end{aligned}$$

with $a = \frac{36}{5}$ and $b = \frac{12}{\sqrt{5}}$ so $c^2 = a^2 - b^2 = \frac{1296}{25} - \frac{144}{5} = \frac{576}{25}$. Thus $c = \frac{24}{5}$ so $e = \frac{c}{a} = \frac{25/4}{36/5} = \frac{2}{3}$.

14. From $r = \dfrac{2\sqrt{3}}{\sqrt{3} + \sin\theta} = \dfrac{2}{1 + \frac{1}{\sqrt{3}}\sin\theta}$, we have $e = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$. Converting to a rectangular equation, we get

$$\begin{aligned} r &= \frac{2\sqrt{3}}{\sqrt{3} + \sin\theta} \\ \sqrt{3}r &= 2\sqrt{3} - r\sin\theta \\ \sqrt{3}\sqrt{x^2 + y^2} &= 2\sqrt{3} - y \\ \frac{x^2}{6} + \frac{(y + \sqrt{3})}{9} &= 1 \end{aligned}$$

with $a = 3$ and $b = \sqrt{6}$ so $c^2 = a^2 - b^2 = 9 - 6 = 3$. Hence $c = \sqrt{3}$ and $e = \dfrac{c}{a} = \dfrac{\sqrt{3}}{3}$.

15. Since $e = 1$, the conic is a parabola. The directrix is 3 units to the right of the focus and perpendicular to the x-axis. Therefore, $r = \dfrac{3}{1 + \cos\theta}$.

16. Since $e = \frac{3}{2}$, the conic is a hyperbola. The directrix is 2 units above the focus and parallel to the x-axis. Therefore, $r = \dfrac{3}{1 + \frac{3}{2}\sin\theta}$.

17. Since $e = \frac{2}{3}$, the conic is an ellipse. The directrix is 2 units below the focus and parallel to the x-axis. Therefore, $r = \dfrac{\frac{4}{3}}{1 - \frac{2}{3}\sin\theta}$.

18. Since $e = \frac{1}{2}$, the conic is an ellipse. The directrix is 4 units to the right of the focus and perpendicular to the x-axis. Therefore, $r = \dfrac{2}{1 + \frac{1}{2}\cos\theta}$.

19. Since $e = 2$, the conic is a hyperbola. The directrix is 6 units to the right of the focus and perpendicular to the x-axis. Therefore, $r = \dfrac{12}{1 + 2\cos\theta}$.

20. Since $e = 1$, the conic is a parabola. The driectrix is 2 units below the focus and parallel to the x-axis. Therefore, $r = \dfrac{2}{1 - \sin\theta}$.

21. $r = \dfrac{3}{1 + \cos\left(\theta + \frac{2\pi}{3}\right)}$

22. $r = \dfrac{3}{1 + \frac{3}{2}\sin\left(\theta - \frac{\pi}{6}\right)}$

23. Since the vertex is $\frac{3}{2}$ units below the focus, the directrix must be 3 units below the focus and parallel to the x-axis. Therefore, $r = \dfrac{3}{1 - \sin\theta}$.

24. Since the vertex is 2 units to the left of the focus, the directrix must be 4 units tot he left of the focus and perpendicular to the x-axis. Therefore, $r = \dfrac{4}{1 - \cos\theta}$.

25. Since the vertex is $\frac{1}{2}$ units to the left of the focus, the directrix must be 1 units to the left of the focus and perpendicular to the x-axis. Therefore, $r = \dfrac{1}{1 - \cos\theta}$.

26. Since the vertex is 2 units to the right of the focus, the directrix must be 4 units to the right of the focus and perpendicular to the x-axis. Therefore, $r = \dfrac{4}{1 + \cos\theta}$.

27. Since the vertex is $\frac{1}{4}$ units below the focus, the directrix must be $\frac{1}{2}$ units below the focus and parallel to the $x-$axis. Therefore, $r = \dfrac{1/2}{1 - \sin\theta}$.

28. Since the vertex is $\frac{3}{2}$ units above the focus, the directrix must be 3 units above the focus and parallel to the x-axis. Therefore, $r = \dfrac{3}{1 + \sin\theta}$.

29. This is the parabola $r = \dfrac{4}{1 + \cos\theta}$ rotated counterclockwise by $\pi/4$. The original parabola had its focus at the origin and its directrix at $x = 4$. The original vertex therefore had polar coordinates $(2, 0)$. After rotation, the vertex is located at $(2, \pi/4)$.

30. This is the ellipse $r = \dfrac{5}{3+2\cos\theta} = \dfrac{5/3}{1+2/3\cos\theta}$ rotated counterclockwise by $\pi/3$. The original ellipse had vertices at $\theta = 0$ and $\theta = \pi$. The polar coordinates of the vertices were $(1,0)$ and $(5,\pi)$. After rotation, the vertices are located at $(1,\pi/3)$ and $(5,4\pi/3)$.

31. This is the ellipse $r = \dfrac{10}{2-\sin\theta} = \dfrac{5}{2-\frac{1}{2}\sin\theta}$ rotated clockwise by $\pi/6$. The original ellipse had vertices at $\theta = \pi/2$ and $\theta = 3\pi/2$. The polar coorinates of the vertices were $(10,\pi/2)$ and $(10.3, 3\pi/2)$. After rotation, the vertices are located at $(10.\pi/3)$ and $(10/3, 4\pi/3)$.

32. This is the hyperbola $r = \dfrac{6}{1+2\sin\theta}$ rotated clockwise by $\pi/3$. The original hyperbola had vertices at $\theta = \pi/2$ and $\theta = 3\pi/2$. The polar coordinates of the vertices were $(2,\pi/2)$ and $(-6, 3\pi/2)$. After rotation, the vertices are located at $(2,\pi/6)$ and $(-6, 7\pi/6)$.

33. Identifying $r_a = 12{,}000$ and $e = 0.2$, we have from (7) in the text $0.2 = \dfrac{12{,}000 - r_p}{12{,}000 + r_p}$. Solving for r_p, we obtain $r_p = 8{,}000$ km.

34. The equation of the orbit is $r = \dfrac{0.2p}{(1-0.2\cos\theta)}$. When $\theta = 0$, $r = 12{,}000$ so $12{,}000 = \dfrac{0.2p}{(1-0.2)} = \frac{p}{4}$. Thus, $p = 48{,}000$ and the equation of the orbit is $r = \dfrac{9{,}600}{(1-0.2\cos\theta)}$.

35. The equation of the orbit is $r = \dfrac{ep}{(1-e\cos\theta)}$. From (7) in the text,

$$e = \frac{1.5\times10^8 - 1.47\times10^8}{1.52\times10^8 + 1.47\times10^8} = \frac{5}{299} \approx 1.67\times10^{-2}.$$

When $\theta = 0$, $r = r_a = 1.52\times10^8 = \dfrac{ep}{(1-1.67\times10^{-2})}$. Thus $ep \approx 1.52\times10^8 - 2.52\times10^6 \approx 1.49\times10^8$ and the equation of the orbit is $r = \dfrac{(1.49\times10^8)}{(1-1.67\times10^{-2}\cos\theta)}$.

36. (a) The equation of the orbit is $r = \dfrac{0.97p}{(1-0.97\cos\theta)}$. The length of the major axis is the sum of $r_a = r(0) = \dfrac{0.97p}{0.03}$ and $r_p = r(\pi) = \dfrac{0.97p}{1+0.97} = \dfrac{0.97p}{1.97}$. That is

$$r_a + r_p = 0.97p\left(\frac{1}{0.03} + \frac{1}{1.97}\right) = 3.34\times10^9.$$

Solving for p we obtain $p = 1.02\times10^8$. The equation of the orbit is

$$r = \frac{0.97(1.02\times10^8)}{(1-0.97\cos\theta)}.$$

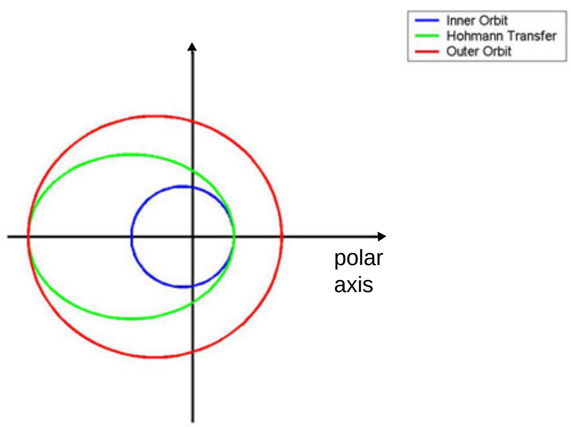
(b) From part (a)

$$r_a = r(0) = \frac{9.87\times10^7}{1-0.97} = \frac{9.87\times10^7}{0.03} \approx 3.29\times10^9 \text{ miles}$$

and

$$r_p = r(\pi) = \frac{9.87 \times 10^7}{1 + 0.97} = \frac{9.87 \times 10^7}{1.97} \approx 5.01 \times 10^7 \text{ miles} .$$

37.



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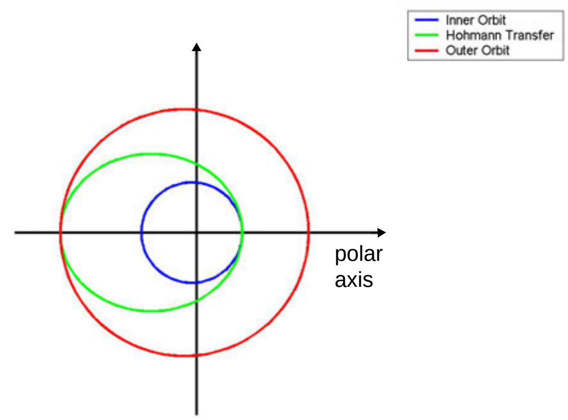

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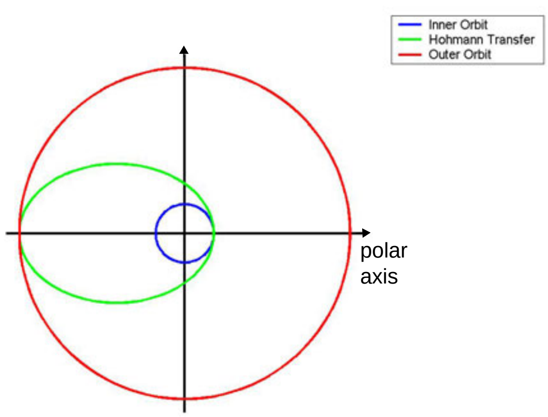

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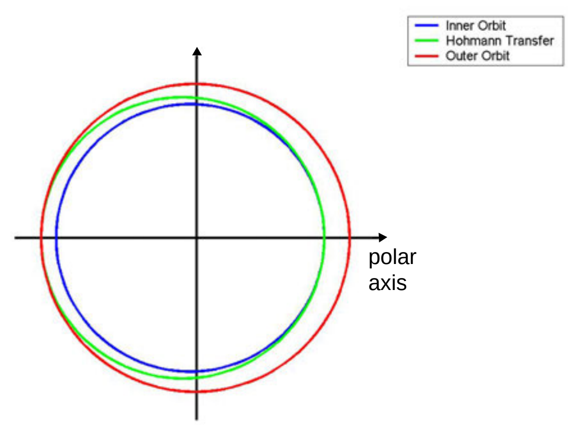

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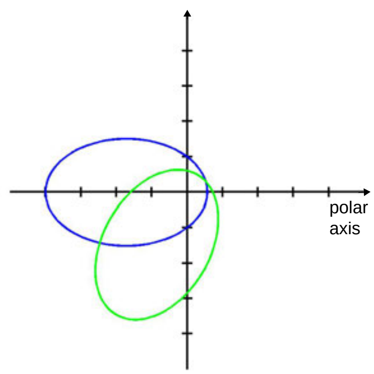
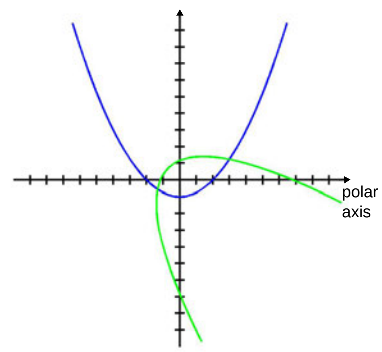


42.



43. Continuing the development in the paragraph following **(3)** in the text, we see that $r = e(d + r\cos\theta)$ yields $(1-e^2)x^2 - 2e^2dx + y^2 = e^2d^2$ which in turn gives

$$x^2 - \frac{2e^2d}{1-e^2}x + \frac{y^2}{1-e^2} = \frac{e^2d^2}{1-e^2}$$
$$x^2 - \frac{2e^2d}{1-e^2}x + \left(\frac{e^2d}{1-e^2}\right)^2 + \frac{y^2}{1-e^2} = \frac{e^2d^2}{1-e^2} + \left(\frac{e^2d}{1-e^2}\right)^2$$
$$\left(x - \frac{e^2d}{1-e^2}\right)^2 + \frac{y^2}{1-e^2} = \frac{e^2d^2(1-e^2)}{(1-e^2)^2} + \frac{e^4d^2}{(1-e^2)^2} = \frac{1}{(1-e^2)^2}$$
$$\frac{\left(x - \frac{e^2d}{1-e^2}\right)^2}{\left[\frac{1}{(1-e^2)^2}\right]} + \frac{y^2}{\left[\frac{1}{1-e^2}\right]} = 1,$$

When $0 < e < 1$, both the denominators are positive and the denominator of the fraction involving the x term is smaller. Therefore, this is in the standard form for an ellipse with center and foci on the x-axis. When $e > 1$, the first denominator is positive while the second is negative. Therefore, this is in the standard form for a hyperbola with center and foci on the x-axis.

44. At $\theta = 0$, $r = r_a$ so $\dfrac{ed}{1-e} = r_a$.
At $\theta = \pi$, $r = r_p$ so $\dfrac{ed}{1-e} = r_p$.
Solving the second equation for d, we have $d = \dfrac{(1+e)r_p}{e}$. Plugging this value for d into the

first equation, we have

$$\begin{aligned} \frac{e\left(\frac{(1+e)r_p}{e}\right)}{r_a} &= r_a \\ \frac{(1+e)r_p}{1-e} &= r_a \\ r_p + er_p &= r_a - er_a \\ r(r_a + r_p) &= r_a - r_p \\ e &= \frac{r_a - r_p}{r_a + r_p} \end{aligned}$$

Chapter 10 in Review

A. True/False

1. True
2. True
3. True
4. False; there are no y-intercepts since $-y^2/b^2 = 1$ has no real solution.
5. True
6. True
7. False; $(-r, \theta)$ and $(r, \theta + \pi)$ are the same point.
8. False; since $x = t^2$, $x \geq 0$ in the parametric form, but $(-1, 2)$ is on the graph of $y = x^2 + 1$.
9. True; solving $x = t^2 + t - 12 = (t-3)(t+4) = 0$ we obtain $t = 3$ and $t = 4$. Since $3^3 - 7(3) = 27 - 21 = 6$, the graph intersects the y-axis at $(0, 6)$.
10. True
11. True
12. False; since 6 is even the graph has 12 petals.
13. False; the same point can be expressed as $(-4, \pi/2)$, which does satisfy the equation.
14. True
15. True
16. True; since $e = 1/15$ is close to 0.
17. True
18. True; since $r = -5\sec\theta$ is equivalent to $r\cos\theta$ or $x = -5$.

19. False; if $r < 0$, the point (r, θ) is in the same quadrant as the terminal side of $\theta + \pi$.
20. True
21. True
22. True
23. True
24. False; this integral will compute the area inside the inner loop of the limacon twice.
25. False
26. False; $r = \cos\theta$ and $r = \sin\theta$ intersect at the pole which is $(0, \pi/2)$ for $r = \cos\theta$ and $(0, 0)$ for $r = \sin\theta$.

B. Fill in the Blanks

1. $4p = 1/2$, $p = 1/8$. The focus is $(0, 1/8)$.
2. $c^2 = 4 + 12 = 16$. The foci are $(\pm 4, 0)$.
3. The center is $(0, 2)$.
4. The asymptotes are $25y^2 - 4x^2 = 0$ or $y = \pm 2x/5$.
5. $4p = 8$, $\;p = 2$. The directrix is $y = -3 - 2 = -5$.
6. $a^2 = 36$, $b^2 = 16$. The vertices are $(-1 \pm 6, -7)$ and $(-1, -7 \pm 4)$ or $(-7, -7)$, $(5, -7)$, $(-1, -11)$, and $(-1, -3)$.
7. Completing the square, $y + 10 = (x + 2)^2$. The vertex is $(-2, -10)$.
8. $b^2 = 9$. The length of the conjugate axis is $2 \cdot 3 = 6$.
9. $a^2 = 4$. The endpoints of the transverse axis are the vertices $(4 \pm 2, -1)$ or $(2, -1)$ and $(6, -1)$.
10. The major axis is on the line $x = 3$.
11. Completing the square, we have $25(x^2 - 8x + 16) + (y^2 + 6y + 9) = -384 + 409 = 25$ or $(x - 4)^2 + (y + 3)^2/25 = 1$. The center of the ellipse is at $(4, -3)$.
12. Setting $y = 0$ and solving, we have $(x + 1)^2 + 64 = 100$, $(x + 1)^2 = 36$, or $x = \pm 6 - 1$. The x-intercepts are -7 and 5.
13. Setting $x = 0$ and solving, we have $y^2 - 4 = 1$, $y^2 = 5$, or $y = \pm\sqrt{5}$. The y-intercepts are $\pm\sqrt{5}$.
14. Using implicit differentiation, $2yy' - y' + 3 = 0$ or $y' = \dfrac{3}{1 - 2y}$. At $(1, 1)$ the slope of the tangent line is $\frac{3}{(-1)} = -3$.
15. line

16. $x = 0$ at $t = \pm 1$, so the y-intercepts occurs at $(0, 3)$ and $(0, -1)$.

17. circle

18. convex limacon

19. $r = 0$ at $\theta = 0,\ \pi/3,\ 2\pi/3,\ \pi,\ 4\pi/3,$ and $5\pi/3$. $\dfrac{dr}{d\theta} = 3\cos 3\theta \neq 0$ at any of the θ values mentioned. Thus, the polar equation $\theta = \dfrac{n\pi}{3}$ defines a tangnet to the graph at the origin for $n = 0, \ldots, 5$.

20. From $r = \dfrac{1}{2 + 5\sin\theta} = \dfrac{\frac{1}{2}}{1 + \frac{5}{2}\sin\theta}$, we have $e = \frac{5}{2}$.

21. The focus is the origin. The directrix is 10 units below the origin and parallel to the x-axis. Therefore, the vertex is 5 units below the origin at $(0, -5)$.

22. From $r = \dfrac{12}{2 + \cos\theta} = \dfrac{6}{1 + \frac{1}{2}\cos\theta}$, we see that the conic is an ellipse with a directrix 12 units to the right of the focus at the origin and perpendicular to the x-axis. The two vertices must therefore occur at $\theta = 0$ and $\theta = \pi$. Thus polar coordinates of the vertices are $(4, 0)$ and $(12, \pi)$. The foci must therefore be at the origin and at $(8, pi)$. The center is at $(4, \pi)$.

C. Exercises

1. $\dfrac{dy}{dx} = \dfrac{\sin t}{1 - \cos t}$. At $t = \pi/2$, $\dfrac{dy}{dx} = \dfrac{(\sqrt{3}/2)}{(1/2)} = \sqrt{3}$. The slope of the normal line is $\frac{-1}{\sqrt{3}}$ and its equation is

$$y - \frac{1}{2} = -\frac{1}{\sqrt{3}}\left[x - \left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)\right] \quad \text{or} \quad y = -\frac{1}{\sqrt{3}}x + \frac{\pi}{3\sqrt{3}} = -\frac{\sqrt{3}}{3}x + \frac{\sqrt{3}\pi}{9}.$$

2. $x'(t) = 1 - \cos t,\ \ y'(t) = \sin t$

$$\begin{aligned}
s &= \int_0^{2\pi} \sqrt{(1-\cos t)^2 + \sin^2 t}\,dt = \int_0^{2\pi} \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}\,dt \\
&= \sqrt{2}\int_0^{2\pi} \sqrt{1 - \cos t}\,dt \quad \boxed{\text{by symmetry}} \\
&= 2\sqrt{2}\int_0^{\pi} \sqrt{1 - \cos t}\,dt = 2\sqrt{2}\int_0^{\pi} \sqrt{1 - \cos t}\frac{\sqrt{1+\cos t}}{\sqrt{1+\cos t}}dt = 2\sqrt{2}\int_0^{\pi} \frac{\sqrt{1-\cos^2 t}}{\sqrt{1+\cos t}}dt \\
&= 2\sqrt{2}\int_0^{\pi} \frac{\sin t}{\sqrt{1+\cos t}}dt \quad \boxed{u = 1 + \cos t,\ \ du = -\sin t\,dt} \\
&= 2\sqrt{2}\int_2^{0} \frac{-du}{\sqrt{u}} = 2\sqrt{2}\lim_{b\to 0^+}\int_b^2 u^{-1/2}du = 2\sqrt{2}\lim_{b\to 0^+}\left(2\sqrt{u}\Big|_b^2\right) = 2\sqrt{2}\lim_{b\to 0^+}(2\sqrt{2} - 2\sqrt{b}) = 8
\end{aligned}$$

3. The slope of the line $6x + y = 8$ is -6 and $\frac{dy}{dx} = \dfrac{(3t^2 - 18t)}{2t} = \frac{3t}{2} - 9,\ \ t \neq 0$. Solving $\frac{3t}{2} - 9 = -6$ we obtain $t = 2$. Since $x(2) = 8$ and $y(2) = -26$, the point on the graph is $(8, -26)$.

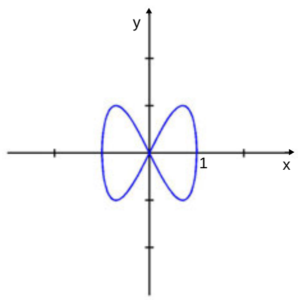
4. $\frac{dy}{dt} = \frac{2}{2t} = \frac{1}{t}$. The slope of the tangent line is $\frac{1}{t}$ and the point on the curve is $(t^2+1, 2t)$. The equation of a tangent line is then $y - 2t = \frac{1}{t}[x - (t^2+1)]$. Since we want the tangent line to pass through $(1,5)$, we have $5 - 2t = \frac{1}{t}(1 - t^2 - 1) = -t$. Solving for t we obtain $t = 5$. The point on the curve is $(x(5), y(5)) = (26, 10)$. Also, the tangent is vertical if $t = 0$. At $t = 0$, the point on the graph is $(1,0)$. The tangent line at this point will also pass through $(1,5)$.

5. (a) Since $4x^2(1-x^2) = y^2 \geq 0$, $1 - x^2 \geq 0$, $x^2 \leq 1$, and $|x| \leq 1$.

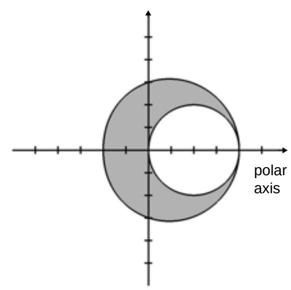
   (b) Letting $x = \sin t$, we have $y^2 = 4\sin^2 t(1 - \sin^2 t) = 4\sin^2 t\cos^2 t = \sin^2 2t$. Parametric equations for the curve are $x = \sin t$, $y = \sin 2t$, for $0 \leq t \leq 2\pi$.

   (c) $\frac{dy}{dx} = \frac{(2\cos 2t)}{\cos t}$. The tangent line is horizontal when $\cos 2t = 0$ or $t = \pi/4,\ 3\pi/4,\ 5\pi/4,\ 7\pi/4$. The points on the graph are $(\sqrt{2}/2, 1)$, $(\sqrt{2}/2, -1)$, $(-\sqrt{2}/2, 1)$ and $(-\sqrt{2}/2, -1)$.

   (d)



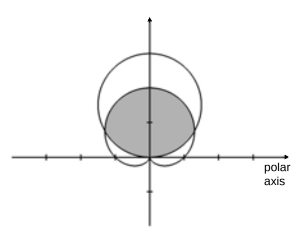
6. The area inside the limacon is



$$\frac{1}{2}\int_0^{2\pi}(3+\cos\theta)^2 d\theta = \frac{1}{2}\int_0^{2\pi}(9 + 6\cos\theta + \cos^2\theta)d\theta$$
$$= \left(\frac{9}{2}\theta + 3\sin\theta + \frac{1}{4}\theta + \frac{1}{8}\sin 2\theta\right)\Bigg|_0^{2\pi} = \frac{19\pi}{2}.$$

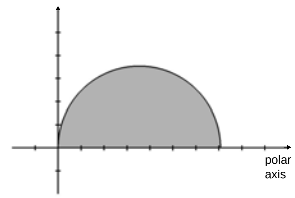
The circle $r = 4\cos\theta$ has radius 2, so its area is 4π. Thus, the area outside the circle and inside the limacon is $\frac{19\pi}{2} - 4\pi = \frac{11\pi}{2}$.

7. Solving $3\sin\theta = 1 + \sin\theta$ in the first quadrant, we have $\sin\theta = 1/2$ and $\theta = \pi/6$. Using symmetry,



$$\begin{aligned} A &= 2\left[\frac{1}{2}\int_0^{\pi/6} 9\sin^2\theta d\theta + \frac{1}{2}\int_{\pi/6}^{\pi/2}(1+\sin\theta)^2 d\theta\right] \\ &= \left(\frac{9}{2}\theta - \frac{9}{4}\sin 2\theta\right)\Big|_0^{\pi/6} + \left(\theta - 2\cos\theta + \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right)\Big|_{\pi/6}^{\pi/2} \\ &= \left(\frac{3\pi}{4} - \frac{9\sqrt{3}}{8}\right) + \left[\frac{3\pi}{4} - \left(\frac{\pi}{4} - \sqrt{3} - \frac{\sqrt{3}}{8}\right)\right] = \frac{5\pi}{4}. \end{aligned}$$

8. Writing $A = \int_0^{\pi/2} 25(1-\sin^2\theta)d\theta = \dfrac{1}{2}\int_0^{\pi/2} 50\cos^2\theta d\theta$, we see that the area corresponds to a semicircle of radius $\dfrac{5\sqrt{2}}{2}$.



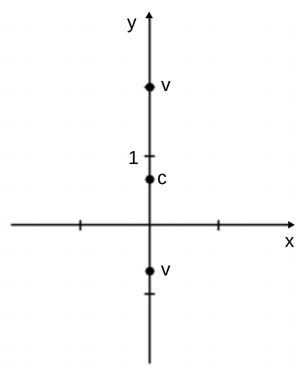
9. From $x = 2\sin 2\theta\cos\theta$, $y = 2\sin 2\theta\sin\theta$, we find

$$\frac{dy}{dx} = \frac{2\sin 2\theta\cos\theta + 4\cos 2\theta\sin\theta}{-2\sin 2\theta\sin\theta + 4\cos 2\theta\cos\theta} \quad \text{and} \quad \frac{dy}{dx}\Big|_{\theta=\pi/4} = \frac{2(1)\left(\frac{\sqrt{2}}{2}\right) + 4(0)\left(\frac{\sqrt{2}}{2}\right)}{-2(1)\left(\frac{\sqrt{2}}{2}\right) + 4(0)\left(\frac{\sqrt{2}}{2}\right)} = -1.$$

(a) At $\theta = \pi/4$, $x = \sqrt{2}$ and $y = \sqrt{2}$. The Cartesian equation of the tangent line is $y - \sqrt{2} = -1(x - \sqrt{2})$ or $y = -x + 2\sqrt{2}$.

(b) Using $x = r\cos\theta$ and $y = r\sin\theta$ in **(a)**, we obtain $r\sin\theta = -r\cos\theta = 2\sqrt{2}$ or $r = \dfrac{2\sqrt{2}}{\sin\theta + \cos\theta}$.

10. By writing $r = \dfrac{1}{(1 - \frac{1}{2}\sin\theta)}$ we see that $e = 1/2$ and the graph is an ellipse with major axis along the y-axis. The vertices on this axis have polar coordinates $(2, \pi/2)$ and $(2/3, 3\pi/2)$. The corresponding rectangular coordinates are $(0, 2)$ and $(0, -2/3)$. Thus, the center of the ellipse is at $(0, 2/3)$. Since once focus is at the origin, $c = 2/3$. From $a = 4/3$ and $c = 2/3$ we find $b^2 = \frac{16}{9} - \frac{4}{9} = \frac{12}{9}$ and $b = \frac{2\sqrt{3}}{3}$. Thus, the vertices on the minor axis are at $\left(-\frac{2\sqrt{3}}{3}, \frac{2}{3}\right)$ and $\left(\frac{2\sqrt{3}}{3}, \frac{2}{3}\right)$.



11. Multiplying both sides of the equation by r, we have $r^2 = r\cos\theta + r\sin\theta$. The corresponding Cartesian equation is $x^2 + y^2 = x + y$.

12. Multipying both sides by $\cos\theta$, we have

$$\begin{aligned} r\cos\theta &= 1 - 5\cos^2\theta \\ x &= 1 - \frac{5x^2}{x^+y^2} \\ x - 1 &= -\frac{5x^2}{x^+y^2} \\ x^2 + y^2 &= -\frac{5x^2}{x-1} \\ y^2 &= -x^2 - \frac{5x^2}{x-1} \\ y^2 &= \frac{-x^3 - 4x^2}{x-1} \end{aligned}$$

13. $2(r\cos\theta)(r\sin\theta) = 5$
$r^2 = \dfrac{5}{2\cos\theta\sin\theta}$

14. Using $x^2 + y^2 = r^2$ and $x = r\cos\theta$, we have

$$\begin{aligned} (r^2 - 2r\cos\theta)^2 = 9r^2 &\Longrightarrow [r(r - 2\cos\theta)]^2 = 9r^2 \Longrightarrow r^2(r - 2\cos\theta)^2 = 9r^2 \\ &\Longrightarrow r - 2\cos\theta = \pm 3 \Longrightarrow r = \pm 3 + 2\cos\theta. \end{aligned}$$

Since replacement of (r, θ) by $(-r, \theta + \pi)$ in $r = -3 + 2\cos\theta$ gives $r = 3 + 2\cos\theta$, we take the polar equation to be $r = 3 + 2\cos\theta$.

15. By writing the equation as $r\cos\theta = -1$ we see that the line is $x = -2$. The curve is then a parabola with axis along the x-axis, directrix $x = -1$, and focus at the origin. Since the vertex is at $(-1/2, 0)$, $p = 1$ and the equation is $r = \dfrac{1}{(1 - \cos\theta)}$.

16. Since the transverse axis lie along the $y - axis$ and $e = 2$, the form of the equation is $r = \dfrac{2p}{(1 - 2\sin\theta)}$. From $4/3 = r(3\pi/2) = \frac{2p}{(1+2)}$ we see that $2p = 4$ and the equation is $r = \dfrac{4}{(1 - 2\sin\theta)}$.

17. $r = 3\sin(10\theta)$

18. $r = 2.8\cos(7\theta)$

19. The form of the equation is $\dfrac{y^2}{100} - \dfrac{x^2}{b^2} = 1$. The asymptotes for the hyperbola are $\dfrac{y^2}{100} - \dfrac{x^2}{b^2} = 0$ or $by = \pm 10x$. Since the given asymptotes are $3y = \pm 5x$, we have the proportion $\frac{b}{3} = \frac{10}{5}$. Thus, $b = 6$ and the equation of the hyperbola is $\dfrac{y^2}{100} - \dfrac{x^2}{36} = 1$.

20. At $\theta = \pi/2$,
$$y = r\sin\theta = \frac{\sin\theta}{1+\cos\theta} = 1$$
$$x = r\cos\theta = \frac{\cos\theta}{1+\cos\theta} = 0$$
$$\frac{dy}{d\theta} = \frac{\cos\theta(1+\cos\theta) - \sin\theta(\sin\theta)}{(1+\cos\theta)^2} = \frac{\cos\theta + \cos^2\theta + \sin^2\theta}{(1+\cos\theta)^2}$$
$$= \frac{1+\cos\theta}{(1+\cos\theta)^2} = 1$$
$$\frac{dx}{d\theta} = \frac{-\sin\theta(1+\cos\theta) - \cos\theta(-\sin\theta)}{(1+\cos\theta)^2}$$
$$= \frac{-\sin\theta - \sin\theta\cos\theta + \sin\theta\cos\theta}{(1+\cos\theta)^2}$$
$$= \frac{-\sin\theta}{(1+\cos\theta)^2} = -1$$
$$\frac{dy}{dx} = \frac{1}{-1} = -1$$
The tangent line therefore has slope -1 and passes through the point $(0, 1)$. Its equations is $y = 1 - (x - 1)$ or $y = 1 - x$.

21. Substituting $y = tx$ into $x^3 + y^3 = 3axy$ we obtain
$$x^3 + y^3 x = 3atx^2 \Longrightarrow (1+t^3)x = 3at \Longrightarrow x = \frac{3at}{1+t^3},$$
and $y = tx = \dfrac{3at^2}{(1+t^3)}$.

22. $\dfrac{dx}{dt} = \dfrac{3a(1-2t^3)}{(1+t^3)^2}$; $\quad \dfrac{dy}{dt} = \dfrac{3at(2-t^3)}{(1+t^3)^2}$
Solving $\frac{dy}{dt} = 0$ we obtain $t = 0$ and $t = 2^{1/3}$. Since $\frac{dx}{dt}$ is not zero at these values, the graph has horizontal tangent lines at $(0, 0)$ and $(\sqrt[3]{2}a, \sqrt[3]{4}a)$.

23. (a) From $x = r\cos\theta$ and $y = r\sin\theta$ we obtain $r^3(\cos^3\theta + \sin^3\theta) = 3ar^2\cos\theta\sin\theta$ or $r = \dfrac{(3a\cos\theta\sin\theta)}{(\cos^3\theta + \sin^3\theta)}$.

(b) The loop is formed from $\theta = 0$ to $\theta = \pi/2$, so the area is
$$A = \frac{1}{2}\int_0^{\pi/2} \frac{9a^2\cos^2\theta\sin^2\theta}{(\cos^3\theta + \sin^\theta)} d\theta = \frac{9a^2}{2}\int_0^{\pi/2} \frac{\cos^2\theta\sin^2\theta}{(1+\tan^3\theta)^2\cos^6\theta} d\theta$$
$$= \frac{9a^2}{2}\int_0^{\pi/2} \frac{\sin^2\theta}{(1+\tan^3\theta)^2\cos^4\theta} d\theta = \frac{9a^2}{2}\int_0^{\pi/2} \frac{\tan^2\theta}{(1+\tan^3\theta)\cos^2\theta} d\theta$$
$$= \frac{9a^2}{2}\int_0^{\pi/2} \frac{\tan^2\theta\sec^3\theta}{(1+\tan^3\theta)\theta} d\theta \quad \boxed{u = \tan\theta, \ \ du = \sec^2\theta d\theta}$$
$$= \frac{9a^2}{2}\int_0^{\infty} \frac{u^2}{(1+u^3)^2} du = \frac{9a^2}{2}\left[-\frac{1}{3}\left(\frac{1}{1+u^3}\right)\right]\Bigg|_0^{\infty} = -\frac{3a^2}{2}\lim_{t\to\infty}\left(\frac{1}{1+t^3} - 1\right) = \frac{3a^2}{2}.$$

24. From Problem 21, $x(t) = \dfrac{3at}{(1+t^3)}$ and $y(t) = \dfrac{3at^2}{(1+t^3)}$. Then, assuming $a > 0$, we have $\lim_{t\to -1^-} x(t) = \infty$, $\lim_{t\to -1^-} y(t) = -\infty$. $\lim_{t\to -1^+} x(t) = -\infty$, and $\lim_{t\to -1^+} y(t) = \infty$. Finally, using L'Hôpital's Rule,

$$\lim_{t\to -1}[x(t)+y(t)] = \lim_{t\to -1} \frac{3at+3at^2}{1+t^3} \overset{h}{=} \lim_{t\to -1} \frac{3a+6at}{3t^2} = -a$$

and $x + y = -a$ or $x + y + a = 0$ is an asymptote.

25. Using symmetry

$$\begin{aligned} A &= 2\left(\frac{1}{2}\right)\int_0^{\pi/2}\left(2\sin\frac{\theta}{3}\right)^2 d\theta = 4\int_0^{\pi/2}\frac{1}{2}\left(1-\cos\frac{2\theta}{3}\right)d\theta \\ &= 2\left(\theta - \frac{3}{2}\sin\frac{2\theta}{3}\right)\Bigg|_0^{\pi/2} = 2\left(\frac{\pi}{2} - \frac{3}{2}\frac{\sqrt{3}}{2}\right) = \pi - \frac{3\sqrt{3}}{2}. \end{aligned}$$

26. The circle centered at $(1, 0)$ has polar equation $r = 2\cos\theta$. Solving $1 = 2\cos\theta$, we obtain $\theta = \pi/3$. Using symmetry,

$$A = 4\left[\frac{1}{2}\int_{\pi/3}^{\pi/2}(1-4\cos^2\theta)d\theta\right] = 2(\theta - 2\theta - \sin 2\theta)|_{\pi/3}^{\pi/2} = 2\left[-\frac{\pi}{2} - \left(-\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)\right] = \sqrt{3} - \frac{\pi}{3}.$$

27. (a) $r = 2\cos\left(\theta - \frac{\pi}{4}\right)$

(b) Note that $r = 2\cos\theta$ defines a circle of radius 1 centered at $(1, 0)$. A rotation of $\pi/4$ puts the center at $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$. The new rectangular equation is therefore $\left(x - \frac{\sqrt{2}}{2}\right)^2 + \left(y - \frac{\sqrt{2}}{2}\right)^2 = 1$

28. (a) $r = \dfrac{1}{1+\cos(\theta + \pi/6)}$

(b) Using the sum of angles identity for cosine, we have $r = \dfrac{1}{1+\cos\theta\cos\pi/6 - \sin\theta\sin\pi/6}$

$r = \dfrac{1}{1+\frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta}$

$r\left(1+\frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta\right) = 1$

$r + \frac{\sqrt{3}}{2}r\cos\theta - \frac{1}{2}r\sin\theta = 1$

$\sqrt{x^2+y^2} + \frac{\sqrt{3}}{2}x - \frac{1}{2}y = 1$

$\sqrt{x^2+y^2} = 1 - \frac{\sqrt{3}}{2}x + \frac{1}{2}y$

$x^2 + y^2 = \frac{3}{4}x^2 - \frac{\sqrt{3}xy}{2} - \sqrt{3}x + \frac{y^2}{4} + y + 1$

$\frac{1}{4}x^2 + \frac{\sqrt{3}xy}{2} + \sqrt{3}x + \frac{3}{4}y^2 - y = 1$

29. Taking the center of the ellipse to be at the origin, we have $a = 5 \times 10^8$ and $b = 3 \times 10^8$, since $c^2 = a^2 - b^2$, $c^2 = 1.6 \times 10^{17}$ and $c = 4 \times 10^8$. The minimum distance is $a - c = 10^8$ m and the maximum distance is $a + c = 9 \times 10^8$m.

30. To find the width, we need to first find the maximum y-value for points on the portion of the petal satisfying $0 \le \theta \le \pi/4$. We know
$y = r\sin\theta = \cos 2\theta \sin 2\theta$ and therefore
$= (\cos^2\theta - \sin^2\theta)\sin\theta$
$\frac{dy}{d\theta} = (2\cos\theta(-\sin\theta) - 2\sin\theta(\cos\theta))\sin\theta + (\cos^2\theta - \sin^2\theta)\cos\theta$ Thus, $\frac{dy}{d\theta} = 0$ when $\cos\theta =$
$= -5\sin^2\theta\cos\theta + \cos^3\theta$
$= \cos\theta(\cos^2\theta - 5\sin^2\theta)$
0 or when $\cos^2\theta - 5\sin^2\theta = 0$. Since $\cos\theta \ne 0$ for $0 \le \theta \le \pi/4$, we need to find where $\cos^2\theta - 5\sin^2\theta = 0$. This occurs when

$$\cos^2\theta = 5\sin^2\theta$$
$$\frac{1}{5} = \frac{\sin^2\theta}{\cos^2\theta}$$
$$\frac{1}{5} = \tan^2\theta$$
$$\frac{1}{\sqrt{5}} = \tan\theta$$
$$\theta = \tan^{-1}\left(\frac{1}{\sqrt{5}}\right) \approx 0.4205$$

A cursory examination of the graph tells us that the critical point $\theta = \tan^{-1}\left(\frac{1}{\sqrt{5}}\right)$ yields a maximum for y on the interval $0 \le \theta \le pi/4$, and this maximum is $y_{max} = \frac{\sqrt{6}}{9}$. The width is therefore $w = 2y_{max} = \frac{2\sqrt{6}}{9}$.

Multivariable Calculus

Complete Solutions Manual

Brian Fulton Melanie Fulton

Fourth Edition

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Chapter 11

Vectors and 3-Space

11.1 Vectors in 2-Space

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2. (a) $\langle 3,3\rangle$ (b) $\langle 3,4\rangle$ (c) $\langle -1,-2\rangle$ (d) 5 (e) $\sqrt{5}$

3. (a) $\langle 12,0\rangle$ (b) $\langle 4,-5\rangle$ (c) $\langle 4,5\rangle$ (d) $\sqrt{41}$ (e) $\sqrt{41}$

4. (a) $\frac{1}{2}\mathbf{i}-\frac{1}{2}\mathbf{j}$ (b) $\frac{2}{3}\mathbf{i}+\frac{2}{3}\mathbf{j}$ (c) $-\frac{1}{3}\mathbf{i}-\mathbf{j}$ (d) $2\sqrt{2}/3$ (e) $\sqrt{10}/3$

5. (a) $-9\mathbf{i}+6\mathbf{j}$ (b) $-3\mathbf{i}+9\mathbf{j}$ (c) $-3\mathbf{i}-5\mathbf{j}$ (d) $3\sqrt{10}$ (e) $\sqrt{34}$

6. (a) $\langle 3,9\rangle$ (b) $\langle -4,-12\rangle$ (c) $\langle 6,18\rangle$ (d) $4\sqrt{10}$ (e) $6\sqrt{10}$

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9. (a) $\langle 4,-12\rangle-\langle -2,2\rangle=\langle 6,-14\rangle$ (b) $\langle -3,9\rangle-\langle -5,5\rangle=\langle 2,4\rangle$

10. (a) $(4\mathbf{i}+4\mathbf{j})-(6\mathbf{i}-4\mathbf{j})=-2\mathbf{i}+8\mathbf{j}$ (b) $(-3\mathbf{i}-3\mathbf{j})-(-15\mathbf{i}-10\mathbf{j})=18\mathbf{i}-17\mathbf{j}$

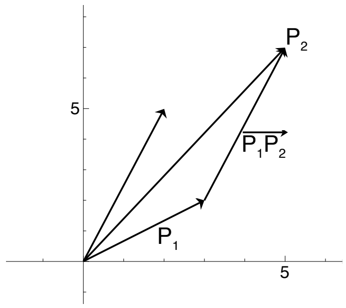
11. (a) $(4\mathbf{i}-4\mathbf{j})-(-6\mathbf{i}+8\mathbf{j})=10\mathbf{i}-12\mathbf{j}$ (b) $(-3\mathbf{i}+3\mathbf{j})-(-15\mathbf{i}+20\mathbf{j})=12\mathbf{i}-17\mathbf{j}$

12. (a) $\langle 8,0\rangle-\langle 0,-6\rangle=\langle 8,6\rangle$ (b) $\langle -6,0\rangle-\langle 0,-15\rangle=\langle -6,15\rangle$

13. (a) $\langle 16,40\rangle-\langle -4,-12\rangle=\langle 20,52\rangle$ (b) $\langle -12,-30\rangle-\langle -10,-30\rangle=\langle -2,0\rangle$

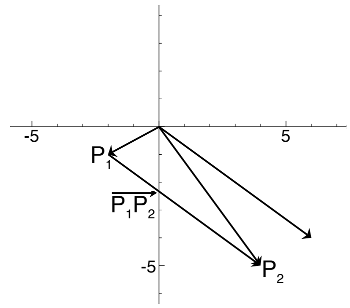
14. (a) $\langle 8,12\rangle-\langle 10,6\rangle=\langle -2,6\rangle$ (b) $\langle -6,-9\rangle-\langle 25,15\rangle=\langle -31,-24\rangle$

15.



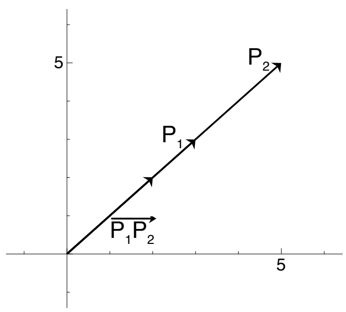
$\overrightarrow{P_1P_2} = \langle 2, 5\rangle$

16.



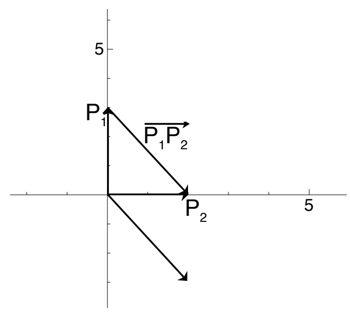
$\overrightarrow{P_1P_2} = \langle 6, -4\rangle$

17.



$\overrightarrow{P_1P_2} = \langle 2, 2\rangle$

18.



$\overrightarrow{P_1P_2} = \langle 2, -3\rangle$

19. Since $\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1}$, $\overrightarrow{OP_2} = \overrightarrow{P_1P_2} + \overrightarrow{OP_1} = (4\mathbf{i} + 8\mathbf{j}) + (-3\mathbf{i} + 10\mathbf{j}) = \mathbf{i} + 18\mathbf{j}$, and the terminal point is $(1, 18)$

20. Since $\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1}$, $\overrightarrow{OP_1} = \overrightarrow{OP_2} + \overrightarrow{P_1P_2} = \langle 4, 7\rangle - \langle -5, -1\rangle = \langle 9, 8\rangle$, and the initial point is $(9, 8)$

21. $a(= -\mathbf{a})$, $b = (-\frac{1}{4}\mathbf{a})$, $c(= \frac{5}{2}\mathbf{a})$, $e(= 2\mathbf{a})$, and $f(= -\frac{1}{2}\mathbf{a})$ are parallel to $\mathbf{a}$.

22. We want $-3\mathbf{b} = \mathbf{a}$, so $\mathbf{c} = -3(9) = -27$

23. $\langle 6, 15\rangle$

24. $\langle 5, 2\rangle$

25. $|\mathbf{a}| = \sqrt{4+4} = 2\sqrt{2}$; (a) $\mathbf{u} = \dfrac{1}{2\sqrt{2}}\langle 2, 2\rangle = \left\langle \dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}} \right\rangle$; (b) $-\mathbf{u} = \left\langle -\dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}} \right\rangle$

26. $|\mathbf{a}| = \sqrt{9+16} = 5$; (a) $\mathbf{u} = \dfrac{1}{5}\langle -3, 4\rangle = \left\langle \dfrac{-3}{5}, \dfrac{4}{5} \right\rangle$; (b) $-\mathbf{u} = \left\langle \dfrac{3}{5}, -\dfrac{4}{5} \right\rangle$

27. $|\mathbf{a}| = 5$; (a) $\mathbf{u} = \dfrac{1}{5}\langle 0, -5\rangle = \langle 0, -1\rangle$; (b) $-\mathbf{u} = \langle 0, 1\rangle$

28. $|\mathbf{a}| = \sqrt{1+3} = 2$; (a) $\mathbf{u} = \dfrac{1}{2}\langle 1, -\sqrt{3}\rangle = \left\langle \dfrac{1}{2}, -\dfrac{\sqrt{3}}{2} \right\rangle$; (b) $-\mathbf{u} = \left\langle -\dfrac{1}{2}, \dfrac{\sqrt{3}}{2} \right\rangle$

29. $|\mathbf{a}+\mathbf{b}| = |\langle 5, 12\rangle| = \sqrt{25+144} = 13; \quad \mathbf{u} = \frac{1}{13}\langle 5, 12\rangle = \left\langle \frac{5}{13}, \frac{12}{13}\right\rangle$

30. $|\mathbf{a}+\mathbf{b}| = |\langle -5, 4\rangle| = \sqrt{25+16} = \sqrt{41}; \quad \mathbf{u} = \frac{1}{\sqrt{41}}\langle -5, 4\rangle = \left\langle -\frac{5}{\sqrt{41}}, \frac{4}{\sqrt{41}}\right\rangle$

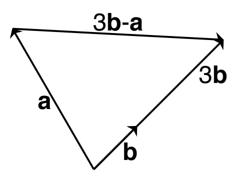
31. $|\mathbf{a}| = \sqrt{9+49} = \sqrt{58}; \quad \mathbf{b} = 2\left((\frac{1}{\sqrt{58}}\right) 3\mathbf{i} + 7\mathbf{j}) = \frac{6}{\sqrt{58}}\mathbf{i} + \frac{14}{\sqrt{58}}\mathbf{j}$

32. $|\mathbf{a}| = \sqrt{\frac{1}{4}+\frac{1}{4}} = \frac{1}{\sqrt{2}}\sqrt{58}; \quad \mathbf{b} = 3\left((\frac{1}{1/\sqrt{2}}\right)(\frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}) = \frac{3\sqrt{2}}{2}\mathbf{i} - \frac{3\sqrt{2}}{2}\mathbf{j}$

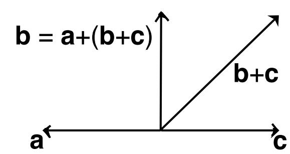
33. $-\frac{3}{4}\mathbf{a} = \langle -3 - 15/2\rangle$

34. $5(\mathbf{a}+\mathbf{b}) = 5\langle 0, 1\rangle = \langle 0, 5\rangle$

35.



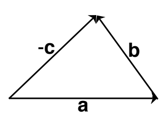
36.



37. $\mathbf{x} = -(\mathbf{a}+\mathbf{b}) = -\mathbf{a} - \mathbf{b}$

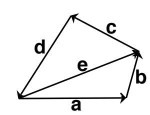
38. $\mathbf{x} = 2(\mathbf{a}-\mathbf{b}) = 2\mathbf{a} - 2\mathbf{b}$

39.



$\mathbf{b} = (-\mathbf{c}) - \mathbf{a}$;
$(\mathbf{b}+\mathbf{c}) + \mathbf{a} = \mathbf{0}$;
$\mathbf{a}+\mathbf{b}+\mathbf{c} = \mathbf{0}$

40.



From Problem 39, $\mathbf{e}+\mathbf{c}+\mathbf{d} = \mathbf{0}$.
but $\mathbf{b} = \mathbf{e} - \mathbf{a}$ and $\mathbf{e} = \mathbf{a} + \mathbf{b}$, so
$(\mathbf{a}+\mathbf{b}) + \mathbf{c} + \mathbf{d} = \mathbf{0}$.

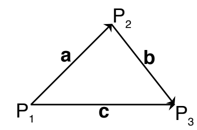
41. From $2\mathbf{i} + 3\mathbf{j} = k_1\mathbf{b} + k_2\mathbf{c} = k_1(\mathbf{i}+\mathbf{j}) + k_2(\mathbf{i}-\mathbf{j}) = (k_1+k_2)\mathbf{i} + (k_1-k_2)\mathbf{j}$ we obtain the system of equations $k_1 + k_2 = 2, \quad k_1 - k_2 = 3$. Solving, we find $k_1 = \frac{5}{2}$ and $k_2 = -\frac{1}{2}$. Then $\mathbf{a} = \frac{5}{2}\mathbf{b} - \frac{1}{2}\mathbf{c}$.

42. From $2\mathbf{i} + 3\mathbf{j} = k_1\mathbf{b} + k_2\mathbf{c} = k_1(-2\mathbf{i}+4\mathbf{j}) + k_2(5\mathbf{i}+7\mathbf{j}) = (-2k_1+5k_2)\mathbf{i} + (4k_1+7k_2)\mathbf{j}$ we obtain the system of equations $-2k_1 + 5k_2 = 2, \quad 4k_1 + 7k_2 = 3$. Solving, we find $k_1 = \frac{1}{34}$ and $k_2 = \frac{7}{17}$.

43. From $y' = \frac{1}{2}x$ we see that the slope of the tangent line at $(2, 2)$ is 1. A vector with slope 1 is $\mathbf{i}+\mathbf{j}$. A unit vector is $(\mathbf{i}+\mathbf{j})/|\mathbf{i}+\mathbf{j}| = (\mathbf{i}+\mathbf{j})/\sqrt{2} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$. Another unit vector tangent to the curve is $-\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$.

44. From $y' = -2x+3$ we see that the slope of the tangent line at $(0, 0)$ is 3. A vector with slope 3 is $\mathbf{i}+3\mathbf{j}$. A unit vector is $(\mathbf{i}+3\mathbf{j})/|\mathbf{i}+3\mathbf{j}| = (\mathbf{i}+3\mathbf{j})/\sqrt{10} = \frac{1}{\sqrt{10}}\mathbf{i} + \frac{1}{\sqrt{10}}\mathbf{j}$. Another unit vector tangent to the curve is $-\frac{1}{\sqrt{10}}\mathbf{i} - \frac{1}{\sqrt{10}}\mathbf{j}$.

45. (a) Since the shortest distance between two point is a straight line, $|\mathbf{a}+\mathbf{b}| \le |\mathbf{a}| + |\mathbf{b}|$.



(b) When P_2 lies on the line segment between P_1 and P_3, $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}| + |\mathbf{b}|$.

46. Since $y = 2a(L^2+y^2)^{3/2}$ is an odd function on $[-a, a]$, $F_y = 0$. Now using the fact that $L/(L^2+y^2)^{3/2}$ is an even function, we have

$$\int_{-a}^{a} \frac{L\ \mathrm{d}y}{2a(L^2+y^2)^{3/2}} = \frac{L}{a}\int_0^a \frac{\mathrm{d}y}{(L^2+y^2)^{3/2}} \qquad \boxed{y = L\tan\theta,\ \ \mathrm{d}y = L\sec^2\theta\ \ \mathrm{d}\theta}$$

$$= \frac{L}{a}\int_0^{\tan^{-1}a/L} \frac{L\sec^2\theta\ \ \mathrm{d}\theta}{L^3(1+\tan^2\theta)^{3/2}} = \frac{1}{La}\int_0^{\tan^{-1}a/L} \frac{\sec^2\theta\ \ \mathrm{d}\theta}{\sec^3\theta}$$

$$= \frac{1}{La}\int_0^{\tan^{-1}a/L} \cos\theta\ \ \mathrm{d}\theta = \frac{1}{La}\sin\theta\Big]_0^{\tan^{-1}a/L}$$

$$= \frac{1}{La}\frac{a}{\sqrt{L^2+a^2}} = \frac{1}{L\sqrt{L^2+a^2}}.$$

Then $F_x = qQ/4\pi\epsilon_0 L\sqrt{L^2+a^2}$ and $\mathbf{F} = (qQ/4\pi\epsilon_0 L\sqrt{L^2+a^2})\mathbf{i}$.

47. (a) Since $\mathbf{F}_f = -\mathbf{F}_g$, $|\mathbf{F}_g| = |\mathbf{F}_f| = \mu|\mathbf{F}_n|$ and $\tan\theta = |\mathbf{F}_g|/|\mathbf{F}_n| = \mu|\mathbf{F}_n|/|\mathbf{F}_n| = \mu$

(b) $\theta = \arctan 0.6 \approx 31°$

48. Since $\mathbf{w} + \mathbf{F_1} + \mathbf{F_2} = \mathbf{0}$,

$$-200\mathbf{j} + |\mathbf{F}_1|\cos 20°\mathbf{i} + |\mathbf{F}_2|\sin 20°\mathbf{j} - |\mathbf{F}_2|\cos 15°\mathbf{i} + |\mathbf{F}_2|\sin 15°\mathbf{j} = \mathbf{0}$$

$$\text{or} \qquad (|\mathbf{F}_1|\cos 20° - |\mathbf{F}_2|\cos 15°)\mathbf{i} + (|\mathbf{F}_1|\sin 20° - |\mathbf{F}_2|\sin 15° - 200)\mathbf{j} = \mathbf{0}.$$

Thus, $|\mathbf{F}_1|\cos 20° - |\mathbf{F}_2|\cos 15° = 0$; $|\mathbf{F}_1|\sin 20° + |\mathbf{F}_2|\sin 15° = 0$. Solving this system for $|\mathbf{F}_1|$ and $|\mathbf{F}_2|$, we obtain

$$|\mathbf{F}_1| = \frac{200\cos 15°}{\sin 15°\cos 20° + \cos 15° sin 20°} = \frac{200\cos 15°}{\sin(15°+20°)} = \frac{200\cos 15°}{\sin 35°} \approx 336.8 \text{ lb}$$

and

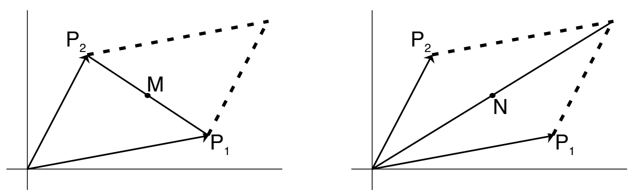
$$|\mathbf{F}_2| = \frac{200\cos 20°}{\sin 15°\cos 20° + \cos 15° sin 20°} = \frac{200\cos 20°}{\sin 35°} \approx 327.7 \text{ lb}.$$

49. Since $\mathbf{F}_2 = 200(\mathbf{i}+\mathbf{j})/\sqrt{2} = 100\sqrt{2}\mathbf{i} + 100\sqrt{2}\mathbf{j}$, $\mathbf{F}_3) = \mathbf{F}_2\mathbf{F}_1 = (100\sqrt{2} - 200)\mathbf{i} + 100\sqrt{2}\mathbf{j}$ and $|\mathbf{F}_3| = \sqrt{(100\sqrt{2}-200)^2 + (100\sqrt{2})^2} = 200\sqrt{2-\sqrt{2}} \approx 153$ lb.

50. We have $\overrightarrow{OA} = 150\cos 20°\mathbf{i} + 150\sin 20°\mathbf{j}$, $\overrightarrow{AB} = 200\cos 113°\mathbf{i} + 200\sin 113°\mathbf{j}$, $\overrightarrow{BC} = 240\cos 190°\mathbf{i} + 240\sin 190°\mathbf{j}$. Then

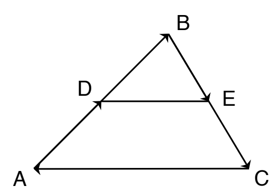
$\mathbf{r} = (150\cos 20° + 200\cos 133° + 240\cos 190°)\mathbf{i} + (150\sin 20° + 200\sin 113° + 240\sin 190°\mathbf{j}$ and
$\approx -173.55\mathbf{i} + 193.73\mathbf{j}$
$|\mathbf{r}| \approx 260.09$ miles.

51.



Place one corner of the parallelogram at the origin, and let two adjacent sides be $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$. Let M be the midpoint of the diagonal connecting P_1 and P_2 and N be the midpoint of the other diagonal. By Problem 37, $\overrightarrow{OM} = \frac{1}{2}(\overrightarrow{OP_1} + \overrightarrow{OP_2})$. Since $\overrightarrow{OP_1} + \overrightarrow{OP_2}$ is the main diagonal of the parallelogram and N is its midpoint, $\overrightarrow{ON} = \frac{1}{2}(\overrightarrow{OP_1} + \overrightarrow{OP_2}$. Thus, $\overrightarrow{OM} = \overrightarrow{ON}$ and the diagonals bisect each other.

52. By Problem 39, $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \mathbf{0}$ and $\overrightarrow{AD} + \overrightarrow{DE} + \overrightarrow{ED} + \overrightarrow{CA} = \mathbf{0}$. From the first equation $\overrightarrow{AB} + \overrightarrow{BC} = -\overrightarrow{CA}$. Since D and E are midpoint, $\overrightarrow{AD} = \frac{1}{2}\overrightarrow{AB}$ and $\overrightarrow{EC} = \frac{1}{2}\overrightarrow{BC}$. Then, $\frac{1}{2}\overrightarrow{AB} + \overrightarrow{DE} + \frac{1}{2}\overrightarrow{BC} + \overrightarrow{CA} = \mathbf{0}$ and

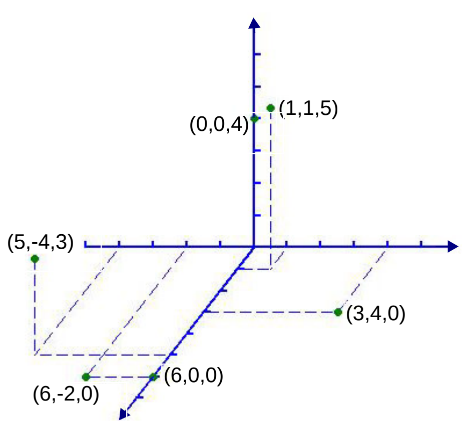


$$\overrightarrow{DE} = -\overrightarrow{CA} - \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{BC}\right) = -\overrightarrow{CA} - \frac{1}{2}\left(-\overrightarrow{CA}\right) = -\frac{1}{2}\overrightarrow{CA}.$$

Thus, the line segment joining the midpoints D and E is parallel to the side AC and half its length.

11.2 3-Space and Vectors

1.- 6.



7. A plane is perpendicular to the z-axis, 5 units above the xy-plane.

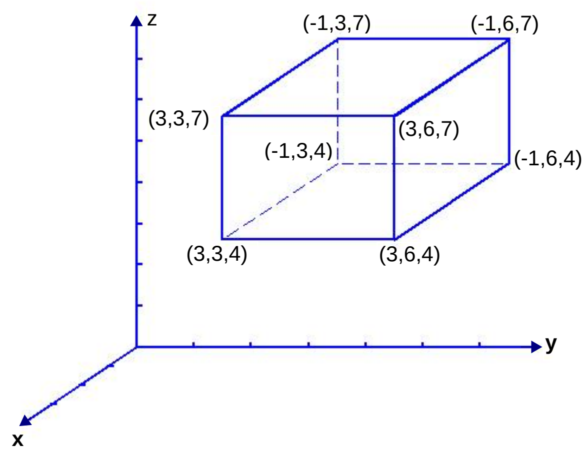
8. A plane perpendicular to the x-axis, 1 unit in front of the yz-plane.

9. A line perpendicular to the xy-plane at $(2, 3, 0)$.

10. A single point located at $(4, -1, 7)$.

11. $(2, 0, 0),\ (2, 5, 0),\ (2, 0, 8),\ (2, 5, 8),\ (0, 5, 0),\ (0, 5, 8),\ (0, 0, 8),\ (0, 0, 0)$

12.



13. (a) xy-plane: $(-2, 5, 0)$, xz-plane: $(-2, 0, 4)$, yz-plane: $(0, 5, 4)$;
    (b) $(-2, 5, -2)$
    (c) Since the shortest distance between a point and a plane is a perpendicular line, the point in the plane $x = 3$ is $(3, 5, 4)$.

14. We find planes that are parallel to coordinate planes.
    (a) $z = -5$; (b) $x = 1$ and $y = -1$; (c) $z = 2$

15. The union of the planes $x = 0,\ y = 0,$ and $z = 0$.

16. The origin $(0, 0, 0)$.

17. The point $(-1, 2, -3)$.

18. The union of the planes $x = 2$ and $z = 8$.

19. The union of the planes $z = 5$ and $z = -5$.

20. The line through the points $(1, 1, 1)$, $(-1, -1, -1)$, and the origin.

21. $d = \sqrt{(3-6)^2 + (-1-4)^2 + (2-8)^2} = \sqrt{70}$

22. $d = \sqrt{(-1-0)^2 + (-3-4)^2 + (5-3)^2} = 3\sqrt{6}$

23. (a) 7; (b) $d = \sqrt{(-3)^2 + (-4)^2} = 5$

24. (a) 2; (b) $d = \sqrt{(-6)^2 + 2^2 + (-3)^2} = 7$

25. $d(P_1, P_2) = \sqrt{(3^2 + 6^2 + (-6)^2} = 9$; $d(P_1, P_3) = \sqrt{2^2 + 1^2 + 2^2} = 3$
$d(P_2, P_3) = \sqrt{((2-3)^2 + (1-6)^2 + (2-(-6))^2} = \sqrt{90}$. The triangle is a right triangle.

26. $d(P_1, P_2) = \sqrt{(1^2 + 2^2 + 4^2} = \sqrt{21}$; $d(P_1, P_3) = \sqrt{3^2 + 2^2 + (2\sqrt{2})^2} = \sqrt{21}$
$d(P_2, P_3) = \sqrt{((3-1)^2 + (2-2)^2 + (2\sqrt{2}-4)^2} = \sqrt{28 - 16\sqrt{2}}$.
The triangle is an isosceles triangle.

27. $d(P_1, P_2) = \sqrt{((4-1)^2 + (1-2)^2 + (3-3)^2} = \sqrt{10}$
$d(P_1, P_3) = \sqrt{(4-1)^2 + (6-2)^2 + (4-3)^2} = \sqrt{26}$
$d(P_2, P_3) = \sqrt{((4-4)^2 + (6-1)^2 + (4-3)^2} = \sqrt{26}$; The triangle is an isosceles triangle.

28. $d(P_1, P_2) = \sqrt{((1-1)^2 + (1-1)^2 + (1-(-1))^2} = 2$
$d(P_1, P_3) = \sqrt{(0-1)^2 + (-1-1)^2 + (1-(-1))^2} = 3$
$d(P_2, P_3) = \sqrt{((0-1)^2 + (-1-1)^2 + (1-1)^2} = \sqrt{5}$; The triangle is a right triangle.

29. $d(P_1, P_2) = \sqrt{((-2-1)^2 + (-2-2)^2 + (-3-0)^2} = \sqrt{34}$
$d(P_1, P_3) = \sqrt{(7-1)^2 + (10-2)^2 + (6-0)^2} = 2\sqrt{34}$
$d(P_2, P_3) = \sqrt{((7-(-2))^2 + (10-(-2))^2 + (6-(-3))^2} = 3\sqrt{34}$
Since $d(P_1, P_2) + d(P_1, P_3) = d(P_2, P_3)$, the points P_1, P_2, and P_3 are collinear.

30. $d(P_1, P_2) = \sqrt{((0-1)^2 + (3-2)^2 + (2-(-1))^2} = \sqrt{11}$
$d(P_1, P_3) = \sqrt{(1-1)^2 + (1-2)^2 + ((-3)-(-1))^2} = \sqrt{5}$
$d(P_2, P_3) = \sqrt{((1-0)^2 + (1-3)^2 + ((-3)-2)^2} = \sqrt{30}$
Since adding any two of the above distances will not result in the third, the points cannot be collinear.

31. $d(P_1, P_2) = \sqrt{((-4)-1)^2 + ((-3)-0)^2 + (5-4)^2} = \sqrt{35}$
$d(P_1, P_3) = \sqrt{((-7)-1)^2 + ((-4)-0)^2 + (8-4)^2} = \sqrt{96}$
$d(P_2, P_3) = \sqrt{((-7)-(-4))^2 + ((-4)-(-3))^2 + (8-5)^2} = \sqrt{19}$
Since adding any two of the above distances will not result in the third, the points cannot be collinear.

32. $d(P_1, P_2) = \sqrt{(1-2)^2 + (4-3)^2 + (4-2)^2} = \sqrt{6}$
$d(P_1, P_3) = \sqrt{(5-2)^2 + (0-3)^2 + (-4-2)^2} = 3\sqrt{6}$
$d(P_2, P_3) = \sqrt{(5-1)^2 + (0-4)^2 + (-4-4)^2} = 4\sqrt{6}$
Since $d(P_1, P_2) + d(P_1, P_3) = d(P_2, P_3)$, the points P_1, P_2, and P_3 are collinear.

33. $\sqrt{(2-x)^2 + (1-2)^2 + (1-3)^2} = \sqrt{21} \longrightarrow x^2 - 4x + 9 = 21 \longrightarrow x^2 - 4x + 4 = 16 \longrightarrow (x-2)^2 = 16 \longrightarrow x = 2 + -4$ or $x = 6, -2$

34. $\sqrt{(0-x)^2 + (3-x)^2 + (5-1)^2} = 5 \longrightarrow 2x^2 - 6x + 25 = 25 \longrightarrow x^2 - 3x = 0 \longrightarrow x = 0, 3$

35. $\left(\dfrac{1+7}{2}, \dfrac{3+(-2)}{2}, \dfrac{1/2+5/2}{2}\right) = (4, 1/2, 3/2)$

36. $\left(\dfrac{0+4}{2}, \dfrac{5+1}{2}, \dfrac{-8+(-6)}{2}\right) = (2, 3, -7)$

37. $(x_1+2)/2 = -1,\ x_1 = -4,\ (y_1+3)/2 = -4,\ y_1 = -11;\quad (z_1+6)/2 = 8,\ \ z_1 = 10$
The coordinates of P_1 are $(-4, -11, 10)$.

38. $(-3+(-5))/2 = x_3 = -4,\ (4+8)/2 = y_3 = 6;\ \ (1+3)/2 = z_3 = 2.$
The coordinates of P_3 are $(-4, 6, 2)$.

(a) $\left(\dfrac{-3+(-4)}{2}, \dfrac{4+6}{2}, \dfrac{1+2}{2}\right) = (-7/2, 5, 3/2)$

(b) $\left(\dfrac{-4+(-5)}{2}, \dfrac{6+8}{2}, \dfrac{2+3}{2}\right) = (-9/2, 7, 5/2)$

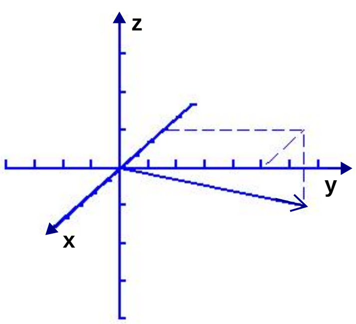
39. $\overrightarrow{P_1P_2} = \langle -3, -6, 1\rangle$

40. $\overrightarrow{P_1P_2} = \langle 8, -5/2, 8\rangle$

41. $\overrightarrow{P_1P_2} = \langle 2, 1, 1\rangle$

42. $\overrightarrow{P_1P_2} = \langle -3, -3, 7\rangle$

43.



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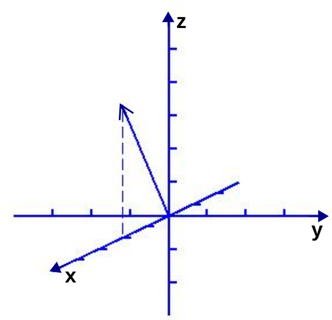

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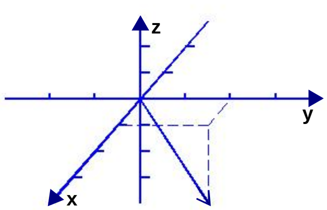

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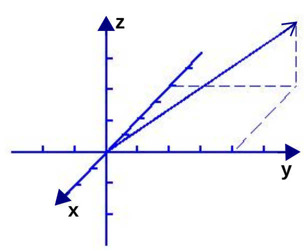


47. Since the **k** component is zero, while the **i** and **j** components are nonzero, the vector lies in the xy-plane.

48. Since the **j** components is the only nonzero component, the vector lies on the y-axis.

49. Since the vector is a scalar multiple of **k**, the vector lies on the z-axis.

50. Since the **i** component is zero while the **j** and **k** components are nonzero, the vector lies in the yz-plane.

51. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = \langle 2, 4, 12 \rangle$

52. $2\mathbf{a} - (\mathbf{b} - \mathbf{c}) = \langle 2, -6, 4 \rangle - \langle -3, -5, -8 \rangle = \langle 5, -1, 12 \rangle$

53. $\mathbf{b} + 2(\mathbf{a} - 3\mathbf{c}) = \langle -1, 1, 1 \rangle + 2\langle -5, -21, -25 \rangle = \langle -11, -41, -49 \rangle$

54. $4(\mathbf{a} + 2\mathbf{c}) - 6\mathbf{b} = 4\langle 5, 9, 20 \rangle - \langle -6, 6, 6 \rangle = \langle 26, 30, 74 \rangle$

55. $|\mathbf{a} + \mathbf{c}| = |\langle 3, 3, 11 \rangle| = \sqrt{9 + 9 + 121} = \sqrt{139}$

56. $|\mathbf{c}||2\mathbf{b}| = (\sqrt{4 + 36 + 81})(2)(\sqrt{1 + 1 + 1}) = 22\sqrt{3}$

57. $\left| \dfrac{a}{|a|} \right| + 5 \left| \dfrac{\mathbf{b}}{|\mathbf{b}|} \right| = \dfrac{1}{|\mathbf{b}|}|\mathbf{b}| = 1 + 5 = 6$

58. $$\begin{aligned} |\mathbf{b}|\mathbf{a} + |\mathbf{a}|\mathbf{b} &= \sqrt{1+1+1}\langle 1, -3, 2 \rangle + \sqrt{1+9+4}\langle -1, 1, 1 \rangle \\ &= \langle \sqrt{3}, -3\sqrt{3}, 2\sqrt{3} \rangle + \langle -\sqrt{14}, \sqrt{14}, \sqrt{14} \rangle \\ &= \langle \sqrt{3} - \sqrt{14}, -3\sqrt{3} + \sqrt{14}, 2\sqrt{3} + \sqrt{14} \rangle \end{aligned}$$

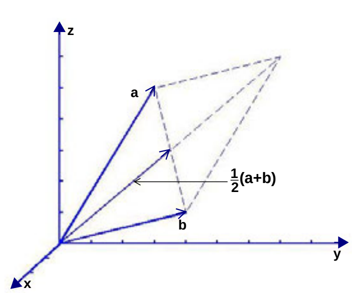
59. $|\mathbf{a}| = \sqrt{100 + 25 + 100} = 15; \quad \mathbf{u} = -\dfrac{1}{15}\langle 10, -5, 10 \rangle = \langle -2/3, 1/3, -2/3 \rangle$

60. $|\mathbf{a}| = \sqrt{1 + 9 + 4} = \sqrt{14}; \quad \mathbf{u} = \dfrac{1}{\sqrt{14}}(\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}) = \dfrac{1}{\sqrt{14}}\mathbf{i} - \dfrac{3}{\sqrt{14}}\mathbf{j} + \dfrac{2}{\sqrt{14}}\mathbf{k}$

61. $\mathbf{b} = 4\mathbf{a} = 4\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}$

62. $|\mathbf{a}| = \sqrt{36 + 9 + 4} = 7; \quad \mathbf{b} = -\dfrac{1}{2}\left(\dfrac{1}{7}\right)\langle -6, 3, -2 \rangle = \left\langle \dfrac{3}{7}, -\dfrac{3}{14}, \dfrac{1}{7} \right\rangle$

63.



64. Following the hint, we first complete the following:

$$d(P_1, M) = \sqrt{\left(\frac{x_1+x_2}{2} - x_1\right)^2 + \left(\frac{y_1+y_2}{2} - y_1\right)^2 + \left(\frac{z_1+z_2}{2} - z_1\right)^2}$$
$$= \sqrt{\left(\frac{x_1-x_2}{2}\right)^2 + \left(\frac{y_1-y_2}{2}\right)^2 + \left(\frac{z_1-z_2}{2}\right)^2}$$
$$= \frac{1}{2}\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$
$$d(M, P_2) = \sqrt{\left(x_2 - \frac{x_1+x_2}{2}\right)^2 + \left(y_2 - \frac{y_1+y_2}{2}\right)^2 + \left(z_2 - \frac{z_1+z_2}{2}\right)^2}$$
$$= \sqrt{\left(\frac{x_1-x_2}{2}\right)^2 + \left(\frac{y_1-y_2}{2}\right)^2 + \left(\frac{z_1-z_2}{2}\right)^2}$$
$$= \frac{1}{2}\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$
$$d(P_1, P_2) = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

From the above, we see that $d(P_1,P_2) = d(P_1,M) + d(M,P_2)$.
Therefore, M is collinear with P_1 and P_2 and lies between them. Since $d(P_1,M) = d(M,P_2)$, we also have that M is equidistant from P_1 and P_2. Therefore, M is the midpoint of the line segment joining P_1 and P_2.

65. $x_P = 1, \quad y_P = \cos 30° + \sin 30° = \dfrac{\sqrt{3}}{2} + \dfrac{1}{2}(\sqrt{3}+1),$

$$x_P = -\sin 30° + \cos 30° = -\frac{1}{2} + \frac{\sqrt{3}}{2} = \frac{1}{2}(\sqrt{3}-1),$$
$$x_R = \cos 45° - \frac{1}{2}(\sqrt{3}-1)\sin 45° = \frac{\sqrt{2}}{2} - \frac{1}{2}(\sqrt{3}-1)\frac{\sqrt{2}}{2} = \frac{1}{4}(3\sqrt{2}-\sqrt{6}), \quad y_r = \frac{1}{2}(\sqrt{3}+1),$$
$$z_r = \sin 45° + \frac{1}{2}(\sqrt{3}-1)\cos 45° = \frac{\sqrt{2}}{2} + \frac{1}{2}(\sqrt{3}-1)\frac{\sqrt{2}}{2} = \frac{1}{4}(\sqrt{2}+\sqrt{6}),$$
$$x_S = \frac{1}{4}(3\sqrt{2}-\sqrt{6})\cos 60° + \frac{1}{2}(\sqrt{3}+1)\sin 60° = \frac{1}{4}(3\sqrt{2}-\sqrt{6})\frac{1}{2} + \frac{1}{2}(\sqrt{3}+1)\frac{\sqrt{3}}{2}$$
$$= \frac{1}{8}(3\sqrt{2}-\sqrt{6}+6+2\sqrt{3}),$$
$$y_S = -\frac{1}{4}(3\sqrt{2}-\sqrt{6})\sin 60° + \frac{1}{2}(\sqrt{3}+1)\cos 60° = -\frac{1}{4}(3\sqrt{2}-\sqrt{6})\frac{\sqrt{3}}{2} + \frac{1}{2}(\sqrt{3}+1)\frac{1}{2}$$
$$= \frac{1}{8}(-3\sqrt{6}+3\sqrt{2}+2\sqrt{3}+2),$$
$$z_s = \frac{1}{4}(\sqrt{2}+\sqrt{6})$$

Thus, $x_S \approx 1.4072, \quad y_S \approx 0.2948, \quad z_S \approx 0.9659.$

66. (a) $M_P = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{vmatrix}, \quad M_R = \begin{vmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{vmatrix}$

(b) $M_\gamma M_R M_P \begin{bmatrix} x \\ y \\ z \end{bmatrix} = M_\gamma M_R \begin{bmatrix} x_P \\ y_P \\ z_P \end{bmatrix} = \begin{bmatrix} x_S \\ y_S \\ z_S \end{bmatrix}$

(c) $$M_P \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 30^\circ & \sin 30^\circ \\ 0 & -\sin 30^\circ & \cos 30^\circ \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2}(\sqrt{3}+1) \\ \frac{1}{2}(\sqrt{3}-1) \end{bmatrix}$$

$$M_R M_P \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos 45^\circ & 0 & -\sin 45^\circ \\ 0 & 1 & 0 \\ \sin 45^\circ & 0 & \cos 45^\circ \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{2}(\sqrt{3}+1) \\ \frac{1}{2}(\sqrt{3}-1) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{2}(\sqrt{3}+1) \\ \frac{1}{2}(\sqrt{3}-1) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4}(3\sqrt{2}-\sqrt{6}) \\ \frac{1}{2}(\sqrt{3}+1) \\ \frac{1}{4}(\sqrt{2}+\sqrt{6}) \end{bmatrix}$$

$$M_\gamma M_R M_P \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos 60^\circ & \sin 60^\circ & 0 \\ \sin \sin 60^\circ & \cos \sin 60^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{4}(3\sqrt{2}-\sqrt{6}) \\ \frac{1}{2}(\sqrt{3}+1) \\ \frac{1}{4}(\sqrt{2}+\sqrt{6}) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{4}(3\sqrt{2}-\sqrt{6}) \\ \frac{1}{2}(\sqrt{3}+1) \\ \frac{1}{4}(\sqrt{2}+\sqrt{6}) \end{bmatrix} = \begin{bmatrix} \frac{1}{8}(3\sqrt{2}-\sqrt{6}+6+2\sqrt{3}) \\ \frac{1}{8}(-3\sqrt{6}+3\sqrt{2}+2\sqrt{3}+2) \\ \frac{1}{4}(\sqrt{2}+\sqrt{6}) \end{bmatrix}$$

11.3 Dot Product

1. $\mathbf{a} \cdot \mathbf{b} = 2(-1) + (-3)2 + 4(5) = 12$
2. $\mathbf{b} \cdot \mathbf{c} = (-1)3 + 2(6) + 5(-1) = 4$
3. $\mathbf{a} \cdot \mathbf{c} = 2(3) + (-3)6 + 4(-1) = -16$
4. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = 2(2) + (-3)8 + 4(4) = -4$
5. $\mathbf{a} \cdot (4\mathbf{b}) = 2(-4) + (-3)8 + 4(20) = 48$
6. $\mathbf{b} \cdot (\mathbf{a} - \mathbf{c}) = (-1)(-1) + 2(-9) = 5(5) = 8$
7. $\mathbf{a} \cdot \mathbf{a} = 2^2 + (-3)^2 + 4^2 = 29$
8. $(2\mathbf{b}) \cdot (3\mathbf{c}) = (-2)9 + 4(18) + 10(-3) = 24$
9. $\mathbf{a} \cdot (\mathbf{a} + \mathbf{b} + \mathbf{c}) = 2(4) + (-3)5 + 4(8) = 25$
10. $(2\mathbf{a}) \cdot (\mathbf{a} - 2\mathbf{b}) = 4(4) + (-6)(-7) + 8(-6) = 10$
11. $\left(\dfrac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}}\right)\mathbf{b} = \left[\dfrac{2(-1) + (-3)2 + 4(5)}{(-1)^2 + 2^2 + 5^2}\right]\langle -1, 2, 5\rangle = \dfrac{12}{30}\langle -1, 2, 5\rangle = \langle -2/5, 4/5, 2\rangle$
12. $(\mathbf{c} \cdot \mathbf{b})\mathbf{a} = [3(-1) + 6(2) + (-1)5]\langle 2, -3, 4\rangle = 4\langle 2, -3, 4\rangle = \langle 8, -12, 16\rangle$
13. $\mathbf{a} \cdot \mathbf{b} = 10(5)\cos(\pi/4) = 25\sqrt{2}$

14. $\mathbf{a}\cdot\mathbf{b} = 6(12)\cos(\pi/6) = 36\sqrt{3}$

15. $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta = (2)(3)\cos(2\pi/3) = 6(-1/2) = -3$

16. $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos(\theta) = (4)(1)\cos(5\pi.6) = 4\left(-\frac{\sqrt{3}}{2}\right) = -2\sqrt{3}$

17. $\sqrt{5}\mathbf{a}\cdot\mathbf{b} = 3(2)+(-1)2 = 4;\;\; |\mathbf{a}| = \sqrt{10},\;\; |\mathbf{b}| = 2\sqrt{2}$
$\cos\theta = \dfrac{4}{(\sqrt{10})(2\sqrt{2}} = \dfrac{1}{\sqrt{5}} \longrightarrow \theta = \arccos\dfrac{1}{\sqrt{5}} \approx 1.11 \text{ rad} \approx 63.45°$

18. $\sqrt{5}\mathbf{a}\cdot\mathbf{b} = 2(-3)+1(-4) = -10;\;\; |\mathbf{a}| = \sqrt{5},\;\; |\mathbf{b}| = 5$
$\cos\theta = \dfrac{-10}{(\sqrt{5})5} = -\dfrac{2}{\sqrt{5}} \longrightarrow \theta = \arccos(-2/\sqrt{5}) \approx 2.68 \text{ rad} \approx 153.43°$

19. $\sqrt{5}\mathbf{a}\cdot\mathbf{b} = 2(-1)+4(-1)+0(4) = -6;\;\; |\mathbf{a}| = 2\sqrt{5},\;\; |\mathbf{b}| = 3\sqrt{2}$
$\cos\theta = \dfrac{-6}{(2\sqrt{5})(3\sqrt{2}} = -\dfrac{1}{\sqrt{10}} \longrightarrow \theta = \arccos(-1/\sqrt{10}) \approx 1.89 \text{ rad} \approx 108.43°$

20. $\sqrt{5}\mathbf{a}\cdot\mathbf{b} = \frac{1}{2}(2)+\frac{1}{2}(-4)+\frac{3}{2}(6) = 8;\;\; |\mathbf{a}| = \sqrt{11}/2,\;\; |\mathbf{b}| = 2\sqrt{14}$
$\cos\theta = \dfrac{8}{(\sqrt{11}/2)(2\sqrt{14})} = \dfrac{8}{\sqrt{154}} \longrightarrow \theta = \arccos(8/\sqrt{154}) \approx 0.87 \text{ rad} \approx 49.86°$

21. a and f, b and e, c and d

22. (a) $\mathbf{a}\times\mathbf{b} = 2\cdot 3+(-c)2+3(4) = 0 \longrightarrow c = 9$
(b) $\mathbf{a}\cdot\mathbf{b} = c(-3)+\frac{1}{2}(4)+c^2 = c^2-3c+2 = (c-2)(c-1) = 0 \longrightarrow c = 1, 2$

23. Solving the system of equations $3x_1+y_1-1=0,\;\; -3x_1+2y_1+2=0$ gives $x_1 = 4/9$ and $y_1 = -1/3$. Thus, $\mathbf{v} = \langle 4/9, -1/3, 1\rangle$.

24. If $\mathbf{a}$ and $\mathbf{b}$ represent adjacent sides of the rhombus, then $|\mathbf{a}| = |\mathbf{b}|$, then diagonals of the rhombus are $\mathbf{a}+\mathbf{b}$ and $\mathbf{a}-\mathbf{b}$, and

$$(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = \mathbf{a}\cdot\mathbf{a}-\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{a}-\mathbf{b}\cdot\mathbf{b} = |\mathbf{a}|^2-|\mathbf{b}|^2 = 0.$$

Thus, the diagonals are perpendicular.

25. Since

$$\mathbf{c}\cdot\mathbf{a} = \left(\mathbf{b}-\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}\right)\cdot\mathbf{a} = \mathbf{a}\cdot\mathbf{a}-\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}(\mathbf{a}\cdot\mathbf{a}) = \mathbf{b}\cdot\mathbf{a}-\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}|\mathbf{a}|^2 = \mathbf{b}\cdot\mathbf{a}-\mathbf{a}\cdot\mathbf{b} = 0,$$

the vectors $\mathbf{c}$ and $\mathbf{a}$ are orthogonal.

26. $\mathbf{a}\cdot\mathbf{b} = 1(1)+c(1) = c+1;\;\; |\mathbf{a}| = \sqrt{1+c^2},\;\; |\mathbf{b}| = \sqrt{2}$
$\cos 45° = \dfrac{1}{\sqrt{2}} = \dfrac{c+1}{\sqrt{1+c^2}\sqrt{2}} \Longrightarrow \sqrt{1+c^2} = c+1 \longrightarrow 1+c^2 = c^2+2c+1 \Longrightarrow c = 0$

27. $|\mathbf{a}| = \sqrt{14};\;\; \cos\alpha - 1/\sqrt{14},\;\; \alpha \approx 74.50°;\;\; \cos\beta = 2/\sqrt{14},\;\; \beta \approx 57.69°;\;\; \cos\gamma = 3/\sqrt{14},$
$\gamma = 36.70°$

28. $|\mathbf{a}| = 9$; $\cos\alpha = 2/3$, $\alpha \approx 48.19°$; $\cos\beta = 2/3$, $\beta \approx 48.19°$; $\cos\gamma = -1/3$, $\gamma = 109.47°$

29. $|\mathbf{a}| = 2$; $\cos\alpha = 1/2$, $\alpha \approx 60°$; $\cos\beta = 0$, $\beta \approx 90°$; $\cos\gamma = -\sqrt{3}/2$, $\gamma = 150°$

30. $|\mathbf{a}| = \sqrt{78}$; $\cos\alpha = 5/\sqrt{78}$, $\alpha \approx 55.52°$; $\cos\beta = 7/\sqrt{78}$, $\beta \approx 37.57°$; $\cos\gamma = 2/\sqrt{78}$, $\gamma = 76.91°$

31. Let θ be the angle between $\overrightarrow{AD}$ and $\overrightarrow{AB}$ and a be the length of an edge of an edge of the cube. Then $\overrightarrow{AD} = a\mathbf{i} + a\mathbf{j} + a\mathbf{k}$, $\overrightarrow{AB} = a\mathbf{i}$ and

$$\cos\theta \frac{\overrightarrow{AD} \cdot \overrightarrow{AB}}{|\overrightarrow{AD}||\overrightarrow{AB}|} = \frac{a^2}{\sqrt{3a^2}\sqrt{a^2}} = \frac{1}{\sqrt{3}}$$

so $\theta \approx 0.955317$ radian or $54.7356°$. Letting ϕ be the angle between $\overrightarrow{AD}$ and $\overrightarrow{AC}$ and noting that $\overrightarrow{AC} = a\mathbf{i} + a\mathbf{j}$ we have

$$\cos\theta \frac{\overrightarrow{AD} \cdot \overrightarrow{AC}}{|\overrightarrow{AD}||\overrightarrow{AC}|} = \frac{a^2 + a^2}{\sqrt{3a^2}\sqrt{2a^2}} = \sqrt{\frac{2}{3}}$$

so $\phi \approx 0.61548$ radian or $35.2644°$

32. $\mathbf{a} = \langle 5, 7, 4\rangle$; $|\mathbf{a}| = 3\sqrt{10}$; $\cos\alpha = 5/3\sqrt{10}$, $\alpha = 58.19°$; $\cos\beta = 7/3\sqrt{10}$, $\beta = 42.45°$; $\cos\gamma = 4/3\sqrt{10}$, $\gamma \approx 65.06°$

33. $\text{comp}_{\mathbf{b}}\mathbf{a} = \mathbf{a} \cdot \mathbf{b}/|\mathbf{b}| = \langle 1, -1, 3\rangle \cdot \langle 2, 6, 3\rangle/7 = 5/7$

34. $\text{comp}_{\mathbf{a}}\mathbf{b} = \mathbf{b} \cdot \mathbf{a}/|\mathbf{a}| = \langle 2, 6, 3\rangle \cdot \langle 1, -1, 3\rangle/\sqrt{11} = 5/\sqrt{11}$

35. $\mathbf{b} - \mathbf{a} = \langle 1, 7, 0\rangle$; $\text{comp}_{\mathbf{a}}(\mathbf{b} - \mathbf{a}) = (\mathbf{b} - \mathbf{a}) \cdot \mathbf{a}/|\mathbf{a}| = \langle 1, 7, 0\rangle \cdot \langle 1, -1, 3\rangle/\sqrt{11} = -6/\sqrt{11}$

36. $\mathbf{a} + \mathbf{b} = \langle 3, 5, 6\rangle$; $2\mathbf{b} = \langle 4, 12, 6\rangle$; $\text{comp}_{2\mathbf{b}}(\mathbf{a} + \mathbf{b}) \cdot 2\mathbf{a}/|2\mathbf{b}| = \langle 3, 5, 6\rangle \cdot \langle 4, 12, 6\rangle/14 = 54/7$

37. $\overrightarrow{OP} = 3\mathbf{i} + 10\mathbf{j}$; $|\overrightarrow{OP}| = \sqrt{109}$; $\text{comp}_{\overrightarrow{OP}}\mathbf{a} = \mathbf{a} \cdot \overrightarrow{OP}/|\overrightarrow{OP}| = (4\mathbf{i} + 6\mathbf{j}) \cdot (3\mathbf{i} + 10\mathbf{j})/\sqrt{109} = 72/\sqrt{109}$

38. $\overrightarrow{OP} = \langle 1, -1, 1\rangle$; $|\overrightarrow{OP}| = \sqrt{3}$; $\text{comp}_{\overrightarrow{OP}}\mathbf{a} = \mathbf{a} \cdot \overrightarrow{OP}/|\overrightarrow{OP}| = \langle 2, 1, -1\rangle \cdot \langle 1, -1, 1\rangle/\sqrt{3} = 0$

39. (a) $\text{comp}_{\mathbf{b}}\mathbf{a} = \mathbf{a} \cdot \mathbf{b}/|\mathbf{b}| = (-5\mathbf{i} + 5\mathbf{j}) \cdot (-3\mathbf{i} + 4\mathbf{j})/5 = 7$
    $\text{proj}_{\mathbf{b}}\mathbf{a} = (\text{comp}_{\mathbf{b}}\mathbf{a})\mathbf{b}/|\mathbf{b}| = 7(-3\mathbf{i} + 4\mathbf{j})/5 = -\frac{21}{5}\mathbf{i} + \frac{28}{5}\mathbf{j}$
    (b) $\text{proj}_{\mathbf{b}^\perp}\mathbf{a} = \mathbf{a} - \text{proj}_{\mathbf{b}}\mathbf{a} = (-5\mathbf{i} + 5\mathbf{j}) - (-\frac{21}{5} + \frac{28}{5}\mathbf{j}) = -\frac{4}{5}\mathbf{i} - \frac{3}{5}\mathbf{j}$

40. (a) $\text{comp}_{\mathbf{b}}\mathbf{a} = \mathbf{a} \cdot \mathbf{b}/|\mathbf{b}| = (4\mathbf{i} + 2\mathbf{j}) \cdot (-3\mathbf{i} + \mathbf{j})/\sqrt{10} = -\sqrt{10}$
    $\text{proj}_{\mathbf{b}}\mathbf{a} = (\text{comp}_{\mathbf{b}}\mathbf{a})\mathbf{b}/|\mathbf{b}| = -\sqrt{10}(-3\mathbf{i} + \mathbf{j})/\sqrt{10} = 3\mathbf{i} - \mathbf{j}$
    (b) $\text{proj}_{\mathbf{b}^\perp}\mathbf{a} = \mathbf{a} - \text{proj}_{\mathbf{b}}\mathbf{a} = (4\mathbf{i} + 2\mathbf{j}) - (3\mathbf{i} - \mathbf{j}) = \mathbf{i} + 3\mathbf{j}$

41. (a) $\text{comp}_{\mathbf{b}}\mathbf{a} = \mathbf{a} \cdot \mathbf{b}/|\mathbf{b}| = (-\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}) \cdot (6\mathbf{i} - 3\mathbf{j} - 2\mathbf{k})/7 = -2$
    $\text{proj}_{\mathbf{b}}\mathbf{a} = (\text{comp}_{\mathbf{b}}\mathbf{a})\mathbf{b}/|\mathbf{b}| = -2(6\mathbf{i} - 3\mathbf{j} - 2\mathbf{k})/7 = -\frac{12}{7}\mathbf{i} + \frac{6}{7}\mathbf{j} + \frac{4}{7}\mathbf{k}$
    (b) $\text{proj}_{\mathbf{b}^\perp}\mathbf{a} = \mathbf{a} - \text{proj}_{\mathbf{b}}\mathbf{a} = (-\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}) - (-\frac{12}{7}\mathbf{i} - \frac{6}{7}\mathbf{j} + \frac{4}{7}\mathbf{k}) = \frac{5}{7}\mathbf{i} - \frac{20}{7}\mathbf{j} + \frac{45}{7}\mathbf{k}$

42. (a) $\text{comp}_{\mathbf{b}}\mathbf{a} = \mathbf{a}\cdot\mathbf{b}/|\mathbf{b}| = \langle 1,1,1\rangle\cdot\langle -2,2,-1\rangle/3 = -1/3$
$\text{proj}_{\mathbf{b}}\mathbf{a} = (\text{comp}_{\mathbf{b}}\mathbf{a})\mathbf{b}/|\mathbf{b}| = -\frac{1}{3}\langle -2,2,-1\rangle/3 = \langle 2/9,-2/9,1/9\rangle$

(b) $\text{proj}_{\mathbf{b}^\perp}\mathbf{a} = \mathbf{a} - \text{proj}_{\mathbf{b}}\mathbf{a} = \langle 1,1,1\rangle - \langle 2/9,-2/9,1/9\rangle = \langle 7/9,11/9,8/9\rangle$

43. $\mathbf{a}+\mathbf{b} = 3\mathbf{i}+4\mathbf{j}$; $|\mathbf{a}+\mathbf{b}| = 5$; $\text{comp}_{\mathbf{a}+\mathbf{b}}\mathbf{a} = \mathbf{a}\cdot(\mathbf{a}+\mathbf{b})/|\mathbf{a}+\mathbf{b}| = (4\mathbf{i}+3\mathbf{j})\cdot(3\mathbf{i}+4\mathbf{j})/5 = 24/5$; $\text{proj}_{(\mathbf{a}+\mathbf{b})}\mathbf{a} = (\text{comp}_{(\mathbf{a}+\mathbf{b})}\mathbf{a})(\mathbf{a}+\mathbf{b})/|\mathbf{a}+\mathbf{b}| = \frac{24}{5}(3\mathbf{i}+4\mathbf{j})/5 = \frac{72}{25}\mathbf{i}+\frac{96}{25}\mathbf{j}$

44. $\mathbf{a}-\mathbf{b} = 5\mathbf{i}+2\mathbf{j}$; $|\mathbf{a}-\mathbf{b}| = \sqrt{29}$; $\text{comp}_{(\mathbf{a}-\mathbf{b})}\mathbf{b} = \mathbf{b}\cdot(\mathbf{a}-\mathbf{b})/|\mathbf{a}-\mathbf{b}| = (-\mathbf{i}+\mathbf{j})/\sqrt{29} = -3/\sqrt{29}$
$\text{proj}_{(\mathbf{a}-\mathbf{b})}\mathbf{b} = (\text{comp}_{(\mathbf{a}-\mathbf{b})})(\mathbf{a}-\mathbf{b})/|\mathbf{a}-\mathbf{b}| = -\dfrac{3}{\sqrt{29}}(5\mathbf{i}+2\mathbf{j})/\sqrt{29} = -\dfrac{15}{29}\mathbf{i} - \dfrac{6}{29}\mathbf{j}$
$\text{proj}_{(\mathbf{a}-\mathbf{b})^\perp}\mathbf{b} = \mathbf{b} - \text{proj}_{(\mathbf{a}-\mathbf{b})}\mathbf{b} = (-\mathbf{i}+\mathbf{j}) - \left(-\dfrac{15}{29}\mathbf{i} - \dfrac{6}{29}\mathbf{j}\right) = -\dfrac{14}{29}\mathbf{i} + \dfrac{35}{29}\mathbf{j}$

45. We identify $\mathbf{F} = 29$, $\theta = 60°$ and $|\mathbf{d}| = 100$. Then $W = |\mathbf{F}||\mathbf{d}|\cos\theta = 20(100)(\frac{1}{2}) = 1000$ ft-lb.

46. $\mathbf{W} = \mathbf{F}\cdot\mathbf{d} = |\mathbf{F}||\mathbf{d}|\cos\theta$ Since the force is acting at a 45° angle to the direction of motion, we have $\theta = 45°$. Therefore,

$$\mathbf{W} = (3000)(400)\cos 45° = 600,000\sqrt{2} \text{ ft-lb.}$$

47. We identify $\mathbf{d} = -\mathbf{i}+3\mathbf{j}+8\mathbf{k}$. Then $W = \mathbf{F}\cdot\mathbf{d} = \langle 4,3,5\rangle\cdot\langle -1,3,8\rangle = 45$ N-m.

48. (a) Since $\mathbf{w}$ and $\mathbf{d}$ are orthogonal, $W = \mathbf{w}\cdot\mathbf{d} = 0$.

(b) We identify $\theta = 0°$. Then $W = |\mathbf{F}||\mathbf{d}|\cos\theta = 30(\sqrt{4^2+3^2}) = 150$ N-m.

49. Using $\mathbf{d} = 6\mathbf{i}+2\mathbf{j}$ and $\mathbf{F} = 3\left(\frac{3}{5}\mathbf{i}+\frac{4}{5}\mathbf{j}\right)$, $W = \mathbf{F}\cdot\mathbf{d} = \left\langle \frac{9}{5},\frac{12}{5}\right\rangle\cdot\langle 6,2\rangle = \frac{78}{5}$ ft-lb.

50. Let $\mathbf{a}$ and $\mathbf{b}$ be vectors from the center of the carbon atom to the center of two distinct hydrogen atoms. The distance between two hydrogen atoms is then

$$\begin{aligned}|\mathbf{b}-\mathbf{a}| &= \sqrt{(\mathbf{b}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{a})} = \sqrt{\mathbf{b}\cdot\mathbf{b} - 2\mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{a}} \\ &= \sqrt{|\mathbf{b}|^2 + |\mathbf{a}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos\theta} \\ &= \sqrt{(1.1)^2 + (1.1)^2 - 2(1.1)(1.1)\cos 109.5°} \\ &= \sqrt{1.21 + 1.21 - 2.42(-0.333807)} \approx 1.80 \text{ angstroms.}\end{aligned}$$

51. If $\mathbf{a}$ and $\mathbf{b}$ are orthogonal, then $\mathbf{a}\cdot\mathbf{b} = 0$ and

$$\begin{aligned}\cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2 &= \frac{a_1}{\mathbf{a}}\frac{b_1}{\mathbf{b}} + \frac{a_2}{\mathbf{a}}\frac{b_2}{\mathbf{b}} + \frac{a_3}{\mathbf{a}}\frac{b_3}{\mathbf{b}} \\ &= \frac{1}{|\mathbf{a}||\mathbf{b}|}(a_1b_1 + a_2b_2a_3b_3) = \frac{1}{|\mathbf{a}||\mathbf{b}|}(\mathbf{a}\cdot\mathbf{b}) = 0.\end{aligned}$$

52. We want $\cos\alpha = \cos\beta = \cos\gamma$ or $a_1 = a_2 = a_3$. Letting $a_1 = a_2 = a_3 = 1$ we obtain the vector $\mathbf{i}+\mathbf{j}+\mathbf{k}$. A unit vector in the same direction is $\dfrac{1}{\sqrt{3}}\mathbf{i} + \dfrac{1}{\sqrt{3}}\mathbf{j} + \dfrac{1}{\sqrt{3}}\mathbf{k}$.

53. For the following, let $\mathbf{a} = \langle a_1, a_2, a_3\rangle$, $\quad \mathbf{b} = \langle b_1, b_2, b_3\rangle$, and let k be any scalar.
Proof of (i): If $\mathbf{a} = \mathbf{0} = \langle 0, 0, 0\rangle$, then $\mathbf{a}\cdot\mathbf{b} = (0)b_1 + (0)b_2 + (0)b_3 = 0$. Similarly, if $\mathbf{b} = \mathbf{0}$, then $\mathbf{a}\cdot\mathbf{b} = a_1(0) + a_2(0) + a_3(0) = 0$
Proof of (ii): Using the Commutative Property of real numbers, we have

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= a_1b_1 + a_2b_2 + a_3b_3\\ &= b_1a_1 + b_2a_2 + b_3a_3\\ &= \mathbf{b}\cdot\mathbf{a}\end{aligned}$$

Proof of (iv): $\begin{aligned}[t]\mathbf{a}\cdot(k\mathbf{b}) &= \langle a_1, a_2, a_3\rangle\cdot\langle kb_2, kb_2, kb_3\rangle\\ &= a_1kb_1 + a_2kb_2 + a_3kb_3\\ &= k(a_1b_1 + a_2b_2 + a_3b_3)\\ &= k(\mathbf{a}\cdot\mathbf{b})\\ &= \langle la_1, ka_2, ka_3\rangle\cdot\langle b_1, b_2, b_3\rangle\\ &= ka_1b_1 + ka_2b_2 + ka_3b_3\\ &= k(a_1b_1 + a_2b_2 + a_3b_3)\\ &= k(\mathbf{a}\cdot\mathbf{b})\end{aligned}$
Therefore, $\mathbf{a}\cdot(k\mathbf{b}) = (k\mathbf{a})\cdot\mathbf{b} = k(\mathbf{a}\cdot\mathbf{b})$
Proof of (v): Since $x^2 \geq 0$ for any real number x, we have $\mathbf{a}\cdot\mathbf{a} = a_1^2 + a_2^2 + a_3^2 \geq 0$.

54. Using the fact that $|\cos\theta| < 1$, we have $|\mathbf{a}\cdot\mathbf{b}| = |\mathbf{a}||\mathbf{b}||\cos\theta| \leq |\mathbf{a}||\mathbf{b}|$.

55. $|\mathbf{a}+\mathbf{b}|^2 = (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) = \mathbf{a}\cdot\mathbf{a} + 2\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{b} = |\mathbf{a}|^2 + 2\mathbf{a}\cdot\mathbf{b} + |\mathbf{b}|^2$
$\leq |\mathbf{a}|^2 + 2|\mathbf{a}\cdot\mathbf{b}| + |\mathbf{b}|^2 \quad \boxed{\text{since } x \leq |x|}$
$\leq |\mathbf{a}|^2 + 2|\mathbf{a}||\mathbf{b}| + |\mathbf{b}|^2 = (|\mathbf{a}| + |\mathbf{b}|)^2 \quad \boxed{\text{by Problem 54}}$
Thus, since $|\mathbf{a}+\mathbf{b}|$ and $|\mathbf{a}| + |\mathbf{b}|$ are positive, $|\mathbf{a}+\mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$.

56. Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be distinct points on the line $ax + by = -c$. Then

$$\mathbf{n}\cdot\overrightarrow{P_1P_2} = \langle a, b\rangle\cdot\langle x_2 - x_1, y_2 - y_1\rangle = ax_2 - ax_1 = by_2 - by_1(ax_2 + by_2) - (ax_1 + by_1) = -c - (-c) = 0,$$

and the vectors are perpendicular. Thus, $\mathbf{n}$ is perpendicular to the line.

57. Let θ be the angle between $\mathbf{n}$ and $\overrightarrow{P_2P_1}$. Then

$$\begin{aligned}d = ||\overrightarrow{P_1P_2}|\cos\theta| &= \frac{|\mathbf{n}\cdot\overrightarrow{P_2P_1}|}{|\mathbf{n}|} = \frac{|\langle a, b\rangle\cdot\langle x_1 - x_2, y_1 - y_2\rangle|}{\sqrt{a^2+b^2}} = \frac{|ax_1 - ax_2 + by_1 = by_2|}{\sqrt{a^2+b^2}}\\ &= \frac{|ax_1 + by_1 - (ax_2 + by_2)|}{\sqrt{a^2+b^2}} = \frac{|ax_1 + by_1 - (-c)|}{\sqrt{a^2+b^2}} = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}.\end{aligned}$$

58. (a) Since $\mathbf{N} = \langle x, y\rangle$ is a unit normal, $\mathbf{T} = \langle -y, x\rangle$ is a unit tangent at $P(x, y)$. Now we

compute

$$\begin{aligned}\mathbf{N}\cdot\overrightarrow{PO} &= \langle x,y\rangle\cdot\langle c-x,d-y\rangle = cx+dy-(x^2+y^2) = cx+dy-1\\ \mathbf{T}\cdot\overrightarrow{OP} &= \langle -y,x\rangle\cdot\langle c-x,d-y\rangle = dx-cy\\ \mathbf{N}\cdot\overrightarrow{PS} &= \langle x,y\rangle\cdot\langle a-x,b-y\rangle = ax+by-(x^2+y^2) = ax+by-1\\ -\mathbf{T}\cdot\overrightarrow{PS} &= \langle y,-x\rangle\cdot\langle a-x,b-y\rangle = ay-bx.\end{aligned}$$

Now, since $|\mathbf{N}| = |\mathbf{T}| = 1$,

$$\frac{\mathbf{N}\cdot\overrightarrow{PO}}{|\overrightarrow{PO}|} = \cos\theta = \frac{\mathbf{N}\cdot\overrightarrow{PS}}{|\overrightarrow{PS}|}, \text{ and } \frac{\mathbf{T}\cdot\overrightarrow{PO}}{|\overrightarrow{PO}|} = \cos\phi = \frac{-\mathbf{T}\cdot\overrightarrow{PS}}{|\overrightarrow{PS}|},$$

we have

$$\frac{\mathbf{N}\cdot\overrightarrow{PO}}{\mathbf{T}\overrightarrow{PO}} = \frac{\mathbf{N}\cdot\overrightarrow{PS}}{-\mathbf{T}\overrightarrow{PS}} \text{ or } \frac{cx+dy-1}{dx-cy} = \frac{ax+by-1}{ay-bx}.$$

(b) With $a=2,\quad b=0,\quad c=0$, and $d=3$ the equation in **(a)** becomes $(3y-1)/3x = (2x-1)/2y$ or $6x^2-3x = 6y^2-2y$. Substituting $y^2 = 1-x^2$ we obtain $6x^2-3x = 6(1-x^2)-2\sqrt{1-x^2}$ or $12x^2-3x-6 = -2\sqrt{1-x^2}$. Squaring both sides, we obtain $144x^4-72x^3+36x+32=0$.

(c) Newton's method gives us the roots $-0.6742,\quad -0.48302,;\ 0.76379$, and 0.89343. Since S and O are on the positive x- and y-axes, respectively, we can ignore the negative roots. Computing $y = \sqrt{1-0.76379^2} \approx 0.645465$ and $y = \sqrt{1-0.89343^2} \approx 0.449202$ we see that only $P(x,y) = (0.76379, 0.645465)$ satisfies $(3y-1)/3x = (2x-1)/2y$.

11.4 Cross Product

1. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & -1 & 0\\ 0 & 3 & 5\end{vmatrix} = \begin{vmatrix}-1 & 0\\ 3 & 5\end{vmatrix}\mathbf{i} - \begin{vmatrix}1 & 0\\ 0 & 5\end{vmatrix}\mathbf{j} + \begin{vmatrix}1 & -1\\ 0 & 3\end{vmatrix}\mathbf{k} = -5\mathbf{i}-5\mathbf{j}+3\mathbf{k}$

2. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 1 & 0\\ 4 & 0 & -1\end{vmatrix} = \begin{vmatrix}1 & 0\\ 0 & -1\end{vmatrix}\mathbf{i} - \begin{vmatrix}2 & 0\\ 4 & -1\end{vmatrix}\mathbf{j} + \begin{vmatrix}2 & 1\\ 4 & 0\end{vmatrix}\mathbf{k} = -\mathbf{i}+2\mathbf{j}-4\mathbf{k}$

3. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & -3 & 1\\ 2 & 0 & 4\end{vmatrix} = \begin{vmatrix}-3 & 1\\ 0 & 4\end{vmatrix}\mathbf{i} - \begin{vmatrix}1 & 1\\ 2 & 4\end{vmatrix}\mathbf{j} + \begin{vmatrix}1 & -3\\ 2 & 0\end{vmatrix}\mathbf{k} = \langle -12,-2,6\rangle$

4. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 1 & 1\\ -5 & 2 & 3\end{vmatrix} = \begin{vmatrix}1 & 1\\ 2 & 3\end{vmatrix}\mathbf{i} - \begin{vmatrix}1 & 1\\ -5 & 3\end{vmatrix}\mathbf{j} + \begin{vmatrix}1 & 1\\ -5 & 2\end{vmatrix}\mathbf{k} = \langle 1,-8,7\rangle$

5. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & -1 & 2\\ -1 & 3 & -1\end{vmatrix} = \begin{vmatrix}-1 & 2\\ 3 & -1\end{vmatrix}\mathbf{i} - \begin{vmatrix}2 & 2\\ -1 & -1\end{vmatrix}\mathbf{j} + \begin{vmatrix}2 & -1\\ -1 & 3\end{vmatrix}\mathbf{k} = -5\mathbf{i}+5\mathbf{k}$

6. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 1 & -5 \\ 2 & 3 & -1 \end{vmatrix} = \begin{vmatrix} 1 & -5 \\ 3 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 4 & -5 \\ 2 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 4 & 1 \\ 2 & 3 \end{vmatrix}\mathbf{k} = 14\mathbf{i} - 6\mathbf{j} + 10\mathbf{k}$

7. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1/2 & 0 & 1/2 \\ 4 & 6 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1/2 \\ 6 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1/2 & 1/2 \\ 4 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1/2 & 0 \\ 4 & 6 \end{vmatrix}\mathbf{k} = \langle -3, 2, 3\rangle$

8. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 5 & 0 \\ 2 & -3 & 4 \end{vmatrix} = \begin{vmatrix} 5 & 0 \\ -3 & 4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 0 & 0 \\ 2 & 4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 0 & 5 \\ 2 & -3 \end{vmatrix}\mathbf{k} = \langle 20, 0, -10\rangle$

9. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & -4 \\ -3 & -3 & 6 \end{vmatrix} = \begin{vmatrix} 2 & -4 \\ -3 & 6 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & -4 \\ -3 & 6 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 2 \\ -3 & -3 \end{vmatrix}\mathbf{k} = \langle 0, 0, 0\rangle$

10. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 8 & 1 & -6 \\ 1 & -2 & 10 \end{vmatrix} = \begin{vmatrix} 1 & -6 \\ -2 & 10 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 8 & -6 \\ 1 & 10 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 8 & 1 \\ 1 & -2 \end{vmatrix}\mathbf{k} = \langle -2, -86, -17\rangle$

11. $\overrightarrow{P_1P_2} = (-2, 2, -4)$; $\overrightarrow{P_1P_3} = (-3, 1, 1)$
$\overrightarrow{P_1P_2}\times\overrightarrow{P_1P_3} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 2 & -4 \\ -3 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 2 & -4 \\ 1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -2 & -4 \\ -3 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -2 & 2 \\ -3 & 1 \end{vmatrix}\mathbf{k} = 6\mathbf{i} + 14\mathbf{j} + 4\mathbf{k}$

12. $\overrightarrow{P_1P_2} = (0, 1, 1)$; $\overrightarrow{P_1P_3} = (1, 2, 2)$
$\overrightarrow{P_1P_2}\times\overrightarrow{P_1P_3} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 1 \\ 1 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix}\mathbf{k} = \mathbf{j} - \mathbf{k}$

13. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 7 & -4 \\ 1 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 7 & -4 \\ 1 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & -4 \\ 1 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 7 \\ 1 & 1 \end{vmatrix}\mathbf{k} = -3\mathbf{i} - 2\mathbf{j} - 5\mathbf{k}$
is perpendicular to both **a** and **b**.

14. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -2 & 4 \\ 4 & -1 & 0 \end{vmatrix} = \begin{vmatrix} -2 & 4 \\ -1 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -1 & 4 \\ 4 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -1 & -2 \\ 4 & -1 \end{vmatrix}\mathbf{k} = \langle 4, 16, 9\rangle$
is perpendicular to both **a** and **b**.

15. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & -2 & 1 \\ 2 & 0 & -7 \end{vmatrix} = \begin{vmatrix} -2 & 1 \\ 0 & -7 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 5 & 1 \\ 2 & -7 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 5 & -2 \\ 2 & 0 \end{vmatrix}\mathbf{k} = \langle 14, 37, 4\rangle$
$\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b}) = \langle 5, -2, 1\rangle\cdot\langle 14, 37, 4\rangle = 70 - 74 + 4 = 0$
$\mathbf{b}\cdot(\mathbf{a}\times\mathbf{b}) = \langle 2, 0, -7\rangle\cdot\langle 14, 37, 4\rangle = 28 + 0 - 28 = 0$

16. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1/2 & -1/4 & -4 \\ 2 & -2 & 6 \end{vmatrix} = \begin{vmatrix} -1/4 & -4 \\ -2 & 6 \end{vmatrix}\mathbf{i} + \begin{vmatrix} 1/2 & -1/4 \\ 2 & -2 \end{vmatrix}\mathbf{k}$

$= -\frac{19}{2}\mathbf{i} - 11\mathbf{j} - \frac{1}{2}\mathbf{k}$

$\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b}) = \left(\frac{1}{2}\mathbf{i} - \frac{1}{4}\mathbf{j} - 4\mathbf{k}\right)\cdot(-\frac{19}{2}\mathbf{i} - 11\mathbf{j} - \frac{1}{2}\mathbf{k}\bigg) = -\frac{19}{4} + \frac{11}{4} + 2 = 0$

$\mathbf{b}\cdot(\mathbf{a}\times\mathbf{b}) = (2\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})\cdot\left(-\frac{19}{2}\mathbf{i} - 11\mathbf{j} - \frac{1}{2}\mathbf{k}\right) = -19 + 22 - 3 = 0$

17. (a) $\mathbf{b}\times\mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ 3 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix}\mathbf{k} = \mathbf{j} - \mathbf{k}$

$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ 0 & 1 & -1 \end{vmatrix} = \begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix}\mathbf{k} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$

(b) $\mathbf{a}\cdot\mathbf{c} = (\mathbf{i} - \mathbf{j} + 2\mathbf{k})\cdot(3\mathbf{i} + \mathbf{j} + \mathbf{k}) = 4;\ \ (\mathbf{a}\cdot\mathbf{c})\mathbf{b} = 4(2\mathbf{i} + \mathbf{j} + \mathbf{k}) = 8\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$
$\mathbf{a}\cdot\mathbf{b} = (\mathbf{i} - \mathbf{j} + 2\mathbf{k})\cdot(2\mathbf{i} + \mathbf{j} + \mathbf{k}) = 3;\ \ (\mathbf{a}\cdot\mathbf{b})\mathbf{c} = 3(3\mathbf{i} + \mathbf{j} + \mathbf{k}) = 9\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$
$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} = (8\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}) - (9\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}) = -\mathbf{i} + \mathbf{j} + \mathbf{k}$

18. (a) $\mathbf{b}\times\mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ -1 & 5 & 8 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 5 & 8 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -1 \\ -1 & 8 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ -1 & 5 \end{vmatrix}\mathbf{k} = 21\mathbf{i} - 7\mathbf{j} + 7\mathbf{k}$

$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 0 & -4 \\ 21 & -7 & 7 \end{vmatrix} = \begin{vmatrix} 0 & -4 \\ -7 & 7 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & -4 \\ 21 & 7 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & 0 \\ 21 & -7 \end{vmatrix}\mathbf{k}$

$= -28\mathbf{i} - 105\mathbf{j} - 21\mathbf{k}$

(b) $\mathbf{a}\cdot\mathbf{c} = (3\mathbf{i} - 4\mathbf{k})\cdot(-\mathbf{i} + 5\mathbf{j} + 8\mathbf{k}) = -35;\ \ (\mathbf{a}\cdot\mathbf{c})\mathbf{b} = -35(\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = -35\mathbf{i} - 70\mathbf{j} + 35\mathbf{k}$
$\mathbf{a}\cdot\mathbf{b} = (3\mathbf{i} - 4\mathbf{k})\cdot(\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = 7;\ \ (\mathbf{a}\cdot\mathbf{b})\mathbf{c} = 7(-\mathbf{i} + 5\mathbf{j} + 8\mathbf{k}) = -7\mathbf{i} + 35\mathbf{j} + 56\mathbf{k}$
$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} = (-35\mathbf{i} - 70\mathbf{j} + 35\mathbf{k}) - (-7\mathbf{i} + 35\mathbf{j} + 56\mathbf{k}) = -28\mathbf{i} - 105\mathbf{j} - 21\mathbf{k}$

19. $(2\mathbf{i})\times\mathbf{j} = 2(\mathbf{i}\times\mathbf{j}) = 2\mathbf{k}$

20. $\mathbf{i}\times(-3\mathbf{k}) = -3(\mathbf{i}\times\mathbf{k}) = -3(-\mathbf{j}) = 3\mathbf{j}$

21. $\mathbf{k}\times(2\mathbf{i} - \mathbf{j}) = \mathbf{k}\times(2\mathbf{i}) + \mathbf{k}\times(-\mathbf{j}) = 2(\mathbf{k}\times\mathbf{i}) - (\mathbf{k}\times\mathbf{j}) = 2\mathbf{j} - (-\mathbf{i}) = \mathbf{i} + 2\mathbf{j}$

22. $\mathbf{i}\times(\mathbf{j}\times\mathbf{k}) = \mathbf{i}\times\mathbf{i} = \mathbf{0}$

23. $[(2\mathbf{k})\times(3\mathbf{j})]\times(4\mathbf{j}) = [2\cdot3(\mathbf{k}\times\mathbf{j})\times(4\mathbf{j})] = 6(-\mathbf{i})\times4\mathbf{j} = (-6)(4)(\mathbf{i}\times\mathbf{j}) = -24\mathbf{k}$

24. $(2\mathbf{i} - \mathbf{j} + 5\mathbf{k})\times\mathbf{i} = (2\mathbf{i}\times\mathbf{i}) + (-\mathbf{j}\times\mathbf{i}) + (5\mathbf{k}\times\mathbf{i}) = 2(\mathbf{i}\times\mathbf{i}) + (\mathbf{i}\times\mathbf{j}) + 5(\mathbf{k}\times\mathbf{i}) = 5\mathbf{j} + \mathbf{k}$

25. $(\mathbf{i} + \mathbf{j})\times(\mathbf{i} + 5\mathbf{k}) = [(\mathbf{i} + \mathbf{j})\times\mathbf{i}] + [(\mathbf{i} + \mathbf{j})\times5\mathbf{k}] = (\mathbf{i}\times\mathbf{i}) + (\mathbf{j}\times\mathbf{i}) + (\mathbf{i}\times5\mathbf{k}) + (\mathbf{j}\times5\mathbf{k})$

$= -\mathbf{k} + 5(-\mathbf{j}) + 5\mathbf{i} = 5\mathbf{i} - 5\mathbf{j} - \mathbf{k}$

26. $\mathbf{i}\times\mathbf{k} - 2(\mathbf{j}\times\mathbf{i}) = -\mathbf{j} - 2(-\mathbf{k}) = -\mathbf{j} + 2\mathbf{k}$

27. $\mathbf{k} \cdot (\mathbf{j} \times \mathbf{k}) = \mathbf{k} \cdot \mathbf{i} = 0$

28. $\mathbf{i} \cdot [\mathbf{j} \times (-\mathbf{k})] = \mathbf{i} \cdot [-(\mathbf{j} \times \mathbf{k})] = \mathbf{i} \cdot (-\mathbf{i}) = -(\mathbf{i} \cdot \mathbf{i}) = -1$

29. $|4\mathbf{j} - 5(\mathbf{i} \times \mathbf{j})| = |4\mathbf{j} - 5\mathbf{k}| = \sqrt{41}$

30. $(\mathbf{i} \times \mathbf{j}) \cdot (3\mathbf{j} \times \mathbf{i}) = \mathbf{k} \cdot (-3\mathbf{k}) = -3(\mathbf{k} \cdot \mathbf{k}) = -3$

31. $\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = -\mathbf{j}$

32. $(\mathbf{i} \times \mathbf{j}) \times \mathbf{i} = \mathbf{k} \times \mathbf{i} = \mathbf{j}$

33. $(\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = \mathbf{0} \times \mathbf{j} = \mathbf{0}$

34. $(\mathbf{i} \cdot \mathbf{i})(\mathbf{i} \times \mathbf{j}) = 1(\mathbf{k}) = \mathbf{k}$

35. $$\begin{aligned} 2\mathbf{j} \cdot [\mathbf{i} \times (\mathbf{j} - 3\mathbf{k})] &= 2\mathbf{j} \cdot [(\mathbf{i} \times \mathbf{j}) + (\mathbf{i} \times (-3\mathbf{k})] = 2\mathbf{j} \cdot [\mathbf{k} + 3(\mathbf{k} \times \mathbf{i})] = 2\mathbf{j} \cdot (\mathbf{k} + 3\mathbf{j}) = 2\mathbf{j} \cdot \mathbf{k} + 2\mathbf{j} \cdot 3\mathbf{j} \\ &= 2(\mathbf{j} \cdot \mathbf{k}) + 6(\mathbf{j} \cdot \mathbf{j}) = 2(0) + 6(1) = 6 \end{aligned}$$

36. $(\mathbf{i} \times \mathbf{k}) \times (\mathbf{j} \times \mathbf{i}) = (-\mathbf{j}) \times (-\mathbf{k}) = (-1)(-1)(\mathbf{j} \times \mathbf{k}) = \mathbf{j} \times \mathbf{k} = \mathbf{i}$

37. $\mathbf{a} \times (3\mathbf{b}) = 3(\mathbf{a} \times \mathbf{b}) = 3(4\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}) = 12\mathbf{i} - 9\mathbf{j} + 18\mathbf{k}$

38. $\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b} = -(\mathbf{a} \times \mathbf{b}) = -4\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$

39. $(-\mathbf{a}) \times \mathbf{b} = -(\mathbf{a} \times \mathbf{b}) = -4\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$

40. $|\mathbf{a} \times \mathbf{b}| = \sqrt{4^2 + (-3)^2 + 6^2} = \sqrt{61}$

41. $$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -3 & 6 \\ 2 & 4 & -1 \end{vmatrix} = \begin{vmatrix} -3 & 6 \\ 4 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 4 & 6 \\ 2 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 4 & -3 \\ 2 & 4 \end{vmatrix} \mathbf{k} = -21\mathbf{i} + 16\mathbf{j} + 22\mathbf{k}$$

42. $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = 4(2) + (-3)4 + 6(-1) = -10$

43. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = 4(2) + (-3)4 + 6(-1) = -10$

44. $(4\mathbf{a}) \cdot (\mathbf{b} \times \mathbf{c}) = (4\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = 16(2) + (-12)4 + 24(-1) = -40$

45. (a) Let $A = (1, 3, 0), B = (2, 0, 0), C = (0, 0, 4)$, and $D = (1, -3, 4)$. Then $\overrightarrow{AB} = \mathbf{i} - 3\mathbf{j}$, $\overrightarrow{AC} = -\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$, $\overrightarrow{CD} = \mathbf{i} - 3\mathbf{j}$, and $\overrightarrow{BD} = -\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$. Since $\overrightarrow{AB} = \overrightarrow{CD}$ and $\overrightarrow{AC} = \overrightarrow{BD}$, the quadrilateral is a parallelogram.

(b) Computing

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & 0 \\ -1 & -3 & 4 \end{vmatrix} = -12\mathbf{i} - 4\mathbf{j} - 6\mathbf{k}$$

we find that the area is $|-12\mathbf{i} - 4\mathbf{j} - 6\mathbf{k}| = \sqrt{144 + 16 + 36} = 14$.

46. (a) Let $A = (3, 4, 1), B = (-1, 4, 2), C = (2, 0, 2)$, and $D = (-2, 0, 3)$. Then $\overrightarrow{AB} = -4\mathbf{i} + \mathbf{k}$, $\overrightarrow{AC} = -\mathbf{i} - 4\mathbf{j} + \mathbf{k}$, $\overrightarrow{CD} = -4\mathbf{i} + \mathbf{k}$, and $\overrightarrow{BD} = -\mathbf{i} - 4\mathbf{j} + \mathbf{k}$. Since $\overrightarrow{AB} = \overrightarrow{CD}$ and $\overrightarrow{AC} = \overrightarrow{BD}$, the quadrilateral is a parallelogram.

(b) Computing

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 0 & 1 \\ -1 & -4 & 1 \end{vmatrix} = 4\mathbf{i} + 3\mathbf{j} + 16\mathbf{k}$$

we find that the area is $|4\mathbf{i} + 3\mathbf{j} + 16\mathbf{k}| = \sqrt{16 + 9 + 256} = \sqrt{281} \approx 16.76$.

47. $\overrightarrow{P_1P_2} = \mathbf{j}$; $\overrightarrow{P_2P_3} = -\mathbf{j} + \mathbf{k}$

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_2P_3} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 0 & 1 \\ 0 & -1 \end{vmatrix}\mathbf{k} = \mathbf{i}$$

$A = \frac{1}{2}|\mathbf{i}| = \frac{1}{2}$ sq. unit

48. $\overrightarrow{P_1P_2} = \mathbf{j} + 2\mathbf{k}$; $\overrightarrow{P_2P_3} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_2P_3} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 2 \\ 2 & 1 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 1 & -2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 0 & 2 \\ 2 & -2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix}\mathbf{k} = -4\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$$

$A = \frac{1}{2}|-4\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}| = 3$ sq. units

49. $\overrightarrow{P_1P_2} = -3\mathbf{j} - \mathbf{k}$; $\overrightarrow{P_2P_3} = -2\mathbf{i} - \mathbf{k}$

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_2P_3} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -3 & -1 \\ -2 & 0 & -1 \end{vmatrix} = \begin{vmatrix} -3 & -1 \\ 0 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 0 & -1 \\ -2 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 0 & -3 \\ -2 & 0 \end{vmatrix}\mathbf{k} = 3\mathbf{i} + 2\mathbf{j} - 6\mathbf{k}$$

$A = \frac{1}{2}|3\mathbf{i} + 2\mathbf{j} - 6\mathbf{k}| = \frac{7}{2}$ sq. units

50. $\overrightarrow{P_1P_2} = -\mathbf{i} + 3\mathbf{k}$; $\overrightarrow{P_2P_3} = 2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_2P_3} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & 3 \\ 2 & 4 & -1 \end{vmatrix} = \begin{vmatrix} 0 & 3 \\ 4 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -1 & 3 \\ 2 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -1 & 0 \\ 2 & 4 \end{vmatrix}\mathbf{k} = -12\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}$$

$A = \frac{1}{2}|-12\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}| = \frac{\sqrt{185}}{2}$ sq. units

51. $$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 4 & 0 \\ 2 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 4 & 0 \\ 2 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -1 & 0 \\ 2 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -1 & 4 \\ 2 & 2 \end{vmatrix}\mathbf{k} = 8\mathbf{i} + 2\mathbf{j} - 10\mathbf{k}$$

$\mathbf{v} = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = |(\mathbf{i} + \mathbf{j}) \cdot (8\mathbf{i} + 2\mathbf{j} - 10\mathbf{k})| = |8 + 2 + 0| = 10$ cu. units

52. $$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 1 \\ 1 & 1 & 5 \end{vmatrix} = \begin{vmatrix} 4 & 1 \\ 1 & 5 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix}\mathbf{k} = 19\mathbf{i} - 4\mathbf{j} - 3\mathbf{k}$$

$\mathbf{v} = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = |(3\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot (19\mathbf{i} - 4\mathbf{j} - 3\mathbf{k})| = |57 - 4 - 3| = 50$ cu. units

53. $$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 6 & -6 \\ \frac{5}{2} & 3 & \frac{1}{2} \end{vmatrix} = \begin{vmatrix} 6 & -6 \\ 3 & \frac{1}{2} \end{vmatrix}\mathbf{i} - \begin{vmatrix} -2 & -6 \\ \frac{5}{2} & \frac{1}{2} \end{vmatrix}\mathbf{j} + \begin{vmatrix} -2 & 6 \\ \frac{5}{2} & 3 \end{vmatrix}\mathbf{k} = 21\mathbf{i} - 14\mathbf{j} - 21\mathbf{k}$$

$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (4\mathbf{i} + 6\mathbf{j}) \cdot (21\mathbf{i} - 14\mathbf{j} - 21\mathbf{k}) = 84 - 84 + 0 = 0$. The vectors are coplanar.

54. $\mathbf{b}\times\mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 1 \\ 0 & \frac{3}{2} & -2 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ \frac{3}{2} & -2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -2 & 1 \\ 0 & -2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -2 & 1 \\ 0 & \frac{3}{2} \end{vmatrix}\mathbf{k} = -\frac{7}{2}\mathbf{i} - 4\mathbf{j} - 3\mathbf{k}$
$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{i}+2\mathbf{j}-4\mathbf{k})\cdot(-\frac{7}{2}\mathbf{i}-4\mathbf{j}-3\mathbf{k}) = -\frac{7}{2} - 8 + 12 = 0$. The vectors are not coplanar.

55. The four points will be coplanar if the three vectors $\overrightarrow{P_1P_2} = \langle 3,-1,-1\rangle$, $\overrightarrow{P_2P_3} = \langle -3,-5,13\rangle$, and $\overrightarrow{P_3P_4} = \langle -8,7,-6\rangle$ are coplanar.
$\overrightarrow{P_2P_3}\times\overrightarrow{P_3P_4} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & -5 & 13 \\ -8 & 7 & -6 \end{vmatrix} = \begin{vmatrix} -5 & 13 \\ 7 & -6 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -3 & 13 \\ -8 & -6 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -3 & -5 \\ -8 & 7 \end{vmatrix}\mathbf{k} = \langle -61,-122,-61\rangle$
$\overrightarrow{P_1P_2}\cdot(\overrightarrow{P_2P_3}\times\overrightarrow{P_3P_4}) = \langle 3,-1,-1\rangle\cdot\langle -61,-122,-61\rangle = -183+122+61 = 0$
The four points are coplanar.

56. The four points will be coplanar if the three vectors $\overrightarrow{P_1P_2} = \langle -3,3,-1\rangle$, $\overrightarrow{P_2P_3} = \langle 1,2,-6\rangle$, and $\overrightarrow{P_3P_4} = \langle 4,-6,5\rangle$ are coplanar.
$\overrightarrow{P_2P_3}\times\overrightarrow{P_3P_4} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -6 \\ 4 & -6 & 5 \end{vmatrix} = \begin{vmatrix} 2 & -6 \\ -6 & 5 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -6 \\ 4 & 5 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ 4 & -6 \end{vmatrix}\mathbf{k} = \langle -26,-29,-14\rangle$
$\overrightarrow{P_1P_2}\cdot(\overrightarrow{P_2P_3}\times\overrightarrow{P_3P_4}) = \langle -3,3,-1\rangle\cdot\langle -26,-29,-14\rangle = 78-87+14 = 5$
The four points are not coplanar.

57. (a) Since $\theta = 90^\circ$, $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}||\sin 90^\circ| = 6.4(5) = 32$.
(b) The direction of $\mathbf{a}\times\mathbf{b}$ is into the fourth quadrant of the xy-plane or to the left of the plane determined by $\mathbf{a}$ and $\mathbf{b}$ as shown in Figure 11.4.9 in the text. It makes an angle of 30° with the positive x-axis.
(c) We identify $\mathbf{n} = (\sqrt{3}\mathbf{i}-\mathbf{j})/2$. Then $\mathbf{a}\times\mathbf{b} = 32\mathbf{n} = 16\sqrt{3}\mathbf{i} - 16\mathbf{j}$.

58. Using Definition 11.4, $\mathbf{a}\times\mathbf{b} = \sqrt{27}(8)\sin 120^\circ\mathbf{n} = 24\sqrt{3}(\sqrt{3}/2)\mathbf{n} = 36\mathbf{n}$. By the right-hand rule, $\mathbf{n} = \mathbf{j}$ or $\mathbf{n} = -\mathbf{j}$. Thus, $\mathbf{a}\times\mathbf{b} = 36\mathbf{j}$ or $-36\mathbf{j}$.

59. $\mathbf{b}\times\mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 5 & 6 \\ 7 & 8 & 3 \end{vmatrix} = \begin{vmatrix} 5 & 6 \\ 8 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 4 & 6 \\ 7 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}\mathbf{k} = \langle -30,30,-3\rangle$

$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 5 & 6 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}\mathbf{k} = \langle -3,6,-3\rangle$

$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -33 & 30 & -3 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 30 & -3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 3 \\ -33 & -3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ -30 & 30 \end{vmatrix}\mathbf{k} = \langle -96,-93,96\rangle$

$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 6 & -3 \\ 7 & 8 & 3 \end{vmatrix} = \begin{vmatrix} 6 & -3 \\ 8 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -3 & -3 \\ 7 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -3 & 6 \\ 7 & 8 \end{vmatrix}\mathbf{k} = \langle 42,-12,-66\rangle$
Therefore, $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) \neq (\mathbf{a}\times\mathbf{b})\times\mathbf{c}$

60. For the following, let $\mathbf{a} = \langle a_1, a_2, a_3\rangle$, $\mathbf{b} = \langle b_1, b_2, b_3\rangle$, and $\mathbf{c} = \langle c_1, c_2, c_3\rangle$ be arbitrary vectors.
Proof of (iv):

$$\begin{aligned}(\mathbf{a}+\mathbf{b})\times\mathbf{c} &= -\mathbf{c}\times(\mathbf{a}_{\mathbf{b}}) \quad \text{by property (ii)}\\ &= (-\mathbf{c}\times\mathbf{a}) + (-\mathbf{c}\times\mathbf{b}) \quad \text{by property (iii)}\\ &= (\mathbf{a}\times\mathbf{c}) + (\mathbf{b}\times\mathbf{c}) \quad \text{by property (ii)}\end{aligned}$$

Proof of (v): Let k be any scalar. Then

$$\begin{aligned}\mathbf{a}\times(k\mathbf{b}) &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ kb_1 & kb_2 & kb_3\end{vmatrix}\\ &= (a_2kb_3 - a_3kb_2)\mathbf{i} - (a_1kb_3 - a_3kb_1)\mathbf{j} + (a_1kb_2 - a_3kb_1)\mathbf{k}\\ &= k(a_2b_3 - a_3b_2)\mathbf{i} - k(a_1b_3 - a_3b_1)\mathbf{j} + k(a_1b_2 - a_2b_1)\mathbf{k}\\ (k\mathbf{a})\times\mathbf{b} &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ ka_1 & ka_2 & ka_3\\ b_1 & b_2 & b_3\end{vmatrix}\\ &= (ka_2b_3 - ka_3b_2)\mathbf{i} - (ka_1b_3 - ka_3b_1)\mathbf{j} + (ka_1b_2 - ka_2b_1)\mathbf{k}\\ &= k(a_2b_3 - a_3b_2)\mathbf{i} - k(a - 1b_3 - a_3b_1)\mathbf{j} + k(a_1b_2 - a_2b_1)\mathbf{k}\end{aligned}$$

Using Definition 11.4.1, we have

$$\begin{aligned}k(\mathbf{a}\times\mathbf{b}) &= k[(a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}]\\ &= k(a_2b_3 - a_3b_2)\mathbf{i} - k(a_1b_3 - a_3b_1)\mathbf{j} + k(a_1b_2 - a_2b_1)\mathbf{k}\end{aligned}$$

Therefore, $\mathbf{a}\times(k\mathbf{b}) = (k\mathbf{a})\times\mathbf{b} = k(\mathbf{a}\times\mathbf{b})$.
Proof of (vii): From Equation 11.4.7, we have

$$\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b}) = \begin{vmatrix} a_1 & a_2 & a_3\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

However, using Property (ii) of determinants in Appendix I in the text, we see that $\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b} = 0.)$
Proof of (viii): From Equation 11.4.7, we have

$$\mathbf{b}\cdot(\mathbf{a}\times\mathbf{b}) = \begin{vmatrix} b_1 & b_2 & b_3\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

However, using Property (ii) of determinants in Appendix I in the text, we see that $\mathbf{b}\times(\mathbf{a}\times\mathbf{b}) = 0$.

61. Using equation 9 in the text,

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix} \text{ and } (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) = \begin{vmatrix} c_1 & c_2 & c_3\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}.$$

The second determinant can be obtained from the first by an interchange of the second and third rows followed by an interchange of the new first and second rows. Using Property (iii) of determinates in Appendix I in the text, we see that $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$.

62. $\mathbf{b}\times\mathbf{c} = (b_2c_3 - b_3c_2)\mathbf{i} - (b_1c_3 - b_3c_1)\mathbf{j} + (b_1c_2 - b_2c_1)\mathbf{k}$

$$\begin{aligned}\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) &= [a_2(b_1c_2 - b_2c_1) + a_3(b_1c_3 - b_3c_1)]\mathbf{i} - [a_1(b_1c_2 - b_2c_1) - a_3(b_2c_3 - b_3c_2)]\mathbf{j} \\ &\quad + [-a_1(b_1c_3 - b_3c_1) - a_2(b_2c_3 - b_3c_2)]\mathbf{k} \\ &= (a_2b_1c_2 - a_2b_2c_1 + a_3b_3c_1)\mathbf{i} - (a_1b_1c_2 - a_2b_2c_1 - a_3b_2c_3 + a_3b_3c_2)\mathbf{j} \\ &\quad - (a_1b_1c_3 - a_1b_3c_1 + a_2b_2c_3 - a_2b_3c_2)\mathbf{k}\end{aligned}$$

$$\begin{aligned}(\mathbf{a}\times\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} &= (a_1c_1 + a_2c_2 + a_3c_3)(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) - (a_1b_1 + a_2b_2 + a_3b_3)(c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}) \\ &= (a_2b_1c_2 - a_2b_2c_1 + a_3b_1c_3 - a_3b_3c_1)\mathbf{i} - (a_1b_1c_2 - a_2b_2c_1 - a_3b_2c_3 + a_3b_3c_2)\mathbf{j} \\ &\quad - (a_1b_1c_3 - a_1b_3c_1 + a_2b_2c_3 - a_2b_3c_2)\mathbf{k}\end{aligned}$$

63. $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) + \mathbf{b}\times(\mathbf{c}\times\mathbf{a}) + \mathbf{c}\times(\mathbf{a}\times\mathbf{b})$

$$\begin{aligned}&= (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a} + (\mathbf{c}\cdot\mathbf{b})\mathbf{a} - (\mathbf{c}\cdot\mathbf{a})\mathbf{b} \\ &= [(\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{c}\cdot\mathbf{a})\mathbf{b}] + [(\mathbf{b}\cdot\mathbf{a})\mathbf{c} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}] + [(\mathbf{c}\cdot\mathbf{b})\mathbf{a} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}] = \mathbf{0}\end{aligned}$$

64. If either $\mathbf{a}$, $\mathbf{b}$ or $\mathbf{c}$ is the zero vector, the result is trivial. Therefore, assume all three are nonzero. If $\mathbf{b}$ is a scalar multiple of $\mathbf{c}$, then $\mathbf{b}\times\mathbf{c} = \mathbf{0}$ and $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{a}\cdot\mathbf{0} = 0$
If $\mathbf{b}$ is not a scalar multiple of $\mathbf{c}$, then $\mathbf{b}\times\mathbf{c}$ is orthogonal to the plane containing $\mathbf{b}$ and $\mathbf{c}$. This implies $\mathbf{b}\times\mathbf{c}$ is orthogonal to $\mathbf{a}$ since $\mathbf{a}$ lies in the same plane as $\mathbf{b}$ and $\mathbf{c}$. Hence, by Theorem 11.3.3, we have $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = 0$.

65. (a) We first note that $\mathbf{a}\times\mathbf{b} = \mathbf{k}$, $\mathbf{b}\times\mathbf{c} = \frac{1}{2}(\mathbf{i}-\mathbf{k})$, $\mathbf{c}\times\mathbf{a} = \frac{1}{2}(\mathbf{j}-\mathbf{k})$, $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \frac{1}{2}$, $\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \frac{1}{2}$, and $\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) = \frac{1}{2}$. Then

$$\mathbf{A} = \frac{\frac{1}{2}(\mathbf{i}-\mathbf{k})}{\frac{1}{2}} = \mathbf{i}-\mathbf{k},\quad \mathbf{B} = \frac{\frac{1}{2}(\mathbf{j}-\mathbf{k})}{\frac{1}{2}} = \mathbf{j}-\mathbf{k},\quad \text{and } \mathbf{C} = \frac{\mathbf{k}}{\frac{1}{2}} = 2\mathbf{k}.$$

(b) We need to compute $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$. Using the formula from Problem 62 we have

$$\begin{aligned}\mathbf{B}\times\mathbf{C} &= \frac{(\mathbf{c}\times\mathbf{a})\times(\mathbf{a}\times\mathbf{b})}{[\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})][\mathbf{c}\times(\mathbf{a}\times\mathbf{b})]} = \frac{([\mathbf{c}\times\mathbf{a})\cdot\mathbf{b}]\mathbf{a} - [(\mathbf{c}\times\mathbf{a})\cdot\mathbf{a}]\mathbf{b}}{[\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})][\mathbf{c}\times(\mathbf{a}\times\mathbf{b})]} \\ &= \frac{\mathbf{a}}{\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}} \quad \boxed{\text{since } (\mathbf{c}\times\mathbf{a})\cdot\mathbf{a} = 0.}\end{aligned}$$

Then

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \frac{\mathbf{b}\times\mathbf{c}}{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})}\cdot\frac{\mathbf{a}}{\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})} = \frac{1}{\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})}$$

and the volume of the unit cell of the reciprocal lattice is the reciprocal of the volume of the unit cell of the original lattice.

11.5 Lines in 3-Space

1. $\langle x, y, z\rangle = \langle 4, 6, -7\rangle + t\langle 3, \frac{1}{2}, -\frac{3}{2}\rangle$

2. $\langle x, y, z\rangle = \langle 1, 8, -2\rangle + t\langle -7, -7, 0\rangle$

3. $\langle x, y, z\rangle = \langle 0, 0, 0\rangle + t\langle 5, 9, 4\rangle$

4. $\langle x, y, z\rangle = \langle 0, -3, 10\rangle + t\langle 12, -5, -6\rangle$
The equation of a line through P_1 and P_2 is 3-space with $\mathbf{r}_1 = \overrightarrow{OP_1}$ and $\mathbf{r}_2 = \overrightarrow{OP_2}$ can be expressed as $\mathbf{r} = \mathbf{r}_1 + t(k\mathbf{a})$ or $\mathbf{r} = \mathbf{r}_2 + t(k\mathbf{a})$ where $\mathbf{a} = \mathbf{r}_2 - \mathbf{r}_1$ and k is any non-zero scalar. Thus, the form of the equation of a line is not unique. (See the alternative solution to Problem 5.)

5. $\mathbf{a} = \langle 1-3, 2-5, 1-(-2)\rangle = \langle -2, -3, 3\rangle$; $\quad \langle x, y, z\rangle = \langle 1, 2, 1\rangle + t\langle -2, -3, 3\rangle$

   Alternate Solution: $\mathbf{a} = \langle -31, 5-2, -2-1\rangle = \langle 2, 3, -3\rangle$; $\quad \langle x, y, z\rangle = \langle 3, 5, -2\rangle + t\langle 2, 3, -3\rangle$

6. $\mathbf{a} = \langle 0-(-2), 4-6, 5-3\rangle = \langle 2, -2, 2\rangle$; $\quad \langle x, y, z\rangle = \langle 0, 4, 5\rangle + t\langle -2, -2, 2\rangle$

7. $\mathbf{a} = \langle 1/2-(-3/2), -1/2-5/2, 1-(-1/2)\rangle = \langle 2, -3, 3/2\rangle$;
$\langle x, y, z\rangle = \langle 1/2, -1/2, 1\rangle + t\langle 2, -3, 3/2\rangle$

8. $\mathbf{a} = \langle 10-5, 2-(-3), 10-5\rangle = \langle 5, 5, -15\rangle$; $\quad \langle x, y, z\rangle = \langle 10, 2, -10\rangle + t\langle 5, 5, -15\rangle$

9. $\mathbf{a} = \langle 1-(-4), 1-1, -1-(-1)\rangle = \langle 5, 0, 0\rangle$; $\quad \langle x, y, z\rangle = \langle 1, 1, -1\rangle + t\langle 5, 0, 0\rangle$

10. $\mathbf{a} = \langle 3-5/2, 2-1, 1-(-2)\rangle = \langle 1/2, 1, 3\rangle$; $\quad \langle x, y, z\rangle = \langle 3, 2, 1\rangle + t\langle 1/2, 1, 3\rangle$

11. $\mathbf{a} = \langle 2-6, 3-(-1), 5-8\rangle = \langle -4, 4, -3\rangle$; $\ x = 2-4t,\ \ y = 3+4t,\ \ z = 5-3t$

12. $\mathbf{a} = \langle 2-0, 0-4, 0-9\rangle = \langle 2, -4, -9\rangle$; $\ x = 2+2t,\ \ y = -4t,\ \ z = -9t$

13. $\mathbf{a} = \langle 1-3, 0-(-2), 0-(-7)\rangle = \langle -2, 2, 7\rangle$; $\ x = 1-2t,\ \ y = 2t,\ \ z = 7t$

14. $\mathbf{a} = \langle 0-(-2), 0-4, 5-0\rangle = \langle 2, -4, 5\rangle$; $\ x = 2t,\ \ y = -4t,\ \ z = 5+5t$

15. $\mathbf{a} = \langle 4-(-)6, 1/2-(-1/4), 1/3-1/6\rangle = \langle 10, 3/4, 1/6\rangle$; $\ x = 4+10t,\ \ y = \frac{1}{2}+\frac{3}{4}t,\ \ z = \frac{1}{3}+\frac{1}{6}t$

16. $\mathbf{a} = \langle -3-4, 7-(-8), 9-1(-1)\rangle = \langle -7, 15, 10\rangle$; $\ x = -3-7t,\ \ y = 7+15t,\ \ z = 9+10t$

17. $a_1 = 10-1 = 9,\ \ a_2 = 14-4 = 10,\ \ a_3 = -2-(-9) = 7;\ \ \dfrac{x-10}{9} = \dfrac{y-14}{10} = \dfrac{z+2}{7}$

18. $a_1 = 1-2/3 = 1/3,\ \ a_2 = 3-0 = 3,\ \ a_3 = 1/4-(1/4) = 1/2;\ \ \dfrac{x-1}{1/3} = \dfrac{y-3}{3} = \dfrac{z-1/4}{1/2}$

19. $a_1 = -7-4 = -11,\ \ a_2 = 2-2 = 0,\ \ a_3 = 5-1 = 4;\ \ \dfrac{x-7}{-11} = \dfrac{z-5}{4},\ \ y = 2$

20. $a_1 = 1-(-5) = 6,\ \ a_2 = 1-(-2),\ \ a_3 = 2-(-4);\ \ \dfrac{x-1}{6} = \dfrac{y-1}{3} = \dfrac{z-2}{6}$

21. $a_1 = 5-5 = 0,\ \ a_2 = 10-1 = 9,\ \ a_3 = -2-(-14) = 12;\ \ x = 5,\ \ \dfrac{y-10}{9} = \dfrac{z+2}{12}$

22. $a_1 = 5/6 - 1/3 = 1/2$, $a_2 = -1/4 - 3/8 = 5/8$, $a_3 = 1/5 - 1/10 = 1/10$;
$\dfrac{x - 5/6}{1/2} = \dfrac{y + 1/4}{-5/8} = \dfrac{z - 1/5}{1/10}$

23. Writing the given line in the form $x/2 = (y-1)/(-3) = (z-5)/6$, we see that a direction vector is $\langle 2, -3, 6\rangle$. Parametric equations for the lines are $x = 6+2t$, $y = 4-3t$, $z = -2+6t$.

24. A direction vector is $\langle 5, 1/3, -2\rangle$. Symmetric equations for the line are
$(x-4)/5$
$= (y+11)(1/3) = (z+7)/(-2)$.

25. A direction vector parallel to both the xy- and xy-planes is $\mathbf{i} = \langle 1, 0, 0\rangle$. Parametric equations for the line are $x = 2+t$, $y = -2$, $z = 15$.

26. (a) Since the unit vector $\mathbf{j} = \langle 0, 1, 0\rangle$ lies along the y-axis, we have $x = 1$, $y = 2+t$, $z = 8$.
   (b) Since the unit vector $\mathbf{k} = \langle 0, 0, 1\rangle$ is perpendicular to the xy-plane, we have $z = 1$, $y = 2$, $z = 8+t$.

27. Both lines go through the points $(0, 0, 0)$ and $(6, 6, 6)$. Since two points determine a line, the lines are the same.

28. The direction vector of line L_1 is $\mathbf{v}_1 = \langle 3, 6, -9\rangle$. The direction vector of line

L_2 is $\mathbf{v}_2 = \langle -1, -2, 3\rangle$. Since $\mathbf{v}_1 = -3\mathbf{v}_2$, lines L_1 and L_2 are parallel. Hence, if we can find a point that lies on both lines, then they must be parallel. Letting $t = 0$ for L_1 and $t = 3$ for L_2, we see that the point $(2, -5, 4)$ lies on both lines. Therefore L_1 and L_2 are the same.

29. (a) Equating the x components, we have $x = 3+2t, = -7$, which gives $t = \dfrac{-7-3}{2} = -5$. We can check our work by plugging this value of t into the y and z components to get $y = 4 - (-5) = 9$ and $z = -1 + 6(-5) = -31$
   (b) Equating the x components, we have $x = 5 - x = -7$ which gives $s = 5 + 7 = 12$. We can check our work by plugging this value of s into the y and z components to get $y = 3 + \dfrac{1}{2}(12) = 9$ and $z = 5 - 3(12) = -31$

30. $\mathbf{a}$ and $\mathbf{f}$ are parallel since $\langle 9, -12, 6\rangle = -3\langle -3, 4, 2\rangle$. $\mathbf{c}$ and $\mathbf{d}$ are orthogonal since $\langle 2, -3, 4\rangle \cdot \langle 1, 4, 5/2\rangle = 0$.

31. In the xy-plane, $z = 9+3t = 0$ and $t = -3$. Then $x = 4-2(-3) = 10$ and $y = 1+2(-3) = -5$. The point is $(10, -5, 0)$. In the xz-plane, $y = 1+2t = 0$ and $t = -1/2$. Then $x = 4-2(-1/2) = 5$ and $z = 9+3(-1/2) = 15/2$. The point is $(5, 0, 15/2)$. In the yz-plane, $x = 4-2t = 0$ and $t = 2$. Then $y = 1+2(2) = 5$ and $z = 9+3(2) = 15$. The point is $(0, 5, 15)$.

32. The parametric equations for the line are $x = 1+2t$, $y = -2+3t$, $z = 4+2t$. In the xy-plane, $z = 4+2t = 0$ and $t = -2$. Then $x = 1+2(-2) = -3$ and $y = -2+3(-2) = -8$. The point is $(-3, -8, 0)$. In the xz-plane, $y = -2+3t = 0$ and $t = 2/3$. Then $x = 1+2(2/3) = 7/3$ and $z = 4+2(2/3) = 16/3$. The point is $(7/3, 0, 16/3)$. In the yz-plane, $x = 1+2t = 0$ and $t = -1/2$. Then $y = -2+3(-1/2) = -7/2$ and $z = 4+2(-1/2) = 3$. The point is $(0, -7/2, 3)$.

33. Solving the system $4+t=6+2s$, $5+t=11+4s$, $-1+2t=-3+s$, or $t-2s=2$, $t-4s=6$, $2t-s=-2$ yields $s=-2$ and $t=-2$ in all three equations. Thus, the lines intersect at the point $x=4+(-2)=2$, $y=5+(-2)=3$, $z=-1+2(-2)=-5$, or $(2,3,-5)$.

34. Solving the system $1+t=2-s$, $2-t=1+s$, $3t=6s$, or $t+s=1$, $t+s=1$, $t-2s=0$ yields $s=1/3$ and $t=2/3$ in all three equations. Thus, the lines intersect at the point $x=1+2/3=5/3$, $y=2-2/3=4/3$, $z=3(2/3)=2$, or $(5/3,4/3,2)$.

35. The system of equations $2-t=4+s$, $3+t=1+s$, $1+t=1-s$, or $t+s=-2$, $t-s=-2$, $t+s=0$ has no solution since $-2\neq 0$. Thus, the lines do not intersect.

36. Solving the system $3+t=2+2s$, $2+t=-2+3s$, $8+2t=-2+8s$, or $t+2s=1$, $t-3s=-4$, $2t-8s=-10$ yields $s=1$ and $t=-1$ in all three equations. Thus, the lines intersect at the point $x=3-(-1)=4$, $y=2+(-1)=1$, $z=8+2(-1)=6$, or $(4,1,6)$.

37. Using the first two points, we determine the line $x=4+6t$, $y=3+12t$, $z=-5-6t$. Letting $t=-5/6$ we see that $(-1,-7,0)$ is on the line. Thus, the points lie on the same line.

38. Using the first two points, we determine the line $x=-1-12t$, $y=6+4t$, $z=6-8t$. Setting $x=-2$ in the first equation, we obtain $t=1/4$. Since $z=6-8\left(\frac{1}{4}\right)=4\neq 5$ when $t=1/4$, the points do not lie on the same line.

39. A direction vector for the line is $\langle 6-2,-1-5,3-9\rangle=\langle 4,-6,-6\rangle$. Thus, parametric equations for the line segment are $x=2+4t$, $y=5-6t$, $z=9-6t$, where $0\le t\le 1$.

40. The midpoint of the first line segment, obtained by letting $t=3/2$, is $(4,1/2,-1/2)$. The midpoint of the second line segment, obtained by letting $t=0$, is $(-2,6,5)$. A direction vector for the line segment connecting the midpoints is $\langle -2-4,6-1/2,5-(-1/2)\rangle=\langle -6,11/2,11/2\rangle$. Thus, parametric equations for the line segment are $x=4-6t$, $y=1/2+(11/2)t$, $z=-1/2+(11/2)t$, where $0\le t\le 1$.

41. $\mathbf{a}=\langle -1,2,-2\rangle$, $\quad \mathbf{b}=\langle 2,3,-6\rangle$, $\quad \mathbf{a}\cdot\mathbf{b}=16$, $\quad |\mathbf{a}|=3$, $\quad |\mathbf{b}|=7$; $\quad \cos\theta=\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}=\dfrac{16}{3\cdot 7}$;
$\theta=\arccos\dfrac{16}{21}\approx 40.37^\circ$

42. $\mathbf{a}=\langle 2,7,-1\rangle$, $\quad \mathbf{b}=\langle -2,1,4\rangle$, $\quad \mathbf{a}\cdot\mathbf{b}=-1$, $\quad |\mathbf{a}|=3\sqrt{6}$, $\quad |\mathbf{b}|=\sqrt{21}$;
$\cos\theta=\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}=-\dfrac{1}{(3\sqrt{6})(\sqrt{21})}=-\dfrac{1}{9\sqrt{14}}$; $\quad \theta=\arccos\left(-\dfrac{1}{(3\sqrt{6})(\sqrt{21})}\right)\approx 91.70^\circ$

43. A direction vector perpendicular to the given lines will be $\langle 1,1,1\rangle\times\langle -1,1,-5\rangle=\langle -6,3,3\rangle$. Equations of the lines are $x=4-6t$, $\quad y=1+3t$, $\quad z=6+3t$.

44. The direction vectors of the given lines are $\langle 3,2,4\rangle$ and $\langle 6,4,8\rangle=2\langle 3,2,4\rangle$. These are parallel, so we need a third vector parallel to the plane containing the lines which is not parallel to them. The point $(1,-1,0)$ is on the first line and $(-4,6,10)$ is on the second line. A third vector is then $\langle 1,-1,0\rangle-\langle -4,6,10\rangle=\langle 5,-7,-10\rangle$. Now a direction vector perpendicular to the plane is $\langle 3,2,4\rangle\times\langle 5,-,7-,10\rangle=\langle 8,50,-31\rangle$. Equations of the line through $(1,-1,0)$ and perpendicular to the plane are $x=1+8t$, $\quad y=-1+50t$, $\quad z=-31t$.

45. In the system $-3+t=4+s$, $\quad 7+3t=8-2s$, $\quad 5+2t=10-4s$, or $t-s=7$, $\quad 3t+2s=1$, $\quad 2t+4s=5$, the first and second equations have solution $t=3$ and $s=-4$. Substituting into the third equation, we find $2(3)=4(-4)=6-16=-10\neq 5$. The direction vectors of the lines are $\langle 1,3,2\rangle$ and $\langle 1,-2,-4\rangle$, so the lines are not parallel. Thus, the lines are skew.

46. In the system $6+2t=7+8s$, $\quad 6t=4-4s$, $\quad -8+10t=3-24s$, or $2t-8s=1$, $\quad 6t+4s=4$, $\quad 10t+24s=11$, the second and third equations have solution $t=1/2$ and $s=1/4$. Substituting into the first equation, we find $2(1/2)-8(1/4)=-1\neq 1$. The direction vectors of the lines are $\langle 2,6,10\rangle$ and $\langle 8,-4,,-24\rangle$, so the lines are not parallel. Thus, the lines are skew.

47. The vector $(\overrightarrow{P_1P_2}\times\overrightarrow{P_3P_4})/|\overrightarrow{P_1P_2}\times\overrightarrow{P_3P_4}|$ is a unit vector perpendicular to the two planes. To find the shortest distance between the planes we compute the absolute value of the component of $\overrightarrow{P_1P_3}$ on this unit vector. Then

$$d=\left|\overrightarrow{P_1P_3}\cdot\frac{\overrightarrow{P_1P_2}\times\overrightarrow{P_3P_4}}{|\overrightarrow{P_1P_2}\times\overrightarrow{P_3P_4}|}\right|=\frac{|\overrightarrow{P_1P_3}\cdot\overrightarrow{P_1P_2}\times\overrightarrow{P_3P_4}|}{|\overrightarrow{P_1P_2}\times\overrightarrow{P_3P_4}|}$$

48. We take $P_1=(-3,7,5)$, $\quad P_2=(-2,10,7)$, $\quad P_3+(4,8,10)$, and $P_4=(5,6,6)$. Then $\overrightarrow{P_1P_3}=\langle 7,1,5\rangle$, $\quad \overrightarrow{P_1P_2}=\langle 1,3,2\rangle$, $\quad \overrightarrow{P_3P_4}=\langle 1,-2,-4\rangle$, and $\overrightarrow{P_1P_2}\times\overrightarrow{P_3P_4}=\langle -8,6,-5\rangle$. The distance between the lines is then

$$d=\frac{|\langle 7,1,5\rangle\cdot\langle -8,6,-5\rangle|}{|\langle -8,6,-5\rangle|}=\frac{75}{5\sqrt{5}}=3\sqrt{5}.$$

11.6 Planes

1. $2(x-5)-3(y-1)+4(z-3)=0;\quad 2x-3y+4z=19$

2. $4(x-1)-2(y-2)+0(z-5)=0;\quad 4x-2y=0$

3. $-5(x-6)+0(y-10)+3(z+7)=0;\quad -5x+3z=-51$

4. $6x-y+3z=0$

5. $6(x-1/2)+8(y-3/4)-4(z-1/2)=0;\quad 6x+8y-4z=11$

6. $-(x+1)+(y-1)-(z-0)=0;\quad -x+y-z=2$

7. From the points $(3,5,2)$ and $(2,3,1)$ we obtain the vector $\mathbf{u}=\mathbf{i}+2\mathbf{j}+\mathbf{k}$. From the points $(2,3,1)$ and $(-1,-1,4)$ we obtain the vector $\mathbf{v}=3\mathbf{i}+4\mathbf{j}-3\mathbf{k}$. From the points $(-1,-1,4)$ and (x,y,z) we obtain the vector $\mathbf{w}=(x+1)\mathbf{i}+(y+1)\mathbf{j}+(z-4)\mathbf{k}$. Then, a normal vector is

$$\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&2&1\\3&4&-3\end{vmatrix}=-10\mathbf{i}+6\mathbf{j}-2\mathbf{k}$$

A vector equation of the plane is $-10(x+1)+6(y+1)-2(z-4)=0$ or $5x-3y+z=2$.

8. From the points $(0,1,0)$ and $(0,1,1)$ we obtain the vector $\mathbf{u} = \mathbf{k}$. From the points $(0,1,1)$ and $(1,3,-1)$ we obtain the vector $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$. From the points $(1,3,-1)$ and (x,y,z) we obtain the vector $\mathbf{w} = (x-1)\mathbf{i} + (y-3)\mathbf{j} + (z+1)\mathbf{k}$. Then, a normal vector is

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 1 & 2 & -2 \end{vmatrix} = -2\mathbf{i} + \mathbf{j}$$

A vector equation of the plane is $-2(x-1) + (y-3) + 0(z+1) = 0$ or $-2x + y = 1$.

9. From the points $(0,0,0)$ and $(1,1,1)$ we obtain the vector $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$. From the points $(1,1,1)$ and $(3,2,-1)$ we obtain the vector $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$. From the points $(3,2,-1)$ and (x,y,z) we obtain the vector $\mathbf{w} = (x-3)\mathbf{i} + (y-2)\mathbf{j} + (z+1)\mathbf{k}$. Then, a normal vector is

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & 1 & -2 \end{vmatrix} = -3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$$

A vector equation of the plane is $-3(x-3) + 4(y-2) - (z+1) = 0$ or $-3x + 4y - z = 0$.

10. The three points are not collinear and all satisfy $x = 0$, which is the equation of the plane.

11. From the points $(1,2,-1)$ and $(4,3,1)$ we obtain the vector $\mathbf{u} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$. From the points $(4,3,1)$ and $(7,4,3)$ we obtain the vector $\mathbf{v} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$. From the points $(7,4,3)$ and (x,y,z) we obtain the vector $\mathbf{w} = (x-7)\mathbf{i} + (y-4)\mathbf{j} + (z-3)\mathbf{k}$. Since $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, the points are collinear.

12. From the points $(2,1,2)$ and $(4,1,0)$ we obtain the vector $\mathbf{u} = 2\mathbf{i} - 2\mathbf{k}$. From the points $(4,1,0)$ and $(5,0,-5)$ we obtain the vector $\mathbf{v} = \mathbf{i} - \mathbf{j} - 5\mathbf{k}$. From the points $(5,0,-5)$ and (x,y,z) we obtain the vector $\mathbf{w} = (x-5)\mathbf{i} + y\mathbf{j} + (z+5)\mathbf{k}$. Then, a normal vector is

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -2 \\ 1 & -1 & -5 \end{vmatrix} = -2\mathbf{i} + 8\mathbf{j} - 2\mathbf{k}$$

A vector equation of the plane is $-2(x-5) + 8y - 2(z+5) = 0$ or $x - 4y = z = 0$.

13. A normal vector to $x + y - 4z = 1$ is $\langle 1, 1, -4 \rangle$. The equation of the parallel plane is $(x-2) + (y-3) - 4(z+5) = 0$ or $x + y - 4z = 25$.

14. A normal vector to $5x - y + z = 6$ is $\langle 5, -1, 1 \rangle$. The equation of the parallel plane is $5(x - 0) - (y-0) + (z-0) = 0$ or $5x - y + z = 0$.

15. A normal vector to the xy-plane is $\langle 0, 0, 1 \rangle$. The equation of the parallel plane is $z - 12 = 0$ or $z = 12$.

16. A normal vector is $\langle 0, 1, 0 \rangle$. The equation of the plane is $y + 5 = 0$ or $y = -5$.

17. Direction vectors of the lines are $\langle 3,-1,1\rangle$ and $\langle 4,2,1\rangle$. A normal vector to the plane is $\langle 3,-1,1\rangle \times \langle 4,2,1\rangle = \langle -3,1,10\rangle$. A point on the first line, and thus in the plane, is $(1,1,2)$. The equation of the plane is $-3(x-1)+(y-1)+10(z-2)=0$ or $-3x+y+10z=18$.

18. Direction vectors of the lines are $\langle 2,-1,6\rangle$ and $\langle 1,1,-3\rangle$. A normal vector to the plane is $\langle 2,-1,6\rangle \times \langle 1,1,-3\rangle = \langle -3,12,3\rangle$. A point on the first line, and thus in the plane, is $(1,-1,5)$. The equation of the plane is $-3(x-1)+12(y+1)+3(z-5)=0$ or $-x+4y+z=0$.

19. A direction vector for the two lines is $\langle 1,2,1\rangle$. Points on the lines are $(1,1,3)$ and $(3,0,-2)$. Thus, another vector parallel to the plane is $\langle 1-3,1-0,3+2\rangle = \langle -2,1,5\rangle$. A normal vector to the plane is $\langle 1,2,1\rangle \times \langle -2,1,5\rangle = \langle 9,-7,5\rangle$. Using the point $(3,0,-2)$ in the plane, the equation of the plane is $9(x-3)-7(y-0)+5(z+2)=0$ or $9x-7y+5z=17$.

20. A direction vector for the line is $\langle 3,2,-1\rangle$. Letting $t=0$, we see that the origin is on the line and hence in the plane. Thus, another vector parallel to the plane is $\langle 4-0,0-0,-6-0\rangle = \langle 4,0,-6\rangle$. A normal vector to the plane is $\langle 3,2,-2\rangle \times \langle 4,0,-6\rangle = \langle -12,10,-8\rangle$. The equation of the plane is $-12(x-0)+10(y-0)-8(z-0)=0$ or $6x-5y+4z=0$.

21. A direction vector for the line, and hence a normal vector for the plane, is $\langle -3,1,-1/2\rangle$. The equation of the plane is $-3(x-2)+(y-4)-\frac{1}{2}(z-8)=0$ or $-3x+y-\frac{1}{2}z=-6$.

22. A normal vector to the plane is $\langle 2-1,6-0,-3+2\rangle = \langle 1,6,-1\rangle$. The equation of the plane is $(x-1)+6(y-1)-(z-1)=0$ or $x+6y-z=6$.

23. Normal vectors to the plane are **(a)** $\langle 2,-1,3\rangle$, **(b)** $\langle 1,2,2\rangle$, **(c)** $\langle 1,1,-3/2\rangle$, **(d)** $\langle -5,2,4\rangle$, **(e)** $\langle -8,-8,12\rangle$, **(f)** $\langle -2,1,-2\rangle$. Parallel planes are **(c)** and **(e)**, and **(a)** and **(f)**. Perpendicular planes are **(a)** and **(d)**, **(b)** and **(c)**, **(b)** and **(e)**, and **(d)** and **(f)**.

24. A normal vector to the plane is $\langle -7,2,3\rangle$. This is the direction vector for the line and the equations of the line are $x-4-7t, \quad y=1+2t, \quad z=7+3t$.

25. A direction vector of the line is $\langle -6,9,3\rangle$, and the normal vectors of the plane are **(a)** $\langle 4,1,2\rangle$, **(b)** $\langle 2,-3,1\rangle$, **(c)** $\langle 10,-15,-5\rangle$, **(d)** $\langle -4,6,2\rangle$. Vectors **(c)** and **(d)** are multiples of the direction vector and hence the corresponding planes are perpendicular to the line.

26. A direction vector of the line is $\langle -2,4,1\rangle$, and the normal vectors of the plane are **(a)** $\langle 1,-1,3\rangle$, **(b)** $\langle 6,-3,0\rangle$, **(c)** $\langle 1,-2,5\rangle$, **(d)** $\langle -2,1,-2\rangle$. Since the dot product of each normal vector with the direction vector is non-zero, none of the planes are parallel to the line.

27. Letting $z=t$ in both equations and solving $5x-4y=8+9t, \quad x+4y=4-3t$, we obtain $x=2+t, \quad y=\frac{1}{2}-t, \quad z=t$.

28. Letting $y=t$ in both equations and solving $x-z=2-2t, \quad 3x+2z=1+t$, we obtain $x=1-\frac{3}{5}t, \quad y=t, \quad z=-1+\frac{7}{5}t$ or, letting $t=5s, \quad x=1-3s, \quad y=5s, \quad z=-1+7s$.

29. Letting $z=t$ in both equations and solving $4x-2y=1+t, \quad x+y=1-2t$, we obtain $x=\frac{1}{2}-\frac{1}{2}t, \;\; y=\frac{1}{2}-\frac{3}{2}t, \;\; z=t$.

30. Letting $z=t$ and using $y=0$ in the first equation, we obtain $x=-\frac{1}{2}t, \;\; y=0, \;\; z=t$.

31. Substituting the parametric equations into the equation of the plane, we obtain $2(1+2t)23(2-t)+2(-3t)=-7$ or $t=-3$. Letting $t=-3$ in the equation of the line, we obtain the point of intersection $(-5,5,9)$.

32. Substituting the parametric equations into the equation of the plane, we obtain $(3-2t)+(1+6t)-4(2-\frac{1}{2})=12$ or $2t=0$. Letting $t=0$ in the equation of the line, we obtain the point of intersection $(3,1,2)$.

33. Substituting the parametric equations into the equation of the plane, we obtain $1+2-(1+t)=8$ or $t=-6$. Letting $t=-6$ in the equation of the line, we obtain the point of intersection $(1,2,-5)$.

34. Substituting the parametric equations into the equation of the plane, we obtain $4+t-3(2+t)+2(1+5t)=0$ or $t=0$. Letting $t=0$ in the equation of the line, we obtain the point of intersection $(4,2,1)$.
    In Problems 35 and 26, the cross product of the normal vectors to the two planes will be a vector parallel to both planes, and hence a direction vector for a line parallel to the two planes.

35. Normal vectors are $\langle 1,1,-4\rangle$ and $\langle 2,-1,1\rangle$. A direction vector is

$$\langle 1,1,-4\rangle \times \langle 2,-1,1\rangle = \langle -3,-9,-3\rangle = -3\langle 1,3,1\rangle.$$

    Equations of the line are $x=5+t, \quad y=6+3t, \quad z=-12+t$.

36. Normal vectors are $\langle 2,0,1\rangle$ and $\langle -1,3,1\rangle$. A direction vector is

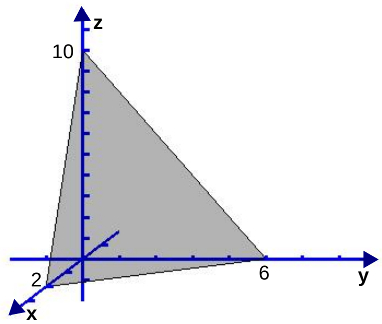
$$\langle 2,0,1\rangle \times \langle -1,3,1\rangle = \langle -3,-3,6\rangle = -3\langle 1,1,-2\rangle.$$

    Equations of the line are $x=-3+t, \quad y=5+t, \quad z=-1-2t$.
    In Problems 37 and 38, the cross product of the direction vector of the line with the normal vector of the given plane will be a normal vector to the desired plane.

37. A direction vector of the line is $\langle 3,-1,5\rangle$ and a normal vector to the given plane is $\langle 1,1,1\rangle$. A normal vector to the desired plane is $\langle 3,-1,5\rangle \times \langle 1,1,1\rangle = \langle -6,2,4\rangle$. A point on the line, and hence in the plane is $\langle 4,0,1\rangle$. The equation of the plane is $-6(x-4)+2(y-0)+4(z-1)=0$ or $3x-y-2x=10$.

38. A direction vector of the line is $\langle 3,5,2\rangle$ and a normal vector to the given plane is $\langle 2,-4,-1\rangle$. A normal vector to the desired plane is $\langle -3,5,2\rangle \times \langle 2,-4,-1\rangle = \langle 3,1,2\rangle$. A point on the line, and hence in the plane is $\langle 2,-2,8\rangle$. The equation of the plane is $3(x-2)+(y+2)+2(z-8)=0$ or $3x+y+2x=20$.

39.



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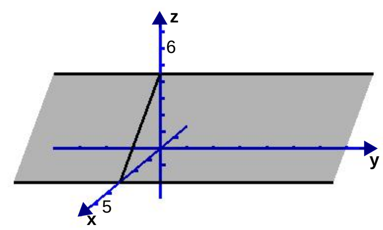

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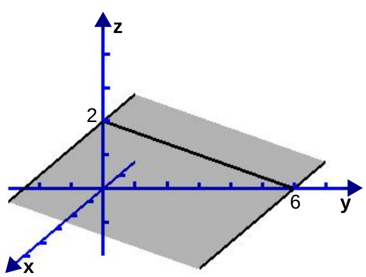

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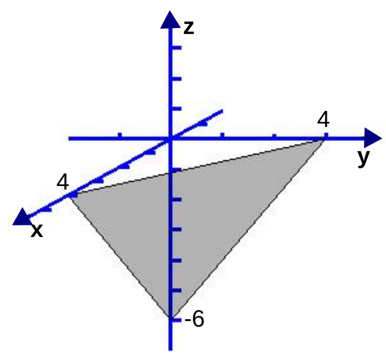

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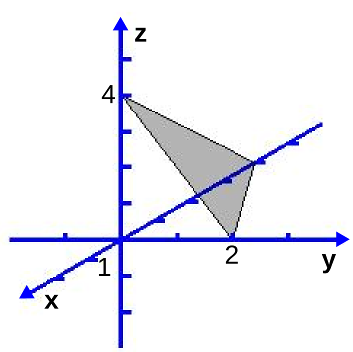

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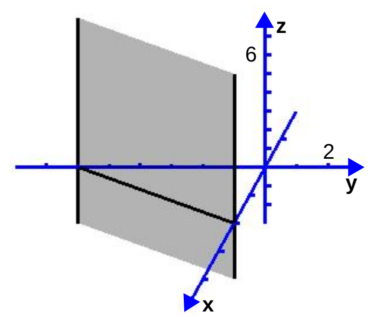


45. (a) A direction vector for the line is $\mathbf{a} = -2\mathbf{i} + \mathbf{j} - \mathbf{k}$ and a normal vector for the plane is $\mathbf{n} = \mathbf{i} + \mathbf{j} - \mathbf{k}$. Since $\mathbf{a} \cdot \mathbf{n} = -2 + 1 + 1 = 0$, the line is perpendicular to $\mathbf{n}$ and thus parallel

to the plane. Since $(0,0,0)$ is on the line and $(0,0,-1)$ is in the plane, the line is above the plane.

(b) A normal vector for the plane is $\mathbf{n} - 3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$. Since $\mathbf{a} \cdot \mathbf{n} = 6 - 4 - 2 = 0$, the line is parallel to the plane. Since $(0,0,0)$ is one the line and $(0,0,4)$ is in the plane, the line is below the plane.

46. The distance D will be the absolute value of $\text{comp}_{\mathbf{n}} \overrightarrow{P_0P_1}$. Thus, using $ax_1 + by_1 + cz_1 = -d$,

$$D = \left| \overrightarrow{P_0P_1} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right| = \left| \frac{\langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle \cdot \langle a, b, c \rangle}{\sqrt{a^2 + b^2 + c^2}} \right|$$
$$= \frac{|ax_2 + by_2 + cz_2 - (zx_1 + by_1 + cz_1)|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|ax_2 + by_2 + cz_2 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

47. Using, Problem 46, $D = \dfrac{|1(2) - 3(1) + 1(4) - 6|}{\sqrt{1 + 9 + 1}} = \dfrac{3}{\sqrt{11}}$.

48. (a) The normal vectors are $\mathbf{n}_1 = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{n}_2 = -4\mathbf{i} + 8\mathbf{j} - 12\mathbf{k} = -4\mathbf{n}$. Since $\mathbf{n}_1$ and $\mathbf{n}_2$ are parallel, the planes are parallel.

(b) To find the distance between the planes we choose $(0,0,1)$ on the first plane. Then, using Problem 46, the distance between the planes is

$$D = \frac{|-4(0) + +8(0) - 12(1) - 7|}{\sqrt{(-4)^2 + 8^2 + (-12)^2}} = \frac{19}{\sqrt{224}} \approx 1.27.$$

49. Normal vectors are $\langle 1, -3, 2 \rangle$ and $\langle -1, 1, 1 \rangle$. Then

$$\cos\theta = \frac{\langle 1, -3, 2 \rangle \cdot \langle -1, 1, 1 \rangle}{|\langle 1, -3, 2 \rangle||\langle -1, 1, 1 \rangle|} = \frac{-2}{\sqrt{14}\sqrt{3}} = -\frac{2}{\sqrt{42}}$$

and $\theta = \arccos(-2/\sqrt{42}) \approx 107.98°$

50. Normal vectors are $\langle, 2, 6, 3 \rangle$ and $\langle 4, -2, 4 \rangle$. Then

$$\cos\theta = \frac{\langle 2, 6, 3 \rangle \cdot \langle 4, -2, 4 \rangle}{|\langle 2, 6, 3 \rangle||\langle 4, -2, 4 \rangle|} = \frac{8}{7(6)} = \frac{4}{21}$$

and $\theta = \arccos(-4/21) \approx 79.02°$

51. Let the bottoms of the table legs be represented by points in 3-space. The rocking of a four-legged table occurs when these four points are not coplanar. Hence, not all four legs can rest on the plane of the floor simultaneously.
However, a three-legged table cannot have this problem. Given any three points in space, a plane can be found passing through them. Therefore, the bottoms of the legs in a three-legged table are coplanar. This implies that they will all rest on the plane of the floor, even if the legs are of uneven lengths.

52. Let $\mathbf{n}_1 = \langle 1, -1, 2\rangle$ which is normal to the plane $x - y + 2z = 1$. Let $\mathbf{n}_2 = \langle 1, 1, 1\rangle$ which is normal to the plane $x + y + z = 3$. Since L is perpendicular to both $\mathbf{n}_1$ and $\mathbf{n}_2$, L must be parallel to $\mathbf{n}_1 \times \mathbf{n}_2$.

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ 1 & 1 & 1 \end{vmatrix} - 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$$

Therefore, L is parallel to $\mathbf{v} = \langle -3, 1, 2\rangle$.
To completely determine L, we need a point which L passes through. Hence we need a point (x, y, z) which satisfies the equations of both planes. Since it satisfies both equations, it must satisfy their sum:

$$\begin{aligned} x - y + 2z &= 1 \\ x + y + z &= 3 \\ 2x + 3z &= 4 \end{aligned}$$

So the coordinates of the point satisfy $2x + 3z = 4$. Since $\mathbf{v}$ as a nonzero $\mathbf{k}$-component, the line L passes through every possible z-value. This implies the existence of a point on the intersection of the two planes with a z-value of zero. Letting $z = 0$, we must have $x = 2$ since $2x + 3z = 4$ for every point on L. Plugging $x = 2$ and $z = 0$ into the equation of the first place, we get $y = 1$. Therefore $(2, 1, 0)$ lies on the line L.
Using this point and the parallel vector $\mathbf{v}$, the parametric equations of L are

$$x = 2 - 3t; \quad y = 1 + t; \quad z = 2t$$

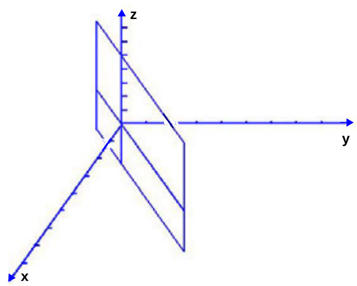
To show that this answer is equivalent to that found in Example 8, first note that both lines pass through $(2, 1, 0)$. Also, the parallel vector used in Example 8 is $\langle -3/2, 1/2, 1\rangle = \frac{1}{2}\mathbf{v}$. Therefore, the two solutions are the same.

53. (a) The plane should pass through the midpoint of the line segment joining $(1, -2, 3)$ and $(2, 5, -2)$. This is given in Problem 11.2.64 as $M = \left(\frac{1+2}{2}, \frac{-2+5}{2}, \frac{3-1}{2}\right) = \left(\frac{3}{2}, \frac{3}{2}, 1\right)$. The vector joining $(1, -2, 3)$ and $(2, 5, -1)$ should be perpendicular to the plane. This vector is $\mathbf{n} = \langle 1, 7, -4\rangle$. Using the point $(3/2, 3/2, 1)$ and the normal vector $\mathbf{n}$, the equation of the plane is given by $z + 7y - 4z = 8$.

(b) The distance from the plane to either of the two points is equal to half the length of the line segment joining the two points. This is given by $\frac{1}{2}\sqrt{1^2 + 7^2 + (-4)^2} = \frac{1}{2}\sqrt{66}$

11.7 Cylinders and Spheres

1.



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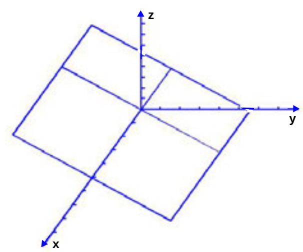

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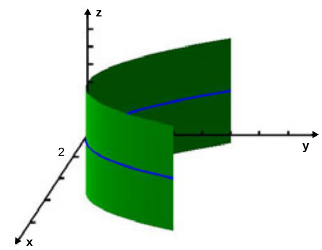

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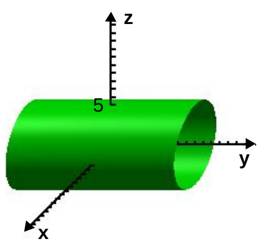

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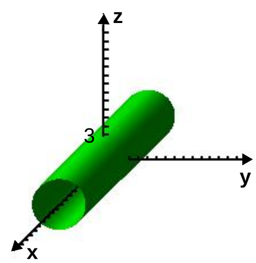

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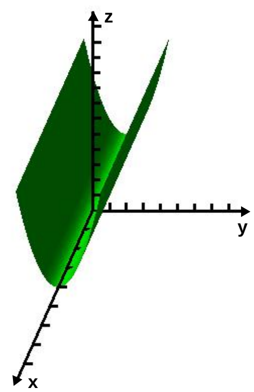

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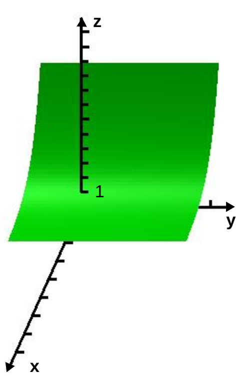

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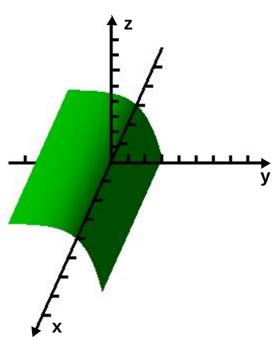

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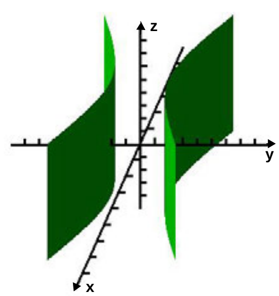

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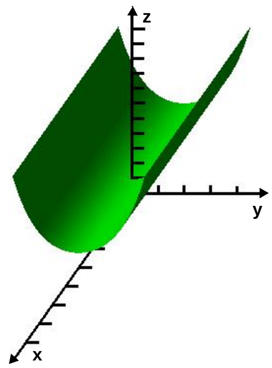

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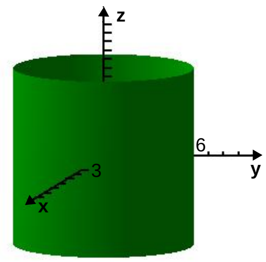

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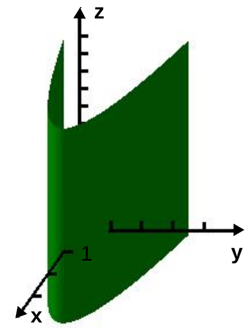

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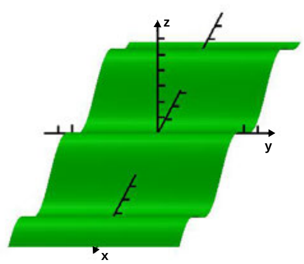

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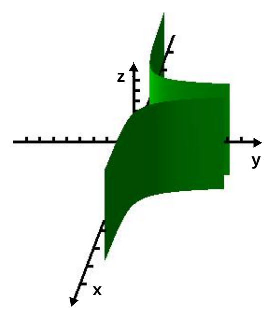

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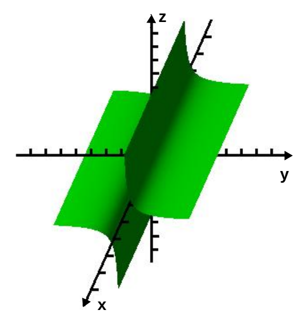

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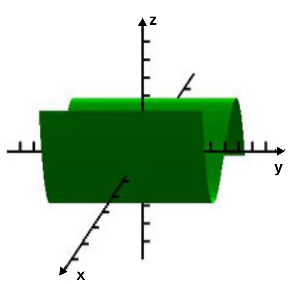

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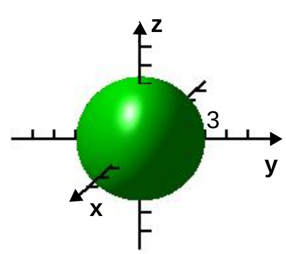

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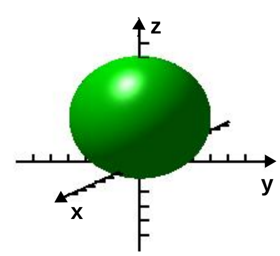
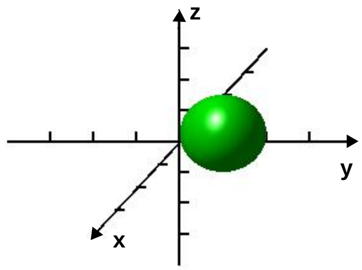


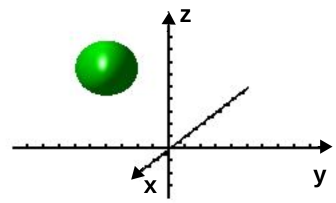
center: $(0, 0, 3)$
radius: 4

19.



center: $(1,1,1)$
radius: 1

20.



center: $(-3,-4,5)$
radius: 2

21. $(x^2+8x+16)+(y^2-6x+9)+(z^2-4z+4)=7+16+9+4$
$(x+4)^2+(y-3)^2+(z-2)^2=36$; center: $(-4,3,2)$; radius: 6

22. $4(x^2+x+1/4)+4y^2+4(z^2-3z+9/4)=-9+1+9$
$(x+1/2)^2+y^2+(z-3/2)^2=1/4$; center: $(-1/2,0,3/2)$; radius: $1/2$

23. $x^2+y^2+(z^2-16z=64)=64$; center: $(0,0,8)$; radius: 8

24. $(x^2-x+1/4)+(y^2+y+1/4)+z^2=1/4+1/4$; $(x-1/2)^2+(y+1/2)^2+z^2=1/2$
center: $(1/2,-1/2,0)$; radius: $\sqrt{2}/2$

25. $(x+1)^2+(y-4)^2+(z-6)^2=3$

26. $x^2+(y-3)^2+z^2=25/16$

27. $(x-1)^2+(y-1)^2+(z-4)^2=16$

28. $(x-5)^2+(y-2)^2+(z-2)^2=5^2$

29. There are two solutions: one sphere is inside the given sphere and the other is outside.
$x^2+(y-8)^2+z^2=4$ or $x^2+(y-4)^2+z^2=4$.

30. $\sqrt{(2t)^2+(3t)^2+(6t)^2}=21$; $t=3$; $a=2t=6$; $b=3t=9$; $c=6t=18$
$(x-6)^2+(y-9)^2+(z-18)^2=25$

31. The center is at $(1,4,2)$ and the radius is $\sqrt{(1-0)^2+(4+4)^2(2-7)^2}=3\sqrt{10}$. The equation is $(x-1)^2+(y-4)^2+(z-2)^2=90$.

32. The radius is $\sqrt{(-3-0)^2+(1-0)^2+(2-0)^2}=\sqrt{14}$. The equation is $(x+3)^2+(y-1)^2+(z-2)^2=14$.

33. The upper half of the sphere $x^2+y^2+(z-1)^2=4$; a hemisphere

34. A circle on the sphere $x^2+y^2+(z-1)^2=4$; the circle is parallel to the xy-plane and has radius $\sqrt{3}$.

35. All points on and outside the unit sphere centered at the origin

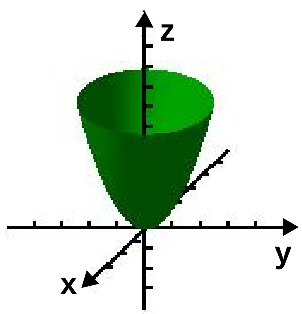
36. All points inside the sphere of radius 1 centered at $(1,2,3)$, except the center

37. $x^2+y^2+z^2=1$ represents a sphere of radius 1 and $x^2+y^2+z^2=9$ represents a sphere of radius 3. Therefore $1 \le x^2+y^2+z^2 \le 9$ represents the set of points lying between these two spheres. Thus, the geometric object is a hollowed out ball with outer radius 3 and inner radius 1.

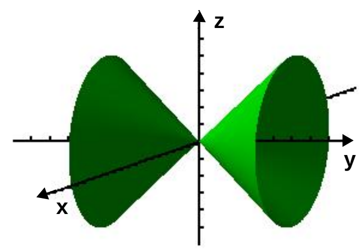
38. This set of points is identical to the found in Problem 11.7.37, with the added restriction $z \ge 0$. This restriction will remove points with negative z-coordinates, leaving only the upper half of the hollowed out ball.

11.8 Quadric Surfaces

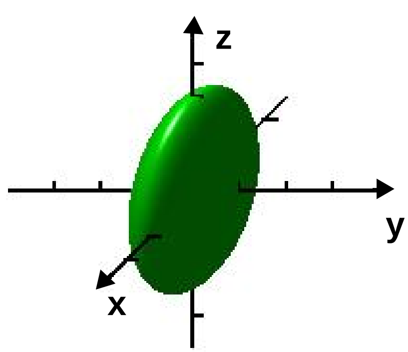
1. paraboloid



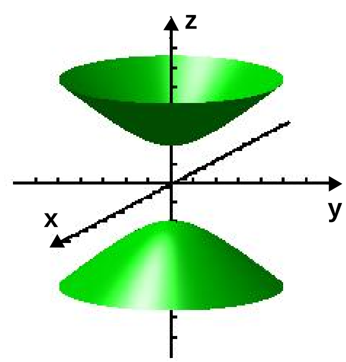
2. elliptical cone



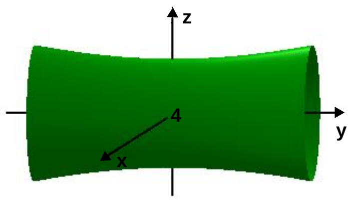
3. $x^2/4+y^2+z^2/9=1$; ellipsoid



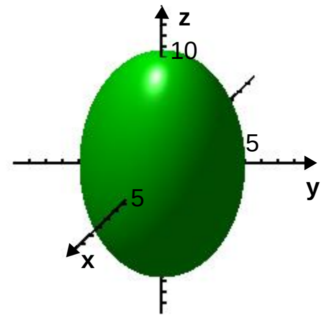
4. $-x^2/4-y^2/4+z^2/4=1$
hyperboloid of two sheets



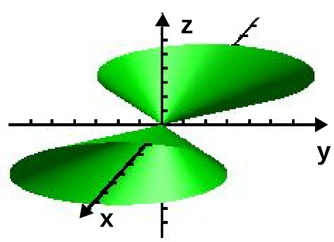
5. $x^2/4 - y^2/144 + z^2/16 = 1$
hyperboloid of one sheet



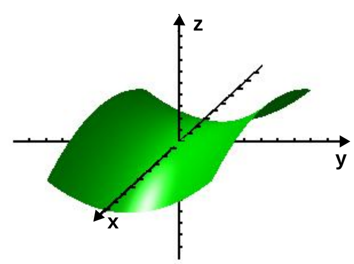
6. $x^2/25 + y^2/25 + z^2/100 = 1$
ellipsoid



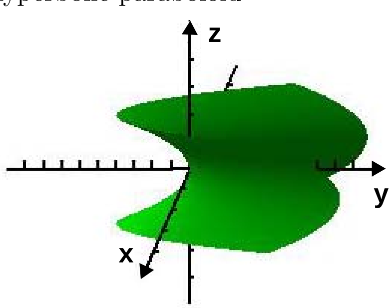
7. elliptical cone



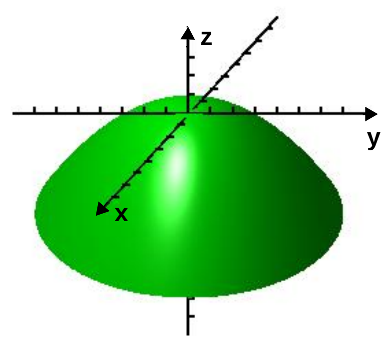
8. $y^2/9 - x^2/16 = z$
hyperbolic paraboloid



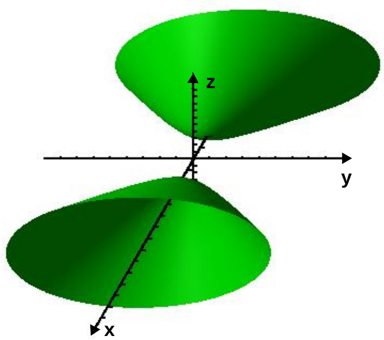
9. hyperbolic paraboloid



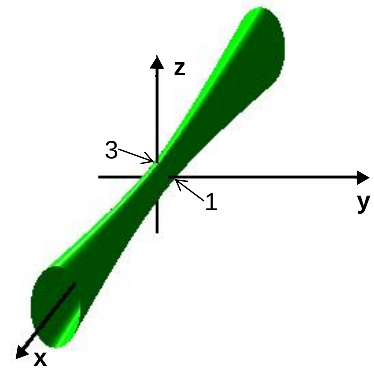
10. $x^2 + y^2 = -9z$
paraboloid



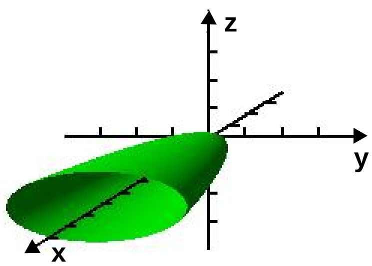
11. $x^2/4 - y^2/4 - z^2/4 = 1$
hyperboloid of two sheets



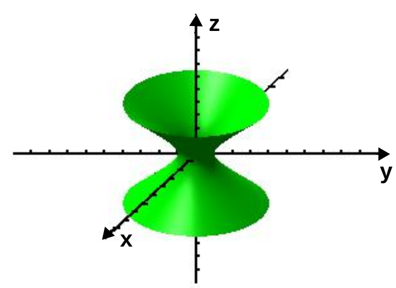
12. $-z^2/9 + y^2 + z^2/9 = 1$
hyperboloid of one sheet



13. $y^2 + \dfrac{z^2}{1/4} = x$
paraboloid



14. hyperboloid of one sheet



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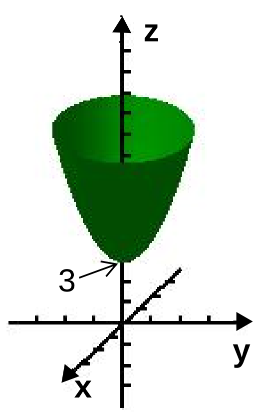

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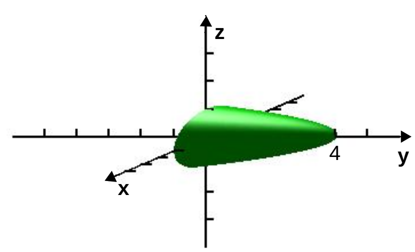

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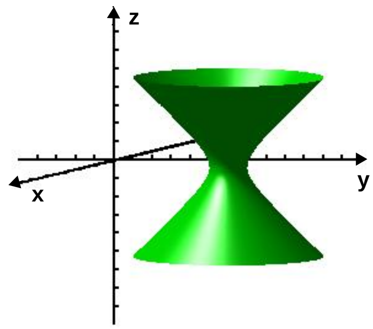

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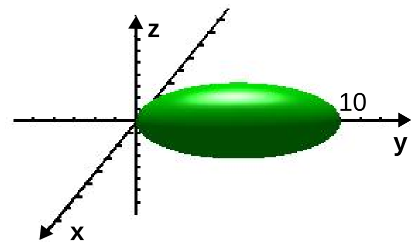


19. The equation can be written as $x^2 + (\pm\sqrt{y^2 + z^2})^2 = 1$. The surface is generated by revolving the circles $x^2 + y^2 = 1$ or $x^2 + z^2 = 1$ about the x-axis. [Alternatively, the surface is generated by revolving the circles $x^2 + y^2 = 1$ or $y^2 + z^2 = 1$ about the y-axis, or the circles $x^2 + z^2 = 1$ or $y^2 + z^2 = 1$ about the z-axis.]

20. The equation can be written as $-9x^2 + (\pm 2\sqrt{y^2 + z^2 0^2} = 36$. The surface is generated by revolving the hyperbolas $= 9x^2 + 4y^2 = 36$ or $-9x^2 + 4z^2 = 36$ about the x-axis.

21. The equation can be written as $y = e^{\pm\sqrt{x^2+z^2})^2}$. The surface is generated by revolving the curves $y = e^{x^2}$ or $y = e^{z^2}$ about the y-axis.

22. The equation can be written as $(\pm\sqrt{x^2 + y^2})^2 = \sin^2 z$. The surface is generated by revolving the curves $x^2 = \sin^2 z$ or $y^2 = \sin^2 z$ about the z-axis.

23. Replacing x by $\pm\sqrt{x^2 + y^2}$ we have $y = \pm 2\sqrt{x^2 + z^2}$ or $y^2 = 4x^2 + 4z^2$.

24. Replacing z by $\sqrt{x^2 + z^2}$ we have $y = (\sqrt{x^2 + z^2})^{1/2}$ or $y^4 = x^2 + z^2$; $y \geq 0$.

25. Replacing z by $\pm\sqrt{y^2 + z^2}$ we have $\pm\sqrt{y^2 + z^2} = 9 - x^2$ or $y^2 + z^2 = (9 - x^2)^2$, $x \geq 0$.

26. Replacing y by $\sqrt{x^2 + y^2}$ we have $z = 1 + (\sqrt{x^2 + y^2})^2$ or $z = 1 + x^2 + y^2$.

27. Replacing z by $\sqrt{y^2 + z^2}$ we have $x^2 - (\pm\sqrt{y^2 + z^2})^2 = 4$ or $x^2 - y^2 - z^2 = 4$.

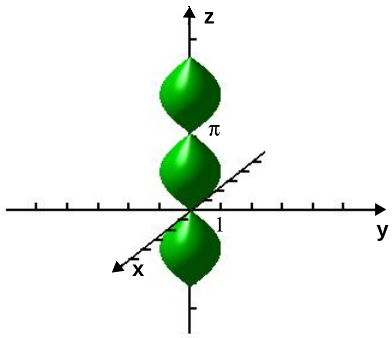
28. Replacing x by $\pm\sqrt{x^2 + y^2}$ we have $3(\pm\sqrt{x^2 + y^2})^2 + 4z^2 = 12$ or $3x^2 + 3y^2 + 4z^2 = 12$.

29. Replacing y by $\sqrt{x^2 + y^2}$ we have $z = \ln\sqrt{x^2 + y^2}$.

30. Replacing y by $\pm\sqrt{y^2 + z^2}$ we have $x(\pm\sqrt{y^2 + z^2}) = 1$ or $x^2(y^2 + z^2) = 1$.

31. The surface is Problem 11 is a surface of revolution about the x-axis. The surface in Problem 2 is a surface of revolution about the y-axis. The surface is Problems 1, 4, 6, 10, and 14 are surfaces of revolution about the z

32.



33. The first equation is the lower nappe of the cone $(z+2)^2 = x^2 + y^2$ whose axis of revolution is the z-axis and whose vertex is at $(0,0,-2)$.

34. The first equation is the right-hand of the cone $(y-1)^2 = x^2 + z^2$ whose axis of revolution is the y-axis and whose vertex is at $(0,1,0)$.

35. (a) Writing the equation of the ellipse in the form $x^2/(c-z)a^2 + y^2/(c-z)b^2 = 1$ we see that the area of a cross-section is $\pi a\sqrt{c-z}b\sqrt{c-z} = \pi ab(c-z)$.

    (b) $V = \int_0^c \pi ab(c-x)\mathrm{d}z = \pi ab\left(-\frac{1}{2}\right)(c-z)^2\Big|_0^c = \dfrac{1}{2}\pi abc^2$

36. (a) Using the formula for the area of an ellipse given in Problem 35(a) we see that a horizontal cross-sectional area of the ellipsoid is $\pi ab(1 - z^2/c^2)$. Then

$$V = 2\int_0^c \pi ab\left(1 - \frac{z}{c^2}\right)\mathrm{d}z = 2\pi ab\left(z - \frac{z^3}{3c^2}\right)\Bigg|_0^c = \frac{4}{3}\pi abc.$$

    (b) When $a = b = c$ the volume is $\frac{4}{3}\pi a^3$, which is the formula for the volume of a sphere.

37. Expressing the line in the form $(x-2)/4 = (y+2)/(-6) = (z-6)/3$ we see that parametric equations for the line are $x = 2+4t$, $\quad y = -2 = 6t$, $\quad z = 6+3t$. Writing the equation of the ellipse as $36x^2 + 9y^2 + 4z^2 = 324$ and substituting, we obtain $36(2+4t)^2 + 9(-2-6t)^2 + 4(6+3t)^2 = 936t^2 + 936t + 324$ or $936t(t+1) = 0$. When $t = 0$ we obtain the point $(2,-2,6)$, and when $t = -1$ we obtain the point $(-2,4,3)$.

Chapter 11 in Review

A. True/False

1. True

2. False; the points must be non-collinear.

3. False; since a normal to the plane is $\langle 2,3,-4\rangle$ which is not a multiple of the direction vector $\langle 5,-2,2\rangle$ of the line.

4. True
5. True
6. True
7. True
8. True
9. True

10. True; since $\mathbf{a}\times\mathbf{b}$ and $\mathbf{c}\times\mathbf{d}$ are both normal to the plane and hence parallel (unless $\mathbf{a}\times\mathbf{b}=\mathbf{0}$ or $\mathbf{c}\times\mathbf{d}$.)

11. True. The normal vector of the first plane is $\langle 1,2,-1\rangle$ while the normal vector of the second plane is $\langle -2,-4,2\rangle$. Since the second vector is a scalar multiple of the first, the planes are parallel.

12. False. Look at Figure 11.5.3 in the text.

13. True. This is a parabolic cylinder similar to that shown in Figure 11.7.6.

14. True. In the yz-plane, we have $x=0$. Therefore, the equation of the surface becomes $\frac{y^2}{2}+\frac{z^2}{2}=1$ or $y^2+z^2=2$.

15. False. Find the equation of the plane containing the first three points, $P_1(0,1,2)$, $P_2(1,-1,1)$, and $P_3(3,2,6)$. This plane must contain the vectors $\overrightarrow{P_1P_2}=\langle 1,-2,1\rangle$ and $\overrightarrow{P_1P_3}=\langle 3,1,4\rangle$. Define $\mathbf{n}=\overrightarrow{P_1P_2}\times\overrightarrow{P_1P_3}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ 1&-2&-1\\ 3&1&4\end{vmatrix}=\langle -7,-7,-7\rangle$. Then $\mathbf{n}$ must be normal to the plane. Using $\mathbf{n}$ and the point P_1, the equation of the plane becomes $-7x-7y+7z=7$ or $z+y-z=-1$. The fourth point $P_4(2,1,2)$ does not lie on the plane since $(2)+(1)-(2)\neq -1$.

16. True

17. False. The trace in the yz-plane is described by the equation $9y^2+z^2=1$ which represents an ellipse.

18. True. This ellipsoid results from revolving the graph of the ellipse $x^2+9y^2=1$ about the y-axis.

19. True. $|\mathbf{a}\times\mathbf{b}|=|\mathbf{a}||\mathbf{b}||\sin\theta|=|\mathbf{a}||\mathbf{b}|$ since $\theta=90°$

20. False. Let $\mathbf{a}=\mathbf{i}$, $\mathbf{b}=\mathbf{j}$, and $\mathbf{c}=\mathbf{k}$. Then $\mathbf{a}\cdot\mathbf{b}=\mathbf{a}\cdot\mathbf{c}=0$ but $\mathbf{b}\neq\mathbf{c}$.

B. Fill in the Blanks

1. $9\mathbf{i}+2\mathbf{j}+2\mathbf{k}$
2. orthogonal
3. $-5(\mathbf{k}\times\mathbf{j})=-5(-\mathbf{i})=5\mathbf{i}$
4. $\mathbf{i}\cdot(\mathbf{i}\times\mathbf{j})=\mathbf{i}\times\mathbf{k}=\mathbf{0}$
5. $\sqrt{(-12)^2+4^2+6^2}=14$
6. $$\begin{aligned}\mathbf{k}\times(\mathbf{i}+2\mathbf{j}-5\mathbf{k})&=\mathbf{k}\times\mathbf{i}+2(\mathbf{k}\times\mathbf{j})-5(\mathbf{k}\times\mathbf{k})\\&=\mathbf{j}-2\mathbf{i}-5(\mathbf{0})\\&=\langle -2,1,0\rangle\end{aligned}$$

7. $\begin{vmatrix} 2 & -5 \\ 4 & 3 \end{vmatrix} = 2(3) - (-5)(4) = 6 + 20 = 26$

8. $(-1-20)\mathbf{i} - (-2-0)\mathbf{j} + (8-0)\mathbf{k} = -21\mathbf{i} + 2\mathbf{j} + 8\mathbf{k}$

9. $-6\mathbf{i} + \mathbf{j} - 7\mathbf{k}$

10. The smallest component is the $\mathbf{j}$-component with magnitude 3. Therefore, the sphere cannot have a radius larger than 3 or its interior will intersect the xz-plane. Thus, we need a sphere with radius 3 and center $(4, 3, 7)$. The equation is

$$(x-4)^2 + (y-3)^2 + (z-7)^2 = 9$$

11. Writing the line in parametric form, we have $x = 1 + t, \quad y = -2 + 3t \quad z = -1 + 2t$. Substituting into the equation of the plane yields $(1+t) + 2(-2+3t) - (-1+2t) = 13$ or $t = 3$. Thus, the point of intersection is $x = 1 + 3, \quad y = -2 + 3(3) = 7, \quad z = -1 + 2(3) = 5$, or $(4, 7, 5)$.

12. $|\mathbf{a}| = \sqrt{4^2 + 3^2 + (-5)^2} = 5\sqrt{2}; \quad \mathbf{u} = -\dfrac{1}{5\sqrt{2}}(4\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}) = -\dfrac{4}{5\sqrt{2}}\mathbf{i} - \dfrac{3}{5\sqrt{2}}\mathbf{j} + \dfrac{1}{\sqrt{2}}\mathbf{k}$

13. $x_2 - 2 = 3, \quad x_2 = 5; \quad y_2 - 1 = 5, \quad y_2 = 6; \quad z_2 - 7 = -4, \quad z_2 = 3; \quad P_2 = (5, 6, 3)$

14. $(5, 1/2, 5/2)$

15. $(7.2)(10)\cos 135^\circ = -36\sqrt{2}$

16. $2\mathbf{b} = \langle -2, 4, 2\rangle; \quad 4\mathbf{c} = \langle 0, -8, 8\rangle; \quad \mathbf{a} \cdot (2\mathbf{b} + 4\mathbf{c}) = \langle 3, 1, 0\rangle \cdot \langle -2, -4, 10\rangle = -10$

17. $12, -8, 6$

18. $\cos\theta = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \dfrac{1}{\sqrt{2}\sqrt{2}} = 1/2; \quad \theta = 60^\circ$

19. $A = \dfrac{1}{2}|5\mathbf{i} - 4\mathbf{j} - 7\mathbf{k}| = 3\sqrt{10}/2$

20. $(x+5)^2 + (y-7)^2 + (z+9)^2 = 6$

21. $|-5-(-3)| = 2$

22. parallel: $-2c = 5, \quad c = -5/2$; orthogonal: $1(-2) + 3(-6) + c(5) = 0, \quad c = 4$

23. The equation can be transformed into something more recognizable by completing the square:
$x^2 + 2y^2 + 2z^2 - 4y - 12z = 0$
$\Longrightarrow x^2 + 2(y^2 - 2y) + 2(z^2 - 6z) = 0$
$\Longrightarrow x^2 + 2(y^2 - 2y + 1) + 2(z^2 - 6z = 9) = 20$
$\Longrightarrow x^2 + 2(y-1)^2 + 2(z-3)^2 = 20$
This is the equation of an ellipsoid centered at $(0, 1, 3)$.

24. Letting $z = 1$, the trace is described by the equation $y = x^2 - 1$, which is a parabola.

C. Exercises

1. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -2 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix}\mathbf{k} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$

   A unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ is
   $$\frac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|} = \frac{1}{\sqrt{1+1+9}}(\mathbf{i}-\mathbf{j}-3\mathbf{k} = \frac{1}{\sqrt{11}}\mathbf{i} - \frac{1}{\sqrt{11}}\mathbf{j} - \frac{3}{\sqrt{11}}\mathbf{k}.$$

2. The magnitude of $\mathbf{a}$ is given by $|\mathbf{a}| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$. Letting α, β, and γ represent the angles between $\mathbf{a}$ and $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ respectively, we have $\cos\alpha = \frac{\left(\frac{1}{2}\right)}{\left(\frac{\sqrt{3}}{2}\right)} = \frac{1}{\sqrt{3}}$, $\cos\beta = \frac{\left(\frac{1}{2}\right)}{\left(\frac{\sqrt{3}}{2}\right)} = \frac{1}{\sqrt{3}}$, and $\cos\gamma = \frac{\left(-\frac{1}{2}\right)}{\left(\frac{\sqrt{3}}{2}\right)} = -\frac{1}{\sqrt{3}}$. From this we are able to compute: $\alpha = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 0.95532$

   $\beta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 0.95532$

   $\gamma = \cos^{-1}\left(-\frac{1}{\sqrt{3}}\right) \approx 2.18628$

3. $\text{comp}_{\mathbf{b}}\mathbf{a} = \mathbf{a}\cdot\mathbf{b}/|\mathbf{b}| = \langle 1, 2, -2\rangle \cdot \langle 4, 3, 0\rangle/5 = 2$

4. $\text{comp}_{\mathbf{a}}\mathbf{b} = \mathbf{b}\cdot\mathbf{a}/|\mathbf{a}| = \langle 4, 3, 0\rangle \cdot \langle 1, 2, -2\rangle/3 = 10/3$
   $\text{proj}_{\mathbf{a}}\mathbf{b} = (\text{comp}_{\mathbf{a}}\mathbf{b}\mathbf{a}/|\mathbf{a}| = (10/3)\langle 1, 2, -2\rangle/3 = \langle 10/9, 20/9, -20/9\rangle$

5. First we compute $2\mathbf{a} = \langle 2, 4, -4\rangle$, $|\mathbf{b}| = \sqrt{16+9} = 5$, and $2\mathbf{a}\cdot\mathbf{b} = 20$. So $\text{proj}_{\mathbf{b}}2\mathbf{a} = \frac{2\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\mathbf{b} = \frac{20}{25}\langle 4, 3, 0\rangle = \langle \frac{16}{5}, \frac{12}{5}, 0\rangle$.

6. $\text{comp}_{\mathbf{b}}(\mathbf{a}-\mathbf{b}) = (\mathbf{a}-\mathbf{b})\cdot\mathbf{b}/|\mathbf{b}| = \langle -3, -1, -2\rangle \cdot \langle 4, 3, 0\rangle/5 = -3$
   $\text{proj}_{\mathbf{b}}(\mathbf{a}-\mathbf{b}) = (\text{comp}_{\mathbf{b}}(\mathbf{a}-\mathbf{b}))\mathbf{b}/|\mathbf{b}| = -3\langle 4, 3, 0\rangle/5 = \langle -12/5, -9/5, 0\rangle$
   $\text{proj}_{\mathbf{b}^\perp}(\mathbf{a}-\mathbf{b}) = (\mathbf{a}-\mathbf{b}) - \text{proj}_{\mathbf{b}}(\mathbf{a}-\mathbf{b}) = \langle -3, -1, -2\rangle - \langle -12/5, -9/5, 0\rangle = \langle -3/5, 4/5, -10/5\rangle$

7. $\frac{x^2}{16} + \frac{y^2}{4} = 1$; elliptical cylinder

8. $\frac{x^2}{2} + z^2 = -\frac{1}{4}y$; paraboloid

9. $-\frac{x^2}{9} - \frac{y^2}{9/4} + \frac{z^2}{9} = 1$; hyperboloid of two sheets

10. $\frac{x^2}{25} + \frac{y^2}{25} + \frac{(z-5)^2}{25} = 1$; sphere

11. $x^2 - y^2 = 9z$; hyperbolic paraboloid

12. plane

13. Replacing x by $\pm\sqrt{x^2+z^2}$ we have $(\pm\sqrt{x^2+z^2})^2 - y^2 = 1$ or $x^2+z^2-y^2=1$, which is a hyperboloid of one sheet. Replacing y by $(\pm\sqrt{y^2+z^2})$ we have $x^2-(\pm\sqrt{y^2+z^2})^2 = 1$ or $x^2-y^2-z^2=1$, which is a hyperboloid of two sheets.

14. The surface is generated by revolving $y = 1+x, \quad x \geq 0$, about the y-axis or $y = 1+z, \quad z \geq 0$ about the z-axis. The restrictions on x and z are required since $y = 1+\sqrt{x^2+z^2} \geq 1$.

15. Let $\mathbf{a} = \langle a, b, c\rangle$ and $\mathbf{r} = \langle x, y, z\rangle$. Then

    (a) $(\mathbf{r}-\mathbf{a})\cdot\mathbf{r} = \langle x-a, y-b, z-c\rangle \cdot \langle x, y, z\rangle = x^2 - ax + y^2 - by + z^2 - ac = 0$ implies $\left(x-\dfrac{a}{2}\right)^2 + \left(y-\dfrac{b}{x}\right)^2 + \left(z-\dfrac{c}{2}\right)^2 = \dfrac{a^2+b^2+c^2}{4}$. The surface is a sphere.

    (b) $(\mathbf{r}-\mathbf{a})\cdot\mathbf{a} = \langle x-a, y-b, z-c\rangle \cdot \langle a, b, c\rangle = a(x-a)+b(y-b)+c(z-c) = 0$
    The surface is a plane.

16. $\langle 4, 2, -2\rangle - \langle 2, 4, -3\rangle = \langle 2, -2, 1\rangle; \quad \langle 2, 4, -3\rangle - \langle 6, 7, -5\rangle = \langle -4, -3, 2\rangle;$
$\langle 2, -2, 1\rangle \cdot \langle -4, -3, 2\rangle = 0$. The points are the vertices of a right triangle.

17. A direction vector of the given line is $\langle 4, -2, 6\rangle$. A parallel line containing $(7, 3, -5)$ is $(x-7)/4 = (y-3)/(-2) = (z+5)/6$.

18. A normal to the plane is $\langle 8, 3, -4\rangle$. The line with this direction vector and through $(5, -9, 3)$ is $x = 5+8t, \;\; y = -9+3t, \;\; z = 3-4t$.

19. The direction vectors are $\langle -2, 3, 1\rangle$ and $\langle 2, 1, 1\rangle$. Since $\langle -2, 3, 1\rangle \cdot \langle 2, 1, 1\rangle = 0$, the lines are orthogonal. Solving $1-2t = x = 1+2s, \;\; 3t = y = -4+s$, we obtain $t = -1$ and $s = 1$. The point $(3, -3, 0)$ obtained by letting $t = -1$ and $s = 1$ is common to the two lines, so they do intersect.

20. Vectors in the plane are $\langle 2, 3, 1\rangle$ and $\langle 1, 0, 2\rangle$. A normal vector is $\langle 2, 3, 1\rangle \times \langle 1, 0, 2\rangle = \langle 6, -3, -3\rangle = 3\langle 2, -1, -1\rangle$. An equation of the plane is $2x - y - z = 0$.

21. The lines are parallel with direction vector $\langle 1, 4, -2\rangle$. Since $(0, 0, 0)$ is on the first line and $(1, 1, 3)$ is on the second line, the vector $\langle 1, 1, 3\rangle$ is in the plane. A normal vector to the plane is thus $\langle 1, 4, -2\rangle \times \langle 1, 1, 3\rangle = \langle 14, -5, -3\rangle$. An equation of the plane is $14x - 5y - 3z = 0$.

22. Letting $z = t$ in the equations of the plane and solving $-x+y = 4+8t, \;\; 3x-y = -2t$, we obtain $x = 2+3t, \;\; y = 6+11t, \;\; z = t$. Thus, a normal to the plane is $\langle 3, 11, 1\rangle$ and an equation of the plane is $3(x-1)+11(y-7)+(z+1) = 0$ or $3x+11y+z = 79$.

23. A normal vector is $(\mathbf{i}-2\mathbf{j}) \times (2\mathbf{i}+3\mathbf{k}) = -6\mathbf{i}-3\mathbf{j}+4\mathbf{k}$. Thus, an equation of the plane is $-6(z-1)-3(y+1)+4(z-2) = 0$ or $6x+3y-4z = -5$.

24. The points at the ends of the diameter, obtained from $t = -1$ and $t = 0$ are $(2, 4, 2)$ and $(4, 7, 8)$. The center of the sphere is the midpoint of the line segment or $(3, 11/2, 5)$. The diameter of the sphere is $\sqrt{2^2+3^2+6^2} = 7$. The equation is $(x-3)^2+(y-11/2)^2+(z-5)^2 = 49/4$.

25. We compute $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$. First $\mathbf{a}\times\mathbf{b} = -3\mathbf{i}+3\mathbf{j}-3\mathbf{k}$. Then $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = -3(4)+3(5)-3(1) = 0$, and the vectors are coplanar.

26. Let $\mathbf{d}$ be the vector from the right angle to M. We want to show that $|\mathbf{d}| = \dfrac{1}{2}|\mathbf{c}|$. Since $\mathbf{a}\cdot\mathbf{b} = 0$ we have $|\mathbf{b}-\mathbf{a}|^2 = (\mathbf{b}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{a}) = \mathbf{a}\cdot\mathbf{a}+2\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{b} = |\mathbf{a}|^2+|\mathbf{b}|^2$, we have $|\mathbf{d}| = \frac{1}{2}|\mathbf{a}+\mathbf{b}| = \dfrac{1}{2}\sqrt{|\mathbf{a}|^2+|\mathbf{b}|^2} = \dfrac{1}{2}|\mathbf{b}-\mathbf{a}| = \dfrac{1}{2}|\mathbf{c}|$.

27. (a) We have $\mathbf{v} = v\mathbf{j}$ and $\mathbf{B} = B\mathbf{i}$. Then $\mathbf{F} = q(\mathbf{v}\times\mathbf{B} = q(v\mathbf{j}\times B\mathbf{i}) = q(-vB\mathbf{k}) = -qvB\mathbf{k}$.

    (b) We first note that $\mathbf{L} = m\mathbf{r}\times\mathbf{v}$ and $\mathbf{r}\times\mathbf{v} = 0$. Then

$$\mathbf{r}\times\mathbf{L} = \mathbf{r}\times(m\mathbf{r}\times\mathbf{v}) = m[\mathbf{r}\times(\mathbf{r}\times\mathbf{v})] = m[(\mathbf{r}\cdot\mathbf{v})\mathbf{r}-(\mathbf{r}\cdot\mathbf{r})\mathbf{v}] = -m|\mathbf{r}|^2\mathbf{v},$$

    and so $\mathbf{v} = -\dfrac{1}{m|\mathbf{r}|^2}(\mathbf{r}\times\mathbf{L}) = \dfrac{1}{m|\mathbf{r}|^2}(\mathbf{L}\times\mathbf{r})$.

28. $\mathbf{F} = 10\dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{10}{\sqrt{2}}(\mathbf{i}+\mathbf{j}) = 5\sqrt{2}\mathbf{i}+5\sqrt{2}\mathbf{j}$; $\quad \mathbf{d} = \langle 7,4,0\rangle - \langle 4,1,0\rangle = 3\mathbf{i}+3\mathbf{j}$
    $W = \mathbf{F}\cdot\mathbf{d} = 15\sqrt{2}+15\sqrt{2} = 30\sqrt{2}$ N-m

29. $\mathbf{F} = 5\sqrt{2}\mathbf{i}+5\sqrt{2}\mathbf{j}+50\mathbf{i} = (5\sqrt{2}+50)\mathbf{i}+5\sqrt{2}\mathbf{j}$; $\quad \mathbf{d} = 3\mathbf{i}+3\mathbf{j}$

    $W = 15\sqrt{2}+150+15\sqrt{2} = 30\sqrt{2}+150$N-m ≈ 192.4 N-m

30. Let $|\mathbf{F}_1| = F_1$ and $|\mathbf{F}_2| = F_2$. Then $\mathbf{F}_1 = F_1[(\cos 45^\circ)\mathbf{i}+(\sin 45^\circ)\mathbf{j}]$ and $\mathbf{F}_2 = F_2[(\cos 120^\circ)\mathbf{i}+(\sin 120^\circ)\mathbf{j}]$ or $\mathbf{F}_1 = F_1\left(\dfrac{1}{\sqrt{2}}\mathbf{i}+\dfrac{1}{\sqrt{2}}\right)$ and $\mathbf{F}_2 = F_2\left(-\dfrac{1}{2}\mathbf{i}+\dfrac{\sqrt{3}}{2}\mathbf{j}\right)$. Since $\mathbf{w}+\mathbf{F}_1+\mathbf{F}_2 = \mathbf{0}$,

$$F_1\left(\frac{1}{\sqrt{2}}\mathbf{i}+\frac{1}{\sqrt{2}}\right)+F_2\left(-\frac{1}{2}\mathbf{i}+\frac{\sqrt{3}}{2}\mathbf{j}\right) = 50\mathbf{j}, \quad \left(\frac{1}{\sqrt{2}}F_1-\frac{1}{2}F_2\right)\mathbf{i}+\left(\frac{1}{\sqrt{2}}F_1+\frac{\sqrt{3}}{2}F_2\right)\mathbf{j} = 50\mathbf{j}$$

    and

$$\frac{1}{\sqrt{2}}F_1-\frac{1}{2}F_2 = 0, \quad \frac{1}{\sqrt{2}}F_1+\frac{\sqrt{3}}{2}F_2 = 50.$$

    Solving, we obtain $F_1 = 25(\sqrt{6}-\sqrt{2}) \approx 25.9$lb and $F_2 = 50(\sqrt{3}-1) \approx 36.6$lb.

Chapter 12

Vector-Valued Functions

12.1 Vector Functions

1. Since the square root function is only defined for nonnegative values, we must have $t^2 - 9 \geq 0$. So the domain is $(-\infty, -3) \bigcup [3, \infty)$.

2. Since the natural logarithm is only defined for positive values, we must have $1 - t^2 > 0$. So the domain is $(-1, 1)$.

3. Since the inverse sine function is only defined for values between -1 and 1, the domain is $[-1, 1]$.

4. The vector function is defined for all real numbers.

5. $\mathbf{r}(t) = \sin \pi t\mathbf{i} + \cos \pi t\mathbf{j} - \cos^2 \pi t\mathbf{k}$

6. $\mathbf{r}(t) = \cos^2 \pi t\mathbf{i} + 2\sin^2 \pi t\mathbf{j} + t^2\mathbf{k}$

7. $\mathbf{r}(t) = e^{-t}\mathbf{i} + e^{2t}\mathbf{j} + e^{3t}\mathbf{k}$

8. $\mathbf{r}(t) = -16t^2\mathbf{i} + 50t\mathbf{j} + 10\mathbf{k}$

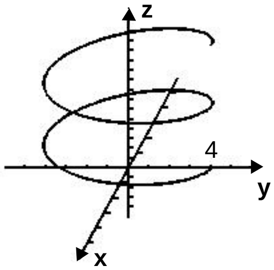
9. $x = t^2, \quad y = \sin t, \quad z = \cos t$

10. $\mathbf{r}(t) = t\sin t(\mathbf{i} + \mathbf{k}) = t\sin t\mathbf{i} + 0\mathbf{j} + t\sin t\mathbf{k}$ so $x = t\sin t, \quad y = 0, \quad z = t\sin t$

11. $x = \ln t, \quad y = 1 + t, \quad z = t^3$

12. $x = 5\sin t\sin 3t, \quad y = 5\cos 3t, \quad z = 5\cos t\sin 3t$

13.



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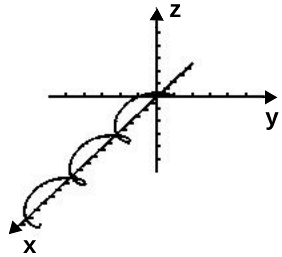

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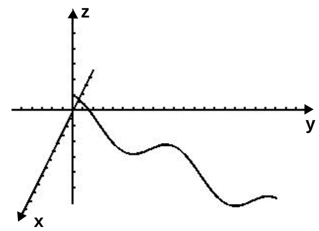

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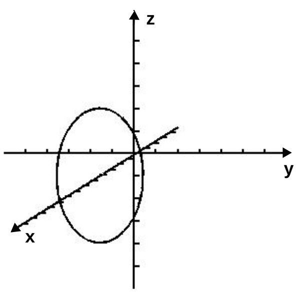

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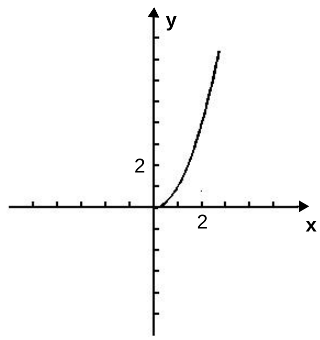

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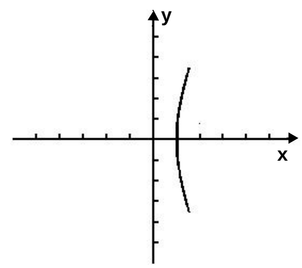

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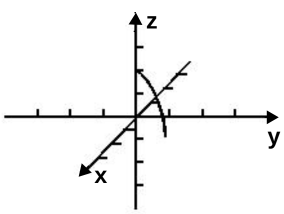
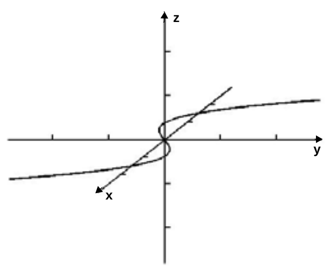
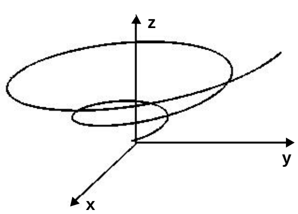


20.

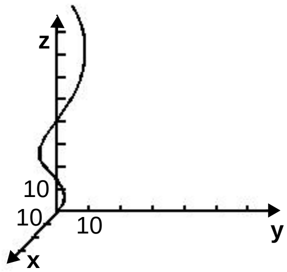


21.



Note: the scale is distorted in this graph. For $t = 0$, the graph starts at $(1, 0, 1)$. The upper loop shown intersects the xz-plane at about $(286751, 0, 286751)$.

22.



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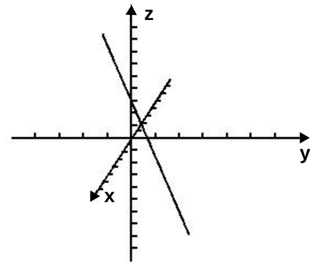

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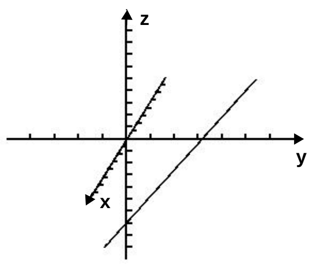
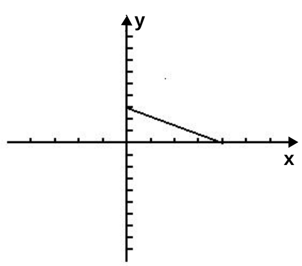
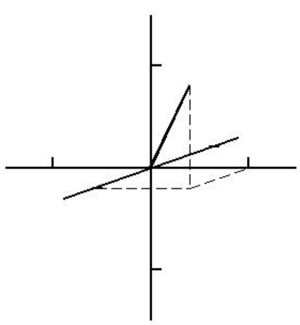


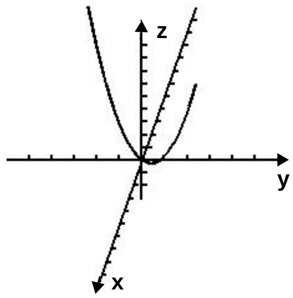
25. $\mathbf{r}(t) = \langle 4, 0\rangle + \langle 0-4, 3-0\rangle t = (4-4t)\mathbf{i} + 3t\mathbf{j}, \quad 0 \le t \le 1$



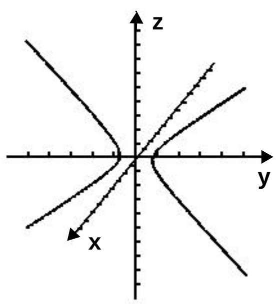
26. $\mathbf{r}(t) = \langle 0, 0, 0\rangle + \langle 1-0, 1-0, 1-0\rangle t = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}, \quad 0 \le t \le 1$



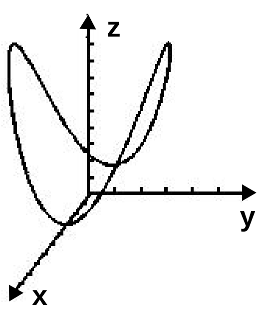
27. $x = t, \;\; y = t, \;\; z = t^2 + t^2 = 2t^2; \;\; \mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + 2t^2\mathbf{k}$



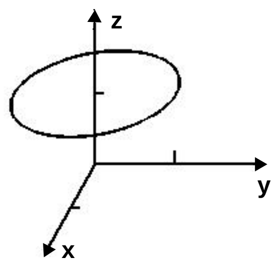
28. $x = t, \;\; y = 2t, \;\; z = \pm\sqrt{t^2 + 4t^2 + 1} = \pm\sqrt{5t^2 - 1}; \;\; \mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} \pm \sqrt{5t^2 - 1}\mathbf{k}$



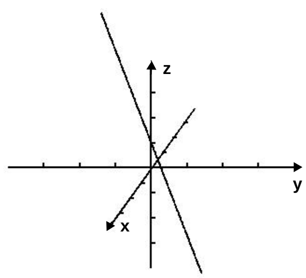
29. $x = 3\cos t, \;\; z = 9 - 9\cos^2 t = 0\sin^2 t; \;\; y = 3\sin t; \mathbf{r}(t) = 3\cos t\mathbf{i} + 3\sin t\mathbf{j} + 9\sin^2 t\mathbf{k}$



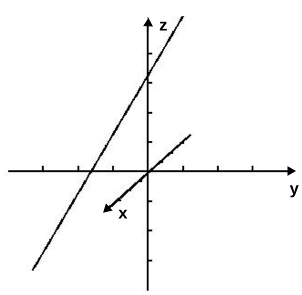
30. $x = \sin t, \;\; z = 1, \;\; y = \cos t; \;\; \mathbf{r}(t) = \sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}$



31. $x = t, \;\; y = t, \;\; z = 1 - 2t; \;\; \mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + (1 - 2t)\mathbf{k}$



32. $x = 11, \quad y = t, \quad z = 3 + 2t; \quad \mathbf{r}(t) = \mathbf{i} + t\mathbf{j} + (3 + 2t)\mathbf{k}$



33. (b); Notice that the y and z values consistently increase while the x values oscillate rapidly between -1 and 1. The only vector fucntion that describes this behavior is (b).

34. (c); The trace of the graph on the $xy-$plane would look like a circle, while the z value oscillates between 0 and 1. The only vector function that describes this behavior is (c).

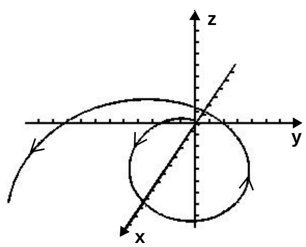
35. (d); Notice that the z value is contant. The only vector function that satisfies this constraint is (d).

36. (a); Notice that the x values consistently increase while the trace of the graph on the yz-plane would look like a circle. The only vector function that describes this behavior is (a).

37. Letting $x = at\cos t$, $y = bt\sin t$, and $z = ct$, we have

$$\begin{aligned}\frac{z^2}{c^2} = \frac{c^2t^2}{c^2} = t^2 &= t^2\cos^2 t + t^2\sin^2 t \\ &= \frac{a^2t^2\cos^2 t}{a^2} + \frac{b^2t^2\sin^2}{b^2} \\ &= \frac{x^2}{a^2} + \frac{y^2}{b^2}\end{aligned}$$

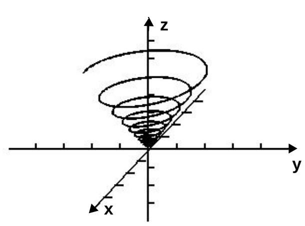
38.



39. Letting $x = ae^{kt}\cos t$, $y = be^{kt}\sin t$, and $z = ce^{kt}$, we have

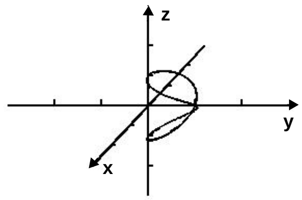
$$\begin{aligned}\frac{z^2}{c^2} = \frac{c^2e^{kt}}{c^2} = e^{2kt} &= e^{2kt}\cos^2 t + e^{2kt}\sin^2 t \\ &= \frac{a^2e^{2kt}\cos^2 t}{a^2} + \frac{b^2e^{2kt}\sin^2 t}{b^2} \\ &\frac{x^2}{a^2} + \frac{y^2}{b^2}\end{aligned}$$

40.

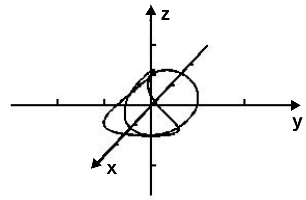


41. $$\begin{aligned}x^2 + y^2 + z^2 &= a^2\sin^2 kt\cos^2 t + a^2\sin^2 kt\sin^2 t + a^2\cos^2 kt \\ &= a^2\sin^2 kt + a^2\cos^2 kt \\ &= a^2\end{aligned}$$

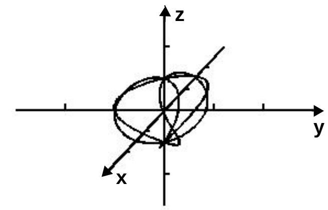
42. $k = 1$



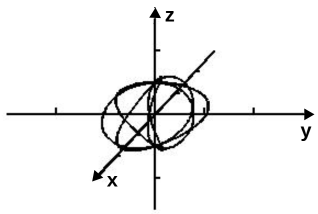
$k = 2$



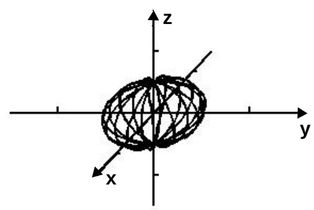
$k = 3$



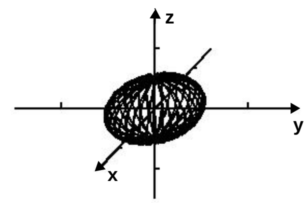
$k = 4$



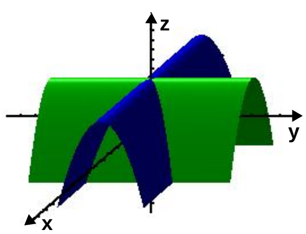
$k = 10$



$k = 20$

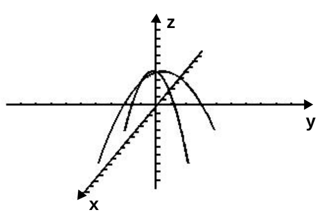


43. (a)



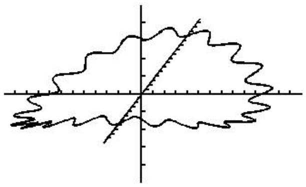
(b) $\mathbf{r}_1(t) = t\mathbf{i} + t\mathbf{j} + (4 - t^2)\mathbf{k}$
$\mathbf{r}_2(t) = t\mathbf{i} - t\mathbf{j} + (4 - t^2)\mathbf{k}$

(c)

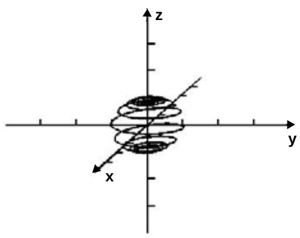


44. C lies on the surface of the sphere of radius a.

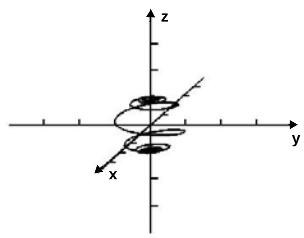
45.



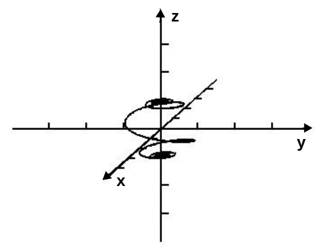
46. $k = 0.1$



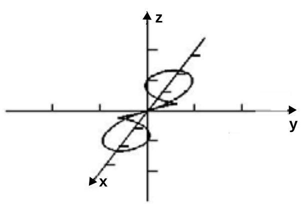
$k = 0.2$



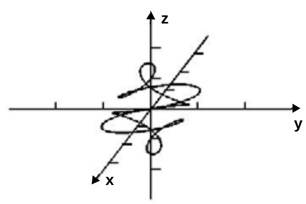
$k = 0.3$



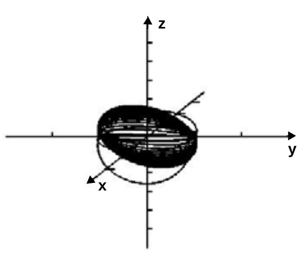
47. $k = 2$



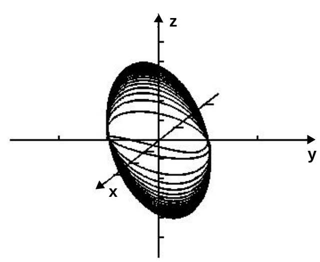
$k = 4$



48. $k = \frac{1}{10}$



$k = 1$



12.2 Calculus of Vector Functions

1. $\lim_{t\to 2}[t^3\mathbf{i} + t^4\mathbf{j} + t^5\mathbf{k}] = 2^3\mathbf{i} + 2^4\mathbf{j} + 2^5\mathbf{k} = 8\mathbf{i} + 16\mathbf{j} + 32\mathbf{k}$

2. $\mathbf{r}(t) = \dfrac{\sin 2t}{t}\mathbf{i} + (t-2)^5\mathbf{k} + \dfrac{\ln t}{1/t}\mathbf{k}$. Using L'Hôpital's Rule,

$$\lim_{t\to 0^+} \mathbf{r}(t) = \left[\frac{2\cos 2t}{1}\mathbf{i} + (t-2)^5\mathbf{j} + \frac{1/t}{-1/t^2}\mathbf{k}\right] = 2\mathbf{i} - 32\mathbf{j}$$

3. Using L'Hôpital's Rule, we have
$\lim_{t\to 1}\left\langle \dfrac{t^2-1}{t-1}, \dfrac{5t-1}{t+1}, \dfrac{2e^{t-1}-2}{t-1}\right\rangle = \lim_{t\to 1} = \left\langle \dfrac{2t}{1}, \dfrac{5t-1}{t+1}, \dfrac{2e^{t-1}}{1}\rangle = \langle 2, 2, 2\right\rangle$

4. Since $\lim_{t\to\infty} \tan^{-1} t = \dfrac{\pi}{2}$, we have
$$\begin{aligned}\lim_{t\to\infty}\left\langle \frac{e^{2t}}{2e^{2t}+t}, \frac{e^{-1}}{2e^{-t}+5}, \tan^{-1} t\right\rangle &= \lim_{t\to\infty}\left\langle \frac{1}{2+te^{-2t}}, \frac{1}{2+5e^{t}}, \tan^{-1} t\right\rangle \\ &= \left\langle \frac{1}{2}, 0, \frac{\pi}{2}\right\rangle\end{aligned}$$
The last equality follows from using L'Hôpital's Rule to get

$$\lim_{t\to\infty} te^{-2t} = \lim_{t\to\infty}\frac{t}{e^{2t}} = \lim_{t\to\infty}\frac{1}{2e^{2t}} = 0$$

5. $\lim_{t\to\alpha}[-4\mathbf{r}_1(t) + 3\mathbf{r}_2(t)] = -4(\mathbf{i} - 2\mathbf{j} + \mathbf{k}) + 3(2\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}) = 2\mathbf{i} + 23\mathbf{j} + 17\mathbf{k}$

6. $\lim_{t\to\alpha} \mathbf{r}_1(t)\cdot\mathbf{r}_2(t) = (\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = (\mathbf{i} - 2\mathbf{j} + \mathbf{k})\cdot(2\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}) = -1$

7. Notice that the $\mathbf{k}$ component $\ln(t-1)$ is not defined at $t = 1$. Therefore, $\mathbf{r}(t)$ is not continuous at $t = 1$.

8. Notice that $\sin \pi t$, $\tan \pi t$, and $\cos \pi t$ are each continuous at $t = 1$ since the sine, cosine, and tangent function are continuous on their domains. Therefore, since each of the component functions are continuous at $t = 1$, we know that $\mathbf{r}(t)$ is continuous at $t = 1$.

9. $\mathbf{r}'(t) = 3\mathbf{i} + 8t\mathbf{j} + (10t-1)\mathbf{k}$
so $\mathbf{r}'(1) = 3\mathbf{i} + 8\mathbf{j} + 9\mathbf{k} = \langle 3, 8, 9\rangle$
while
$$\begin{aligned}\frac{\mathbf{r}(1.1)-\mathbf{r}(1)}{0.1} &= \frac{\langle 3(1.1)-1, 4(1.1)^2, 5(1.1)^2-(1.1)\rangle - \langle 3(1)-1, 4(1)^2, 5(1)^2-(1)\rangle}{0.1} \\ &= \frac{\langle 2.3, 4.84, 4.95\rangle - \langle 2, 4, 4\rangle}{0.1} \\ &= \frac{\langle 0.3, 0.84, 0.95\rangle}{0.1} = \langle 3, 8.4, 9.5\rangle\end{aligned}$$

10. $\mathbf{r}'(t) = \dfrac{-5}{(1+5t)^2}\mathbf{i} + (6t+1)\mathbf{j} - 3(1-t)^2\mathbf{k}$
so $\mathbf{r}'(0) = \dfrac{-5}{1}\mathbf{i} + \mathbf{j} + 3\mathbf{k} = \langle -5, 1, -3\rangle$

$$\text{while } \frac{\mathbf{r}(0.05)-\mathbf{r}(0)}{0.05} = \frac{\left\langle \frac{1}{1+5(0.05)}, 3(0.05)^2+(0.05), (1+0.05)^3 \right\rangle - \left\langle \frac{1}{1+5(0)}, 3(0)^2+(0), (1-0)^3 \right\rangle}{0.05}$$

$$= \frac{\langle 0.8, 0.0575, 0.857375\rangle - \langle 1, 0, 1\rangle}{0.05}$$

$$= \frac{\langle -0.2, 0.0575, -0.142625\rangle}{0.05}$$

$$= \langle -4, 1.15, -2.8525\rangle$$

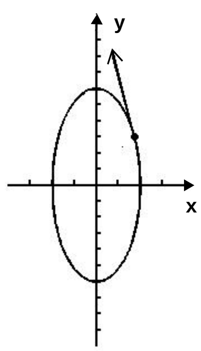
11. $\mathbf{r}'(t) = \frac{1}{t}\mathbf{i} - \frac{1}{t^2}\mathbf{j}$; $\quad \mathbf{r}''(t) = -\frac{1}{t^2}\mathbf{i} + \frac{2}{t^3}\mathbf{j}$

12. $\mathbf{r}'(t) = \langle -t\sin t, 1-\sin t\rangle$; $\quad \mathbf{r}''(t) = \langle -t\cos t - \sin t, -\cos t\rangle$

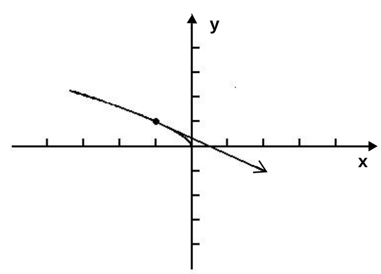
13. $\mathbf{r}'(t) = \langle 2te^{2t}+e^{2t}, 3t^2, 8t-1\rangle$; $\quad \mathbf{r}''(t) = \langle 4te^{2t}+4e^{2t}, 6t, 8\rangle$

14. $\mathbf{r}'(t) = 2t\mathbf{i} + 3t^2\mathbf{j} + \frac{1}{1+t^2}\mathbf{k}$; $\quad \mathbf{r}''(t) = 2\mathbf{i} + 6t\mathbf{j} - \frac{2t}{(1+t^2)^2}\mathbf{k}$

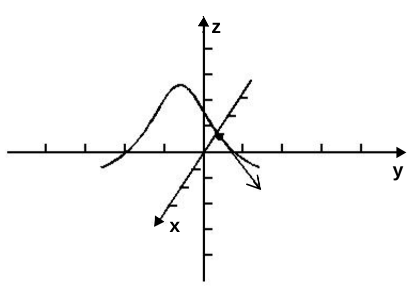
15. $\mathbf{r}'(t) = -2\sin t\mathbf{i} + 6\cos t\mathbf{j}$
$\mathbf{r}'(\pi/6) = -\mathbf{i} + 3\sqrt{3}\mathbf{j}$



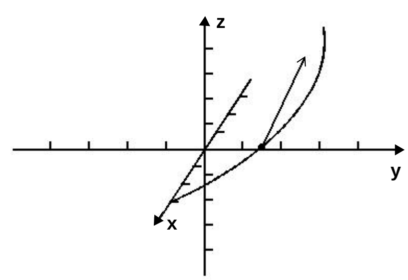
16. $\mathbf{r}'(t) = 3t^2\mathbf{i} + 2t\mathbf{j}$
$\mathbf{r}'(-1) = 3\mathbf{i} - 2\mathbf{j}$



17. $\mathbf{r}'(t) = \mathbf{j} - \frac{8t}{(1+t^2)^2}\mathbf{k}$
$\mathbf{r}'(-1) = \mathbf{j} - 2\mathbf{k}$



18. $\mathbf{r}'(t) = -3\sin t\mathbf{i} + 3\cos t\mathbf{j} + 2\mathbf{k}$
$\mathbf{r}'(\pi/4) = \frac{-3\sqrt{2}}{2}\mathbf{i} + \frac{3\sqrt{2}}{2}\mathbf{j} + 2\mathbf{k}$



19. $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{2}\mathbf{j} + \frac{1}{3}t^3\mathbf{k}$; $\mathbf{r}(2) = 2\mathbf{i} + 2\mathbf{j} + \frac{8}{3}\mathbf{k}$; $\mathbf{r}'(t) = \mathbf{i} + t\mathbf{j} + t^2\mathbf{k}$; $\mathbf{r}'(2) = \mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$
Using the point $(2, 2, 8/3)$ and the direction vector $\mathbf{r}'(2)$, we have $x = 2 + t$, $y = 2 + 2t$, $z = 8/3 + 4t$.

20. $\mathbf{r}(t) = (t^3 - t)\mathbf{i} + \frac{6t}{t+1}\mathbf{j} + (2t+1)^2\mathbf{k}$; $\mathbf{r}(1) = 3\mathbf{j} + 9\mathbf{k}$; $\mathbf{r}'(t) = (3t^2 - 1)\mathbf{i} + \frac{6}{(t+1)^2}\mathbf{j} + (8t+4)\mathbf{k}$;
$\mathbf{r}'(1) = 2\mathbf{i} + \frac{3}{2}\mathbf{j} + 12\mathbf{k}$. Using the point $(0, 3, 9)$ and the direction vector $\mathbf{r}'(1)$, we have $x = 2t$, $y = 3 + \frac{3}{2}$, $z = 9 + 12t$.

21. $\mathbf{r}'(t) = \langle e^t + te^t, 2t + 2, 3t^2 - 1\rangle$ so $\mathbf{r}'(0) = \langle 1, 2, -1\rangle$ and $|\mathbf{r}'(0)| = \sqrt{1^2 + 2^2 + (-1)^2} = \sqrt{6}$
The unit tangent vector at $t = 0$ is given by $\frac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} = \frac{\langle 1, 2, -1\rangle}{\sqrt{6}} = \left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}} \right\rangle$
To find the parametric equations of the tangent line at $t = 0$, we first compute $\mathbf{r}(0) = \langle 0, 0, 0\rangle$. The tangent line is then given in vector form as $\mathbf{p}(t) = \langle 0, 0, 0\rangle + t\left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}} \right\rangle = \left\langle \frac{1}{\sqrt{6}}t, \frac{2}{\sqrt{6}}t, \frac{-1}{\sqrt{6}}t \right\rangle$ or in parametric form as $x = \frac{1}{\sqrt{6}}t$, $y = \frac{2}{\sqrt{6}}t$, $z = \frac{-1}{\sqrt{6}}t$.

22. $\mathbf{r}'(t) = \langle 3\cos 3t, 2\sec^2 2t, 1\rangle$ so $\mathbf{r}'(\pi) = \langle -3, 2, 1\rangle$ and $|\mathbf{r}'(\pi)| = \sqrt{(-3)^2 + (2)^2 + (1)^2} = \sqrt{14}$.
The unit tangent vector at $t = \pi$ is given by $\frac{\mathbf{r}'(\pi)}{|\mathbf{r}'(\pi)|} = \frac{\langle -3, 2, 1\rangle}{\sqrt{14}} = \left\langle \frac{-3}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}} \right\rangle$
To find the parametric equations of the tangent line at $t = \pi$, we first compute $\mathbf{r}'(\pi) = \langle 1, 0, \pi\rangle$.
The tangent line is then given in vector form as
$$\begin{aligned} \mathbf{p}(t) &= \langle 1, 0, \pi\rangle + t\left\langle \frac{-3}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}} \right\rangle \\ &= 1 - \left\langle \frac{-3}{\sqrt{14}}t, \frac{2}{\sqrt{14}}t, \pi + \frac{1}{\sqrt{14}}t \right\rangle \end{aligned}$$
or in parametric form as $x = 1 - \frac{-3}{\sqrt{14}}t$, $y = \frac{2}{\sqrt{14}}t$, $z = \pi + \frac{1}{\sqrt{14}}t$

23. $\mathbf{r}(\pi/3) = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\pi}{3} \right\rangle$
$\mathbf{r}'(t) = \langle -\sin t, \cos t, 1\rangle$
$\mathbf{r}'(\pi/3) = \left\langle -\frac{\sqrt{3}}{2}, \frac{1}{2}, 1 \right\rangle$
so the tangent line is given by
$$\begin{aligned} \mathbf{p}(t) &= \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\pi}{3} \right\rangle + t\left\langle -\frac{\sqrt{3}}{2}, \frac{1}{2}, 1 \right\rangle \\ &= \left\langle \frac{1}{2} - \frac{\sqrt{3}}{2}t, \frac{\sqrt{3}}{2} + \frac{1}{2}t, \frac{\pi}{3} + t \right\rangle \end{aligned}$$

24. $\mathbf{r}(0) = \langle 6, 1, 1\rangle$
$\mathbf{r}'(t) = \langle -3e^{-t/2}, 2e^{2t}, 3e^{3t}\rangle$

$\mathbf{r}'(0) = \langle -3, 2, 3\rangle$ So the tangent line is given by
$\mathbf{r}(t) = \langle 6, 1, 1\rangle + t\langle -3, 2, 2\rangle$
$\quad = \langle 6 - 3t, 1 + 2t, 1 + 3t\rangle$

25. $\dfrac{d}{dt}[\mathbf{r}(t) \times \mathbf{r}'(t)] = \mathbf{r}(t) \times \mathbf{r}''(t) + \mathbf{r}'(t) \times \mathbf{r}'(t) = \mathbf{r}(t) \times \mathbf{r}''(t)$

26. $$\begin{aligned}\frac{d}{dt}[\mathbf{r}(t) \cdot (t\mathbf{r}(t))] &= \mathbf{r}(t) \cdot \frac{d}{dt}(t\mathbf{r}(t)) + = \mathbf{r}(t) \cdot (t\mathbf{r}'(t) + \mathbf{r}(t)) + \mathbf{r}'(t) \cdot (t\mathbf{r}(t))\\ &= \mathbf{r}(t) \cdot (t\mathbf{r}'(t)) + \mathbf{r}(t) \cdot \mathbf{r}(t) + \mathbf{r}'(t) \cdot (t\mathbf{r}(t)) = 2t(\mathbf{r}(t) \cdot \mathbf{r}'(t)) + \mathbf{r}(t) \cdot \mathbf{r}(t)\end{aligned}$$

27. $$\begin{aligned}\frac{d}{dt}[\mathbf{r}(t) \cdot (\mathbf{r}'(t) \times \mathbf{r}''(t))] &= \mathbf{r}(t) \cdot \frac{d}{dt}(\mathbf{r}'(t) \times \mathbf{r}''(t)) + \mathbf{r}'(t) \cdot (\mathbf{r}'(t) \times \mathbf{r}''(t))\\ &= \mathbf{r}(t) \cdot (\mathbf{r}'(t) \times \mathbf{r}'''(t) + \mathbf{r}''(t) \times \mathbf{r}''(t)) + \mathbf{r}'(t) \cdot (\mathbf{r}'(t) \times \mathbf{r}''(t))\\ &= \mathbf{r}(t) \cdot (\mathbf{r}'(t) \times \mathbf{r}'''(t))\end{aligned}$$

28. $$\begin{aligned}\frac{d}{dt}[\mathbf{r}_1(t) \times (\mathbf{r}_2(t) \times \mathbf{r}_3(t))] &= \mathbf{r}_1(t) \times \frac{d}{dt}(\mathbf{r}_2(t) \times \mathbf{r}_3(t)) + \mathbf{r}'(t) \times (\mathbf{r}_2(t) \times \mathbf{r}_3(t))\\ &= \mathbf{r}_1(t) \times (\mathbf{r}_2(t) \times \mathbf{r}_3'(t) + \mathbf{r}_2'(t) \times \mathbf{r}_3(t) + \mathbf{r}_1'(t) \times (\mathbf{r}_2(t) \times \mathbf{r}_3(t))\\ &= \mathbf{r}_1(t) \times (\mathbf{r}_2(t) \times \mathbf{r}_3'(t)) + \mathbf{r}_1(t) \times (\mathbf{r}_2'(t) \times \mathbf{r}_3(t)) + \mathbf{r}_1(t) \times (\mathbf{r}_2(t) \times \mathbf{r}_3(t))\end{aligned}$$

29. $\dfrac{d}{dt}[\mathbf{r}_1(2t) + \mathbf{r}_2(\frac{1}{t})] = 2\mathbf{r}'(2t) - \dfrac{1}{t^2}\mathbf{r}_2'(\frac{1}{t})$

30. $\dfrac{d}{dt}[t^3\mathbf{r}(t^2)] = t^3(2t)\mathbf{r}'(t^2) + 3t^2\mathbf{r}(t^2) = 2t^4\mathbf{r}'(t^2) + 3t^2\mathbf{r}(t^2)$

31. $\displaystyle\int_{-1}^{2} \mathbf{r}(t)\mathrm{d}t = \left[\int_{-1}^{2} t\mathrm{d}t\right]\mathbf{i} + \left[\int_{-1}^{2} 3t^2\mathrm{d}t\right]\mathbf{j} + \left[\int_{-1}^{2} 4t^3\mathrm{d}t\right]\mathbf{k} \quad = \frac{1}{2}t^2\Big|_{-1}^{2}\mathbf{i} + t^3\Big|_{-1}^{2}\mathbf{j} + t^4\Big|_{-1}^{2}\mathbf{k} = \frac{3}{2}\mathbf{i} + 9\mathbf{j} + 15\mathbf{k}$

32. $$\begin{aligned}\int_0^4 \mathbf{r}(t)\mathrm{d}t &= \left[\int_0^4 \sqrt{2t+1}\mathrm{d}t\right]\mathbf{i} + \left[\int_0^4 -\sqrt{t}\mathrm{d}t\right]\mathbf{j} + \left[\int_0^4 \sin\pi t\mathrm{d}t\right]\mathbf{k}\\ &= \frac{1}{3}(2t+1)^{3/2}\Big|_0^4\mathbf{i} - \frac{2}{3}t^{3/2}\Big|_0^4\mathbf{j} - \frac{1}{\pi}\cos\pi t\Big|_0^4\mathbf{k} = \frac{26}{3}\mathbf{i} - \frac{16}{3}\mathbf{j}\end{aligned}$$

33. $$\begin{aligned}\int \mathbf{r}(t)\mathrm{d}t &= \left[\int te^t\mathrm{d}t\right]\mathbf{i} + \left[\int -e^{-2t}\mathrm{d}t\right]\mathbf{j} + \left[\int te^{t^2}\mathrm{d}t\right]\mathbf{k}\\ &= [te^t - e^t + c_1]\mathbf{i} + \left[\frac{1}{2}e^{-2t} + c_2\right]\mathbf{j} + \left[\frac{1}{2}e^{t^2} + d_3\right]\mathbf{k} = e^t(t-1)\mathbf{i} + \frac{1}{2}e^{-2t}\mathbf{j} + \frac{1}{2}e^{t^2}\mathbf{k} + \mathbf{c},\end{aligned}$$
where $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$.

34. $$\begin{aligned}\int \mathbf{r}(t)\mathrm{d}t &= \left[\int \frac{1}{1+t^2}\mathrm{d}t\right]\mathbf{i} + \left[\int \frac{t}{1+t^2}\mathrm{d}t\right]\mathbf{j} + \left[\int \frac{t^2}{1+t^2}\mathrm{d}t\right]\mathbf{k}\\ &= [\tan^{-1} t + c_1]\mathbf{i} + \left[\frac{1}{2}\ln(1+t^2) + c_2\right]\mathbf{j} + \left[\int \left(1 - \frac{1}{1+t^2}\right)\right]\mathbf{k}\\ &= [\tan^{-1} t + c_1]\mathbf{i} + \left[\frac{1}{2}\ln(1+t^2) + c_2\right]\mathbf{j} + [t - \tan^{-1} t + c_3]\mathbf{k}\\ &= \tan^{-1} t\mathbf{i} + \frac{1}{2}\ln(1+t^2)\mathbf{j} + (t - \tan^{-1} t)\mathbf{k} + \mathbf{c},\end{aligned}$$

where $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$.

35. $\mathbf{r}(t) = \int \mathbf{r}'(t)\mathrm{d}t = \left[\int 6\mathrm{d}t\right]\mathbf{i} + \left[\int 6t\mathrm{d}t\right]\mathbf{j} + \left[\int 3t^2\mathrm{d}t\right]\mathbf{k} = [6t + c_1]\mathbf{i} + [3t^2 + c_2]\mathbf{j} + [t^3 + c_3]\mathbf{k}$
Since $\mathbf{r}(0) = \mathbf{i} + 2\mathbf{j} + \mathbf{k} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$, $\quad c_1 - 1$, $\;c_2 = -2$, and $c_3 = 1$. Thus,

$$\mathbf{r}(t) = (6t + 1)\mathbf{i} + (3t^2 - 2)\mathbf{j} + (t^3 + 1)\mathbf{k}$$

36. $\mathbf{r}(t) = \int \mathbf{r}'(t)\mathrm{d}t = \left[\int t\sin t^2\mathrm{d}t\right]\mathbf{i} + \left[\int -\cos 2t\mathrm{d}t\right]\mathbf{j} = -\left[\frac{1}{2}\cos t^2 + c_1\right]\mathbf{i} + \left[-\frac{1}{2}\sin 2t + c_2\right]\mathbf{j}$
Since $\mathbf{r}(0) = \frac{3}{2} = (-\frac{1}{2} + c_1)\mathbf{i} + c_2\mathbf{j}$, $\;c_1 = 2$, and $c_2 = 0$. Thus,

$$\mathbf{r}(t) = \left(-\frac{1}{2}\cos t^2 + 2\right)\mathbf{i} - \frac{1}{2}\sin 2t\mathbf{j}.$$

37. $\mathbf{r}'(t) = \int \mathbf{r}''(t)\mathrm{d}t = \left[\int 12t\mathrm{d}t\right]\mathbf{i} + \left[\int -3t^{-1/2}\mathrm{d}t\right]\mathbf{j} + \left[\int 2\mathrm{d}t\right]\mathbf{k} = [6t^2 + c_1]\mathbf{i} + [-6t^{1/2} + c_2]\mathbf{j} + [2t + c_3]\mathbf{k}$
Since $\mathbf{r}'(1) = \mathbf{j} = (6 + c_1)\mathbf{i} + (-6 + c_2)\mathbf{j} + (2 + c_3)\mathbf{k}$, $\;c_1 = -6$, $\;c_2 = 7$, and $c_3 = -2$. Thus,

$$\mathbf{r}'(t) = (6t^2 - 6)\mathbf{i} + (-6t^{1/2} + 7)\mathbf{j} + (2t - 2)\mathbf{k}.$$

$$\begin{aligned}\mathbf{r}(t) &= \int \mathbf{r}'(t)\mathrm{d}t = \left[\int (6t^2 - 6)\mathrm{d}t\right]\mathbf{i} + \left[\int (-6t^{1/2} + 7)\mathrm{d}t\right]\mathbf{j} + \left[\int (2t - 2)\mathrm{d}t\right]\mathbf{k} \\ &= [2t^3 - 6t + c_4]\mathbf{i} + [-4t^{3/2} + 7t + c_5]\mathbf{j} + [t^2 - 2t + c_6]\mathbf{k}.\end{aligned}$$

Since

$$\mathbf{r}(1) = 2\mathbf{i} - \mathbf{k} = (-4 + c_4)\mathbf{i} + (3 + c_5)\mathbf{j} + (-1 + c_6)\mathbf{k},$$

$c_4 = 6$, $\;c_5 = -3$, and $c_6 = 0$. Thus,

$$\mathbf{r}(t) = (2t^3 - 6t + 6)\mathbf{i} + (-4t^{3/2} + 7t - 3)\mathbf{j} + (t^2 - 2t)\mathbf{k}.$$

38. $$\begin{aligned}\mathbf{r}'(t) &= \int \mathbf{r}''(t)\mathrm{d}t = \left[\int \sec^2 t\mathrm{d}t\right]\mathbf{i} + \left[\int \cos t\mathrm{d}t\right]\mathbf{j} + \left[\int -\sin t\mathrm{d}t\right]\mathbf{k} \\ &= [\tan t + c_1]\mathbf{i} + [\sin t + c_2]\mathbf{j} + [\cos t + c_3]\mathbf{k}\end{aligned}$$

Since $\mathbf{r}'(0) = \mathbf{i} + \mathbf{j} + \mathbf{k} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$, $\;c_1 = 1$, $\;c_2 = 1$, and $c_3 = 0$. Thus,

$$\mathbf{r}'(t) = (\tan t + 1)\mathbf{i} + (\sin t + 1)\mathbf{j} + \cos t\mathbf{k}.$$

$$\begin{aligned}\mathbf{r}(t) &= \int \mathbf{r}'(t)\mathrm{d}t = \left[\int (\tan t + 1)\mathrm{d}t\right]\mathbf{i} + \left[\int (\sin t + 1)\mathrm{d}t\right]\mathbf{j} + \left[\int \cos t\mathrm{d}t\right]\mathbf{k}. \\ &= [\ln|\sec t| + c_4]\mathbf{i} + [-\cos t + t + c_5]\mathbf{j} + [\sin t + c_6]\mathbf{k}\end{aligned}$$

Since $\mathbf{r}(0) = -\mathbf{j} + 5\mathbf{k} = (-1 + c_5)\mathbf{j} + (c_6)\mathbf{k}$, $c_4 = 0$, $\;c_5 = 0$, and $c_6 = 5$. Thus,

$$\mathbf{r}(t) = (\ln|\sec t| + t)\mathbf{i} + (-\cos t + t)\mathbf{j} + (\sin t + 5)\mathbf{k}.$$

39. $\mathbf{r}'(t) = -a\sin t\mathbf{i} + a\cos t\mathbf{j} + c\mathbf{k}$; $\quad |\mathbf{r}'(t)| = \sqrt{(-a\sin t)^2 + (a\cos t)^2 + c^2} = \sqrt{a^2 + c^2}$
$s = \int_0^{2\pi} \sqrt{a^2 + c^2}\mathrm{d}t = \sqrt{a^2 + c^2}t\Big|_0^{2\pi} = 2\pi\sqrt{a^2 + c^2}$

40. $\mathbf{r}'(t) = \mathbf{i} + (\cos t - t\sin t)\mathbf{j} + (\sin t + t\cos t)\mathbf{k}$
$|\mathbf{r}'(t)| = \sqrt{1^2 + (\cos t - t\sin t)^2 + (\sin t + t\cos t)^2} = \sqrt{2+t^2}$
$s = \int_0^{\pi} \sqrt{2+t^2}\mathrm{d}t = \left(\frac{t}{2}\sqrt{2+t^2} + \ln|t + \sqrt{2+t^2}|\right)\Big|_0^{\pi} = \frac{\pi}{2}\sqrt{2+\pi^2} + \ln(\pi + \sqrt{2+\pi^2}) - \ln\sqrt{2}$

41. $\mathbf{r}'(t) = (-2e^t\sin 2t + e^t\cos 2t)\mathbf{i} + (2e^t\cos 2t + e^t\sin 2t)\mathbf{j} + e^t\mathbf{k}$
$|\mathbf{r}'(t)| = \sqrt{5e^{2t}\cos^2 2t + 5e^{2t}\sin^2 2t + e^{2t}} = \sqrt{6e^{2t}} = \sqrt{6}e^t$
$s = \int_0^{3\pi} \sqrt{6}e^t\mathrm{d}t = \sqrt{6}e^t\Big|_0^{3\pi} = \sqrt{6}(e^{3\pi} - 1)$

42. $\mathbf{r}'(t) = 3\mathbf{i} + 2\sqrt{3}t\mathbf{j} + 2t^2\mathbf{k}; \quad |\mathbf{r}'(t)| = \sqrt{3^2 + (2\sqrt{3}t)^2 + (2t^2)^2} = \sqrt{9 + 12t^2 + 4t^4} = 3 + 2t^2$
$s = \int_0^1 (3+2t^2)\mathrm{d}t = \left(3t + \frac{2}{3}t^3\right)\Big|_0^1 = 3 + \frac{2}{3} = \frac{11}{3}$

43. From $\mathbf{r}'(t) = \langle 9\cos t, -9\sin t\rangle$, we find $|\mathbf{r}'(t)| = 9$. Therefore, $s = \int_0^t 9\mathrm{d}u = 9t$ so that $t = \dfrac{s}{9}$. By substituting for t in $\mathbf{r}(t)$, we obtain $\mathbf{r}(s) = \left\langle 9\sin\dfrac{s}{9}, 9\cos\dfrac{s}{9}\right\rangle$. Note that $\mathbf{r}'(s) = \left\langle \sin\dfrac{s}{9}, \cos\dfrac{s}{9}\right\rangle$ so that $\left|\mathbf{r}'(s) = \sqrt{\sin^2\dfrac{s}{9} + \cos\dfrac{s}{9}}\right| = 1$.

44. From $\mathbf{r}(t) = \langle -5\sin t, 12, 5\cos t\rangle$, we find $|\mathbf{r}'(t)| = \sqrt{169} = 13$. Therefore, $s = \int_0^t 13\mathrm{d}u = 13t$ so that $t = \dfrac{s}{13}$. By substituting for t in $\mathbf{r}(t)$, we obtain $\mathbf{r}(s) = \langle 5\cos\frac{s}{13}, \frac{12}{13}s, \frac{5}{13}\cos\frac{s}{13}\rangle$.
Note that $\mathbf{r}'(t) = \left\langle -\frac{5}{13}\sin\frac{s}{13}, \frac{12}{13}, \frac{5}{13}\cos\frac{s}{13}\right\rangle$ so that
$|\mathbf{r}'(s)| = \sqrt{\dfrac{25}{169}\sin\dfrac{25}{13} + \frac{144}{169} + \dfrac{25}{169}\cos^2\frac{s}{13}} = 1$

45. From $\mathbf{r}'(t) = \langle 2, -3, 4\rangle$, we find $|\mathbf{r}'(t)| = \sqrt{29}$. Therefore, $s = \int_0^t \sqrt{29}\mathrm{d}u = \sqrt{20}t$ so that $t = \dfrac{s}{\sqrt{29}}$. By substituting for t in $\mathbf{r}(t)$, we obtain $\mathbf{r}(s) = \left\langle 1 + \dfrac{2}{\sqrt{29}}s, 5 - \dfrac{3}{\sqrt{29}}s, 2 + \dfrac{4}{\sqrt{29}}s\right\rangle$.
Note that $\mathbf{r}'(s) = \left\langle \frac{2}{\sqrt{29}}, -\frac{3}{\sqrt{29}}, \frac{4}{\sqrt{29}}\right\rangle$ so that $\mathbf{r}'(s) = \sqrt{\dfrac{4}{29} + \dfrac{9}{29} + \dfrac{16}{29}} = 1$.

46. From $\mathbf{r}'(t) = \langle e^t\cos t - e^t\sin t, e^t\sin t + e^t\cos t, 0\rangle$ we find
$|\mathbf{r}'(t)| = \sqrt{e^{2t}\cos^t - 2e^{2t}\cos t\sin t + e^{2t}\sin^2 t + e^{2t}\sin^2 t + 2e^{2t}\sin t\cos t + e^{2t}\cos^2 t} = \sqrt{2e^{2t}} = e^t\sqrt{2}$.
Therefore, $s = \int_0^t e^u\sqrt{2}\mathrm{d}u = \sqrt{2}(e^t - 1)$ so that $t = \ln\left(\frac{s}{\sqrt{2}} + 1\right)$. By substituting for t in $\mathbf{r}(t)$, we obtain
$\mathbf{r}(s) = \left\langle \left(\frac{s}{\sqrt{2}} + 1\cos(\ln\left(\frac{s}{\sqrt{2}} + 1\right)\right), \left(\frac{s}{\sqrt{2}} + 1\right)\sin\left(\ln\left(\frac{s}{\sqrt{2}} + 1\right)\right), 1\right\rangle$ Note that
$\mathbf{r}'(s) = \left\langle \frac{1}{\sqrt{2}}\cos\left(\ln\left(\frac{s}{\sqrt{2}}+1\right)\right) - \frac{1}{\sqrt{2}}\sin\left(\ln\left(\frac{s}{\sqrt{2}}+1\right)\right), \frac{1}{\sqrt{2}}\sin\left(\ln\left(\frac{s}{\sqrt{2}}+1\right)\right) + \frac{1}{\sqrt{2}}\cos\left(\ln\left(\frac{s}{\sqrt{2}}+1\right)\right), 0\right\rangle$
so that

$$|\mathbf{r}'(s)| = \sqrt{\begin{array}{l} \frac{1}{2}\cos^2\left(\ln\left(\frac{s}{\sqrt{2}}+1\right)\right) - \cos\left(\ln\left(\frac{s}{\sqrt{2}}+1\right)\right)\sin\left(\ln\left(\frac{s}{\sqrt{2}}+1\right)\right) + \frac{1}{2}\sin^2\left(\ln\left(\frac{s}{\sqrt{2}}+1\right)\right) \\ +\frac{1}{2}\sin^2\left(\ln\left(\frac{s}{\sqrt{2}}+1\right)\right) + \sin\left(\ln\left(\frac{s}{\sqrt{2}}+1\right)\right)\cos\left(\ln\left(\frac{s}{\sqrt{2}}+1\right)\right) + \frac{1}{2}\cos^2\left(\ln\left(\frac{s}{\sqrt{2}}+1\right)\right) \end{array}}$$

$$= \sqrt{\cos^2\left(\ln\left(\frac{s}{\sqrt{2}}+1\right)\right) + \sin^2\left(\ln\left(\frac{s}{\sqrt{2}}+1\right)\right)} = 1$$

47. Since $\dfrac{d}{dt}(\mathbf{r}\cdot\mathbf{r}) = \dfrac{d}{dt}|\mathbf{r}|^2 = \dfrac{d}{dt}c^2 = 0$ and $\frac{d}{dt}(\mathbf{r}\cdot\mathbf{r}) = \mathbf{r}\cdot\mathbf{r}' + \mathbf{r}'\cdot\mathbf{r} = 2\mathbf{r}\cdot\mathbf{r}'$, we have $\mathbf{r}\cdot\mathbf{r}' = 0$. Thus, $\mathbf{r}'$ is perpendicular to $\mathbf{r}$.

48. Let $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ and $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$. Then

$$\int_a^b \mathbf{v}\cdot\mathbf{r}(t)\mathrm{d}t = \int_a^b [ax(t) + by(t)]\mathrm{d}t = a\int_a^b x(t)\mathrm{d}t + b\int_a^b y(t)\mathrm{d}t = \mathbf{v}\cdot\int_a^b \mathbf{r}(t)\mathrm{d}t.$$

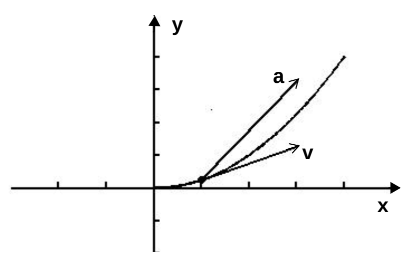
49. From $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$, we get $\mathbf{r}'(t) = \mathbf{v}$ so that $|\mathbf{r}'(t)| = |\mathbf{v}|$. Therfore $s = \int_0^t |\mathbf{r}'(t)|\mathrm{d}u = \int_0^t |\mathbf{v}|\mathrm{d}u = |\mathbf{v}|t$ which gives $t = \dfrac{s}{|\mathbf{v}|}$. Substituting for t in $\mathbf{r}(t)$, we have $\mathbf{r}'(s) = \mathbf{r}_0 + \dfrac{s}{|\mathbf{v}|\mathbf{v} = \mathbf{r}_0 + s\dfrac{\mathbf{v}}{|\mathbf{v}|}}$. Note that $\mathbf{r}'(s) = \dfrac{\mathbf{v}}{|\mathbf{v}|}$ so that $|\mathbf{r}'(s)| = \dfrac{|\mathbf{v}|}{|\mathbf{v}|} = 1$.

50. (a) $|\langle 3,-4\rangle| = \sqrt{3^2 + (-4)^2} = 5$ so $\mathbf{r}(s) = \langle 1,2\rangle + \dfrac{s}{5}\langle 3,-4\rangle = \langle 1,2\rangle + s\left\langle \dfrac{3}{5}, \dfrac{-4}{5}\right\rangle$

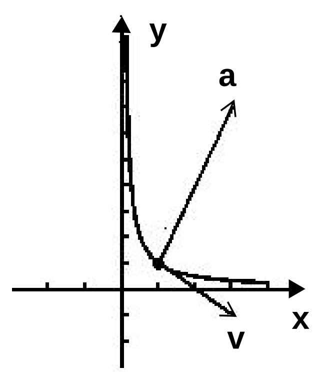
(b) $\mathbf{r}(t) = \langle 1,1,10\rangle + t\langle 1,2,-1\rangle$ and $|\langle 1,2,-1\rangle| = \sqrt{1+4+1} = \sqrt{6}$ so $\mathbf{r}(s) = \langle 1,1,10\rangle + s\left\langle \dfrac{1}{\sqrt{6}}, \dfrac{2}{\sqrt{6}}, \dfrac{-1}{\sqrt{6}}\right\rangle$

12.3 Motion on a Curve

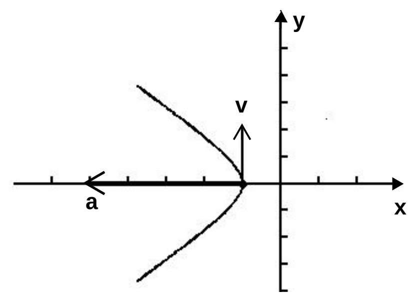
1. $\mathbf{v}(t) = 2t\mathbf{i} + t^3\mathbf{j}$; $\mathbf{v}(1) = 2\mathbf{i} + \mathbf{j}$; $|\mathbf{v}(1)| = \sqrt{4+1} = \sqrt{5}$; $\mathbf{a}(t) = 2\mathbf{i} + 3t^2\mathbf{j}$; $\mathbf{a}(1) = 2\mathbf{i} + 3\mathbf{j}$



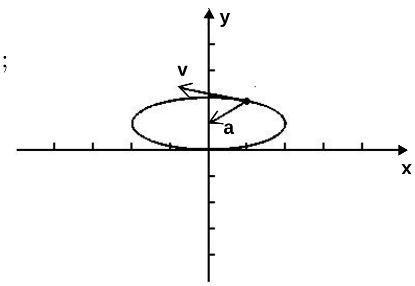
2. $\mathbf{v}(t) = 2t\mathbf{i} - \dfrac{2}{t^3}\mathbf{j}$; $\mathbf{v}(1) = 2\mathbf{i} - 2\mathbf{j}$; $|\mathbf{v}(1)| = \sqrt{4+4} = 2\sqrt{2}$; $\mathbf{a}(t) = 2\mathbf{i} + \dfrac{6}{t^4}\mathbf{j}$; $\mathbf{a}(1) = 2\mathbf{i} + 6\mathbf{j}$



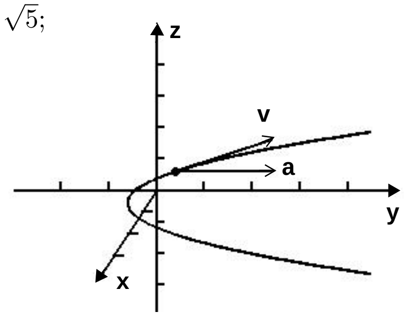
3. $\mathbf{v}(t) = -2\sinh 2t\mathbf{i} + 2\cosh 2t\mathbf{j}$; $\mathbf{v}(1) = 2\mathbf{j}$; $|\mathbf{v}(0)| = 2$;
$\mathbf{a}(t) = -4\cosh 2t\mathbf{i} + +4\sinh 2t\mathbf{j}$; $\mathbf{a}(0) = -4\mathbf{i}$



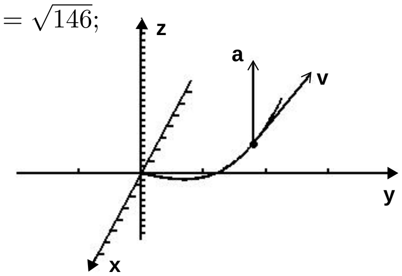
4. $\mathbf{v}(t) = -2\sin t\mathbf{i} + \cos t\mathbf{j}$; $\mathbf{v}(\pi/3) = -\sqrt{3}\mathbf{i} + \dfrac{1}{2}\mathbf{j}$;
$|\mathbf{v}(\pi/3)| = \sqrt{3 + 1/4} = \sqrt{13}/2$; $\mathbf{a}(t) = -2\cos t\mathbf{i} - \sin t\mathbf{j}$;
$\mathbf{a}(\pi/3) = -\mathbf{i} - \dfrac{\sqrt{3}}{2}\mathbf{j}$



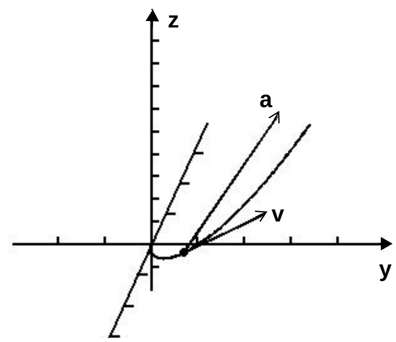
5. $\mathbf{v}(t) = (2t-2)\mathbf{i} + \mathbf{k}$; $\mathbf{v}(2) = 2\mathbf{j} + \mathbf{k}$; $|\mathbf{v}(2)| = \sqrt{4+1} = \sqrt{5}$;
$\mathbf{a}(t) = 2\mathbf{j}$; $\mathbf{a}(2) = 2\mathbf{j}$



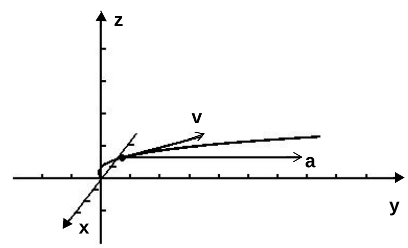
6. $\mathbf{v}(t) = \mathbf{i} + \mathbf{j}$; $\mathbf{v}(2) = \mathbf{i} + \mathbf{j} + 12\mathbf{k}$; $|\mathbf{v}(2)| = \sqrt{1+1+144} = \sqrt{146}$;
$\mathbf{a}(t) = 6t\mathbf{k}$; $\mathbf{a}(2) = 12\mathbf{k}$



7. $\mathbf{v}(t) = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}$;
$mathbfv(1) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$; $|\mathbf{v}(1)| = \sqrt{1+1+9} = \sqrt{14}$;
$\mathbf{a}(t) = 2\mathbf{j} + 6t\mathbf{k}$; $\mathbf{a}(1) = 2\mathbf{j} + 6\mathbf{k}$



8. $\mathbf{v}(t) = \mathbf{i} + 3t^2\mathbf{j} + \mathbf{k}$;
$\mathbf{v}(1) = \mathbf{i} + 3\mathbf{j} + \mathbf{k}$; $|\mathbf{v}(1)| = \sqrt{1+9+1} = \sqrt{11}$;
$\mathbf{a}(t) = 6t\mathbf{j}$; $\mathbf{a}(1) = 6\mathbf{j}$



9. The particle passes through the xy-plane when $z(t) = t^2 - 5t = 0$ or $t = 0,\ 5$ which gives us the points $(0,0,0)$ and $(25,115,0)$. $\mathbf{v}(t) = 2t\mathbf{i} + (3t^2-2)\mathbf{j} + (2t-5)\mathbf{k}$; $\mathbf{v}(0) = -2\mathbf{j} - 5\mathbf{k}$, $\mathbf{v}(5) = 10\mathbf{i} + 73\mathbf{j} + 5\mathbf{k}$; $\mathbf{a}(t) = 2\mathbf{i} + 6t\mathbf{j} + 2\mathbf{k}$; $\mathbf{a}(0) = 2\mathbf{i} + 30\mathbf{j} + 2\mathbf{k}$

10. If $\mathbf{a}(t) = \mathbf{0}$, then $\mathbf{v}(t) = \mathbf{c}_1$ and $\mathbf{r}(t) = \mathbf{c}_1 t + \mathbf{c}_2$. The graph of this equation is a straight line.

11. Initially we are given $s_0 = \mathbf{0}$ and $\mathbf{v}_0 = (480\cos 30°)\mathbf{i} + (480\cos 30°)\mathbf{j} = 240\sqrt{3}\mathbf{i} + 240\mathbf{j}$. Using $\mathbf{a}(t) = -32\mathbf{j}$ we find

$$\mathbf{v}(t) = \int \mathbf{a}(t)\mathrm{d}t = -32t\mathbf{j} + \mathbf{c}$$

$$240\sqrt{3}\mathbf{i} + 240\mathbf{j} = \mathbf{v}(0) = \mathbf{c}$$

$$\mathbf{v}(t) = -32t\mathbf{j} + 240\sqrt{3}\mathbf{i} + 240\mathbf{j} = 240\sqrt{3}\mathbf{i} + (240 - 32t)\mathbf{j}$$

$$\mathbf{r}(t) = \int \mathbf{v}(t)\mathrm{d}t = 240\sqrt{3}t\mathbf{i} + (240t - 16t^2)\mathbf{j} + \mathbf{b}$$

$$\mathbf{0} = \mathbf{r}(0) = \mathbf{b}.$$

(a) The shell's trajectory is given by $\mathbf{r}(t) = 240\sqrt{3}t\mathbf{i} + (240t - 16t^2)\mathbf{j}$ or $x = 240\sqrt{3}t$, $y = 240 - 16t^2$.

(b) Solving $dy/dt = 240 - 32t = 0$, we see that y is maximum when $t = 15/2$. The maximum altitude is $y(15/2) = 900$ ft.

(c) Solving $y(t) = 240t - 16t^2 = 16t(15 - t) = 0$, we see that the shell is at ground level when $t = 0$ and $t = 15$. The range of the shell is $s(15) = 3600\sqrt{3} \approx 6235$ ft.

(d) From (c), impact is when $t = 15$. The speed at impact is

$$|\mathbf{v}(15)| = |240\sqrt{3}\mathbf{i} + (240 - 32 \cdot 15)\mathbf{j}| = \sqrt{240^2 \cdot 3 + (-240)^2} = 480 \text{ ft/s}.$$

12. Initially we are given $s_0 = 1600\mathbf{j}$ and $v_0 = (480\cos 30°)\mathbf{i} + (480\sin 30°)\mathbf{j} = 240\sqrt{3}\mathbf{i} + 240\mathbf{j}$. Using $\mathbf{a}(t) = -32\mathbf{j}$ we find

$$\mathbf{v}(t) = \int \mathbf{a}(t)\mathrm{d}t = -32t\mathbf{j} + \mathbf{c}$$

$$240\sqrt{3}\mathbf{i} + 240\mathbf{j} = \mathbf{v}(0) = \mathbf{c}$$

$$\mathbf{v}(t) = -32t\mathbf{j} + 240\sqrt{3}\mathbf{i} + 240\mathbf{j} = 240\sqrt{3}\mathbf{i} + (240 - 32t)\mathbf{j}$$

$$\mathbf{r}(t) = \int \mathbf{v}(t)\mathrm{d}t = 240\sqrt{3}t\mathbf{i} + (240t - 16t^2)\mathbf{j} + \mathbf{b}$$

$$1600\mathbf{j} = \mathbf{r}(0) = \mathbf{b}.$$

(a) The shell's trajectory is given by $\mathbf{r}(t) = 240\sqrt{3}t\mathbf{i} + (240t - 16t^2 + 1600)\mathbf{j}$ or $s = 240\sqrt{3}t,\ \ y = 240t - 16t^2 + 1600$.

(b) Solving $dy/dt = 240 - 32t = 0$, we see that y is maximum when $t = 15/2$. The maximum altitude is $y(15/2) = 2400$ ft.

(c) Solving $y(t) = -16t^2 + 240t + 1600 = -16(t - 20)(t + 5) = 0$, we see that the shell hits the ground when $t = 20$. The range of the shell is $x(20) = 4800\sqrt{3} \approx 8314$ ft.

(d) From (c), impact is when $t = 20$. The speed at impact is

$$|\mathbf{v}(20)| = |240\sqrt{3}\mathbf{i} + (240 - 32 \cdot 20)\mathbf{j}| = \sqrt{240^2 \cdot 3 + (-400)^2} = 160\sqrt{13} \approx 577 \text{ ft/s}.$$

13. We are given $s_0 = 81\mathbf{j}$ and $\mathbf{v}_0 = 4\mathbf{i}$. Using $\mathbf{a}(t) = -32\mathbf{j}$, we have

$$\mathbf{v}(t) = \int \mathbf{a}(t)\mathrm{d}t = -32t\mathbf{j} + \mathbf{c}$$

$$4\mathbf{i} = \mathbf{v}(0) = \mathbf{c}$$

$$\mathbf{v}(t) = 4\mathbf{i} - 32t\mathbf{j}$$

$$\mathbf{r}(t) = \int \mathbf{v}(t)\mathrm{d}t = 4t\mathbf{i} - 16t^2\mathbf{j} + \mathbf{b}$$

$$81\mathbf{j} = \mathbf{r}(0) = \mathbf{b}$$

$$\mathbf{r}(t) = 4t\mathbf{i} + (81 - 16t^2)\mathbf{j}.$$

Solving $y(t) = 81 - 16t^2 = 0$, we see that the car hits the water when $t = 9/4$. Then

$$|v(9/4)| = |4\mathbf{i} - 32(9/4)\mathbf{j}| = \sqrt{4^2 + 72^2} = 20\sqrt{13} \approx 72.11\text{ft/s}.$$

14. Let θ be the angle of elevation. Then $\mathbf{v}(0) = 98\cos\theta\mathbf{i} + 98\sin\theta\mathbf{j}$. Using $\mathbf{a}(t) = -9.8\mathbf{j}$, we have

$$\mathbf{v}(t) = \int \mathbf{a}(t)\mathrm{d}t = -9.8t\mathbf{j} + \mathbf{c}$$

$$98\cos\theta\mathbf{i} + 98\sin\theta\mathbf{j} = \mathbf{v}(0) = \mathbf{c}$$

$$\mathbf{v}(t) = 98\cos\theta\mathbf{i} + (98\sin\theta - 9.8t)\mathbf{j}$$

$$\mathbf{r}(t) = 98t\cos\theta\mathbf{i} + (98t\sin\theta - 4.9t^2)\mathbf{j} + \mathbf{b}.$$

Since $\mathbf{r}(0) = \mathbf{0}$, $\mathbf{b} = \mathbf{0}$ and $\mathbf{r}(t) = 98t\cos\theta\mathbf{i} + (98t\sin\theta - 4.9t^2)\mathbf{j}$. Setting $y(t) = 98t\sin\theta - 4.9t^2 = t(98\sin\theta - 4.9t) = 0$, we see that the projectile hits the ground when $t = 20\sin\theta$. Thus, using $x(t) = 98t\cos\theta$, $490 = s(t) = 98(20\sin\theta)\cos\theta$ or $\sin 2\theta = 0.5$. Then $2\theta = 30°$ or $150°$. The angles of elevation are $15°$ and $75°$.

15. Let s be the initial speed. Then $\mathbf{v}(0) = s\cos 45°\mathbf{i} + s\sin 45°\mathbf{j} = \frac{s\sqrt{2}}{2}\mathbf{i} + \frac{s\sqrt{2}}{2}\mathbf{j}$. Using $\mathbf{a}(t) = -32\mathbf{j}$, we have

$$\mathbf{v}(t) = \int \mathbf{a}(t)\mathrm{d}t = -32\mathbf{j} + \mathbf{c}$$

$$\frac{s\sqrt{2}}{2}\mathbf{i} + \frac{s\sqrt{2}}{2}\mathbf{j} = \mathbf{v}(0) = \mathbf{c}$$

$$\mathbf{v}(t) = \frac{s\sqrt{2}}{2}\mathbf{i} + \left(\frac{s\sqrt{2}}{2} - 32t\right)\mathbf{j}$$

$$\mathbf{r}(t) = \frac{s\sqrt{2}}{2}t\mathbf{i} + \left(\frac{s\sqrt{2}}{2}t - 16t^2\right)\mathbf{j} + \mathbf{b}.$$

Since $\mathbf{r}(0) = \mathbf{0}$, $\mathbf{b} = \mathbf{0}$ and

$$\mathbf{r}(t) = \frac{s\sqrt{2}}{2}t\mathbf{i} + \left(\frac{s\sqrt{2}}{2}t - 16t^2\right)\mathbf{j}.$$

Setting $y(t) = s\sqrt{2}t/2 - 16t^2 = t(2\sqrt{2}/2 - 16t) = 0$ we see that the ball hits the ground when $t = \sqrt{2}s/32$. Thus, using $x(t) = s\sqrt{2}t/2$ and the fact that 100 yd = 300 ft, $300 = x(t) = \frac{s\sqrt{2}}{2}(\sqrt{2}s/32) = \frac{s^2}{32}$ and $s = \sqrt{9600} \approx 97.98$ ft/s.

16. Let s be the initial speed and θ the initial angle. Then $\mathbf{v}() = s\cos\theta\mathbf{i} + s\sin\theta\mathbf{j}$. Using $\mathbf{a}(t) = -32\mathbf{j}$, we have

$$\mathbf{v}(t) = \int \mathbf{a}(t)\mathrm{d}t = -32t\mathbf{j} + \mathbf{c}$$

$$s\cos\theta\mathbf{i} + s\sin\theta\mathbf{j} = \mathbf{v}(0) = \mathbf{c}$$

$$\mathbf{v}(t) = s\cos\theta\mathbf{i} + (s\sin\theta - 32t)\mathbf{j}$$

$$\mathbf{r}(t) = st\cos\theta\mathbf{i} + (st\sin\theta - 16t^2)\mathbf{j} + \mathbf{b}.$$

Since $\mathbf{r}(0) = \mathbf{0}$, $\mathbf{b} = \mathbf{0}$ and $\mathbf{r}(t) = st\cos\theta\mathbf{i} + (st\sin\theta - 16t^2)\mathbf{j}$. Setting $y(t) = st\sin\theta - 16t^2 = t(s\sin\theta - 16t) =$, we see that the ball hits the ground when $t = (s\sin\theta)/16$. Using $x(t) = st\cos\theta\mathbf{i}$, we see that the range of the ball is

$$x\left(\frac{s\sin\theta}{16}\right) = \frac{s^2\sin\theta\cos\theta}{16} = \frac{s^2\sin 2\theta}{32}.$$

For $\theta = 30°$, the range is $s^2\sin 60°/32 = \sqrt{3}s^2/64$ and for $\theta = 60°$ the range is $s^2\sin 120°/32 = \sqrt{3}s^2/64$. In general, when the angle is $90° - \theta$ then range is

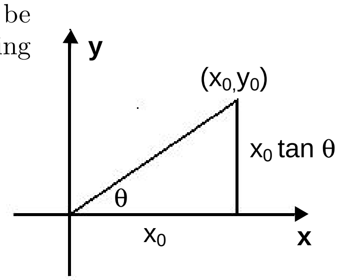
$$[s^2\sin 2(90° - \theta)]/32 = s^2[\sin(180° - 2\theta)]/32 = s^2(\sin 2\theta)/32.$$

Thus, for angles θ and $90° - \theta$, the range is the same.

17. $\mathbf{r}'(t) = \mathbf{v}(t) = -r_0\omega\sin\omega t\mathbf{i} + r_0\omega\cos\omega t\mathbf{j}$; $\quad v = |\mathbf{v}(t)| = \sqrt{r_0^2\omega^2\sin^2\omega t + r_0^2\omega^2\cos^2\omega t} = r_0\omega$
$\omega = v/r_0$; $\quad \mathbf{a}(t) = \mathbf{r}''(t) = -r_0\omega^2\cos\omega t\mathbf{i} - r_0\omega^2\sin\omega t\mathbf{j}$
$a = |\mathbf{a}(t)| = \sqrt{r_0^2\omega^4\cos^2\omega t + r_0^2\omega^4\sin^2\omega t} = r_0\omega^2 = r_0(v/r_0)^2 = v^2/r_0$.

18. (a) $\mathbf{v}(t) = -b\sin t\mathbf{i} + b\cos t\mathbf{j} + c\mathbf{k}$; $\quad |\mathbf{v}(t)| = \sqrt{b^2\sin^2 t + b^2\cos^2 t + c^2} = \sqrt{b^2 + c^2}$
(b) $s = \int_0^t |\mathbf{v}(t)|du = \int_0^t \sqrt{b^2 + c^2}du = t\sqrt{b^2 + c^2}$; $\quad \dfrac{ds}{dt} = \sqrt{b^2 + c^2}$
(c) $\dfrac{d^2s}{dt^2} = 0$; $\quad \mathbf{a}(t) = -b\cos t\mathbf{i} - b\sin t\mathbf{j}$; $\quad |\mathbf{a}(t)| = \sqrt{b^2\cos^2 t + b^2\sin^2 t} = |b|$. Thus, $d^2s/dt^2 \neq |\mathbf{a}(t)|$.

19. Let the initial speed of the projectile be s and let the target be at (x_0, y_0). Then $\mathbf{v}_p(0) = s\cos\theta\mathbf{i} + s\sin\theta\mathbf{j}$ and $\mathbf{v}_t(0) = \mathbf{0}$. Using $\mathbf{a}(t) = -32\mathbf{j}$, we have



$\mathbf{v}_p(t) = \int \mathbf{a}\, dt = -32t\mathbf{j} + \mathbf{c}$
$s\cos\theta\mathbf{i} + s\sin\theta\mathbf{j} = \mathbf{v}_p(0) = \mathbf{c}$
$\mathbf{v}_p(t) = s\cos\theta\mathbf{i} + (s\sin\theta - 32t)\mathbf{j}$
$\mathbf{r}_p(t) = st\cos\theta\mathbf{i} + (st\sin\theta - 16t^2)\mathbf{j} + \mathbf{b}$.

Since $\mathbf{r}_p(0) = \mathbf{0}$, $\mathbf{b} = \mathbf{0}$ and $\mathbf{r}_p(t) = st\cos\theta\mathbf{i} + (st\sin\theta - 16t^2)\mathbf{j}$. Also, $\mathbf{v}_t(t) = -32t\mathbf{j} + \mathbf{c}$ and since $\mathbf{v}_t(0) = \mathbf{0}$, $\mathbf{c} = \mathbf{0}$ and $\mathbf{v}_t(t) = -32t\mathbf{j}$. Then $\mathbf{r}_t(t) = -16t^2t\mathbf{j} + \mathbf{b}$. Since $\mathbf{r}_t(0) = x_0\mathbf{i} + y_0\mathbf{j}$, $\mathbf{b}x_0\mathbf{i} + y_0\mathbf{j}$ and $\mathbf{r}_t(t) = x_0\mathbf{i} + (y_0 - 16t^2)\mathbf{j}$. Now, the horizontal component of $\mathbf{r}_p(t)$ will be x_0 when $t = x_0/s\cos\theta$ at which time the vertical component of $\mathbf{r}_p(t)$ will be

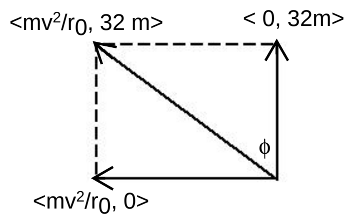
$$(sx_0/s\cos\theta)\sin\theta - 16(x_0/s\cos\theta)^2 = x_0\tan\theta - 16(x_0/s\cos\theta)^2 = y-) - 16(x_0/s\cos\theta)^2.$$

Thus, $\mathbf{r}_p(x_0/s\cos\theta) = \mathbf{r}_t(x_0/s\cos\theta)$ and the projectile will strike the target as it falls.

20. The initial angle is $\theta = 0$, the initial height is 1024 ft, and the initial speed is $s = 180(5280)/3600 = 264$ ft/s. Then $x(t) = 264t$ and $y(t) = -16t^2 + 1024$. Solving $y(t) = 0$ we see that the pack hits the ground at $t = 8$ seconds. The horizontal distance tranvelled is $x(8) = 2112$ feet. From the figure in the text, $\tan\alpha = 1024/2112 = 16/33$ and $\alpha \approx 0.45$ radian or $25.87°$.

21. By Problem 17, $a = v^2/v_0 = 1530^2/(4000 \cdot 5280) \approx 0.1108$. We are given $mg = 192$, so $m = 192/32$ and $w_e = 1192 - (192/32)(0.1108) \approx 191.33$ lb.

22. By problem 17, the centripetal acceleration is v^2/r_0. Then the horizontal force is mv^2/r_0. The vertical force is $32m$. The resultant force is $\mathbf{U} = (mv^2/r_0)\mathbf{i} + 32m\mathbf{j}$. From the figure, we see that $\tan\phi = (mv^2/r_0)/32m = v^2/32r_0$. Using $r_0 = 60$ and $v = 44$ we obtain $\tan\phi = 44^2/32(60) \approx 1.0083$ and $\phi \approx 45.24°$.



23. Solving $x(t) = (v_0\cos\theta)t$ for t and substituting into $y(t) - \frac{1}{2}gt^2 + (v_0\sin\theta)t + s_0$ we obtain

$$y = -\frac{1}{2}g\left(\frac{x}{v_0\cos\theta}\right)^2 + (v_0\sin\theta)\frac{x}{v_0\cos\theta} + s_) = -\frac{g}{2v_0^2\cos^2\theta}x^2 + (\tan\theta)x + s_0,$$

which is the equation of a parabola.

24. Since the projectile is launched from ground level, $s_0 = 0$. To find the maximum height we maximize $y(t) = -\frac{1}{2}gt^2 + (v_0\sin\theta)t$. Solving $y'(t) = -gt + v_0\sin\theta = 0$, we see that $t = (v_0/g)\sin\theta$ is a critical point. Since $y''(t) = -g \le 0$,

$$H = y\left(\frac{v_0\sin\theta}{g}\right) = \frac{1}{2}g\frac{v_0^2\sin^2\theta}{g^2} + v_0\sin\theta\frac{v_0\sin\theta}{g} = \frac{v_0^2\sin^2\theta}{2g}$$

is the maximum height. To find the range we solve $y(t) = -\frac{1}{2}gt^2 + (v_0\sin\theta)t = t(v_0\sin\theta - \frac{1}{2}gt) = 0$. The positive solution to this equation is $t = (2v_0\sin\theta)/g$. The range is thus

$$x(t) = (v_0\cos\theta)\frac{2v_0\sin\theta}{g} = \frac{v_0^2\sin 2\theta}{g}.$$

25. Letting $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, the equation $d\mathbf{r}/dt = \mathbf{v}$ is equivalent to $dx/dt = 6t^2x$, $dy/dt = -4ty^2$, $dz/dt = 2t(z+1)$. Separating the variables and integrating, we obtain x/$x = 6t^2$dt, dy/$y^2 = -4t$dt, $dz/(z+1) = 2t$dt, and $\ln x = 2t^3 + c_1$, $-1/y = 2t^2 + c_2$, $\ln(z+1) + t^2 + c_3$. Thus,

$$\mathbf{r}(t) = k_1e^{2t^3}\mathbf{i} + \frac{1}{2t^2 + k_2}\mathbf{j} + (k_3e^{t^2} - 1)\mathbf{k}.$$

26. We require the fact that $d\mathbf{r}/dt = \mathbf{v}$. Then

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt}(\mathbf{r}\times\mathbf{p} = \mathbf{r}\frac{d\mathbf{p}}{dt} + \frac{d\mathbf{r}}{dt}\times\mathbf{p} = \tau + \mathbf{v}\times\mathbf{p} = \tau + \mathbf{v}\times m\mathbf{v} = \tau + m(\mathbf{v}\times\mathbf{v}) = \tau + \mathbf{0} = \tau.$$

27. (a) Since $\mathbf{F}$ is directed along $\mathbf{r}$ we have $\mathbf{F} = c\mathbf{r}$ for some constant c. Then

$$\tau = \mathbf{r}\times\mathbf{F} = \mathbf{r}\times(c\mathbf{r}) = c(\mathbf{r}\times\mathbf{r}) = \mathbf{0}.$$

(b) If $\tau = \mathbf{0}$ then $d\mathbf{L}/dt = \mathbf{0}$ and $\mathbf{L}$ is constant.

28. (a) Since the cannon is pointing directly to the left, tha parmetric equations describing the path of the cannon ball are given by

$$x(t) = v_0 t, \quad y(t) = -\frac{1}{2}gt^2 + s_0$$

The cannon ball will touch the groun when $y = 0$, which occurs at $t = \sqrt{\frac{2s_0}{g}}$. At that time, x is given by $x = \left(\sqrt{\frac{2s_0}{g}}\right) = -v_0\sqrt{\frac{2s_0}{g}}$. Notice that this x value will be farther to the left with increasing values of $\mathbf{v}_0$. Therefore, the cannon ball travels farther with more gunpowder.

(b) As shown in part (a), the cannon ball will touch the groun when $t = \sqrt{\frac{2s_0}{g}}$. This value of t is independent of v_0. This occurs because v_0 has no vertical component.

(c) If the cannon ball is dropped, we have $\mathbf{v}_0 = \mathbf{0}$. Therefore, the parametric equations describing the cannon ball motion are given by

$$x(t) = 0, \quad y(t) = -\frac{1}{2}gt^2 + s_0.$$

As before, $y = 0$ when $t = \sqrt{\frac{2s_0}{g}}$. Therefore the cannon ball touches the ground at the same time regardless of whether it is fired or dropped.

12.4 Curvature and Acceleration

1. $\mathbf{r}'(t) = -t\sin t\mathbf{i} + t\cos t\mathbf{j} + 2t\mathbf{k}; \quad |\mathbf{r}'(t)| = \sqrt{t^2\sin^2 t + t^2\cos^2 t + 4t^2} = \sqrt{5}t;$
$\mathbf{T} = -\frac{\sin t}{\sqrt{5}}\mathbf{i} + \frac{\cos t}{\sqrt{5}}\mathbf{j} + \frac{2}{\sqrt{5}}\mathbf{k}$

2. $\mathbf{r}'(t) = e^t(-\sin t + \cos t)\mathbf{i} + e^t(\cos t + \sin t)\mathbf{i} + \sqrt{2}e^t\mathbf{k},$
$|\mathbf{r}'(t)| = [e^t(\sin^2 t - 2\sin t\cos t + \cos^2 t) + e^{2t}(\cos^2 t + 2\sin t\cos t + \sin^2 t) + 2e^{2t}]^{1/2} = \sqrt{4e^{2t}} = 2e^t;$
$\mathbf{T}(t) = \frac{1}{2}(-\sin t + \cos t)\mathbf{i} + \frac{1}{2}(\cos t + \sin t)\mathbf{j} + \frac{\sqrt{2}}{2}\mathbf{k}$

3. We assume $a > 0$. $\mathbf{r}'(t) = -a\sin t\mathbf{i} + a\cos t\mathbf{j} + c\mathbf{k}; \quad |\mathbf{r}'(t)| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + c^2} = \sqrt{a^2 + c^2};$
$\mathbf{T}(t) - \frac{a\sin t}{\sqrt{a^2+c^2}}\mathbf{i} + \frac{a\cos t}{\sqrt{a^2+c^2}}\mathbf{j} + \frac{c}{\sqrt{a^2+c^2}}\mathbf{k}; \quad \frac{d\mathbf{T}}{dt} = -\frac{a\cos t}{\sqrt{a^2+c^2}}\mathbf{i} - \frac{a\sin t}{\sqrt{a^2+c^2}}\mathbf{j},$
$\left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\frac{a^2\cos^2 t}{a^2+c^2} + \frac{a^2\sin^2 t}{a^2+c^2}} = \frac{a}{\sqrt{a^2+c^2}}; \quad \mathbf{N} = -\cos t\mathbf{i} - \sin t\mathbf{j};$

$$\mathbf{B} = \mathbf{T}\times\mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\frac{a\sin t}{\sqrt{a^2+c^2}} & \frac{a\cos t}{\sqrt{a^2+c^2}} & \frac{c}{\sqrt{a^2+c^2}} \\ -\cos t & -\sin t & 0 \end{vmatrix} = \frac{c\sin t}{\sqrt{a^2+c^2}}\mathbf{i} - \frac{c\cos t}{\sqrt{a^2+c^2}} + \frac{a}{\sqrt{a^2+c^2}}\mathbf{k};$$

$\kappa = \frac{|d\mathbf{T}/dt|}{\mathbf{r}'(t)} = \frac{a/\sqrt{a^2+c^2}}{\sqrt{a^2+c^2}} = \frac{a}{a^2+c^2}$

4. $\mathbf{r}'(t) = \mathbf{i} + t\mathbf{j} + t^2\mathbf{k}$; $\quad |\mathbf{r}'(t)| = \sqrt{1+t^2+t^4}$, $\quad |\mathbf{r}';(1)| = \sqrt{3}$;
$\mathbf{T}(t) = (1+t^2+t^4)^{-1/2}(\mathbf{i}+t\mathbf{j}+t^2\mathbf{k})$, $\quad \mathbf{T}(1)\dfrac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k})$;
$$\frac{d\mathbf{T}}{dt} = -\frac{1}{2}(1+t^2+t^4)^{-3/2}(2t+4t^3)\mathbf{i} + [(1+t^2+t)^{-1/2} - \frac{t}{2}(1+t^2+t)^{-3/2}(2t+4t^3)]\mathbf{j}$$
$$[2t(1+t^2+t^4)^{-1/2}\frac{t^2}{2}(1+t^2+t^4)^{-3/2}(2t+4t^3)]\mathbf{k};$$
$$\frac{d}{dt}\mathbf{T}(1) = -\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{k}, \quad \left|\frac{d}{dt}\mathbf{T}(1)\right| = \sqrt{\frac{1}{3}+\frac{1}{3}} = \frac{\sqrt{2}}{\sqrt{3}}; \quad \mathbf{N}(1) = -\frac{1}{\sqrt{2}}(\mathbf{i}-\mathbf{k})k,$$
$$\mathbf{B}(1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{vmatrix} = \frac{1}{\sqrt{6}}(\mathbf{i}-2\mathbf{j}+\mathbf{k});$$
$$\kappa = \left|\frac{d}{dt}\mathbf{T}(1)\right| = |\mathbf{r}'(1)| = \frac{\sqrt{2}/\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{2}}{3}$$

5. From Example 1 in the text, a normal to the osculating plane is $\mathbf{B}(\pi/4) = \frac{1}{26}(3\mathbf{i}-3\mathbf{j}+2\sqrt{2}\mathbf{k})$. The point on the curve when $t=\pi/4$ is $(\sqrt{2},\sqrt{2},3\pi/4)$. An equation of the plane is $3(x-\sqrt{2})-3(y-\sqrt{2})+2\sqrt{2}(z-3\pi/4(=0$, $\;3x-3y+2\sqrt{2}z = 3\pi/2$, or $3\sqrt{2}x-3\sqrt{2}y+4z=3\pi$.

6. From Problem 4, a normal to the osculating plane is $\mathbf{B}(1) = \frac{1}{\sqrt{6}}(\mathbf{i}-2\mathbf{j}+\mathbf{k})$. The point on the curve when $t=1$ is $(1,1/2,1/3)$. An equaiton of the plane is $(x-1)-2(y-1/2)+(z-1/3)=0$ or $x-2y+z=1/3$.

7. $\mathbf{v}(t) = \mathbf{j}+2t\mathbf{k}$, $\quad |\mathbf{v}(t)| = \sqrt{1+4t^2}$; $\quad \mathbf{a}(t) = 2\mathbf{k}$; $\quad \mathbf{v}\cdot\mathbf{a} = 4t$, $\quad \mathbf{v}\times\mathbf{a} = 2\mathbf{i}$, $\quad |\mathbf{v}\times\mathbf{a}| = 2$;
$$a_{\mathbf{T}} = \frac{4t}{\sqrt{1+4t^2}}, \quad a_{\mathbf{N}} = \frac{2}{\sqrt{1+4t^2}}$$

8. $\mathbf{v}(t) = -3\sin t\mathbf{i} + 2\cos t\mathbf{j} + \mathbf{k}$,
$|\mathbf{v}(t)| = \sqrt{9\sin^2 t + 4\cos^2 t = 1} = \sqrt{5\sin^2 t + 4\sin^2 t + 4\cos^2 t + 1} = \sqrt{5}\sqrt{\sin^2 +1}$;
$\mathbf{a}(t) = -3\cos t\mathbf{i} - 2\sin t\mathbf{j}$; $\quad \mathbf{v}\cdot\mathbf{a} = 9\sin t\cos t - 4\sin t\cos t = 5\sin t\cos t$,
$\mathbf{v}\times\mathbf{a} = 2\sin t\mathbf{i} - 3\cos t\mathbf{j} + 6\mathbf{k}$, $\quad |\mathbf{v}\times\mathbf{a}| = \sqrt{4\sin^2 + (\cos^2 t + 36} = \sqrt{5}\sqrt{\cos^2 t + 8}$;
$$a_{\mathbf{T}}\frac{\sqrt{5}\sin t\cos t}{\sqrt{\sin^2 t+1}}, \quad a_{\mathbf{N}} = \sqrt{\frac{\cos^2 t+8}{\sin^2 t+1}}$$

9. $\mathbf{v}(t) = 2t\mathbf{i}+2t\mathbf{j}+4t\mathbf{k}$, $\quad |\mathbf{v}(t)| = 2\sqrt{6}t$, $\; t>0$; $\;\mathbf{a}(t) = 2\mathbf{i}+2\mathbf{j}+4\mathbf{k}$; $\quad \mathbf{v}\cdot\mathbf{a} = 24t$, $\quad \mathbf{v}\times\mathbf{a} = \mathbf{0}$;
$$a_{\mathbf{T}} = \frac{24t}{2\sqrt{6}t} = 2\sqrt{6}, \quad a_{\mathbf{N}} = 0, \;\; t>0$$

10. $\mathbf{v}(t) = 2t\mathbf{i} - 3t^2\mathbf{j} = 4t^3\mathbf{k}$, $\quad |\mathbf{v}(t)| = t\sqrt{4+9t^2+16t^4}$, $\; t>)$; $\quad \mathbf{a}(t) = 2\mathbf{i} - 6t\mathbf{j} + 12t^2\mathbf{k}$;
$\mathbf{v}\cdot\mathbf{a} = 4t+18t^3+48t^5$; $\quad \mathbf{v}\times\mathbf{a} = -12t^4\mathbf{i} - 16t^3\mathbf{j} - 6t^2\mathbf{k}$, $\quad |\mathbf{v}\times\mathbf{a}| = 2t^2\sqrt{36t^4+64t^2+9}$;
$$a_T = \frac{4+18t^2+48t^4}{\sqrt{4+9t^2+16t^4}}, \quad a_N = \frac{2t\sqrt{36t^4+64t^2+9}}{\sqrt{4+9t^2+16t^4}} \quad t>0$$

11. $\mathbf{v}(t) = 2\mathbf{i}+2t\mathbf{j}$, $\quad |\mathbf{v}(t)| = 2\sqrt{1+t^2}$; $\quad \mathbf{a}(t) = 2\mathbf{j}$; $\quad \mathbf{v}\times\mathbf{a} = 4\mathbf{k}$, $\quad |\mathbf{v}\times\mathbf{a}| = 4$;
$$a_T = \frac{2t}{\sqrt{1+t^2}}, \quad a_N = \frac{2}{\sqrt{1+t^2}}$$

12. $\mathbf{v}(t)=\dfrac{1}{1+t^2}\mathbf{i}+\dfrac{t}{1+t^2}\mathbf{j}$, $\quad|\mathbf{v}(t)|=\dfrac{\sqrt{1+t^2}}{1+t^2}$; $\quad\mathbf{a}(t)=-\dfrac{2t}{(1+t^2)^2}\mathbf{i}+\dfrac{1-t^2}{(1+t^2)^2}\mathbf{j}$;
$\mathbf{v}\cdot\mathbf{a}=-\dfrac{2t}{(1+t^2)^3}+\dfrac{t-t^3}{(1+t^2)^3}$; $\quad\mathbf{v}\times\mathbf{a}=\dfrac{1}{(1+t^2)^2}\mathbf{k}$, $\quad|\mathbf{v}\times\mathbf{a}|=\dfrac{1}{(1+t^2)^2}$;
$a_T=-\dfrac{t/(1+t^2)^3}{\sqrt{1+t^2})/(1+t^2}=-\dfrac{t}{(1+t^2)3/2}$, $\quad a_N=\dfrac{a/(1+t^2)^2}{\sqrt{1+t^2}/(1+t^2)}=\dfrac{1}{(1+t^2)^{3/2}}$

13. $\mathbf{v}(t)=-5\sin t\mathbf{i}+5\cos t\mathbf{j}$, $\quad|\mathbf{v}(t)|=5$; $\quad\mathbf{a}(t)=-5\cos t\mathbf{i}-5\sin t\mathbf{j}$; $\quad\mathbf{v}\cdot\mathbf{a}=0$,
$\mathbf{v}\times\mathbf{a}=25\mathbf{k}$, $\quad|\mathbf{v}\times\mathbf{a}|=25; a_T=0$, $\quad a_N=5$

14. $\mathbf{v}(t)=\sinh t\mathbf{i}+\cosh t\mathbf{j}$, $\quad|\mathbf{v}(t)|=\sqrt{\sinh t^2+\cosh^2 t}'$ $\quad\mathbf{a}(t)=\cosh t\mathbf{i}+\sinh t\mathbf{j}$
$\mathbf{v}\cdot\mathbf{a}=2\sinh t\cosh t$; $\quad\mathbf{v}\times\mathbf{a}=(\sinh^2 t-\cosh^2 t)\mathbf{k}=-\mathbf{k}$, $\quad|\mathbf{v}\times\mathbf{a}|=1$;
$a_T=\dfrac{2\sinh t\cosh t}{\sqrt{\sinh^2+\cosh^2}}$, $\quad a_N=\dfrac{1}{\sqrt{\sinh^2+\cosh^2}}$

15. $\mathbf{v}(t)=e^t(\mathbf{i}+\mathbf{j}+\mathbf{k})$, $\quad|\mathbf{v}(t)|=\sqrt{3}e^{-t}$; $\quad\mathbf{a}(t)=e^{-t}(\mathbf{i}+\mathbf{j}+\mathbf{k})$; $\quad\mathbf{v}\cdot\mathbf{a}=-3e^{-2t}; \mathbf{v}\times\mathbf{a}=\mathbf{0}$,
$|\mathbf{v}\times\mathbf{a}|=0$; $\quad a_T=-\sqrt{3}e^{-t}$, $\quad a_N=0$

16. $\mathbf{v}(t)=\mathbf{i}+2\mathbf{j}+4\mathbf{k}$, $\quad|\mathbf{v}(t)|=\sqrt{21}$; $\quad\mathbf{a}(t)=\mathbf{0}$; $\quad\mathbf{v}\cdot\mathbf{a}=0$, $\quad\mathbf{v}\times\mathbf{a}=\mathbf{0}$, $\quad|\mathbf{v}\times\mathbf{a}|=0$; $\quad a_T=0$, $\quad a_N=0$

17. $\mathbf{v}(t)=-a\sin t\mathbf{i}+b\cos t\mathbf{j}+c\mathbf{k}$, $\quad|\mathbf{v}(t)|=\sqrt{a^2\sin^2 t+b^2\cos^2+c^2}; \mathbf{a}(t)=-a\cos t\mathbf{i}-b\sin t\mathbf{j}$;
$\mathbf{v}\times\mathbf{a}=bc\sin t\mathbf{i}-ac\cos t\mathbf{j}+ab\mathbf{k}$, $\quad|\mathbf{v}\times\mathbf{a}|=\sqrt{b^2c^2\sin^2 t+a^2c^2\cos^2 t+a^2b^2}$
$\kappa=\dfrac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3}=\dfrac{\sqrt{b^2c^2\sin^2 t+a^2c^2\cos^2 t+a^2b^2}}{(a^2\sin^2 t+b^2\cos^2 t+c^2)^{3/2}}$

18. (a) $\mathbf{v}(t)=-a\sin t\mathbf{i}+b\cos t\mathbf{j}$, $\quad|\mathbf{v}(t)|=\sqrt{a^2\sin^2 t+b^2\cos^2 t}$; $\quad\mathbf{a}(t)=-a\cos t\mathbf{i}-b\sin t\mathbf{j}$;
$\mathbf{v}\times\mathbf{a}=ab\mathbf{k}$; $\quad|\mathbf{v}\times\mathbf{a}|=ab$; $\quad\kappa=\dfrac{ab}{(a^2\sin^2 t+b^2\cos^2 t)^{3/2}}$
(b) When $a=b$, $|\mathbf{v}(t)|=a$, $|\mathbf{v}\times\mathbf{a}|=a^2$, and $\kappa=a^2/a^3=1/a$.

19. The equation of a line is $\mathbf{v}(t)=\mathbf{b}+t\mathbf{c}$, when $\mathbf{b}$ and $\mathbf{c}$ are constant vectors.
$\mathbf{v}(t)=\mathbf{c}$, $|\mathbf{v}(t)|=|\mathbf{c}|$; $\mathbf{a}(t)=\mathbf{0}$; $\quad\mathbf{v}\times\mathbf{a}=\mathbf{0}$; $\quad\kappa=|\mathbf{v}\times\mathbf{a}|/|\mathbf{v}|^3=0$

20. $\mathbf{v}(t)=a(1-\cos t)\mathbf{i}+a\sin t\mathbf{j}$; $\quad\mathbf{v}(\pi)=2a\mathbf{i}$, $\quad|\mathbf{v}(\pi)|=2a$; $\quad\mathbf{a}(t)=a\sin t\mathbf{i}+a\cos t\mathbf{j}$,
$\mathbf{a}(\pi)=-a\mathbf{j}$; $\;|\mathbf{v}\times\mathbf{a}|=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2a&0&0\\0&-a&0\end{vmatrix}=-2a^2\mathbf{k}$; $\quad|\mathbf{v}\times\mathbf{a}|=2a^2$; $\quad\kappa=\dfrac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3}=\dfrac{2a^2}{8a^3}=\dfrac{1}{4a}$

21. $\mathbf{v}(t)=f'(t)\mathbf{i}+g'(t)\mathbf{j}$, $\quad|\mathbf{v}(t)|=\sqrt{[f'(t)]^2+[g'(t)]^2}$; $\quad\mathbf{a}(t)=f''(t)\mathbf{i}+g''(t)\mathbf{j}$;
$\mathbf{v}\times\mathbf{a}=[f'(t)g''(t)-g'(t)f''(t)]\mathbf{k}$, $\quad|\mathbf{v}\times\mathbf{a}|=|f'(t)g''(t)-g'(t)f''(t)|$;
$\kappa=\dfrac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3}=\dfrac{|f'(t)g''(t)-g'(t)f''(t)|}{([f"(t)]^2+[g'(t)]^2)^{3/2}}$

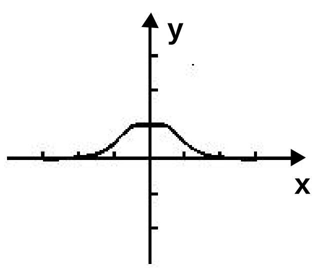
22. For $y=F(x)$, $\;\mathbf{r}=x\mathbf{i}+F(x)\mathbf{j}$. We identify $f(x)=x$ and $g(x)=F(x)$ in Problem 21. Then $f'(x)=1$, $\;f''(x)=0$, $\;g'(x)=F'(x)$, $\;g''(x)=F''(x)$, and $\kappa=|F''(x)|/(1+[F'(x)]^2)^{3/2}$.

23. $F(x) = x^2$, $F(0) = 0$, $F(1) = 1$; $F'(x) = 2x$, $F'(0) = 0$, $F'(1) = 2$; $F''(x) = 2$, $F''(0) = 2$, $F''(1) = 2$; $\kappa(0) = \dfrac{2}{(1+0^2)^{3/2}} = 2$; $\rho(0) = \dfrac{1}{2}$;
$\kappa(1) = \dfrac{2}{(1+2^2)^{3/2}} = \dfrac{2}{5\sqrt{5}} \approx 0.18$;
$\rho(1) = \dfrac{5\sqrt{5}}{2} \approx 5.59$; Since $2 > 2/5\sqrt{5}$, the curve is "sharper" at $(0,0)$.

24. $F(x) = x^3$, $F(-1) = -1$, $F(1/2) = 1/8$; $F'(x) = 3x^2$, $F'(-1) = 3$, $F'(1/2) = 3/4$; $F''(x) = 6x, F''(-1) = -6$, $F''(1/2) = 3$; $\kappa(-1) = \frac{|-6|}{(1+3^2)^{3/2}} = \dfrac{6}{10\sqrt{10}} = \dfrac{3}{5\sqrt{10}} \approx 0.19$;
$\rho(-1) = \dfrac{5\sqrt{10}}{3} \approx 5.27$;
$\kappa(\frac{1}{2}) = \dfrac{3}{[1+(3/4)^2]^{3/2}} = \dfrac{3}{125/64} \approx 1.54$; $\rho(\frac{1}{2}) = \dfrac{125}{192} \approx 0.65$
Since $1.54 > 0.19$, the curve is "sharper" at $(1/2, 1/8)$.

25. Letting $F(x) = x^2$, we can use Problem 22 to get $\kappa(x) = \dfrac{|F''(x)|}{|1+(F'(x))^2|^{3/2}}$.
Now, $F'(x)2x$, $F''(x) = 2$, and $(F'(x))^2 = 4x^2$ so that $\kappa = \dfrac{2}{(1+4x^2)^{3/2}}$.
As $x \to \pm\infty$, the denominator grows without bound. Therefore, $\kappa(x) \to 0$ as $x \to \pm\infty$.

26. (a)



(b) $\kappa'(t) = \dfrac{2t(t^2+2)}{(t^4+t^2+1)^{3/2}\sqrt{t^4+4t^2+1}} - \dfrac{3t(2t^2+1)\sqrt{t^4+4t^2+1}}{(t^4+t^2+1)^{5/2}}$;
critical numbers occur at $t = -.271469$, $t = 0$, and $t = .271469$.

(c) Maximum of 1.017182 occurs at $t = -.271469$ and $t = .271469$.

27. Since $(c, F(c))$ is an inflection point and F'' exists on an interval containg c, we must have $F''(c) = 0$. Therefore, using the formula from Problem 22, we see that the curvature is zero.

28. We use the fact that $\mathbf{T}\cdot\mathbf{N} = 0$ and $\mathbf{T}\cdot\mathbf{T} = \mathbf{N}\cdot\mathbf{N} = 1$. Then
$|\mathbf{a}(t)|^2 = \mathbf{a}\cdot\mathbf{a} = (a_n\mathbf{N} + a_t\mathbf{T})\cdot(a_n\mathbf{N} + a_t\mathbf{T}) = a_N^2\mathbf{N}\cdot\mathbf{N} + 2a_na_t\mathbf{N}\cdot\mathbf{T} + a_T^2\mathbf{T}\cdot\mathbf{T} = a_N^2 + a_T^2$.

Chapter 12 in Review

A. True/False

1. True; $|\mathbf{v}(t)| = \sqrt{2}$
2. True; the curvature of a circle of radius a is $\kappa = \frac{1}{a}$.
3. True
4. False; consider $\mathbf{r}(t) = t^2\mathbf{i}$. In this case, $\mathbf{v}(t) = 2t\mathbf{i}$ and $\mathbf{a}(t) = 2\mathbf{i}$. Since $\mathbf{v} \cdot \mathbf{a} = 4t$, the velocity and acceleration vectors are not orthogonal for $t \neq 0$.
5. True
6. False; see Problem 20c in Section 14.2
7. True
8. True
9. False; consider $\mathbf{r}_1(t) = \mathbf{r}_2(t) = \mathbf{i}$.
10. True, $\dfrac{d}{dt}|\mathbf{r}(t)|^2 = \dfrac{d}{dt}(\mathbf{r} \cdot \mathbf{r}) = \mathbf{r} \cdot \dfrac{d\mathbf{r}}{dt} + \dfrac{d\mathbf{r}}{dt} \cdot \mathbf{r} = 2\mathbf{r} \cdot \dfrac{d\mathbf{r}}{dt}$.

B. Fill in the Blanks

1. $y = 4$
2. 0
3. $\mathbf{r}'(t) = \langle 1, 2t, t^2 \rangle$ so $\mathbf{r}'(1) = \langle 1, 2, 1 \rangle$
4. $\mathbf{r}''(t) = \langle 0, 2, 2t \rangle$ so $\mathbf{r}''(1) = \langle 0, 2, 2 \rangle$
5. $\mathbf{r}'(1) \times \mathbf{r}''(1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{j} \\ 1 & 2 & 1 \\ 0 & 2 & 2 \end{vmatrix} = \langle 2, -2, 2 \rangle$ so $\mathbf{r}'(1) \times \mathbf{r}''(1) = \sqrt{12}$.

   Since $\mathbf{r}'(1)| = \sqrt{6}$, we have $\kappa(1) = \dfrac{\mathbf{r}'(1) \times \mathbf{r}''(1)}{|\mathbf{r}'(1)|^3} = \dfrac{\sqrt{12}}{6\sqrt{6}} = \dfrac{\sqrt{2}}{6}$.
6. $\mathbf{T}(1) = \dfrac{\mathbf{r}'(1)}{|\mathbf{r}'(1)|} = \dfrac{\langle 1, 2, 1 \rangle}{\sqrt{6}} = \left\langle \dfrac{1}{\sqrt{6}}, \dfrac{2}{\sqrt{6}}, \dfrac{1}{\sqrt{6}} \right\rangle$
7. $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{\langle 1, 2t, t^2 \rangle}{\sqrt{1 + 4t^2 + t^4}} = \left\langle \dfrac{1}{\sqrt{1 + 4t^2 + t^4}}, \dfrac{2t}{\sqrt{1 + 4t^2 + t^4}}, \dfrac{t^2}{\sqrt{1 + 4t^2 + t^4}} \right\rangle$

   So $\mathbf{T}'(t) = \left\langle \dfrac{-2(t^2 + 2)}{(t^4 + 4t^2 + 1)^{3/2}}, \dfrac{-2(t^4 - 1)}{(t^4 + 4t^2 + 1)^{3/2}}, \dfrac{2t(2t^2 + 1)}{(t^4 + 4t^2 + 1)^{3/2}} \right\rangle$.

   This gives $\mathbf{T}'(1) = \left\langle \dfrac{-6}{6^{3/2}}, 0, \dfrac{6}{6^{3/2}} \right\rangle = \left\langle \dfrac{-1}{\sqrt{6}}, 0, \dfrac{1}{\sqrt{6}} \right\rangle$ and $|\mathbf{T}'(1)| = \sqrt{\frac{1}{6} + \frac{1}{6}} = \dfrac{1}{\sqrt{3}}$.

   Therefore $\mathbf{N}(1) = \dfrac{\mathbf{T}'(1)}{|\mathbf{T}'(1)|} = \dfrac{\left\langle \frac{-1}{\sqrt{6}}, 0, \frac{1}{\sqrt{6}} \right\rangle}{(\frac{1}{\sqrt{3})}} = \langle \dfrac{-1}{\sqrt{2}}, 0, \dfrac{1}{\sqrt{2}} \rangle$.

8. $\mathbf{B}(1) = \mathbf{T}(1) \times \mathbf{N}(1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{vmatrix} = \left\langle \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$

9. A normal to the normal plane is $\mathbf{T}(1) = \left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle$ so we can use $\mathbf{n} = \langle 1, 2, 1 \rangle$ as a vector normal to the plane. Since $\mathbf{r}(1) = \langle 1, 1, \frac{1}{3} \rangle$, the point $(1, 1, \frac{1}{3})$ lies on the normal plane at $t = 1$. Thus an equation of the normal plane is $(x-1) + 2(y-1) + (z - \frac{1}{3}) = 0$ or $x + 2y + z = \frac{1)}{3}$ or $3x + 6y + 3z = 10$

10. A normal to the osculating plane is $\mathbf{B}(1) = \left\langle \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$. So we can use $\mathbf{n} = \langle 1, -1, 1 \rangle$ as a normal vector. Using the point $(1, 1, \frac{1}{3})$, an equation of the osculating plane is $(z-1) - (y-1) + (z - \frac{1}{3}) = 0$ or $x - y + z = \frac{1}{3}$ or $3x - 3y + 3z = 1$.

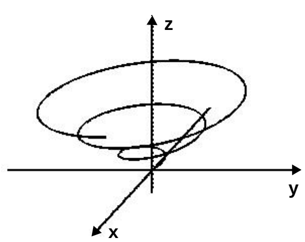
C. Exercises

1. $\mathbf{r}'(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + \mathbf{k}$; $\quad s = \int_0^{\pi} \sqrt{\cos^2 t + \sin^2 + 1}dt = \int_0^{\pi} \sqrt{2}dt = \sqrt{2}\pi$

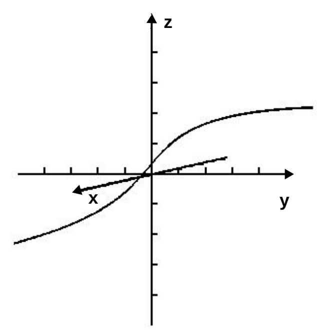
2. $\mathbf{r}'(t) = 5\mathbf{i} + \mathbf{j} + 7\mathbf{k}$; $\quad s(t) = \int_0^t \sqrt{25 + 1 + 49}du = 5\sqrt{3}t$; $\quad s(3) = 15\sqrt{3}$. Solving $5\sqrt{3}t = 80\sqrt{3}$, we see that the distance traveled will be $80\sqrt{3}$ when $t = 16$ or at the point $(80, 17, 112)$.

3. $\mathbf{r}(3) = -27\mathbf{i} + 8\mathbf{j} + \mathbf{k}$; $\quad \mathbf{r}'(t) = -6t\mathbf{i} = \dfrac{2}{\sqrt{t+1}} + \mathbf{k}$; $\quad \mathbf{r}'(2) = -18\mathbf{i} + \mathbf{j} + \mathbf{k}$. The tangent line is $x = -27 - 18t$, $\ y = 8 + t$, $\ z = 1 + t$.

4.



5.



6. $$\begin{aligned} \frac{d}{dt}[\mathbf{r}_1(t) \times \mathbf{r}_2(t)] &= \mathbf{r}_1(t) \times \frac{d}{dt}\mathbf{r}_2(t) + \frac{d}{dt}\mathbf{r}_1(t) \times \mathbf{r}_2(t) \\ &= (t^2\mathbf{i} + 2t\mathbf{j} + t^3\mathbf{k}) \times (-\mathbf{i} + 2t\mathbf{j} + 2t\mathbf{k}) + (2t\mathbf{i} + 2\mathbf{j} + 2t^2\mathbf{k}) \times [-t\mathbf{i} + t^2\mathbf{j} + (t^2+1)\mathbf{k}] \\ &= (4t^2 - 2t^4)\mathbf{i} - 3t^3\mathbf{j} + (2t^3 + 2t)\mathbf{k} + (2t^2 + 2 - 3t^4)\mathbf{i} - (5t^3 + 2t)\mathbf{j} + (2t^3 + 2t)\mathbf{k} \\ &= (2 + 6t^2 - 5t^4)\mathbf{i} - (8t^3 + 2t)\mathbf{j} + (4t^3 + 4t)\mathbf{k} \\ \frac{d}{dt}[\mathbf{r}_1(t) \times \mathbf{r}_2(t)] &= \frac{d}{dt}[(2t^3 + 2t - t^5)\mathbf{i} - (2t^4 + t^2)\mathbf{j} + (t^4 + 2t^2)\mathbf{k}] \\ &= (2 + 6t^2 - 5t^4)\mathbf{i} - (8t^3 + 2t)\mathbf{j} + (4t^3 + 4t)\mathbf{k} \end{aligned}$$

7. $$\begin{aligned}\frac{d}{dt}[\mathbf{r}_1(t)\cdot\mathbf{r}_2(t)] &= \mathbf{r}_1(t)\cdot\frac{d}{dt}\mathbf{r}_2(t)+\frac{d}{dt}\mathbf{r}_1(t)\cdot\mathbf{r}_2(t)\\ &= (\cos t\mathbf{i}-\sin t\mathbf{j}+4t^3\mathbf{k})\cdot(2t\mathbf{i}+\sin t\mathbf{j}+2e^{2t}\mathbf{k})\\ &\quad(-\sin t\mathbf{i}-\cos t\mathbf{j}+12t^2\mathbf{k})\cdot(t^2\mathbf{i}+\sin t\mathbf{j}+e^{2t}\mathbf{k})\\ &= (2t\cos t-\sin t\cos t+8t^3e^{2t}-t^2\sin t-\sin t\cos t+12t^2e^{2t}\\ &= 2t\cos t-t^2\sin t-2\sin t\cos t+8t^3e^{2t}+12t^2e^{2t}\end{aligned}$$

$$\frac{d}{dt}[\mathbf{r}_1(t)\cdot\mathbf{r}_2(t)]=\frac{d}{dt}[t^2\cos t-\sin^2 t+4t^3e^{2t}]=-t^2\sin t+2t\cos t-2\sin t\cos t+8t^3e^{2t}+12t^2e^{2t}$$

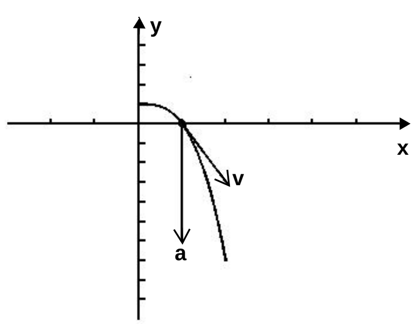
8. $$\begin{aligned}\frac{d}{dt}[\mathbf{r}_1(t)\cdot(\mathbf{r}_2(t)\times\mathbf{r}_3(t))] &= \mathbf{r}_1(t)\cdot\frac{d}{dt}[\mathbf{r}_2(t)\times\mathbf{r}_3(t)]+\mathbf{r}'(t)\cdot[\mathbf{r}_2(t)\times\mathbf{r}_3(t)]\\ &= \mathbf{r}_1(t)\cdot[(\mathbf{r}_2(t)\times\mathbf{r}_3'(t))+(\mathbf{r}_2'(t)\times\mathbf{r}_3(t))]+\mathbf{r}_1'(t)\cdot(\mathbf{r}_2(t)\times\mathbf{r}_3(t))\\ &= \mathbf{r}_1(t)\cdot(\mathbf{r}_2(t)\times\mathbf{r}_3'(t))+\mathbf{r}_1(t)\cdot\mathbf{r}_2'(t)\times\mathbf{r}_3(t))=\mathbf{r}_1'(t)\cdot(\mathbf{r}_2(t)\times\mathbf{r}_3(t))\end{aligned}$$

9. We are given $\mathbf{F}=m\mathbf{a}=2\mathbf{j}$; $\mathbf{v}(0)=\mathbf{i}+\mathbf{j}+\mathbf{k}$. and $\mathbf{r}(0)=\mathbf{i}+\mathbf{j}$. Then

$$\mathbf{v}(t)=\int\mathbf{a}(t)dt=\int\frac{2}{m}\mathbf{j}dt=\frac{2}{m}t\mathbf{j}+\mathbf{c}$$
$$\mathbf{i}=\mathbf{j}+\mathbf{k}=\mathbf{v}(0)=\mathbf{c}$$
$$\mathbf{v}(t)=\mathbf{i}+\left(\frac{2}{m}t+1\right)\mathbf{j}+\mathbf{k}$$
$$\mathbf{r}(t)=t\mathbf{i}+\left(\frac{1}{m}t^2+t\right)\mathbf{j}+t\mathbf{k}+\mathbf{b}$$
$$\mathbf{i}+\mathbf{j}=\mathbf{r}(0)=\mathbf{b}$$
$$\mathbf{r}(t)=(t+1)\mathbf{i}+\left(\frac{1}{m}t^2+t+1\right)\mathbf{j}+t\mathbf{k}$$

The parametric equations are $x=t,\ \ y=\dfrac{1}{m}t^2+t+1,\ \ z=t.$

10.



$\mathbf{v}(t)=\mathbf{i}-3t^2\mathbf{j},\ \ \mathbf{v}(1)=\mathbf{i}-3\mathbf{j};\ \ \mathbf{a}(t)=-6t\mathbf{j},\ \ \mathbf{a}(1)=-6\mathbf{j}$
$|\mathbf{v}(1)|=|\mathbf{i}-3\mathbf{j}|=\sqrt{1+9}=\sqrt{10}$

11. $\mathbf{v}(t) = 6\mathbf{i} + \mathbf{j} + 2t\mathbf{k}$; $\mathbf{a}(t) = 2\mathbf{k}$. To find when the particle passes through the plane, we solve $-6t + t + t^2 = -4$ or $t^2 - 5t + 4 = 0$. This gives $t = 1$ and $t = 4$. $\mathbf{v}(1) = 6\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, $\mathbf{a}(1) = 2\mathbf{k}$; $\mathbf{v}(4) = 6\mathbf{i} + \mathbf{j} + 8\mathbf{k}$, $\mathbf{a}(4) = 2\mathbf{k}$

12. We are given $\mathbf{r}(0) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

$$\mathbf{r}(t) = \int \mathbf{v}(t)dt = \int (-10t\mathbf{i} + (3t^2 - 4t)\mathbf{j} + \mathbf{k})dt = -5t^2\mathbf{i} + (t^3 - 2t^2)\mathbf{j} + t\mathbf{k} + \mathbf{c}$$
$$\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} = \mathbf{r}(0) = \mathbf{c}$$
$$\mathbf{r}(t) = (1 - 5t^2)\mathbf{i} + (t^3 - 2t^2 + 2)\mathbf{j} + (t + 3)\mathbf{k}$$
$$\mathbf{r}(t) = -19\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$$

13. $\mathbf{v}(t) = \int \mathbf{a}(t)dt = \int (\sqrt{2}\sin t\mathbf{i} + \sqrt{2}\cos t\mathbf{j})dt = -\sqrt{2}\cos t\mathbf{i} + \sqrt{2}\sin t\mathbf{j} + \mathbf{c}$;
$-\mathbf{i} + \mathbf{j} + \mathbf{k} = \mathbf{v}(\pi/4) = -\mathbf{i} + \mathbf{j} + \mathbf{c}$, $\mathbf{c} = \mathbf{k}$; $\mathbf{v}(t) = -\sqrt{2}\cos t\mathbf{i} + \sqrt{2}\sin t\mathbf{j} + \mathbf{k}$;
$\mathbf{r}(t) = -\sqrt{2}\sin t\mathbf{i} - \sqrt{2}\cos t\mathbf{j} + t\mathbf{k} + \mathbf{b}$; $\mathbf{i} + 2\mathbf{j} + (\pi/4)\mathbf{k} = \mathbf{r}(\pi/4) = -\mathbf{i} - \mathbf{j} + (\pi/4)\mathbf{k} + \mathbf{b}$, $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j}$;
$\mathbf{r}(t) = (2 - 2\sqrt{2}\sin t)\mathbf{i} + (3 - \sqrt{2}\cos t)\mathbf{j} + t\mathbf{k}$; $\mathbf{r}(3\pi/4) = \mathbf{i} + 4\mathbf{j} + (3\pi/4)\mathbf{k}$

14. $\mathbf{v}(t) = t\mathbf{i} + t^2\mathbf{j} - t\mathbf{k}$; $|\mathbf{v}| = t\sqrt{t^2 + 2}$, $t > 0$; $\mathbf{a}(t) = \mathbf{i} + 2t\mathbf{j} - \mathbf{k}$; $\mathbf{v} \cdot \mathbf{a} = t + 2t^3 + t = 2t + 2t^3$; $\mathbf{v} \times \mathbf{a} = t^2 bi + t^2\mathbf{k}$, $|\mathbf{v} \times \mathbf{a} t^2\sqrt{2}$; $a_T = \dfrac{2t + 2t^3}{t\sqrt{t^2 + 2}} = \dfrac{2 + 2t^2}{\sqrt{t^2 + 2}}$, $a_N = \dfrac{t^2\sqrt{2}}{t\sqrt{t^2 + 2}} = \dfrac{\sqrt{2}t}{\sqrt{t^2 + 2}}$;
$\kappa = \dfrac{t^2\sqrt{2}}{t^3(t^2 + 2)^{3/2}} = \dfrac{\sqrt{2}}{t(t^2 + 2)^{3/2}}$

15. $\mathbf{r}'(t) = \sinh t\mathbf{i} + \cosh t\mathbf{j} + \mathbf{k}$, $\mathbf{r}'(1) = \sinh 1\mathbf{i} + \cosh 1\mathbf{j} + \mathbf{k}$;
$|\mathbf{r}'(t)| = \sqrt{\sinh^2 t + \cosh^2 t + 1} = \sqrt{2\cosh^2 t} = \sqrt{2}\cosh t$; $|\mathbf{r}'(1)| = \sqrt{2}\cosh 1$;
$\mathbf{T} = \dfrac{1}{\sqrt{2}}\tanh t\mathbf{i} + \dfrac{1}{\sqrt{2}}\mathbf{j} + \dfrac{1}{\sqrt{2}}\operatorname{sech} t\mathbf{k}$, $\mathbf{T}(1) = \dfrac{1}{\sqrt{2}}(\tanh 1\mathbf{i} + \mathbf{j} + \operatorname{sech} 1\mathbf{k})$;
$\dfrac{d\mathbf{T}}{dt} = \dfrac{1}{\sqrt{2}}\operatorname{sech}^2 t\mathbf{i} - \dfrac{1}{\sqrt{2}}\operatorname{sech} t\tanh t\mathbf{k}$; $\dfrac{d}{dt}\mathbf{T}(1) = \dfrac{1}{\sqrt{2}}\operatorname{sech}^2 1\mathbf{i} - \dfrac{1}{\sqrt{2}}\operatorname{sech} 1\tanh 1\mathbf{k}$,
$\left|\dfrac{d}{dt}\mathbf{T}(1)\right| == \dfrac{\operatorname{sech} 1}{\sqrt{2}}\sqrt{\operatorname{sech}^2 1 + \tanh^2 + 1} = \dfrac{1}{\sqrt{2}}\operatorname{sech} 1$; $\mathbf{N}(1) = \operatorname{sech} 1\mathbf{i} - \tanh 1\mathbf{k}$;
$$\begin{aligned}\mathbf{B}(1) = \mathbf{T}(1) \times \mathbf{N}(1) &= -\frac{1}{\sqrt{2}}\tanh 1\mathbf{i} + \frac{1}{\sqrt{2}}(\tanh^2 1 + \operatorname{sech}^2 1)\mathbf{j} - \frac{1}{\sqrt{2}}\operatorname{sech} 1\mathbf{k} \\ &= \frac{1}{\sqrt{2}}(-\tanh 1\mathbf{i} + \mathbf{j} - \operatorname{sech} 1\mathbf{k})\end{aligned}$$
$\kappa = \left|\dfrac{d}{dt}\mathbf{T}(1)\right| / |\mathbf{r}'(1)| = \dfrac{(\operatorname{sech} 1)/\sqrt{2}}{\sqrt{2}\cosh 1} = \dfrac{1}{2}\operatorname{sech}^2 1$

16. The parametric equations describing the path of the ball are

$$x(t) = 66\cos(30^\circ)t = 33\sqrt{3}ty(t) = -16t^2 + 66\sin(30^\circ)t + 148 = -16t^2 + 33t + 148$$

The ball touches the ground when $y(t) = 0$ or $-16t^2 + 33t + 148 = 0$. This occurs when $t \approx 4.243$. The ball therefore strikes the ground at $x(4.243) = 242.52$ ft.
The velocity of the ball at time t is $\mathbf{v}(t) = \langle 33\sqrt{3}, -32t + 33\rangle$. The impact velocity is given by $\mathbf{v}(4.243) = \langle 33\sqrt{3}, -32(4.243) + 33\rangle \approx \langle 57.158, -102.776\rangle$. The impact speed is then $|\mathbf{v}(4.243)| \approx 117.6$ ft/s.

Chapter 13

Partial Derivatives

13.1 Functions of Several Variables

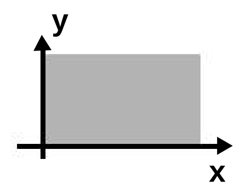
1. $\{(x, y)|(x, y) \neq (0, 0)\}$
2. $\{(x, y)|x \neq x \pm 3y\}$
3. $\{(t, Y)|y \neq x^2\}$
4. $\{(x, y)|y \geq -4\}$
5. $\{(s, t)|s,\ t$ any real numbers$\}$
6. $\{(u, b)|(u, v(\neq (0, 0)\} \bigcup \{(u, v)|u^2 + v^2 \neq 1\}$
7. $\{(r, s)|\ |s| \geq 1\}$
8. $\{(\theta, \phi)\ \mid\ \tan\theta\tan\phi\ \neq\ 1\} \bigcap \{(\theta, \phi)\quad \theta\ \neq\ \pi/2 + k\pi,\ \ k$ an integer$\} \bigcap \{(\theta, \phi)\ \mid\ \phi\ \neq\ \pi/2 + k\pi,\ k$ an integer$\}$
9. $(u, v, w)|u^2 + v^2 + q^2 \geq 16$
10. $\{(x, y, z)|x^2 + y^2 + < 25$ and $z \neq 5\}$
11. **(c)**; The domain of $f(x, y) = \sqrt{x} + \sqrt{y - x}$ is $\{(x, y)|x \geq 0,\ \ y - x \geq 0\} = \{(x, y)|x \geq 0,\ \ y \geq x\}$
12. **(e)**; The domain of $f(x, y) = \sqrt{xy}$ is $\{(x, y)|xy \geq 0\} = \{(x, y)|x \geq 0,\ \ y \geq 0$ or $x \leq 0,\ \ y \leq 0\}$
13. **(b)**; The domain of $f(x, y) = \ln(x - y^2)$ is $\left\{(x, y)|x - y^2 > 0\right\} = \left\{(x, y)|x > y^2\right\}$
14. **(h)**; The domain of $f(x, y) = \dfrac{\sqrt{x^2 + y^2 - 1}}{y - x}$ is $\left\{(x, y)|x^2 + y^2 - 1 \geq 0,\ \ y \neq x\right\} = \left\{(x, y)|x^2 + y^2 \geq 1,\ \ y \neq x\right\}$
15. **(d)**; The domain of $f(x, y) = \sqrt{\frac{x}{y} - 1}$ is $\left\{(x, y)|\frac{x}{y} - 1 \geq 0\right\} = \left\{(x, y)|\frac{x}{y} \geq 1\right\}$

16. **(g)**; The domain of $f(x,y) = \dfrac{x^4+y^4}{xy}$ is $\{(x,y)|xy \neq 0\} = \{(x,y)|x \neq 0,\ \ y \neq 0\}$

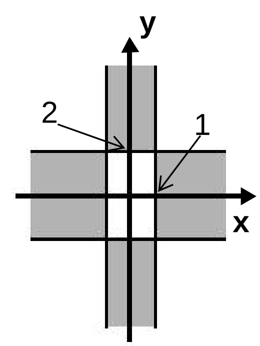
17. **(f)**; The domain of $f(x,y) = \sin^{-1}(xy)$ is $\{(x,y)||xy| \leq 1\}$

18. **(a)**; The domain of $f(x,y) = \sqrt{y-x^2}$ is $\{(x,y)|y-x^2 \geq 0\} = \big\{*x,y)y \geq x^2\big\}$

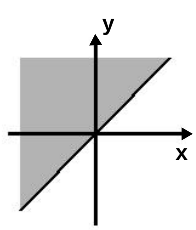
19. $\{(x,y)|x \geq 0 \text{ and } y \geq 0\}$



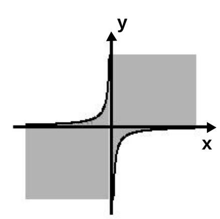
20. $\{(x,y)|\ x^2 \leq 1 \text{ and } y^2 \geq 4\} \bigcap \{(x,y)|\ x^2 \geq 1 \text{ and } y^2 \leq 4\}$
$\{(x,y)|\ |x| \leq 1 \text{ and } |y| \geq 2\} \bigcup \{(x,y)|\ |x| \geq 1 \text{ and } |y| \leq 2\}$



21. $\{(x,y)|y-x \geq 0\}$



22. $\{(x,y)|xy \geq -1\}$



23. $\{z \mid z \geq 10\}$

24. all real numbers

25. $\{w\ ||; -1 \leq w \leq 1\}$

26. $\{x \mid w < 7\}$

27. $f(2,3) = \int_2^4 (2t-1)\mathrm{d}t = (t^2-t)|_2^4 = 12-2 = 10$
$f(-1,1) = \int_{-1}^1 (2t-1)\mathrm{d}t = (t^2-t)|_{-1}^1 = 0-2 = -2$

28. $f(3,0) = \ln 9/9 = \ln 1 = 0$; $\quad f(5,-5) = \ln \dfrac{25}{25+25} = \ln \dfrac{1}{2} = -\ln 2$

29. $f(-1,1,-1) = (-2)^2 = 4$; $\quad f(2,3,-2) = 2^2 = 4$

30. $f(\sqrt{3},\sqrt{3},\sqrt{6}) = 1/3 + 1/2 + 1/6 = 1$; $\quad f(1/4,1/5,1/3) = 16 + 25 + 9 = 50$

31. A plane through the origin perpendicular to the xz-plane

32. A parabolic cylinder perpendicular to the yz-plane

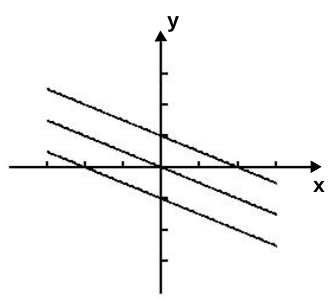
33. The upper half of a cone lying above the xy-plane with axis along the positive z-axis

34. The upper half of a hyperboloid of two sheets with axis lying along the positive z-axis

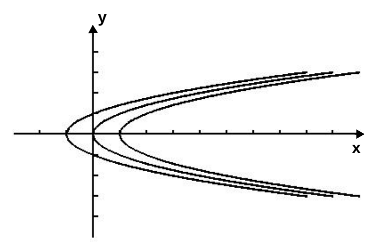
35. The upper half of an ellipsoid

36. A hemisphere lying below the yy-plane

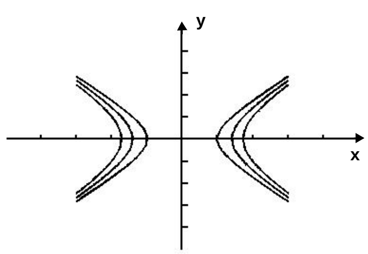
37. $y = -\frac{1}{2}x + C$



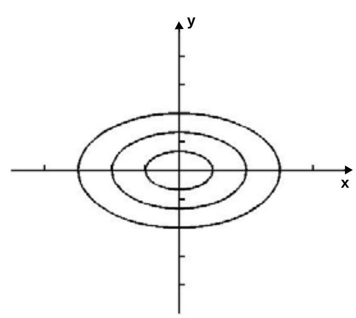
38. $x = y^2 - c$



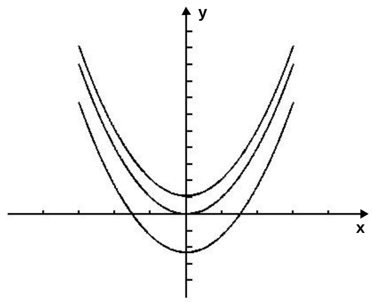
39. $x^2 - y^2 = 1 + c^2$



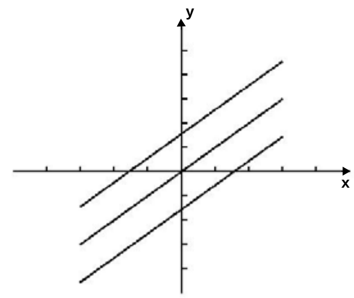
40. $4x^2 + 9y^2 = 36 - c^2, \;\; -6 \le c \le 6$



41. $y = x^2 + \ln c,\ \ c > 0$



42. $y = x + \tan c,\ \ -\pi/2 < c < \pi/2$



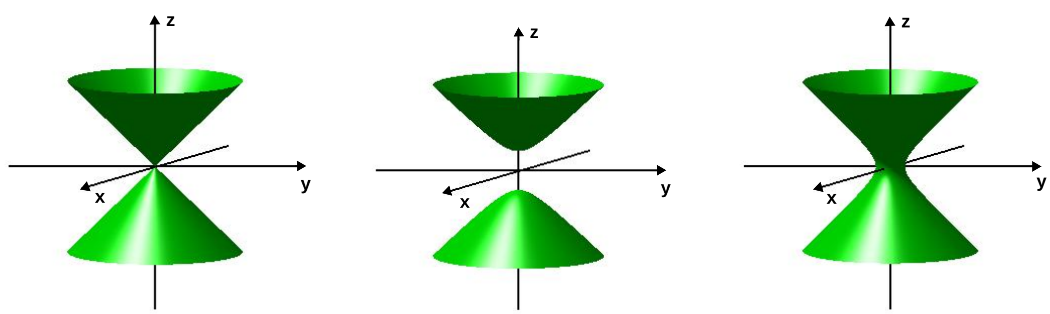
43. $x^2/9 + z^2/4 = c$; elliptical cylinder

44. Setting $f(x, y, z)$ equal to a constant c, we have $(x-1)^2 + (y-2)^2 + (z-2)^2 = c$ which is the equation of a sphere of radius $\sqrt{c}$ centered at $(1, 2, 3)$. Therefore, the level curves are concentric spheres centered at $(1, 2, 3)$.

45. $x^2 + 3y^2 + 6z^2 = c$; ellipsoid

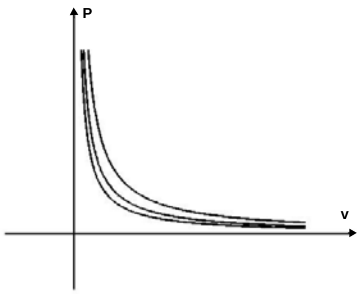
46. $4y - 2z + 1 = c$; plane

47. $c = 0$ $\qquad$ $c < 0$ $\qquad$ $c > 0$



48. Setting $x = -4,\ \ y = 2$, and $z = -3$ in $x^2/16 + y^2/4 + z^2/9 = c$ we obtain $c = 3$. The equation of the surface is $x^2/16 + y^2/4 + z^2/9 = 3$. Setting $y = z = 0$ we find the x-intercepts are $\pm 4\sqrt{3}$. Similarly, the y-intercepts are $\pm 2\sqrt{3}$ and the z-intercepts are $\pm 3\sqrt{3}$.

49.



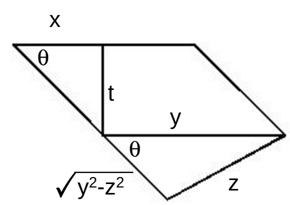
50. From $V = s^2 h$ we obtain $h = V/s^2$.

51. $C(r,h) = \pi r^2(1.8) + \pi r^2(1) + 2\pi h(2.3) = 2.8\pi r^2 + 4.6\pi rh$

52. Let the height of the box be h. Then $2xy + 2xh + 2yh = 500$ and $h = \dfrac{250 - xy}{x+y}$. Thus, $V = xyh = \dfrac{250xy + x^2y^2}{x+y}$.

53. $V + \pi r^2 g + \frac{1}{3}\pi r^2 \left(\frac{2}{3}h\right) = \frac{11}{9}\pi r^2 h$

54. From the figure, we see that $t = x\tan\theta = x\left(\dfrac{z}{\sqrt{y^2 - z^2}}\right) = \dfrac{xz}{\sqrt{y^2 - z^2}}$



55. $X = 2(156)(50) = 15,600$ sq cm

56. $h(20, -6.67) + (10\sqrt{20} - 20 + 10.5)(33 + 6.67) = (20\sqrt{5} - 9.5)(39.67) \approx 1397$ kcal/m^2h

57. (a) The distance the water falls in time t is $s(t) = \frac{1}{2}gt^2 + vt$ where $v$is the velocity of the water at the top level ($t = 0$). The velocity of the water at time t is $v(t) = gt + v$. If t_1 is the time it takes a cross-section of water to fall from the top level to the bottom level, then $V = gt_1 + v$ and $t_1 = (V - v)/g$. The distance traveled in time t_1 is

$$h = \frac{1}{2}gt_1^2 + vt_1 = \frac{1}{2}g\left(\frac{V-v}{g}\right)^2 + v\left(\frac{V-v}{g}\right)$$

Simplifying the equation we obtain $2gh = V^2 - v^2$. Now the rates at the top and bottom levels are $Z = v\pi r^2$ and $Q = V\pi r^2$ (recall that the flow rate is constant). Solving for v and V and substituting into $2gh = V^2 - v^2$ we obtain $2gh = (Q/\pi r^2)^2 - (Q/\pi R^2)^2$. Solving for Q we find $Q = \dfrac{\pi r^2 R^2\sqrt{2gh}}{\sqrt{R^4 - r^4}}$.

(b) When $r = 0.2$ cm, $R = 1$ cm, and $h = 10$, $Q \approx 7.61$ cm^3/s.

13.2 Limits and Continuity

1. $\displaystyle\lim_{(x,y)\to(5,-1)}(x^2+y^2) = 25+1 = 26$

2. $\displaystyle\lim_{(x,y)\to(2,1)}\frac{x^2-y}{x-y} = \frac{4-1}{2-1} = 3$

3. On $y=0$, $\displaystyle\lim_{(x,y)\to(0,0)}\frac{5x^2+y^2}{x^2+y^2} = \lim_{(x,y)\to(0,0)}\frac{5x^2}{x^2} = 5.$
   On $x=0$, $\displaystyle\lim_{(x,y)\to(0,0)}\frac{5x^2+y^2}{x^2+y^2} = \lim_{(x,y)\to(0,0)}\frac{y^2}{y^2} = 1.$ The limit does not exist.

4. $\displaystyle\lim_{(x,y)\to(1,2)}\frac{4x^2+y^2}{16x^4+y^4} = \frac{4+4}{16+16} = \frac{1}{4}$

5. $\displaystyle\lim_{(x,y)\to(1,1)}\frac{4-x^2-y^2}{x^2+y^2} = \frac{4-1-1}{1+1} = 1$

6. On $x=0$, $\displaystyle\lim_{(x,y\to(0,0)}\frac{2x^2-y}{x^2+2y^2} = \lim_{(x,y\to(0,0)}\frac{-y}{2y^2} = \infty.$
   On $y=0$, $\displaystyle\lim_{(x,y\to(0,0)}\frac{2x^2-y}{x^2+2y^2} = \lim_{(x,y\to(0,0)}\frac{2x^2}{x^2} = 2.$. The limit does not exist.

7. On $y=x$, $\displaystyle\lim_{(x,y\to(0,0)}\frac{x^2y}{x^4+y^2} = \lim_{(x,y\to(0,0)}\frac{x^3}{x^4+x^2} = \lim_{(x,y\to(0,0)}\frac{x}{x^2+1} = 0.$
   On $y=x^2$, $\displaystyle\lim_{(x,y\to(0,0)}\frac{x^2y}{x^4+y^2} = \lim_{(x,y\to(0,0)}\frac{x^4}{x^4+x^4} = \frac{1}{2}.$ The limit does not exist.

8. On $y=x$, $\displaystyle\lim_{(x,y\to(0,0)}\frac{6xy^2}{x^2+y^4} = \lim_{(x,y\to(0,0)}\frac{6x^3}{x^2+x^4} = \lim_{(x,y\to(0,0)}\frac{6x}{1+x^2} = 0.$
   On $x=y^2$, $\displaystyle\lim_{(x,y\to(0,0)}\frac{x^2y}{x^4+y^2} = \lim_{(x,y\to(0,0)}\frac{6y^4}{y^4+y^4} = 3.$ The limit does not exist.

9. $\displaystyle\lim_{(x,y)\to(1,2)} x^3y^2(x+y)^3 = 1(4)(27) = 108$

10. $\displaystyle\lim_{(x,y)\to(2,3)}\frac{xy}{x^2-y^2} = \frac{6}{4-9} = -\frac{6}{5}$

11. $\displaystyle\lim_{(x,y)\to(0,0)}\frac{e^{xy}}{x+y+1} = \frac{1}{1} = 1$

12. On $y=mx$, $\displaystyle\lim_{(x,y)\to(0,0)}\frac{\sin xy}{x^2+y^2} = \lim_{(x,y)\to(0,0)}\frac{\sin mx^2}{(1+m^2)x^2}$
    $$= \lim_{(x,y)\to(0,0)}\frac{m}{1+m^2}\frac{\sin mx^2}{mx^2} = \frac{m}{1+m^2}.$$
    The limit does not exist.

13. $\lim\limits_{(x,y)\to(2,2)} \dfrac{xy}{x^3+y^2} = \dfrac{4}{8+4} = \dfrac{1}{3}$

14. $\lim\limits_{(x,y)\to(\pi,\pi/4)} \cos(3x+y) = \cos(3\pi+\pi/4) = \cos 13\pi/4 = -\sqrt{2}/2$

15. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-3y+1}{x+5y-3} = -\dfrac{1}{3}$

16. On $y = mx$, $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^2}{x^4+5y^4} = \lim\limits_{(x,y)\to(0,0)} \dfrac{x^2m^2x^2}{x^4+5m^4x^4} = \dfrac{m^2}{1+5m^4}$.
The limit does not exist.

17. $\lim\limits_{(x,y)\to(4,3)} xy^2\dfrac{x+2y}{x-y} = 4(9)\dfrac{4+6}{4-3} = 360$

18. $\lim\limits_{(x,y)\to(1,0)} \dfrac{x^2y}{x^+y^3} = \dfrac{0}{1+0} = 0$

19. $\lim\limits_{(x,y)\to(1,1)} \dfrac{xy-x-y+1}{x^2+y^2-2x-2y+2} = \lim\limits_{(x,y)\to(1,1)} \dfrac{(x-1)(y-1)}{(x-1)^2+m^2(x-1)^2}$
On $y - x = m(x-1)$,
$\lim\limits_{(x,y)\to(1,1)} \dfrac{(x-1)(y-1)}{(x-1)^2+m^2(x-1)^2} = \lim\limits_{(x,y)\to(1,1)} \dfrac{(x-1)m(x-1)}{(x-1)^2+m^2(x-1)^2} = \dfrac{m}{1+m^2}$.
The limit does not exist.

20. On $x = 0$, $\lim\limits_{(x,y)\to(0,3)} \dfrac{xy-3y}{x^2+y^2-6y+9} = \lim\limits_{(x,y)\to(0,3)} \dfrac{-3y}{(y-3)^2}$. The limit does not exist.

21.
$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{x^3y+xy^3-3x^2-3y^2}{x^2+y^2} &= \lim_{(x,y)\to(0,0)} \frac{xy(x^2+y^2)-3(x^2+y^2}{x^2+y^2}\\ &= \lim_{(x,y)\to(0,0)} (xy-3) = -3\end{aligned}$$

22. $\lim\limits_{(x,y)\to(-2,2)} \dfrac{y^3+2x^3}{x+5xy^2} = \dfrac{8-16}{-2-40} = \dfrac{4}{21}$

23. $\lim\limits_{(x,y)\to(1,1)} \ln(2x^2-y^2) = \ln(2-1) = 0$

24. $\lim\limits_{(x,y)\to(1,2)} \dfrac{\sin^{-1}(x/y)}{\cos^{-1}(x-y} = \dfrac{\sin^{-1}(1/2)}{\cos^{-1}(-1)} = \dfrac{\pi/6}{\pi} = \dfrac{1}{6}$

In Problems 25-30 let $x = r\cos\theta$ and $y = r\sin\theta$. Then $x^2+y^2 = r^2$ and $(x,y)\to(0,0)$ if and only if $r\to 0$. We also use the facts that $|\cos\theta| \le 1$ and $|\sin\theta| \le 1$ for all θ.

25.
$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{(x^2-y^2)^2}{x^2+y^2} &= \lim_{r\to 0} \frac{(r^2\cos^2\theta - r^2\sin^2\theta)^2}{r^2} = \lim_{r\to 0} \frac{r^4(\cos^2\theta-\sin^2\theta)^2}{r^2}\\ &= \lim_{r\to 0} r^2\cos^2 2\theta = 0\end{aligned}$$

26. $$\lim_{(x,y)\to(0,0)} \frac{\sin(3x^2+3y^2)}{x^2+y^2} = \lim_{r\to 0} \frac{\sin 3r^2}{r^2} \quad \boxed{\text{Use L'Hôpital's Rule}}$$
$$= \lim_{r\to 0} \frac{6r\cos 3r^2}{2r} = \lim_{r\to 0} 3\cos 3r^2 = 3$$

27. $$\lim_{(x,y)\to(0,0)} \frac{6xy}{\sqrt{x^2+y^2}} = \lim_{r\to 0} \frac{6r^2\cos\theta\sin\theta}{\sqrt{r^2}} = \lim_{r\to 0} 3|r|\sin 2\theta = 0$$

28. $$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{\sqrt{x^2+y^2}} = \lim_{r\to 0} \frac{r^2\cos^2\theta - r^2\sin^2\theta}{\sqrt{r^2}} = \lim_{r\to 0} |r|\cos 2\theta = 0$$

29. $$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = \lim_{r\to 0} \frac{r^3\cos^3\theta}{r^2} = \lim_{r\to 0} r\cos^3\theta = 0$$

30. $$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = \lim_{r\to 0} \frac{r^3\cos^3\theta + r^3\sin^3\theta}{r^2} = \lim_{r\to 0} r(\cos^3\theta + \sin^3\theta) = 0$$

31. $\{(x,y) \mid x \geq 0 \text{ and } y \geq -x\}$

32. $\{(x,y) \mid x \neq 0 \text{ and } y \neq 0\}$

33. $\{(x,y) \mid y \neq 0 \text{ and } x/y \neq \pi/2 + k\pi,\ k \text{ and integer}\}$

34. $\{(x,y) \mid x \text{ and } y \text{ are real}\}$

35. (a) For $x^2+y^2<1$, $f(x,y)=0$ is continuous
 (b) For $x \geq 0$, $f(x,y)$ is not continuous since it is discontinuous at $(2,0)$.
 (c) For $y > x$, $f(x,y)$ is not continuous since it is discontinuous at $(2,3)$.

36. (a) For $y \geq 3$, $f(x,y)$ is not continuous since it is not defined at $(0,3)$.
 (b) For $|x|+|y|<1$, $f(x.y)$ is discontinuous since it is not defined at $(0,0)$.
 (c) For $(x-2)^2+y^2<1$, $f(x,y)$ is discontinuous since it is not defined at $(2,0)$.

37. Since
$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{6x^2y^3}{(x^2+y^2)^2} = \lim_{r\to 0} \frac{6r^5\cos^2\theta\sin^3\theta}{r^4} = \lim_{r\to 0} 6r\cos^2\theta\sin^3\theta = 0 = f(0,0)$$
the function is continuous at $(0,0)$.

38. Since $f(x,0)=0$ for all x and $f(0,y)=0$ for all y, $f(x,0)$ and $f(0,y)$ are continuous at $x=0$ and $y=0$, respectively. On $y=x$,
$$\lim_{(x,y)\to 0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2}{2x^2+2x^2} = \frac{1}{4},$$
so $f(x,y)$ is not continuous at $(0,0)$.

39. Choose $\epsilon > 0$. Using $x = r\cos\theta$ and $y = r\sin\theta$ we have

$$\frac{3xy^2}{2x^2+2y^2} = \frac{3t\cos\theta r^2\sin^2\theta}{2r^2}\frac{3}{2}r\cos\theta\sin^2\theta.$$

Let $\delta = \frac{2\epsilon}{3}$. Now, whenever $r = \sqrt{x^2+y^2} < \delta$, we have

$$|\frac{3xy^2}{2x^2+2y^2}| = \frac{3}{2}|r\cos\theta\sin^2\theta| \leq \frac{3}{2}|r| < \frac{3}{2}\delta = \frac{3}{2}\left(\frac{2\epsilon}{3}\right) = \epsilon.$$

Thus $\lim\limits_{(x,y)\to(0,0)} \frac{3xy^2}{x^2+y^2} = 0$.

40. Choose $\epsilon > 0$. Using $x = r\cos\theta$ and $y = r\sin\theta$ we have

$$\frac{x^2y^2}{x^2+y^2} = \frac{r^2\cos^2\theta r^2\sin^2\theta}{r^2} = r^2\cos^2\theta\sin^2\theta.$$

Now, whenever $r = \sqrt{x^2+y^2} < \sqrt{\epsilon}$ (for $\delta = \sqrt{\epsilon}$), $\left|\frac{x^2y^2}{x^2+y^2}\right| = r^2\cos^2\theta\sin^2\theta \leq r^2 \leq \epsilon$. Thus, $\lim\limits_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2} = 0$.

41. Where $y \neq x$, we have

$$f(x,y) = \frac{x^3-y^3}{x-y} = \frac{(x-y)(x^2+xy+y^2)}{x-y} = x^2+xy+y^2.$$

When $y = x$, we have

$$x^2+xy+y^2 = x^2+x^2+x^2 = 3x^2 = f(x,y).$$

Therefore, $f(x,y) = x^2+xy+y^2$ throughout the entire plane. Since x^2+xy+y^2 is a polynomial, f must be continuous throughout the plane and thus has no discontinuities.

42. Choose $\epsilon > 0$. Then for $\delta = \epsilon$, whenever $0 < \sqrt{(x-a)^2+(y-b)^2} < \delta$, we have

$$|f(x,y)-b| = |y-b| \leq \sqrt{(x-a)^2+(y-b)^2} < \delta = \epsilon.$$

Thus, $\lim\limits_{(x,y)\to(a,b)} y = b$.

13.3 Partial Derivatives

1. $\frac{\partial z}{\partial x} = \lim\limits_{\triangle\to 0}\frac{7(x = \triangle x)+8y^2-7x-8y^2}{\triangle x} = \lim\limits_{\triangle\to 0}\frac{7\triangle x}{\triangle x} = 7$

$\frac{\partial z}{\partial y} = \lim\limits_{\triangle y\to 0}\frac{7x+8(y+\triangle y)^2-7x-8y^2}{\triangle y} = \lim\limits_{\triangle y\to 0}\frac{16y\triangle y+8(\triangle y)^2}{\triangle y}$

$= \lim\limits_{\triangle y\to 0}(16y+8\triangle y) = 16y$

2. $\dfrac{\partial z}{\partial x} = \lim\limits_{\triangle x \to 0} \dfrac{(x+\triangle x)y - xy}{\triangle x} = \lim\limits_{\triangle x \to 0} \dfrac{y \triangle x}{\triangle x} = y;$
$\dfrac{\partial z}{\partial y} = \lim\limits_{\triangle y \to 0} \dfrac{x(y+\triangle y) - xy}{\triangle y} = \lim\limits_{\triangle y \to 0} \dfrac{x \triangle y}{\triangle y} = x$

3. $$\begin{aligned}\frac{\partial z}{\partial x} &= \lim_{\triangle x \to 0} \frac{3(x+\triangle x)^2 y + 4x + \triangle x)y^2 - 3x^2 y - 4xy^2}{\triangle x} \\ &= \lim_{\triangle x \to 0} \frac{3x^2 y + 6x(\triangle x)y + 3(\triangle x)^2 y + 4xy^2 + 4(\triangle x)y^2 - 3x^2 y - 4xy^2}{\triangle x} \\ &= \lim_{\triangle x \to 0} \frac{6x(\triangle x)y + 3(\triangle x)^2 y + 4(\triangle x)y^2}{\triangle x} = \lim_{\triangle x \to 0}(6xy + 3(\triangle x)y + 4y^2) = 6xy + 4y^2 \\ \frac{\partial z}{\partial y} &= \lim_{\triangle y \to 0} \frac{3x^2(y+\triangle y) + 4x(y+\triangle y)^2 - 3x^2 y - 4xy^2}{\triangle y} \\ &= \lim_{\triangle y \to 0} \frac{3x^2 y + 3x^2 \triangle y + 4xy^2 + 8xy \triangle y + 4x(\triangle y)^2 - 3x^2 y - 4xy^2}{\triangle y} \\ &= \lim_{\triangle y \to 0} \frac{3x^2 \triangle y + 8xy \triangle y + 4x(\triangle y)^2}{\triangle y} = \lim_{\triangle y \to 0}(3x^2 + 8xy + 4x \triangle y) = 3x^2 + 8xy\end{aligned}$$

4. $$\begin{aligned}\frac{\partial z}{\partial x} &= \lim_{\triangle x \to 0} \frac{\dfrac{x+\triangle}{x+\triangle+y} - \dfrac{x}{x+y}}{\triangle x} = \lim_{\triangle x \to 0} \frac{x^2 + x \triangle x + xy + (\triangle x)y - x^2 - x \triangle x - xy}{(x+\triangle x+y)(x+y) \triangle x} \\ &= \lim_{\triangle x \to 0} \frac{(\triangle x)y}{(x+\triangle x+y)(x+y) \triangle x} = \frac{y}{(x+y)^2} \\ \frac{\partial z}{\partial y} &= \lim_{\triangle y \to 0} \frac{\dfrac{x}{x+y+\triangle y} - \dfrac{x}{x+y}}{\triangle y} = \lim_{\triangle y \to 0} \frac{x^2 + xy - x^2 - xy - x \triangle y}{(x+y+\triangle y)(x+y) \triangle y} \\ &= \lim_{\triangle y \to 0} \frac{-x \triangle y}{(x+y+\triangle y)(x+y) \triangle y} = -\frac{x}{(x+y)^2}\end{aligned}$$

5. $z_x = 2x - y^2;\quad z_y = -2xy + 20y^4$

6. $z_x = -3x^2 + 12xy^3;\quad z_y = 18x^2y^2 + 10y$

7. $z_x = 20x^3y^3 - 2xy^6 + 30x^4;\quad z_y = 15x^4y^2 - 6x^2y^5 - 4$

8. $z_x = 3x^2y^2 \sec^2(x^3y^2);\quad z_y = 2x^3 \sec^2(x^3y^2)$

9. $z_x = \dfrac{2}{\sqrt{x}(3y^2+1)};\quad z_y = -\dfrac{24y\sqrt{x}}{(3y^2+1)^2}$

10. $z_x = 12x^2 - 10x + 8;\quad z_y = 0$

11. $z_x = -(x^3 - y^2)^{-2}(3x^2) = -3x^2(x^3 - y^2)^{-2};\quad z_y = -(x^3 - y^2)^{-2}(-2y) = 2y(x^3 - y^2)^{-2}$

12. $z_x = 6(-x^4 + 7y^2 + 3y)^5(-4x^) = -24x^(-x^4 + 7y^2 - 3y)^5;\quad z_y = 6(-x^4 + 7y^2 + 3y)^5(14y + 3)$

13. $z_x = 2(\cos 5x)(-\sin 5x)(5) = -10 \sin 5x \cos 5x;\quad z_y = 2(\sin 5y)(\cos 5y)(5) = 10 \sin 5y \cos 5y$

14. $z_x = (2x \tan^{-1} y^2)e^{x^2 \tan^{-1} y^2}$; $z_y = \dfrac{2x^2 y}{1+y^4} e^{x^2 \tan^{-1} y^2}$

15. $f_x = x(3x^2 y e^{x^3 y} + e^{x^3 y}$; $f_y = x^4 e^{x^3 y}$

16. $f_\theta = \phi^2 \left(\cos \dfrac{\theta}{\phi}\right)\left(\dfrac{1}{\phi}\right)$; $f_\phi = \phi^2 \left(\cos \dfrac{\theta}{\phi}\right)\left(-\dfrac{\theta}{\phi^2}\right) + 2\phi \sin \dfrac{\theta}{\phi} = -\theta \cos \dfrac{\theta}{\phi} + 2\phi \sin \dfrac{\theta}{\phi}$

17. $f_x = \dfrac{(x+2y)3 - (3x-y)}{(x+2y)^2} = \dfrac{7y}{(x+2y)^2}$; $f_y = \dfrac{(x+2y)(-1) - (3x-y)(2)}{(x+2y)^2} = \dfrac{-7x}{(x+2y)^2}$

18. $f_x = \dfrac{(x^2-y^2)^2 y - xy\left[2(x^2-y^2)2x\right]}{(x^2-y^2)^4} = \dfrac{-3x^2 y - y^3}{(x^2-y^2)^3}$;
$f_y = \dfrac{(x^2-y^2)x - xy\left[2(x^2-y^2)(-2y)\right]}{(x^2-y^2)^4} = \dfrac{3xy^2 + x^3}{(x^2-y^2)^3}$

19. $g_u = \dfrac{8u}{4u^2 - 5v^3}$; $g_v = \dfrac{15v^2}{4u^2 + 5v^3}$

20. $h_r = \dfrac{1}{2s\sqrt{r}} + \dfrac{\sqrt{s}}{r^2}$; $h_x = -\dfrac{\sqrt{r}}{s^2} - \dfrac{1}{2s\sqrt{r}}$

21. $w_x = \dfrac{y}{\sqrt{x}}$; $w_y = 2\sqrt{x} - y\left(\dfrac{1}{z}e^{y/z}\right) = 2\sqrt{x} - \left(\dfrac{y}{z} + 1\right)e^{y/z}$; $w_z = -ye^{y/z}\left(-\dfrac{y}{z^2}\right) = \dfrac{y^2}{z^2}e^{y/z}$

22. $w_x = xy\left(\dfrac{1}{x}\right) + (\ln xz)y = y + y \ln xz$; $w_y = x \ln xz$; $w_z = \dfrac{xy}{z}$

23. $F_u = 2uw^2 - v^3 - vwt^2 \sin(ut^2)$; $F_v = -3uv^2 + w\cos(ut^2)$;
$F_x = 3(2x^2t)^3(4xt) = 16xt(2x^2t)^3 = 128x^7t^4$; $F_t = -2uvwt \sin(ut^2) + 64x^8t^3$

24. $G_p = 2pq^3 e^{2r^4 s^5}$
$G_q = 3p^2q^2 e^{2r^4 s^5}$
$G_r = p^2q^3(8r^3s^5)e^{2r^4 s^5} = 8p^2q^3r^3s^5 e^{2r^4 s^5}$
$G_s = p^2q^3(10r^4s^4)e^{2r^4 s^5} = 10p^1q^3r^4s^4 e^{2r^4 s^5}$

25. $z_y = 16x^3y^3$, $z_y(1,-1) = -16$

26. $z_x = 12x^2y^4$, $z_x(1,-1) = 12$

27. $f_y = \dfrac{(x+y)18x - 18xy}{(x+y)^2} = \dfrac{18x^2}{(x+y)^2}$, $f_y(-1,4) = 2$. An equation of the tangent line is given by $x = -1$ and $z + 24 = 2(y-4)$. Parametric equations of the line are $x = -1$, $y = 4 + t$, $z = -24 + 2t$.

28. $f_x = \dfrac{(x+y)18y - 18xy}{(x+y)^2} = \dfrac{18y^2}{(x+y)^2}$, $f_x(-1,4) = 32$. An equation of the tangent line is given by $y = 4$ and $z + 24 = 32(x+1)$. Symmetric equations of the line are $x + 1 = \dfrac{z+24}{32}$, $y = 4$.

29. $z_x = \dfrac{-x}{\sqrt{9 - x^- y^2}}, \quad z_x(2,2) = -2$

30. $z_y = \dfrac{-y}{\sqrt{9 - x^2 - y^2}}, \quad z_y(\sqrt{2}, \sqrt{3}) = -\dfrac{\sqrt{3}}{2}$

31. $\dfrac{\partial z}{\partial x} = ye^{xy}; \quad \dfrac{\partial^2 z}{\partial x^2} = y^2 e^{xy}$

32. $\dfrac{\partial z}{\partial y} = -2x^4 y^{-3}; \quad \dfrac{\partial^2 z}{\partial y^2} = 6x^4 y^{-4}; \quad \dfrac{\partial^3 z}{\partial y^3} = -24x^4 y^{-5}$

33. $f_x = 10xy^2 - 2y^3; \quad f_{xy} = 20xy - 6y^2$

34. $f(p,q) = \ln(p+q) - 2\ln q, \quad f_q = \dfrac{1}{p+q} - \dfrac{2}{q}, \quad f_{qp} = -\dfrac{1}{(p+q)^2}$

35. $w_t = 3u^2 v^3 t^2, \quad w_{tu} 6uv^3 t^2; \quad w_{tuv} 18uv^2 t^2$

36. $w_v = -\dfrac{u^2 \sin(u^2 v)}{t^3}; \quad w_{vv} - \dfrac{u^4 \cos(u^2 x)}{t^3}; \quad w_{vvt} = \dfrac{3u^4 \cos(u^2 v)}{t^4}$

37. $F_r = 2re^{r^2} \cos\theta; \quad F_{r\theta} - 2re^{r^2} \sin\theta; \quad F_{r\theta r} - 2r(2re^{r^2}) \sin\theta - 2e^{r^2} \sin\theta = -2e^{r^2}(2r^2+1)\sin\theta$

38. $H_t = \dfrac{(s-t) - (s+t)(-1)}{(s-t)^2} = \dfrac{2s}{(s-t)^2}; \quad H_{tt} = \dfrac{4s}{(s-t)^3};$
$H_{tts} = \dfrac{(s-t)^4 - 4x(3)(s-t)^2}{(x-t)^6} = \dfrac{-8s-4t}{(s-t)^4}$

39. $\dfrac{\partial z}{\partial y} = -5x^4 y^2 + 8xy; \quad \dfrac{\partial^2 z}{\partial x \partial y} = -60x^3 y^2 + 8y; \quad \dfrac{\partial z}{\partial x} = 6x^5 - 20x^3 y^3 + 4y^2; \quad \dfrac{\partial^2 z}{\partial y \partial x} = -60x^3 y^2 + 8y$

40. $\dfrac{\partial z}{\partial y} = \dfrac{2x}{1+4x^2y^2}; \quad \dfrac{\partial z}{\partial x \partial y} = \dfrac{(1+4x^2y^2)2 - 2x(8xy^2)}{(1+4x^2y^2)^2} = \dfrac{2-8x^2y^2}{(1+4x^2y^2)^2}; \quad \dfrac{\partial z}{\partial x} = \dfrac{2y}{1+4x^2y^2}$
$\dfrac{\partial z}{\partial y \partial x} = \dfrac{(1+4x^2y^2)2 - 2y(8x^2y)}{(1+4x^2y^2)^2} = \dfrac{2-8x^2y^2}{(1+4x^2y^2)^2}$

41. $w_u = 3u^2 v^4 - 8uv^2 t^3, \quad w_{uv} = 12u^2 v^3 - 16uvt^3, \quad w_{uvt} = -48uvt^2; \quad w_t = -12u^2 v^2 t^2 + v^2,$
$w_{tv} = -24u^2 vt^2 + 2v, \quad w_{tvu} = -48uvt^2; \quad w_v = 4u^3 v^3 - 8u^2 vt^3 + 2vt, \quad w_{vu} = 12u^2 v^3 - 16uvt^3,$
$w_{vut} = -48uvt^2$

42. $F_\eta = 6\eta^2(\eta^3 + \xi^2 + \tau) = 6\eta^5 + 6\eta^2\xi^2 + 6\eta^2\tau, \quad F_{\eta\xi} = 12\eta^2\xi, \quad F_{\eta\xi\eta} = 24\eta\xi; \quad F_\xi = 4\xi(\eta^3 + \xi^3 + \tau) =$
$4\eta^3\xi + 4xi^3 + 4xi\tau, \quad F_{\xi\eta} = 12\eta^2\xi, \quad F_{\xi\eta\eta} 24\eta\xi; \quad F_{\eta\eta} = 30\eta^4 + 12\eta\xi^2, \quad F_{\eta\eta\xi} = 24\eta\xi$

43. $2x + 2zz_x = 0, \quad z_x = -x/z; \quad 2y + 2zz_y = 0, \quad z_y = -y/z$

44. $2zz_x = 2x + y^2 z_x \Longrightarrow (2z - y^2) z_x = 2x \Longrightarrow z_x = \dfrac{2x}{2z - y^2};$
$2zz_y = y^2 z_y + 2yz \Longrightarrow (2z - y^2) z_y = 2yz \Longrightarrow z_y = \dfrac{2yz}{2z - y^2}$

45. $2zz_u + 2uv^3 - uvz_u - vz = 0 \Longrightarrow (2z - uv)z_u = vz - 2uv^3 \Longrightarrow z_u = \dfrac{vz - 2uv^3}{2z - uv}$;
$2zz_v + 3u^2v^2 - uvz_v - uz = 0 \Longrightarrow (2z - uv)z_v = uz - 3u^2v^2 \Longrightarrow z_v = \dfrac{uz - 3u^2v^2}{2z - uv}$

46. $se^z z_s + e^z - te^{st} + 12s^2t = z_s \Longrightarrow (se^z - 1)z_s = te^s - e^z - 12s^2t \Longrightarrow z_s = \dfrac{te^{st} - e^z - 12s^2t}{se^z - 1}$;
$se^z z_t - se^{st} + 4s^3 = z_t \Longrightarrow (se^z - 1)z_t = se^{st} - 4s^3 \Longrightarrow z_t = \dfrac{se^{st} - 4s^3}{se^z - 1}$

47. $a_x = y\sin\theta,\quad A_y = x\sin\theta,\quad A_\theta = xy\cos\theta$

48. $V_h = (\pi/3)(r^2 + rR + R^2),\quad V_r = (\pi/3)h(2r + R),\quad V_R = (\pi/3)h(r + 2R)$

49. $\dfrac{\partial u}{\partial x} = 2\pi(\cosh 2\pi y + \sinh 2\pi y)\cos 2\pi x;\quad \dfrac{\partial^2 u}{\partial x^2} = -4\pi^2(\cosh 2\pi y + \sinh 2\pi y)\sin 2\pi x;$
$\dfrac{\partial u}{\partial y} = (2\pi\sinh 2\pi y + 2\pi\cosh 2\pi y)\sin 2\pi x;\quad \dfrac{\partial^2 u}{\partial y^2} = (4\pi^2\cosh 2\pi y + 4\pi^2\sinh 2\pi y)\sin 2\pi x;$
$\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = -4\pi^2(\cosh 2\pi y + \sinh 2\pi y)\sin 2\pi x + 4\pi^2(\cosh 2\pi y + \sinh 2\pi y)\sin 2\pi x = 0$

50. $\dfrac{\partial u}{\partial x} = -\dfrac{n\pi}{L}e^{-(n\pi x/L}\sin\left(\dfrac{n\pi}{L}\right);\quad \dfrac{\partial^2 u}{\partial x^2} = \dfrac{n^2\pi^2}{L^2}e^{-(n\pi x/L)}\sin\left(\dfrac{n\pi}{L}\right)y;$
$\dfrac{\partial u}{\partial y} = \dfrac{n\pi}{L}e^{-(n\pi x/L)}\cos\left(\dfrac{n\pi}{L}\right)y;\quad \dfrac{\partial^2 u}{\partial y^2} = -\dfrac{n^2\pi^2}{L^2}e^{-(n\pi x/L)}\sin\left(\dfrac{n\pi}{L}\right)y;$
$\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = \dfrac{n^2\pi^2}{L^2}e^{-(n\pi x/L)}\sin\left(\dfrac{n\pi}{L}\right) - \dfrac{n^2\pi^2}{L^2}e^{-(n\pi x/L)}\sin\left(\dfrac{n\pi}{L}\right) = 0$

51. $\dfrac{\partial z}{\partial x} = \dfrac{2x}{x^2 + y^2},\quad \dfrac{\partial^2 z}{\partial x^2} = \dfrac{(x^2 + y^2)2 - 2x(2x)}{(x^2 + y^2)^2} = \dfrac{2y^2 - 2x^2}{(x^2 + y^2)};\quad \dfrac{\partial z}{\partial y} = \dfrac{2y}{x^2 + y^2},$
$\dfrac{\partial^2 z}{\partial y^2} = \dfrac{(x^2 + y^2)2 - 2y(2y)}{(x^2 + y^2)^2} = \dfrac{2x^2 - 2y^2}{(x^2 + y^2)^2};\quad \dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = \dfrac{2y^2 - 2x^2 + 2x^2 - 2y^2}{(x^2 + y^2)^2} = 0$

52. $\dfrac{\partial z}{\partial x} = 2ye^{x^2 - y^2}\sin 2xy + 2xe^{x^2 - y^2}\cos 2xy = 2e^{x^2 - y^2}(x\cos 2xy - y\sin 2xy),$
$\dfrac{\partial^2 z}{\partial x^2} = 2e^{x^2 - y^2}(-2xy\sin 2xy + \cos 2xy - 2y^2\cos 2xy) + 4xe^{x^2 - y^2}(x\cos 2xy - y\sin 2xy);$
$\dfrac{\partial z}{\partial y} = -2xe^{x^- y^2}\sin 2xy - 2ye^{x^- y^2}\cos 2xy = -2e^{x^- y^2}(x\sin 2xy + y\cos 2xy),$
$\dfrac{\partial^2 z}{\partial y^2} = -2e^{x^- y^2}(2x^2\cos 2xy - 2xy\sin 2xy + \cos 2xy) + 4ye^{x^- y^2}(x\sin 2xy + y\cos 2xy);$

$$\begin{aligned}\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} =& 2e^{x^- y^2}(-2xy\sin 2xy + \cos 2xy - 2y^2\cos 2xy + 2x^2\cos 2xy - 2xy\sin 2xy \\ &- 2x^2\cos 2xy + 2xy\sin 2xy - \cos 2xy + 2xy\sin 2xy - 2y^2\cos 2xy) = 0\end{aligned}$$

53. $\dfrac{\partial u}{\partial x} = -\dfrac{x}{(x^2 + y^2 + z^2)^{3/2}};\quad \dfrac{\partial u}{\partial y} = -\dfrac{y}{(x^2 + y^2 + z^2)^{3/2}};\quad \dfrac{\partial u}{\partial z} = -\dfrac{z}{(x^2 + y^2 + z^2)^{3/2}};$
$\dfrac{\partial^2 u}{\partial x^2} = \dfrac{2x^2 - y^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}};\quad \dfrac{\partial^2 u}{\partial y^2} = \dfrac{-x^2 + 2y^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}};\quad \dfrac{\partial^2 u}{\partial z^2} = \dfrac{-x^2 - y^2 + 2z^2}{(x^2 + y^2 + z^2)^{5/2}};$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{2x^2-y^2-z^2-x^2+2y^2-z^2-x^2-y^2+2z^2}{(x^2+y^2+z^2)^{5/2}}=0$$

54. $\dfrac{\partial u}{\partial x}=\sqrt{m^2+n^2}e^{\sqrt{m^2+n^2}x}\cos my\sin nz$; $\dfrac{\partial^2 u}{\partial x^2}=(m^2+n^2)e^{\sqrt{m^2+n^2}x}\cos my\sin nz$;
$\dfrac{\partial u}{\partial y}=-me^{\sqrt{m^2+n^2}x}\sin my\sin nz$; $\dfrac{\partial^2 u}{\partial y^2}=-m^2e^{\sqrt{m^2+n^2}x}\cos my\sin nz$;
$\dfrac{\partial u}{\partial z}=ne^{\sqrt{m^2+n^2}x}\cos my\cos nz$; $\dfrac{\partial^2 u}{\partial z^2}=-n^2e^{\sqrt{m^2+n^2}x}\cos my\sin nz$;
$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=(m^2+n^2)e^{\sqrt{m^2+n^2}x}\cos my\sin nz-m^2e^{\sqrt{m^2+n^2}x}\cos my\sin nz$$
$$-n^2e^{\sqrt{m^2+n^2}x}\cos my\sin nz=0$$

55. $\dfrac{\partial u}{\partial x}=\cos at\cos x$, $\dfrac{\partial^2 u}{\partial x^2}=-\cos at\sin x$; $\dfrac{\partial u}{\partial t}=-a\sin at\sin x$, $\dfrac{\partial^2 u}{\partial t^2}=-a^1\cos at\sin x$;
$a^2\dfrac{\partial^2 u}{\partial x^2}=a^2(-\cos at\sin x)=\dfrac{\partial^2 u}{\partial t^2}$

56. $\dfrac{\partial u}{\partial x}=-\sin(x+at)+\cos(x-at)$, $\dfrac{\partial^2 u}{\partial x^2}=-\cos(x+at)-\sin(x-at)$;
$\dfrac{\partial u}{\partial t}=-a\sin(x+at)-a\cos(x-at)$, $\dfrac{\partial^2 u}{\partial t^2}=-a^2\cos(x+at)-a^2\sin(x-at)$; $a^2\dfrac{\partial^2 u}{\partial x^2}=$
$-a^2\cos(x+at)-a^2\sin(x-at)=\dfrac{\partial^2 u}{\partial t^2}$

57. $\dfrac{\partial C}{\partial x}=-\dfrac{2x}{kt}t^{-/12}e^{-x/kt}$, $\dfrac{\partial^2 C}{\partial x^2}=\dfrac{4x^2}{k^2t^2}t^{-1/2}e^{-x^2/kt}-\dfrac{2}{kt}t^{-1/2}e^{-x^2/kt}$;
$\dfrac{\partial C}{\partial t}=t^{-1/2}\dfrac{x^2}{kt^2}e^{-x^2/kt}-\dfrac{t^{-3/2}}{2}e^{-x^2/kt}$; $\dfrac{k}{4}\dfrac{\partial^2 C}{\partial x^2}=\dfrac{x^2}{kt^2}t^{-1/2}e^{-x^2/kt}-\dfrac{t^{-1/2}}{2t}e^{-x^2/kt}=\dfrac{\partial C}{\partial t}$

58. (a) $P_v=-k(T/V^2)$
(b) $PV=kt$, $PV_T=k$, $V_T=k/P$
(c) $PV=kT$, $V=kT_p$, $T_p=V/k$

59. (a) $\dfrac{\partial u}{\partial t}=\begin{cases}-gx/z, & 0\le x\le at\\ -gt, & x>at\end{cases}$
For $x>at$, the motion is that of a freely falling body.
(b) For $x>at$, $\dfrac{\partial u}{\partial x}=0$. For $x>at$, the string is horizontal.

60. $\dfrac{\partial S}{\partial h}=0.0790975w^{0.425}h^{-0.275}$; $S_h(60,36)+0.0790975(60)^{0.425}(36)^{-0.275}\approx 0.1682$
The approximate increase in skin-area as h increases from 36 to 37 inches is 0.1682 sq ft.

61. (a) $\dfrac{\partial^2 z}{\partial x^2}=\lim\limits_{\Delta x\to 0}\dfrac{f_x(x+\Delta x,y)-f_x(x,y)}{\Delta x}$
(b) $\dfrac{\partial^2 z}{\partial y^2}=\lim\limits_{\Delta y\to 0}\dfrac{f_y(x,y+\Delta y)-f_x(x,y)}{\Delta y}$

(c) $\dfrac{\partial^2 z}{\partial x \partial y} = \lim_{\Delta x \to 0} \dfrac{f_y(x+\Delta x, y) - f_y(x,y)}{\Delta x}$

62. Integrating $z_x = 2xy^3 + 2y + 1/x$ with respect to x, we obtain $z = x^2y^3 + 2xy + \ln x + \phi(y)$. Then $3x^2y^2 + 2x + 1 = z_y = 3x^2y^2 + 2x + \phi'(y)$. Since $\phi'(y) = 1$, $\phi(y) = y + C$, and $z = x^2y^3 + 2xy + \ln x + y + C$.

63. Consider the mixed partials:

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = 2y \quad \text{and} \quad \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = 2x.$$

Since $\dfrac{\partial z}{\partial x}$, $\dfrac{\partial z}{\partial y}$, $\dfrac{\partial^2 z}{\partial y \partial x}$, and $\dfrac{\partial^2 z}{\partial x \partial y}$ are all continuous on an open set, we should have $\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial^2 z}{\partial x \partial y}$ on that set. But the mixed partials are equal only on the line $y = x$, which contains no open set in the plane. Therefore, such a function cannot exist.

64. (a) There are 10 different third-order partial derivatives: F_{xxx}, F_{xxy}, F_{xxz}, F_{xyy}, F_{xyz}, F_{xzz}, F_{yyy}, F_{yyz}, F_{yzz}, F_{zzz}

(b) Since the mixed partials are equal, the order in which differentiation occurs is irrelevant. The n^{th} order partial derivatives are given by

$$\frac{\partial^n z}{\partial x^n}, \ \frac{\partial^n z}{\partial x^{n-1}\partial y}, \ \frac{\partial^n z}{\partial x^{n-2}\partial y^2}, \ldots, \frac{\partial^n z}{\partial x \partial y^{n-2}}, \ \frac{\partial^n z}{\partial y^n}.$$

Hence, there are $n+1$ different n^{th} order partial derivatives.

65. (a) There slopes of the surface in the x and y directions are zero everywhere. This implies that the surface must have constant height everywhere. Therefore f must have the form $f(x,y) = c$.

(b) Since the mixed partials are both zero, we have

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = 0 \quad \text{and} \quad dfrac\partial\partial y\left(\frac{\partial z}{\partial x}\right) = 0$$

which implies $\dfrac{\partial z}{\partial y}$ is a function of y alone and $\dfrac{\partial z}{\partial x}$ is a function of x alone. Therefore, z has no term that depends on both x and y. Hence z is of the form $z = g(x) + h(y) + c$ where g and h are twice continuously differentiable functions of a single variable.

66. The level curves suggest that the surface height is decreasing as we move slightly to the right of the point, and increasing as we move slightly up from the point. This implies $\dfrac{\partial z}{\partial x} < 0$ and $\dfrac{\partial z}{\partial y} > 0$.

67. $\left.\dfrac{\partial z}{\partial x}\right|_{(0,0)} = \lim_{\Delta x \to 0} \dfrac{f(0+\Delta x, 0) - f(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \dfrac{0/2(\Delta x)^2}{\Delta x} = 0;$

$\left.\dfrac{\partial z}{\partial y}\right|_{(0,0)} = \lim_{\Delta y \to 0} \dfrac{f(0, 0+\Delta y) - f(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \dfrac{0/2(\Delta y)^2}{\Delta y} = 0$

68. (a) $\dfrac{\partial z}{\partial x} = \dfrac{y^5 - 4x^2y^3 - x^4}{(x^2+y^2)^2}$; $\left.\dfrac{\partial z}{\partial x}\right|_{(0,y)} = y$; $\dfrac{\partial z}{\partial y} = \dfrac{-x^5 + 4x^3y^2 + xy^4}{x^2+y^2)^2}$; $\left.\dfrac{\partial z}{\partial y}\right|_{(x,0)} = -x$

(b) $\dfrac{\partial^2 z}{\partial y \partial x} = 1$; $\dfrac{\partial^2 z}{\partial x \partial y} = -1 \Longrightarrow \dfrac{\partial^2 z}{\partial y \partial x} \neq \dfrac{\partial^2 z}{\partial x \partial y}$

13.4 Linearization and Differentials

1. $\dfrac{\partial f}{\partial x} = 4y^2 - 6x^2y$ so $\dfrac{\partial f}{\partial x}(1,1) = -2$
$\dfrac{\partial f}{\partial y} = 8zy - 2x^3$ so $\dfrac{\partial f}{\partial y}(1,1) = 6$
$f(1,1) = 2$ The linearization is $L(x,y) = 2 - 2(x-1) + 6(y-1) = -2x + 6y - 2$

2. $\dfrac{\partial f}{\partial x} = \dfrac{3x^2y}{2\sqrt{x^3y}}$ so $\dfrac{\partial f}{\partial x}(2,2) = 3$
$\dfrac{\partial f}{\partial y} = \dfrac{x^3}{2\sqrt{x^3y}}$ so $\dfrac{\partial f}{\partial y}(2,2) = 1$
$f(2,2) = 4$ The linearization is $L(x,y) = 4 + 3(x-2) + (y-2) = 3x + y - 4$

3. $\dfrac{\partial f}{\partial x} = \sqrt{x^2+y^2} + \dfrac{x^2}{\sqrt{x^2+y^2}}$ so $\dfrac{\partial f}{\partial x}(8,15) = \dfrac{353}{17}$
$\dfrac{\partial f}{\partial y} = \dfrac{xy}{\sqrt{x^2+y^2}}$ so $\dfrac{\partial f}{\partial y}(8,15) = \dfrac{120}{17}$
$f(8,15) = 136$ The linearization is $L(x,y) = 136 + \frac{353}{17}(x-8) + \frac{120}{17}(y-15) = \frac{353}{17}x + \frac{120}{17}y - 136$

4. $\dfrac{\partial f}{\partial x} = 3\cos x \cos y$ so $\dfrac{\partial f}{\partial x}\left(\frac{\pi}{4}, \frac{3\pi}{4}\right) = \dfrac{-3}{2}$
$\dfrac{\partial f}{\partial y} = 3\sin x \sin y$ so $\dfrac{\partial f}{\partial y}\left(\frac{\pi}{4}, \frac{3\pi}{4}\right) = \dfrac{-3}{2}$
$f\left(\frac{\pi}{4}, \frac{3\pi}{4}\right) = \dfrac{-3}{2}$ The linearization is $L(x,y) = \frac{-3}{2} - \frac{3}{2}\left(x - \frac{\pi}{4}\right) - \frac{3}{2}\left(y - \frac{3\pi}{4}\right) = \frac{-3}{2}x - \frac{3}{2}y + \frac{3}{2}(\pi - 1)$

5. $\dfrac{\partial f}{\partial x} = \dfrac{2x}{x^2+y^3}$ so $\dfrac{\partial f}{\partial x}(-1,1) = -1$
$\dfrac{\partial f}{\partial y} = \dfrac{3y^2}{x^2+y^3}$ so $\dfrac{\partial f}{\partial y}(-1,1) = \dfrac{3}{2}$
$f(-1,1) = \ln(2)$ The linearization is $L(x,y) = \ln(2) - (x+1) + \frac{3}{2}(y-1) = -x + \dfrac{3}{2}y - \frac{5}{2} + \ln(2)$

6. $\dfrac{\partial f}{\partial x} = 3e^{-2y}\cos 3x$ so $\dfrac{\partial f}{\partial x}\left(0, \frac{\pi}{3}\right) = 3e^{-\frac{2\pi}{3}}$
$\dfrac{\partial f}{\partial y} = -2e^{-2y}\sin 3x$ so $\dfrac{\partial f}{\partial y}\left(0, \frac{\pi}{3}\right) = 0$
$f\left(0, \frac{\pi}{3}\right) = 0$ The linearization is $L(x,y) = 3e^{-\frac{2\pi}{3}}(x-0) = 3xe^{-\frac{2\pi}{3}}$

7. Note that we are trying to approximate $f(102,80)$ where $f(x,y) = \sqrt{x} + \sqrt[4]{y}$. Since $(102,80)$ is reasonably close to $(100,81)$, we can use the linearization of f at $(100,81)$ to approximate

the value at $(102, 80)$. To do this, we compute
$\dfrac{\partial f}{\partial x} = \dfrac{1}{2\sqrt{x}}$, $\dfrac{\partial f}{\partial x}(100, 81) = \dfrac{1}{20}$, $\dfrac{\partial f}{\partial y} = \dfrac{1}{4y^{3/2}}$, $\dfrac{\partial f}{\partial y}(100, 81) = \dfrac{1}{4(27)} = \dfrac{1}{108}$, and $f(100, 81) = 13$
The linearization is $L(x, y) = 13 + \frac{1}{20}(x - 100) + \frac{1}{108}(y - 81)$. For the approximation, we have $L9102, 80) = 13 + \frac{1}{20}(102 - 100) + \frac{1}{108}(80 - 81) = 13 + \frac{1}{10} - \frac{1}{108} = \frac{7069}{540} \approx 13.0907$

8. We are trying to approximate $f(36, 63)$ where $f(x, y) = \dfrac{\sqrt{x}}{\sqrt{y}}$. Use the linearization of f at the point $(36, 64)$. To do this, compute
$\dfrac{\partial f}{\partial x} = \dfrac{1}{2\sqrt{x}\sqrt{y}}$, $\dfrac{\partial f}{\partial x}(36, 64) = \dfrac{1}{96}$, $\dfrac{\partial f}{\partial y} = \frac{-\sqrt{x}}{2y^{3/2}}$; $\dfrac{\partial f}{\partial y}(35, 64) = -\dfrac{3}{512}$, and $f(36, 64) = \dfrac{3}{4}$.
The linearization if $L(x, y) = \dfrac{3}{4} + \dfrac{1}{96}(x - 36) - \dfrac{3}{512}(y - 64)$. For the approximation, we have $L(36, 63) = \dfrac{3}{4} + \dfrac{1}{96}(36 - 36) - \dfrac{3}{512}(63 - 64) = \dfrac{387}{512} \approx .7559$.

9. First, linearize f at $(2, 2)$. To do this, compute
$\dfrac{\partial f}{\partial x} = 2(x^2 + y^2)2x$, $\dfrac{\partial f}{\partial x}(2, 2) = 64$, $\dfrac{\partial f}{\partial y} = 2(x^2 + y^2)2y$, $\dfrac{\partial f}{\partial y}(2, 2) = 64$, and $f(2, 2,) = 64$.
The linearizationis $L(x, y) = 64 + 64)x - 2) + 64)y - 2)$. For the approximation, we have $L(1.95, 2.01) = 64 + 64(-0.05) + 64(0.01) \approx 61.44$.

10. First, linearize f at $\left(\frac{1}{2}, 3\right)$. To do this, compute
$\dfrac{\partial f}{\partial x} = -\pi y \sin(\pi xy)$, $\dfrac{\partial f}{\partial x}\left(\frac{1}{2}, 3\right) = 3\pi$, $\dfrac{\partial f}{\partial y} = -\pi x \sin(\pi xy)$, $\dfrac{\partial f}{\partial y}\left(\frac{1}{2}, 3\right) = \dfrac{\pi}{2}$, and $f\left(\frac{1}{2}, 3\right) = 0$.
The linearization is $L(x, y) = 3\pi\left(x - \dfrac{1}{2}\right) + \dfrac{\pi}{2}(y - 3)$. For the approximation, we have $L(0.52, 2.96) = 3\pi(0.02) + \dfrac{\pi}{2}(-0.04) \approx 0.1257$.

11. $dz = 2x \sin 4y dx + 4x^2 \cos 4y dy$

12. $dz = \left[x(2xe^{x^2-y^2}) + e^{x^2-y^2}\right] dx - 2yxe^{x^2-y^2} dy = (2x^2 + 1)e^{x^2-y^2} dx - 2xye^{x^2-y^2} dy$

13. $dz = \dfrac{2x}{\sqrt{2x^2 - 4y^3}} dx - \dfrac{6y^2}{\sqrt{2x^2 - 4y^3}} dy$

14. $dz = 45x^2y(5x^3y + 4y^5)^2 dx + (15x^3 + 60y^4)(5x^3y + 4y^5)^2 dy$

15. $df = \dfrac{(s + 3t)2 - (2s - t)}{(s + 3t)^2} ds + \dfrac{(s + t)(-1) - (2s - t)3}{(s + 3t)^2} dt = \dfrac{7t}{(s + 3t)^2} - \dfrac{7s}{(s + 3t)^2} dt$

16. $dg = 2r \cos 3\theta dr - 3r^2 \sin 3\theta d\theta$

17. $dw = 2xy^4z^{-5} dx + 4x^2y^3z^{-5} dy - 5x^2y^4z^{-6} dz$

18. $dw = -2xe^{-z^2} \sin(x^2 + y^4) dx - 4y^3e^{-z^2} \sin(x^2 + y^4) dy - 2ze^{-z^2} \cos(x^2 + y^4) dz$

19. $dF = 3r^2 dr - 2s^{-3} ds - 2t^{-1/2} dt$

20. $dG = \sin\phi\cos\theta d\rho - \rho\sin\phi\sin\theta d\theta + \rho\cos\phi\cos\theta d\phi$

21. $w = \ln u + \ln v - \ln s - \ln t;\ \ dw = \dfrac{du}{u} + \dfrac{dv}{v} - \dfrac{ds}{s} - \dfrac{dt}{t}$

22. $dw = \dfrac{u}{\sqrt{u^2+s^2t^2-v^2}}du - \dfrac{v}{\sqrt{u^2+s^2t^2-v^2}}dv + \dfrac{st^2}{\sqrt{u^2+s^2t^2-v^2}}ds + \dfrac{s^2t}{\sqrt{u^2+s^2t^2-v^2}}dt$

23. $\Delta z = z(2.2, 3.9) - z(2, 4) = (6.6 + 15.6 + 8) - (6 + 16 + 8) = 0.1;\ \ dz = 3dx + 4dy$
When $x = 2,\ \ y = 4,\ \ dx = 0.2$, and $dy = -0.1,\ \ dz = 3(0.2) + 4(-0.1) = 0.2$

24. $\Delta z = z(0.2, -0.1) - z(0, 0) = 2(0.2)^2(-0.1) + 5(-0.1) - 0 = -0.508;\ \ dz = 4xydx + (2x^{+}5)dy$
When $x = y = 0,\ \ dx = 0.2$, and $dy = -0.1,\ \ dz = 5(-0.1) = -0.5$.

25. $\Delta z = z(3.1, 0.8) - z(3, 1) = (3.1 + 0.8)^2 - (3 + 1)^2 = 15.21 - 16 = -0.79;$
$dz = 2(x + y)dx + 2(x + y)dy$. When $x = 3,\ \ y = 1,\ \ dx = 0.1$, and $dy = -0.2,\ \ dz = 2(3 + 1)(0.1) + 2(3 + 1)(0.2) = 0.8 - 1.6 = -0.8$

26. $\Delta z = z(0.9, 1.1) - z(1, 1) = [(0.9)^2 + (0.9)^2(1.1)^2 + 2] - [1 + 1 + 2] = 3.7901 - 4 = -0.2099;$
$dz = (2x + 2xy^2)dx + 2x^y dy$. When $x = y = 1,\ \ dx = -0.1$, and $dy = 0.1,\ \ dz = 4(-0.1) + 2(0.1) = -0.2$.

27. $$\begin{aligned}\Delta z &= 5(x+\Delta x)^2 + 3(y+\Delta y) - (x+\Delta x)(y+\Delta y) - (5x^2+3y-xy)\\ &= 10x\Delta x + 5(\Delta x)^2 + 3\Delta y - x\Delta y - y\Delta x - \Delta x\Delta y\\ &= (10x-y)\Delta x + (3-z)\Delta y + (5\Delta x)\Delta x - (\Delta x)\Delta y\end{aligned}$$
$\epsilon_1 = 5\Delta x,\ \ \epsilon_2 = -\Delta x$

28. $$\begin{aligned}\Delta z &= 10(y+\Delta y)^2 + 3(x+\Delta x) - (x+\Delta x) - (10y^2+3x-x^2)\\ &= 20y\Delta y + 10(\Delta y)^2 + 3\Delta x - 2x\Delta x - (\Delta x)^2\\ &= (3-2x)\Delta x + 20y\Delta y - (\Delta x)\Delta x + (10\Delta y)\Delta y\end{aligned}$$
$\epsilon_1 = -\Delta x,\ \ \epsilon_2 = 10\Delta y$

29. $$\begin{aligned}\Delta x &= (x+\Delta x)^2(y+\Delta y)^2 - x^2y^2 = [x^2 + 2x\Delta x + (\Delta x)^2][y^2 + 2y\Delta y + (\Delta y)^2] - x^2y^2\\ &= 2x^2y\Delta y + x^2(\Delta y)^2 + 2xy^2\Delta x + 4xy(\Delta x)\Delta y + 2x(\Delta x)(\Delta y)^2 + y^2(\Delta x)^2 + 2y(\Delta x)^2\Delta y + (\Delta x)^2(\Delta y)^2\\ &= 2xy^2\Delta x + 2x^2y\Delta y + [4xy\Delta y + 2x(\Delta y)^2 + y^2x]\Delta x + [x^2\Delta y + 2y(\Delta x)^2 + (\Delta x)^2\Delta y]\Delta y\end{aligned}$$
$\epsilon_1 = 4xy\Delta y + 2x(\Delta y)^2 + y^2x,\ \ \epsilon_2 = x^2\Delta y + 2y(\Delta x)^2 + (\Delta x)^2\Delta y$ (Several other choices of ϵ_1 and ϵ_2 are possible.)

30. $$\begin{aligned}\Delta z &= (x+\Delta x)^3 - (y+\Delta y)^3 - (x^3-y^3) = 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - 3y^2\Delta y - 3y(\Delta y)^2 - (\Delta y)^3\\ &= 3x^2\Delta x - 3y^2\Delta y + [3x\Delta x + (\Delta x)^2] - [3y\Delta y + (\Delta y)^2]\Delta y\end{aligned}$$
$\epsilon_1 = 3x\Delta x + (\Delta x)^2,\ \ \epsilon_2 = -3y\Delta y - (\Delta y)^2$

31. $R = \dfrac{R_1R_2R_3}{R_2R_3 + R_1R_3 + R_1R_2};\ \ \Delta R_1 = \pm 0.009R_1,\ \ \Delta R_! = \pm 0.009R_2,\ \ \Delta R_3 = \pm 0.0009R_3$

$$\begin{aligned}|\Delta R| \approx |dR| &\leq \left|\frac{R_2^2R_3^2}{(R_2R_3+R_1R_3+R_1R_2)^2}(\pm 0.009R_1)\right| + \left|\frac{R_1^2R_3^2}{(R_2R_3+R_1R_3+R_1R_2)^2}(\pm 0.009R_2)\right| \\ &\quad + \left|\frac{R_1^2R_2^2}{(R_2R_3+R_1R_3+R_1R_2)^2}(\pm 0.009R_3)\right| \\ &= 0.009R\left(\frac{R_2R_3+R_1R_3+R_1R_2}{R_2R_3+R_1R_3+R_1R_2}\right) = 0.009R\end{aligned}$$

The maximum percentage error is approximately 0.9%.

32. We are given $\Delta T = \pm 0.006T$ and $\Delta V = \pm 0.008V$. Then

$$|\Delta P| \approx |dP| = \left|\frac{k}{V}(\pm 0.006T) - \frac{kT}{V^2}(\pm 0.008V)\right| \leq \frac{kT}{V}(0.006) + \frac{kT}{V}(0.008) = P(0.014).$$

Thus, the approximate maximum percentage error in P is 1.4%.

33. $dT = mg\dfrac{(2r^2+R^2-R(2R)}{(2r^2+R^2)^2}dR + mg\dfrac{-R(4r)}{2r^2+R^2)^2}dr = mg\dfrac{2r^2-R^2}{(2r^+R^2)^2}dR - mg\dfrac{4rR}{(2r^+R^2)^2}dr$
When $R=4,\ \ r=0.8,\ \ dR=0.1$, and $dr=0.1$,

$$\begin{aligned}\Delta T \approx dT &= mg\left[\frac{2(0.8)^2-4^2}{[2(0.8)^2+4^2]^2}(0.1) - \frac{4(0.8)4}{[2(0.8)^2+4^2]^2}(0.1)\right] \\ &= mg\left(\frac{-1.472-1.28}{298.598}\right) \approx -0.009 \text{ mg}.\end{aligned}$$

The tension decreases.

34. $V=\pi r^2h,\ \ dV = 2\pi rhdr + \pi r^2dh$. When $r=5,\ \ h=10,\ \ dr=0.3$, and $dh=0.5$,

$$\Delta V \approx dV = 2\pi(5)(1-)(0.3) + \pi(5^2)(0.5) = 42.5\pi \text{ cm}^3.$$

Since $V(5,10) = 250\pi$ cm^3,

$$V(5.3, 10.5) = V(5,10) + \Delta V \approx V(5,10) + dV = 250\pi + 42.5\pi = 292.5\pi \text{ cm}^3.$$

35. $V = lwh,\ \ dV = whdl + lhdq + lwdh$. With $dl = \pm 0.02l,\ \ dw = \pm 0.05w$, and $dh = \pm 0.08h$,

$$|\Delta V| \approx |dV| = |wh(\pm 0.02l) + lh(\pm 0.05w) + lw(\pm 0.08h)| \leq lwh(0.02+0.5+0.8) = 0.15V.$$

The approximate percentage increase in volume is 15%.

36. $S = 2lw + 2lh + 2wh,\ \ dS = (2w+2h)dl + (2l+2h)dw + (2l+2w)dh$
With $l=3,\ \ w=1,\ \ h=2,\ \ sl=0.06,\ \ dw=0.05$, and $dh=0.16$,

$$\Delta S \approx dS = (2+4)(0.06) + (6+4)(0.05) + (6+2)(0.16) = 2.14 \text{ ft}^2.$$

Since $S(3,1,2) = 22$ ft^2, the new surface area is approximately $S(3,1,2) + dS = 24.14$ ft^2.

37. $dS = 0.1091(0.425)w^{-0.575}h^{0.725}dw + 0.1091(0.725)w^{0.425}h^{-0.275}dh$
With $dw = \pm 0.03w$ and $dh = \pm 0.05h$,

$$\begin{aligned} |\Delta S| \approx |dS| &= 0.1091|0.425w^{-0.575}h^{0.725}(\pm 0.03w) + 0.725w^{0.425}h^{-0.275}(\pm 0.05h)| \\ &\leq 0.1091[0.425w^{0.425}h^{0.725}(0.03)] + 0.1091[0.725w^{0.425}h^{0.725}(0.05)] \\ &= 0.1091w^{0.425}h^{0.725}(0.013 + 0.036) = 0.049S. \end{aligned}$$

The approximate maximum percentage error is 4.9%.

38. $Z = \left[R^2 + \left(100L - \dfrac{1}{1000c}\right)^2\right]^{1/2}$;

$$\begin{aligned} dZ &= \frac{1}{2}\left[r^2 + \left(100L - \frac{1}{1000c}\right)^2\right]^{-1/2}\left[2RdR + 2\left(100L - \frac{1}{1000c}\right)(100)dL\right. \\ &\left. 2\left(100L - \frac{1}{1000c}\right)\left(\frac{1}{1000c^2}\right)dC\right] \\ &= (R^2 + X^2)^{-1/2}\left(RdR + 1000XdL + \frac{X}{1000c^2}dC\right) \end{aligned}$$

With $R = 4000$, $L + 0.4$, $C = 10^{-5}$, $dR = 25$, $d := 0.05$, and $dC = 1.1 \times 10^{-5} - 10^{-5} = 10^{-6}$, we have $X = 300$ and

$$\begin{aligned} dZ &= (400^2 + 300^2)^{-1/2}\left[400(25_+ 100(300)(0.05) + \frac{300}{1000(10)^{-10}}10^{-6}\right] \\ &= \frac{1}{500}(10000 + 15000 + 3000) = 56 \text{ ohms.} \end{aligned}$$

The new impedance is approximately $Z(400, 0.4, 10^{-5}) + dZ = 500 + 56 = 556$ ohms.

39. (a) If a function $w = f(x, y, z)$ is differentiable at a point (x_0, y_0, z_0), then the function

$$L(x, y, z) = f(x_0, y_0, z_0) + f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0)$$

is a linearization of f at (x_0, y_0, z_0).

(b) Let $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$. Then we wish to approximate $f(9.1, 11.75, 19.98)$. To do this, linearize f at $(9, 12, 20)$. Compute

$$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}, \quad \frac{\partial f}{\partial x}(9, 12, 20) = \frac{9}{25}$$

$$\frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \quad \frac{\partial f}{\partial y}(9, 12, 20) = \frac{12}{25}$$

$$\frac{\partial f}{\partial z} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}, \quad \frac{\partial f}{\partial z}(9, 12, 20) = \frac{4}{5} \text{and } f(9, 12, 20) = 25$$

The linearization is $L(x, y, z) = 25 + \frac{9}{25}(x - 9) + \frac{12}{25}(y - 12) + \frac{4}{5}(z - 20)$. For the approximation, we have $L(9.1, 11.75, 19.98) = 25 + \frac{9}{25}(0.1) + \frac{12}{25}(-0.25) + \frac{4}{5}(-0.02) =$ 24.9

40. According to Theorem 13.4.3, if f were differentiable at $(0, 0)$, then f would have to be continuous at $(0, 0)$. However, as shown in Problem 38 in Exercises 13.2, f is not continuous at $(0, 0)$. Therefore, f cannot be differentiable at $(0, 0)$.

41. (a) The graph of $z = f(x, y)$ is an inverted cone with vertex at the origin. Since the graph comes to a sharp "point" at the origin, there is no possible increment formula for Δz that will work in every direction there.

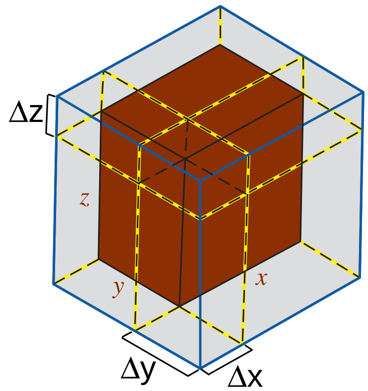
(b) We show that the partial derivative f_x does not exist at $(0, 0)$. If $h > 0$,

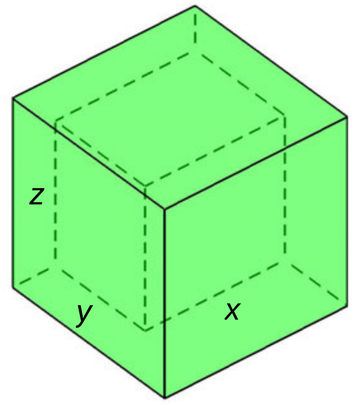
$$\begin{aligned}\frac{f(0 + h) - f(0, 0)}{h} &= \frac{\sqrt{h^2 + 0^2} - \sqrt{0^2 + 0^2}}{h} \\ &= \frac{\sqrt{h^2}}{h} = \frac{|h|}{h} = 1\end{aligned}$$

But if $h < 0$, then $\dfrac{f(0 + h) - f(0, 0)}{h} = -1$.

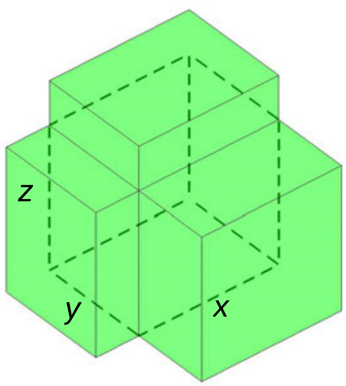
Therefore, $\lim\limits_{h\to 0} \dfrac{f(0 + h) - f(0, 0)}{h}$ does not exist. But this means f_x does not exist at $(0, 0)$ and thus f is not differentiable at $(0, 0)$.

42.

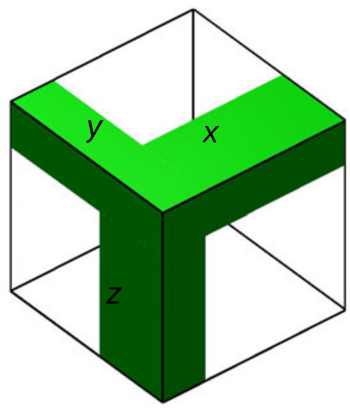




ΔV



dV

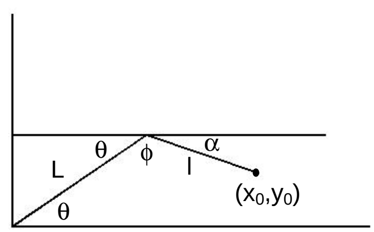


$\Delta V - dV$

43. (a) From the figure we see that $\alpha = \pi - (\theta + \phi)$. Then

$$\begin{aligned} x_h &= L\cos\theta + l\cos\alpha = L\cos\theta + l\cos(\pi - \theta - \phi) \\ &= L\cos\theta - l\cos(\theta + \phi) \end{aligned}$$

and
$$\begin{aligned} y_h &= L\sin\theta - l\sin\alpha = L\sin\theta - l\sin(\pi - \theta - \phi) \\ &= L\sin\theta - l\sin(\theta + \phi). \end{aligned}$$



(b) Using $l\sin(\theta+\phi) = y_e - y_h$ and $l\cos(\theta+\phi) = x_e - x_h$, we have

$$\begin{aligned} dx_h &= (-L\sin\theta + l\sin(\theta+\phi))d\theta + l\sin(\theta+\phi)d\phi = -y_h d\theta + (y_e - y_h)d\phi \\ dy_h &= (L\cos\theta - l\cos(\theta+\phi))d\theta - l\cos(\theta+\phi)d\phi \ = x_h - (x_e - x_h)d\phi. \end{aligned}$$

(c) One position has the lower arm reaching straight up, with the elbow on the x-axis, so that $\theta = 0$ and $\phi = 270°$. The other position has the lower arm reaching straight across, with the elbow on the y-axis, so that $\theta = \phi = 90°$. In both cases, $(x_h, y_h) = (L, L)$. In the first case, $(x_e, y_e) = (L, 0)$, and in the second case $(x_e, y_e) = (0, L)$. In general, the approximate maximum error in x_h is

$$|dx_h| = |-h_h d\theta + (y_e y_h)d\phi| \le L|d\theta| + |y_e - L||d\phi| = (L + |y_e - L|)\frac{\pi}{180}.$$

Thus, in the first case the approximate maximum error is $2\pi L/180$, while in the second case it is only $\pi L/180$.

44. (a) The horizontal and vertical components of velocity are $v\cos\theta$ and $v\sin\theta$, respectively. The projectile strikes the cliff wall at time $t = D/v\cos\theta$. At this time its height is

$$H = tv\sin\theta - \frac{1}{2}gt^2 = D\tan\theta - \frac{1}{2}g\left(\frac{D}{v\cos\theta}\right)^2 = D\tan\theta - \frac{1}{2}g\frac{D^2}{v^2}\sec^2\theta.$$

(b) $dH = \dfrac{\partial H}{\partial v}dv + \dfrac{\partial H}{\partial \theta}d\theta = g\dfrac{D^2}{v^3}\sec^2\theta dv + \left(D\sec^2\theta - g\dfrac{d^2}{v^2}\sec^2\theta\tan\theta\right)d\theta$

(c) When $D = 100$, $g = 32$, $v = 100$, and $\theta = 45°$, we have $H = 68$ ft.

(d) Taking $|dv| \le 1$ and $|d\theta| \le \pi/180$ we find
$|dH| \le 32\left(\dfrac{100^2}{100^3}\right)\sec^2\frac{\pi}{4}(1) + \left[100\sec^2\frac{\pi}{4} - 32\dfrac{100^2}{100^2}\sec^2\frac{\pi}{4}\tan\frac{\pi}{4}\right]\left(\frac{\pi}{180}\right) = \dfrac{16}{25} + \dfrac{34\pi}{45} \approx$
3.01 ft.

(e) We have

$$dH = \frac{\partial H}{\partial v} + \frac{\partial H}{\partial \theta} + \frac{\partial H}{\partial D} \quad = g\frac{D^2}{v^3}\sec^2\theta dv + \left(D\sec^2\theta - g\frac{D^2}{v^2}\sec^2\theta\tan\theta\right) + \left(\tan\theta - g\frac{D}{v^2}\sec^2\theta\right)dD.$$

With $|dD| \le 2$, we obtain $dH \le \dfrac{16}{25} + \dfrac{34\pi}{45} + \dfrac{18}{25} = \dfrac{34}{25} + \dfrac{34\pi}{45} \approx 3.73$ ft.

13.5 Chain Rule

1. $$\begin{aligned} \frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \frac{2x}{x^2+y^2}(2t) + \frac{2y}{x^2+y^2}(-2t^{-3}) \\ &= \frac{4xt - 4yt^{-3}}{x^2+y^2} \end{aligned}$$

2. $$\begin{aligned} \frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\ &= (3x^2y - y^4)(5e^{5t}) + (x^3 - 4xy^3)(5\sec(t)\tan(t)) \end{aligned}$$

3. $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt} = -3\sin(3x+4y)(2) - 4\sin(3x+4y)(-1)$

At $t = \pi,\ \ x = \dfrac{5\pi}{2}$ and $y = \frac{-5\pi}{4}$

so $\left.\dfrac{dz}{dt}\right|_{t=\pi} = -6\sin\left(\dfrac{15\pi}{2} - 5\pi\right) + 4\sin\left(\dfrac{15\pi}{2} - 5\pi\right) = -6+4 = -2$

4. $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt} = ye^{xy}\left(\dfrac{-8}{(2t+1)^2}\right) + xe^{xy}(3)$

At $t = 0,\ \ x = 4$ and $y = 5$

so $\left.\dfrac{dz}{dt}\right|_{t=0} = -40e^{20} + 12e^{20} = -28e^{20}$

5. $\dfrac{dp}{du} = \dfrac{1}{2s+t}(2u) - \dfrac{2r}{(2s+t)^2}\left(-\dfrac{2}{u^3}\right) - \dfrac{r}{(2s+t)^2}\left(\dfrac{1}{2\sqrt{u}}\right) = \dfrac{2u}{2s+t} + \dfrac{4r}{u^3(2s+t)^2} - \dfrac{r}{2\sqrt{u}(2s+t)^2}$

6. $\dfrac{dr}{ds} = \dfrac{y^2}{z^3}(-\sin s) + \dfrac{2xy}{z^3}(\cos s) - \dfrac{3xy^2}{z^4}(\sec^2 s) = -\dfrac{y^2\sin s}{z^3} + \dfrac{2xy\cos s}{z^3} - \dfrac{3xy^2\sec^2 s}{z^4}$

7. $z_u = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial u} = y^2e^{xy^2}(3u^2) + 2xye^{xy^2}(1)$

$= 3u^2y^2e^{xy^2} + 2xye^{xy^2}$

$z_v = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial v} = y^2e^{xy^2}(0) + 2xye^{xy^2}(-2v)$

$= -4vxye^{xy^2}$

8. $z_u = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial u} = 2x\cos 4y(2uv^3) - 4x^2\sin 4y(3u^2)$

$= 4uv^3x\cos 4y - 12u^2x^2\sin 4y$

$z_v = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial v} = 2x\cos 4y(3u^2v^2) - 4x^2\sin 4y(3v^2)$

$= 6u^2v^2x\cos 4y - 12v^2x^2\sin 4y$

9. $z_u = 4(4u^3) - 10y[2(2u-v)(2)] = 16u^3 - 40(2u-v)y$

$z_v = 4(-24v^2) - 10y[2(2u-v)(-1)] = -96v^2 + 20(2u-v)y$

10. $z_u = \dfrac{2y}{(x+y)^2}\dfrac{1}{v} + \dfrac{-2x}{(x+y)^2}\left(-\dfrac{v^2}{u^2}\right) = \dfrac{2y}{v(x+y)^2} + \dfrac{2xv^2}{u^2(x+y)^2}$

$z_v = \dfrac{2y}{(x+y)^2}\left(-\dfrac{u}{v^2}\right) + \dfrac{-2x}{(x+y)^2}\left(\dfrac{2v}{u}\right) = -\dfrac{2yu}{v^2(x+y)^2} - \dfrac{4xv}{u(x+y)^2}$

11. $w_t = \dfrac{3}{2}(u^2+v^2)^{1/2}(2u)(-e^{-t}\sin\theta) + \dfrac{3}{2}(u^2+v^2)^{1/2}(2v)(-e^{-t}\cos\theta)$

$= -3u(u^2+v^2)^{1/2}e^{-t}\sin\theta - 3v(u^2+v^2)^{1/2}e^{-t}\cos\theta$

$w_\theta = \dfrac{3}{2}(u^2+v^2)^{1/2}(2u)e^{-t}\cos\theta + \dfrac{3}{2}(u^2+v^2)^{1/2}(2v)(-e^{-t}\sin\theta)$

$= 3u(u^2+v^2)^{1/2}e^{-t}\cos\theta - 3v(u^2+v^2)^{1/2}e^{-t}\sin\theta$

12. $w_r = \dfrac{v/2\sqrt{uv}}{1+uv}(2r) + \dfrac{u/2\sqrt{uv}}{1+uv}(2rs^2) = \dfrac{rv}{\sqrt{uv}(1+uv)} + \dfrac{rs^2u}{\sqrt{uv}(1+uv)}$

$w_s = \dfrac{v/2\sqrt{uv}}{1+uv}(-2s) + \dfrac{u/2\sqrt{uv}}{1+uv}(2r^2s) = \dfrac{-sv}{\sqrt{uv}(1+uv)} + \dfrac{r^2su}{\sqrt{uv}(1+uv)}$

13. $R_u = s^2t^4(e^{v^2}) + 2rst^4(-2uve^{-u^2}) + 4rs^2t^3(2uv^2e^{u^2v^2}) = s^2t^4e^{v^2} - 4uvrst^4e^{-u^2} + 8uv^2rs^2t^3e^{u^2v^2}$

$R_v = s^2t^4(2uve^{v^2}) + 2rst^4(e^{-u^2}) + 4rs^2t^3(2u^2ve^{u^2v^2}) = 2s^2t^4uve^{v^2} + 2rst^4e^{-u^2} + 8rs^2t^3u^2ve^{u^2v^2}$

14. $Q_x = \dfrac{1}{P}\left(\dfrac{t^2}{\sqrt{1-x^2}}\right) + \dfrac{1}{q}\left(\dfrac{1}{t^2}\right) + \dfrac{1}{r}\left(\dfrac{1/t}{1+(x/t)^2}\right) = \dfrac{t^2}{p\sqrt{1-x^2}} + \dfrac{1}{qt^2} + \dfrac{t}{r(t^2+x^2)}$

$Q_t = \dfrac{1}{P}(2t\sin^{-1}x) + \dfrac{1}{q}\left(-\dfrac{2x}{t^3}\right) + \dfrac{1}{r}\left(\dfrac{-x/t^2}{1+(x/t)^2}\right) = \dfrac{2t\sin^{-1}x}{p} - \dfrac{2x}{qt^3} - \dfrac{x}{r(t^2+x^2)}$

15. $w_t = \dfrac{2x}{2\sqrt{x^2+y^2}}\dfrac{u}{rs+tu} + \dfrac{2y}{2\sqrt{x^2+y^2}}\dfrac{\cosh rs}{u} = \dfrac{xu}{\sqrt{x^2+y^2}(rs+tu)} + \dfrac{y\cosh rs}{u\sqrt{x^2+y^2}}$

$w_r = \dfrac{2x}{2\sqrt{x^2+y^2}}\dfrac{s}{rs+tu} + \dfrac{2y}{2\sqrt{x^2+y^2}}\dfrac{st\sinh rs}{u} = \dfrac{xs}{\sqrt{x^2+y^2}(rs+tu)} - \dfrac{yst\sinh rs}{u\sqrt{x^2+y^2}}$

$w_u = \dfrac{2x}{2\sqrt{x^2+y^2}}\dfrac{t}{rs+tu} + \dfrac{2y}{2\sqrt{x^2+y^2}}\dfrac{-t\cosh rs}{u^2} = \dfrac{xt}{\sqrt{x^2+y^2}(rs+tu)} - \dfrac{yt\cosh rs}{u^2\sqrt{x^2+y^2}}$

16. $s_\phi = 2pe^{3\theta} + 2q[-\sin(\phi+\theta)] - 2r\theta^2 + 4(2) = 2pe^{3\theta} - 2q\sin(\phi+\theta) - 2r\theta^2 + 8$

$s_\theta = 2p(3e^{3\theta}) + 2q[-\sin(\phi+\theta)] - 2r(2\phi\theta) + 4(8) = 6p\phi e^{3\theta} - 2q\sin(\phi+\theta) - 4r\phi\theta + 32$

17. (a) $3x^2 - 2x^2(2yy') - 4xy^2 + y' = 0 \Longrightarrow (1-4x^2y)y' = 4xy^2 - 3x^2 \Longrightarrow y' = \dfrac{4xy^2-3x^2}{1-4x^2y}$

(b) $f_x = 3x^2 - 4xy^2,\ \ f_y = -4x^2y + 1;\ \ y' = -\dfrac{3x^2-4xy^2}{-4x^2y+1} = \dfrac{4xy^2-3x^2}{1-4x^2y}$

18. (a) $1 + 4yy' = e^yy' \Longrightarrow 1 = (e^y - 4y)y' \Longrightarrow y' = \dfrac{1}{e^y-4y}$

(b) $f(x,y) = x + 2y^2 - e^y;\ \ f_x = 1,\ f_y = 4y - e^y;\ \ y' = -\dfrac{1}{4y-e^y} = \dfrac{1}{e^y-4y}$

19. (a) $y' = (\cos xy)(xy'+y) \Longrightarrow (1 - x\cos xy)y' = y\cos xy \Longrightarrow y' = \dfrac{y\cos xy}{1-x\cos xy}$

(b) $f(x,y) = y - \sin xy;\ \ f_x = -y\cos xy,\ f_y = 1 - x\cos xy;$

$y' = -\dfrac{-y\cos xy}{1-x\cos xy} = \dfrac{y\cos xy}{1-x\cos xy}$

20. (a) $\dfrac{2}{3}(x+y)^{-1/3}(1+y') = xy' + y \Longrightarrow 2(x+y)^{-1/3} + 2(x+y)^{-1/3}y' = 3xy' + 3y$

$\Longrightarrow \left[2(x+y)^{-1/3} - 3x\right]y' = 3y - 2(x+y)^{-1/3} \Longrightarrow y' = \dfrac{3y-2(x+y)^{-1/3}}{2(x+y)^{-1/3}-3x}$

(b) $f(x,y) = (x+y)^{2/3} - xy;\ \ f_x = \dfrac{2}{3}(x+y)^{-1/3} - y,\ \ f_y = \dfrac{2}{3}(x+y)^{-1/3} - x;$

$y' = -\dfrac{\frac{2}{3}(x+y)^{-1/3} - y}{\frac{2}{3}(x+y)^{-1/3} - x} = \dfrac{3y-2(x+y)^{-1/3}}{2(x+y)^{-1/3}-3x}$

21. $F_x = 2x,\ F_y = 2y,\ F_z = -2z;\ \dfrac{\partial z}{\partial x} = -\dfrac{2x}{-2z} = \dfrac{x}{z};\ \dfrac{\partial z}{\partial y} = -\dfrac{2y}{-2z} = \dfrac{y}{z}$

22. $F_x = \dfrac{2}{3}x^{-1/3},\ F_y = \dfrac{2}{3}y^{-1/3},\ F_z = \dfrac{2}{3}z^{-1/3};\ \dfrac{\partial z}{\partial x} = -\dfrac{(2/3)x^{-1/3}}{(2/3)z^{-1/3}} = -\dfrac{z^{1/3}}{x^{1/3}};$
$\dfrac{\partial z}{\partial y} = -\dfrac{(2/3)y^{-1/3}}{(2/3)z^{-1/3}} = -\dfrac{z^{1/3}}{y^{1/3}}$

23. $F(x,y,z) = xy^2z^3 + x^2 - y^2 - 5z^2,\quad F_x = y^2z^3 + 2x,\quad F_y = 2xyz^3 - 2y,\quad F_z = 3xy^2z^2 - 10z$
$\dfrac{\partial z}{\partial x} = -\dfrac{y^2z^3 + 2x}{3xy^2z^2 - 10z} = \dfrac{y^2z^3 + 2x}{10z - 3xy^2z^2};\quad \dfrac{\partial z}{\partial y} = -\dfrac{2xyz^3 - 2y}{3xy^2z^2 - 10z} = \dfrac{2xyz^3 - 2y}{10z - 3xy^2z^2}$

24. $F(x,y,z) = z - \ln(xyz);\quad F_x = -\dfrac{1}{x},\quad F_y = -\dfrac{1}{y},\quad F_z = 1 - \dfrac{1}{z};\quad \dfrac{\partial z}{\partial x} = -\dfrac{-1/x}{1 - 1/z} = \dfrac{z}{xz - x};$
$\dfrac{\partial z}{\partial y} = -\dfrac{-1/y}{1 - 1/z} = \dfrac{z}{yz - y}$

25. Let $y = x + at$ and $z = x - at$. Then $u(x,t) = F(y) + G(z)$ and

$$\frac{\partial u}{\partial x} = \frac{dF}{dy}\frac{\partial y}{\partial x} + \frac{dG}{dz}\frac{\partial z}{\partial x} = \frac{dF}{dy} + \frac{dG}{dz};\quad \frac{\partial^2 u}{\partial x^2} = \frac{d^2F}{dy^2}\frac{\partial y}{\partial x} + \frac{d^2G}{dz^2}\frac{\partial z}{\partial x} = \frac{d^2F}{dy^2} + \frac{d^2G}{dz^2};$$

$$\frac{\partial u}{\partial t} = \frac{dF}{dy}\frac{\partial y}{\partial t} + \frac{dG}{dz}\frac{\partial z}{\partial t} = a\frac{dF}{dy} - a\frac{dG}{dz};\quad \frac{\partial^2 u}{\partial xt^2} = a\frac{d^2F}{dy^2}\frac{\partial y}{\partial t} - a\frac{d^2G}{dz^2}\frac{\partial z}{\partial t} = a^2\frac{d^2F}{dy^2} + a^2\frac{d^2G}{dz^2}.$$

Thus, $a^2\dfrac{\partial^2 u}{\partial x^2} = a^2\dfrac{d^2F}{dy^2} + a^2\dfrac{d^2G}{dz^2} = \dfrac{\partial^2 u}{\partial t^2}$.

26. Solving $\eta = x + at$ and $\xi = x - at$ for x and t, we obtain $x = (\eta + \xi)/2$ and $t = (\eta - \xi)/2a$. Then

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial \xi} = \frac{1}{2}\frac{\partial u}{\partial x} - \frac{1}{2a}\frac{\partial u}{\partial t}$$

and

$$\frac{\partial^2 u}{\partial \eta \partial \xi} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial \eta} - \frac{1}{2a}\frac{\partial^2 u}{\partial t^2}\frac{\partial t}{\partial \eta} = \frac{1}{4}\frac{\partial^2 u}{\partial x^2} - \frac{1}{4a^2}\frac{\partial^2 u}{\partial t^2}$$

Setting $\dfrac{\partial^2 u}{\partial \eta \partial \xi} = 0$, we have $\dfrac{1}{4}\dfrac{\partial^2 u}{\partial x^2} - \dfrac{1}{4a^2}\dfrac{\partial^2 u}{\partial t^2} = 0$ or $a^2\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$.

27. With $x = r\cos\theta$ and $y = r\sin\theta$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta$$

$$\frac{\partial^2 u}{\partial r^2} = \frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial r}\cos\theta + \frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial r}\sin\theta = \frac{\partial^2 u}{\partial x^2}\cos^2\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\theta$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial u}{\partial y}(r\sin\theta) + \frac{\partial u}{\partial y}(-r\sin\theta) + \frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial \theta}(r\cos\theta)$$

$$= -r\frac{\partial u}{\partial x}\cos\theta + r^2\frac{\partial^2 u}{\partial x^2}\sin^2\theta - r\frac{\partial u}{\partial y}\sin\theta - r^2\frac{\partial^2 u}{\partial y^2}\cos^2\theta.$$

Using $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$, we have

$$\begin{aligned}\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}&=\frac{\partial^2 u}{\partial x^2}\cos^2\theta+\frac{\partial^2 u}{\partial y^2}\sin^2\theta+\frac{1}{r}\left(\frac{\partial u}{\partial x}\cos\theta+\frac{\partial u}{\partial y}\sin\theta\right)\\&+\frac{1}{r^2}\left(-r\frac{\partial u}{\partial x}+r^2\frac{\partial^2 u}{\partial x^2}\sin^2\theta-r\frac{\partial u}{\partial y}\sin\theta+r^2\frac{\partial^2 u}{\partial y^2}\cos^2\theta\right)\\&=\frac{\partial^2 u}{\partial x^2}(\cos^2\theta+\sin^2\theta)+\frac{\partial^2 u}{\partial y^2}(\cos^2\theta+\sin^2\theta)+\frac{\partial u}{\partial x}\left(\frac{1}{r}\cos\theta-\frac{1}{r}\cos\theta\right)\\&+\frac{\partial u}{\partial y}\left(\frac{1}{r}\sin\theta-\frac{1}{r}\sin\theta\right)\\&=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0\end{aligned}$$

28. $\dfrac{\partial z}{\partial x}=\dfrac{dz}{du}\dfrac{\partial u}{\partial x},\quad \dfrac{\partial z}{\partial y}=\dfrac{dz}{du}\dfrac{\partial u}{\partial y}$

29. Letting $u=y/x$ in Problem 28, we have

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=x\frac{dz}{du}\frac{\partial u}{\partial x}+y\frac{dz}{du}\frac{\partial u}{\partial y}=x\frac{dz}{du}\left(-\frac{y}{x^2}\right)+y\frac{dz}{du}\left(\frac{1}{x}\right)=\frac{dz}{du}\left(\frac{-y}{x}+\frac{y}{x}\right)=0.$$

30. We first compute

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}=\frac{\partial u}{\partial r}\frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial r}\frac{y^2}{(x^2+y^2)^{3/2}}+\frac{x}{\sqrt{x^2+y^2}}\frac{\partial^2 u}{\partial r^2}=\frac{\partial u}{\partial r}\frac{y^2}{(x^2+y^2)^{3/2}}+\frac{\partial^2 u}{\partial r^2}\frac{x^2}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial y^2}=\frac{\partial u}{\partial r}\frac{x^2}{(x^2+y^2)^{3/2}}\frac{\partial^2 u}{\partial r^2}\frac{y^2}{x^2+y^2}.$$

Then

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial u}{\partial r}\frac{x^2+y^2}{(x^2+y^2)^{3/2}}+\frac{\partial^2 u}{\partial r^2}\frac{x^2+y^2}{x^2+y^2}=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}.$$

31. We first compute

$$\frac{\partial u}{\partial x}=B\frac{\partial}{\partial x}erf\left(\frac{x}{\sqrt{4kt}}\right)=B\frac{\partial}{\partial x}\left(\frac{2}{\pi}\int_0^{x/\sqrt{4kt}}e^{-v^2}\mathrm{d}v\right)=B\frac{2}{\sqrt{\pi}}\frac{1}{\sqrt{4kt}}e^{-x^2/4kt}$$

$$\frac{\partial^2 u}{\partial x^2}=B\frac{2}{\sqrt{\pi}}\frac{1}{\sqrt{4kt}}\left(-\frac{x}{2kt}\right)e^{-x^2/4kt}=-B\frac{x}{2k\sqrt{\pi}\sqrt{kt^3}}e^{-x^2/4kt}$$

$$\begin{aligned}\frac{\partial u}{\partial t}&=B\frac{\partial}{\partial t}erf\left(\frac{x}{\sqrt{4kt}}\right)=B\frac{\partial}{\partial t}\left(\frac{2}{\pi}\int_0^{x/\sqrt{4kt}}e^{-v^2}\mathrm{d}v\right)=B\frac{2}{\sqrt{\pi}}\left[-\frac{x}{2\sqrt{k}}\left(-\frac{1}{2}t^{-3/2}\right)\right]e^{-x/4kt}\\&=-B\frac{x}{2\sqrt{\pi}\sqrt{kt^3}}e^{-x^2/4kt}\end{aligned}$$

Then $k\dfrac{\partial^2 u}{\partial x^2}=\dfrac{\partial u}{\partial t}$.

32. We are given $dE/dt = 2$ and $dR/dt = -1$. Then $\dfrac{dI}{dt} = \dfrac{\partial I}{\partial E}\dfrac{dE}{dt} + \dfrac{\partial I}{\partial R}\dfrac{dR}{dt} = \dfrac{1}{R}(2) - \dfrac{E}{R^2}(-1)$, and when $E = 60$ and $R = 50$, $\dfrac{dI}{dt} = \dfrac{2}{50} + \dfrac{60}{50^2} = \dfrac{1}{25} + \dfrac{3/5}{25} = \dfrac{8}{125}$ amp/min.

33. Since the height of the triangle is $x\sin\theta$, the area is given by $A = \frac{1}{2}xy\sin\theta$. Then

$$\frac{dA}{dt} = \frac{\partial A}{\partial x}\frac{dx}{dt} + \frac{\partial A}{\partial y}\frac{dy}{dt} + \frac{\partial A}{\partial \theta}\frac{d\theta}{dt} = \frac{1}{2}y\sin\theta\frac{dx}{dt} + \frac{1}{2}x\sin\theta\frac{dy}{dt} + \frac{1}{2}xy\cos\theta\frac{d\theta}{dt}.$$

When $x = 10$, $y = 8$, $\theta = \pi/6$, $dz/dt = 0.3$, $dy/dt = 0.5$, and $d\theta/dt = 0.1$,

$$\begin{aligned}\frac{dA}{dt} &= \frac{1}{2}(8)\left(\frac{1}{2}\right)(0.3) + \frac{1}{2}(10)\left(\frac{1}{2}\right)(0.5) + \frac{1}{2}(10)(8)\left(\frac{\sqrt{3}}{2}\right)(0.1)\\ &= 0.6 + 1.25 + 2\sqrt{3} = 1.85 + 2\sqrt{2} \approx 5.31\ \text{cm}^2/\text{s}.\end{aligned}$$

34. $$\begin{aligned}\frac{dP}{dt} &= \frac{(V-0.0427)(0.08)dT/dt}{(V-0.0427)^2} - \frac{0.08T(dV/dt)}{(V-0.0427)^2} + \frac{3.6}{V^3}\frac{dV}{dt}\\ &= \frac{0.08}{V-0.0427}\frac{dT}{dt} + \left(\frac{3.6}{V^3} - \frac{0.08T}{(V-0.0427)^2}\right)\frac{dV}{dt}\end{aligned}$$

35. $\dfrac{dS}{dt} = 0.1091\left(0.425w^{-0.575}h^{0.725}\dfrac{dw}{dt} + 0.725w^{0.425}h^{-0.275}\dfrac{dh}{dt}\right)$
When $w = 25$, $h = 29$, $dw/dt = 4.2$, and $dh/dt = 2$,

$$\frac{dS}{dt} = 0.1091[0.425(25)^{-0.575}(29(^{0.725}(4.2) + 0.725(25)^{0.425}(29)^{-0.275}(2)] \approx 0.5976\text{in}^2/\text{yr}.$$

36. $$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} = \frac{xdx/dt + ydy/dt + zdz/dt}{\sqrt{x^2+y^2+z^2}} = \frac{-4x\sin t + 4y\cos t + 5z}{\sqrt{16\cos^2 t + 16\sin^2 t + 25t^2}}\\ &= \frac{-16\sin t\cos t + 16\sin t\cos t + 25t}{\sqrt{16+25t^2}} = \frac{25t}{\sqrt{16+25t^2}}\\ \left.\frac{dw}{dt}\right|_{t=5\pi/2} &= \frac{125\pi/2}{\sqrt{16+625\pi^2/4}} = \frac{125\pi}{\sqrt{64+625\pi^2}} \approx 4.9743\end{aligned}$$

37. Since $dT/dT = 1$ and $\dfrac{\partial P}{\partial T} = 0$,

$$0 = F_T = \frac{\partial F}{\partial P}\frac{\partial P}{\partial T} + \frac{\partial F}{\partial V}\frac{\partial V}{\partial T} + \frac{\partial F}{\partial T}\frac{\partial T}{\partial T} \Longrightarrow \frac{\partial V}{\partial T} = -\frac{\partial F/\partial T}{\partial F/\partial V} = -\frac{1}{\partial T/\partial V}.$$

38. (a) From the law of sines, $\dfrac{r}{\sin\phi} = \dfrac{500}{\sin(\pi-\theta-\phi)}$ so $r = \dfrac{500\sin\phi}{\sin(\theta+\phi)}$.

(b) $r = 500\sin 75^\circ/\sin 137^\circ \approx 708$ yds

(c) Using the chain rule, we obtain

$$\begin{aligned}
\frac{dr}{dt} &= \frac{\partial r}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial r}{\partial \phi}\frac{d\phi}{dt} \\
&= -\frac{500\sin\phi\cos(\theta+\phi)}{\sin^2(\theta+\phi)}\frac{d\theta}{dt} + 500\frac{\sin(\theta+\phi)\cos\phi - \sin\phi\cos(\theta+\phi)}{\sin^2(\theta+\phi)}\frac{d\phi}{dt} \\
&= -\frac{500\sin\phi\cos(\theta+\phi)}{\sin^2(\theta+\phi)}\frac{d\theta}{dt} \\
&\quad + 500\frac{[\sin\theta\cos\phi + \cos\theta\sin\phi]\cos\phi - \sin\phi[\cos\theta\cos\phi - \sin\theta\sin\phi]}{\sin^2(\theta+\phi)}\frac{d\phi}{dt} \\
&= -\frac{500\sin\phi\cos(\theta+\phi)}{\sin^2(\theta+\phi)}\frac{d\theta}{dt} + \frac{500\sin\theta}{\sin^2(\theta+\phi)}\frac{d\phi}{dt}.
\end{aligned}$$

When $d\theta/dt = 5^\circ = 5\pi/180$ and $d\phi/dt = -10\pi/180$, we have

$$\frac{dr}{dt} = -\frac{500\sin 75^\circ\cos 137^\circ}{\sin^2 137^\circ}\left(\frac{5\pi}{180}\right) + \frac{500\sin 62^\circ}{\sin^2 137^\circ}\left(-\frac{10\pi}{180}\right) \approx -99.4\text{yd/min}.$$

The distance from C to A is decreasing.

39. (a) Using $f = \pi$, $l = 6$, $V = 100$, and $c = 330,000$ we obtain $f \approx 380.04$ cycles per second.

(b) We first note that $\dfrac{\partial f}{\partial V} = \dfrac{c}{4\pi}\left(\dfrac{A}{lV}^{-1/2}\right)\left(-\dfrac{A}{lv^2}\right) = -\dfrac{c}{4\pi}\sqrt{\dfrac{A}{lV}}\dfrac{1}{V} = -\dfrac{1}{2V}f$ and $\dfrac{\partial f}{\partial l} = -\dfrac{1}{2l}f.$

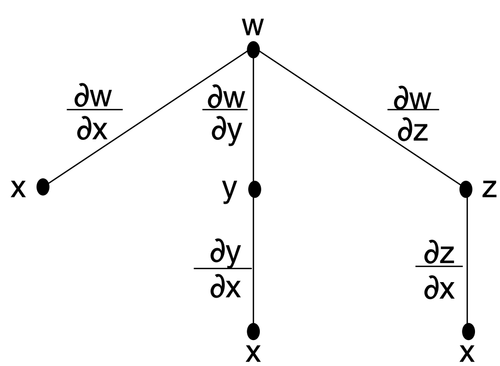
Then $\dfrac{df}{dt} = \dfrac{\partial f}{\partial V}\dfrac{dV}{dt} + \dfrac{\partial f}{\partial l}\dfrac{dl}{dt} = -\dfrac{1}{2V}f\dfrac{dV}{dt} - \dfrac{1}{2l}f\dfrac{dl}{dt} = -\dfrac{f}{2}\left(\dfrac{1}{V}\dfrac{dV}{dt} + \dfrac{1}{l}\dfrac{dl}{dt}\right).$

Using $dV/dt = -10$, $dl/dt = 1$, $V = 100$, and $l = 6$ we find

$$\frac{df}{dt} = -\frac{f}{2}\left[\frac{1}{100}(-10) + \frac{1}{6}(1)\right] = -\frac{f}{2}\left(\frac{1}{6} - \frac{1}{10}\right) < 0.$$

The frequency is decreasing.

40. (a)

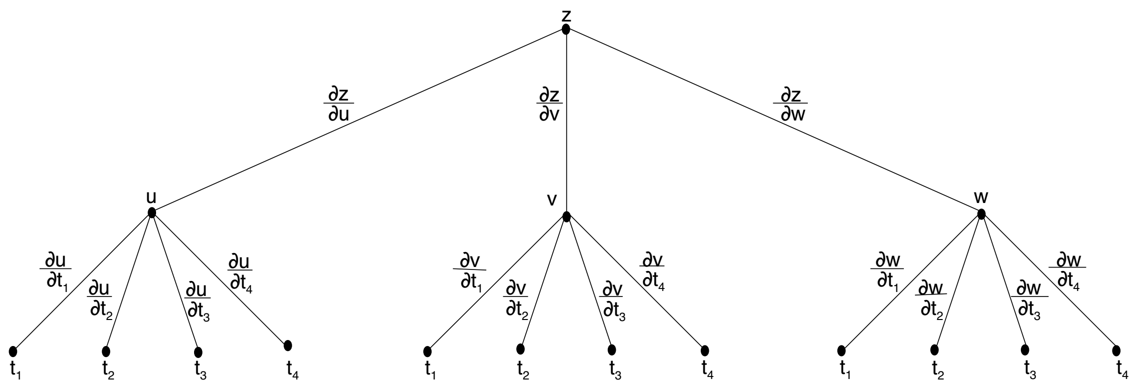


$$\frac{dw}{dx} = \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}\frac{y}{x} + \frac{\partial w}{\partial z}\frac{dz}{dx}$$

(b) Using the formula from Part **(a)**, we have

$$\frac{dw}{dx} = (y^2 + 1) + (2xy - 2z)\left(\frac{1}{x}\right) + (-2y)(e^x)$$

41.



$$\frac{\partial z}{\partial t_2} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial t_2} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial t_2} + \frac{\partial z}{\partial w}\frac{\partial w}{\partial t_2}$$
$$\frac{\partial z}{\partial t_4} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial t_4} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial t_4} + \frac{\partial z}{\partial w}\frac{\partial w}{\partial t_4}$$

42. Since $w = F(x, y, z, u) = 0$, $\partial w/\partial x = 0$. Also $dx/dx = 1$, $\partial y/\partial x = 0$, and $\partial z/\partial x = 0$. Then

$$\frac{\partial w}{\partial x} = F_x(x,y,z,u)\frac{dx}{dx} + F_u(x,y,z,u)\frac{\partial u}{\partial x} + F_Z(x,y,z,u)\frac{\partial z}{\partial x} + F_u(x,y,z,u)\frac{\partial u}{\partial x}$$

implies $\partial u/\partial x = -F_x(x,y,z,u)/F_u(x,y,z,u)$. Similarly, $\partial u/\partial y = -F_Y(x,y,z,u)/F_u(x,y,z,u)$ and $\partial u/\partial z = -F_Z(x,y,z,u)/F_u(x,y,z,u)$.

43. Letting $F(x,y,z,u) = -xyz + x^2yu + 2xy^3u - u^4 - 8$ we find $F_Z = -yz + 2xyu + 2y^3u$, $F_y = -xz + x^2u + 6xy^2u$, $F_z = -xy$, and $F_u = x^2y + 2xy^3 - 4u^3$. Then

$$\frac{\partial u}{\partial x} = -\frac{-yz + 2xyu + 2y^3u}{x^2y + 2xy^3 - 4u^3}, \quad \frac{\partial u}{\partial y} = -\frac{-xz + x^2y + 6xy^2u}{x^2y + 2xy^3 - 4u^3}, \quad \frac{\partial u}{\partial z} = \frac{xy}{x^2y + 2xy^3 - 4u^3}.$$

44. (a) Let $u = \lambda x$ and $v = \lambda y$. Then $f(u,v) = \lambda^n f(x,y)$, and differentiating both sides with respect to λ, we have

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial \lambda} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial \lambda} = n\lambda^{n-1}f(x,y) \text{ or } xf_u(u,v) + yf_u(u,v) = n\lambda^{n-1}f(x,y).$$

Letting $\lambda = 1$, we have $u = x$ and $y = v$, so $xf_x(x,y) + yf_y(x,y) = nf(x,y)$.

(b) $f(\lambda x, \lambda y) = 4(\lambda x)^2(\lambda y^3) - 3(\lambda x)(\lambda y)^4 + (\lambda x)^5 = \lambda^5 f(x,y)$

(c) $$\begin{aligned} xf_x + yf_y &= x(8xy^3 - 3y^4 + 5x^4) + y(12x^2y^2 - 12xy^3) \\ &= 8x^2y^3 - 3xy^4 + 5x^5 + 12x^2y^3 - 12xy^4 = 20x^2 - 15xy^4 + 5x^5 \\ &= 5(4x^2y^3 - 3xy^4 + x^5) = 5f(x,y) \end{aligned}$$

(d) By observing that $f\left(\dfrac{\lambda y}{\lambda x}\right) = f\left(\dfrac{y}{x}\right) = \lambda^0 f\left(\dfrac{y}{x}\right)$, we see that $z = f\left(\frac{y}{x}\right)$ is homogeneous of degree zero.

13.6 Directional Derivative

1. $\nabla f = (2x - 3x^2y^2)\mathbf{i} + (4y^3 - 2x^3y)\mathbf{j}$

2. $\nabla f = 4xye^{-2x^2y}\mathbf{i} + (1 + 2x^2e^{-2x^2y})\mathbf{j}$

3. $\nabla F = \dfrac{y^2}{z^3}\mathbf{i} + \dfrac{2xy}{z^3}\mathbf{j} - \dfrac{3xy^2}{z^4}\mathbf{k}$

4. $\nabla F = y\cos yz\mathbf{i} + (x\cos yz - xyz\sin yz)\mathbf{j} - xy^2\sin yz\mathbf{k}$

5. $\nabla f = 2x\mathbf{i} - 8y\mathbf{j}$; $\nabla f(2,4) = 4\mathbf{i} - 32\mathbf{j}$

6. $\nabla f = \dfrac{3x^2}{2\sqrt{x^3y - y^4}}\mathbf{i} + \dfrac{x^3 - 4y^3}{2\sqrt{x^3y - y^4}}\mathbf{j}$; $\nabla f(3,2) = \dfrac{27}{\sqrt{38}}\mathbf{i} - \dfrac{5}{2\sqrt{38}}\mathbf{j}$

7. $\nabla F = 2xz^2\sin 4y\mathbf{i} + 4x^2z^2\cos 4y\mathbf{j} + 2x^2z\sin 4y\mathbf{k}$
   $\nabla F(-2,\pi/3,1) = -4\sin\dfrac{4\pi}{3}\mathbf{i} + 16\cos\dfrac{4\pi}{3}\mathbf{j} + 8\sin\dfrac{4\pi}{3}\mathbf{k} = 2\sqrt{3}\mathbf{i} - 8\mathbf{j} - 4\sqrt{3}\mathbf{k}$

8. $\nabla F = \dfrac{2x}{x^2+y^2+z^2}\mathbf{i} + \dfrac{2y}{x^2+y^2+z^2}\mathbf{j} + \dfrac{2z}{x^2+y^2+z^2}\mathbf{k}$; $\nabla F(-4,3,5) = -\dfrac{4}{25}\mathbf{i} + \dfrac{3}{25}\mathbf{j} + \dfrac{1}{25}\mathbf{k}$

9. $$\begin{aligned} D_uf(x,y) &= \lim_{h\to 0}\frac{f(x + h\sqrt{3}/2, y + h/2) - f(x,y)}{h} = \lim_{h\to 0}\frac{(x + h\sqrt{3}/2)^2 + (y + h/2)^2)_x^2 - y^2}{h} \\ &= \lim_{h\to 0}\frac{h\sqrt{3}x + 3h^2/4 = hy + h^2/4}{h} = \lim_{h\to 0}(\sqrt{3}x + 3h/4 + y + h/4) = \sqrt{3}x + y \end{aligned}$$

10. $$\begin{aligned} D_uf(x,y) &= \lim_{h\to 0}\frac{f(x + h\sqrt{2}/2, y + h\sqrt{2}/2) - f(x,y)}{h} \\ &= \lim_{h\to 0}\frac{3x + 34\sqrt{2}/2 - (y + h\sqrt{2}/2)^2 - 3x + y^2}{h} \\ &= \lim_{h\to 0}\frac{3h\sqrt{2}/2 - h\sqrt{2}yh^2/2}{h} = \lim_{h\to 0}(3\sqrt{2}/2 - \sqrt{2}y - h/2) = 3\sqrt{2}/2 - \sqrt{2}y \end{aligned}$$

11. $\mathbf{u} = \dfrac{\sqrt{3}}{2}\mathbf{i}\dfrac{1}{2}\mathbf{j}$; $\nabla f = 15x^2y^6\mathbf{i} + 30x^3y^5\mathbf{j}$; $\nabla f(-1,1) = 15\mathbf{i} - 30\mathbf{j}$; $D_\mathbf{u}f(-1,1) = \dfrac{15\sqrt{3}}{2} - 15 = \dfrac{15}{2}(\sqrt{3} - 2)$

12. $\mathbf{u} = \frac{\sqrt{2}}{2}\mathbf{i}\frac{\sqrt{2}}{2}\mathbf{j}$; $\nabla f = (4 + y^2)\mathbf{i}(2xy - 5)\mathbf{j}$; $\nabla f(3, -1) = 5\mathbf{i} - 11\mathbf{j}$;
$D_{\mathbf{u}}f(3, -1) = \frac{5\sqrt{2}}{2} - \frac{11\sqrt{2}}{2} = -3\sqrt{2}$

13. $\mathbf{u} = \frac{\sqrt{10}}{10}\mathbf{i} - \frac{3\sqrt{10}}{10}\mathbf{j}$; $\nabla f = \frac{-y}{x^2 + y^2}\mathbf{i} + \frac{x}{x^2 + y^2}\mathbf{j}$; $\nabla f(2, -2) = \frac{1}{4}\mathbf{i} + \frac{1}{4}\mathbf{j}$;
$D_{\mathbf{u}}f(2, -2) = \frac{\sqrt{10}}{40} - \frac{3\sqrt{10}}{40} = -\frac{\sqrt{10}}{20}$

14. $\mathbf{u} = \frac{6}{10}\mathbf{i} + \frac{8}{10}\mathbf{j}$; $\nabla f = \frac{y^2}{(x + y)^2}\mathbf{i} + \frac{x^2}{(x + y)^2}\mathbf{j}$; $\nabla f(2, -1) = \mathbf{i} + 4\mathbf{j}$;
$D_{\mathbf{u}}f(2, -1) = \frac{3}{5} + \frac{16}{5} = \frac{19}{5}$

15. $\mathbf{u} = (2\mathbf{i} + \mathbf{j})/\sqrt{5}$; $\nabla f = 2y(xy + 1)\mathbf{i} + 2x(xy + 1)\mathbf{j}$; $\nabla f(3, 2) = 28\mathbf{i} + 42\mathbf{j}$;
$D_{\mathbf{u}}f(3, 2) = \frac{2(28)}{\sqrt{5}} + \frac{42}{\sqrt{5}} = \frac{98}{\sqrt{5}}$

16. $\mathbf{u} = -\mathbf{i}$; $\nabla F = 2x\tan y\mathbf{i} + x^2\sec^2 y\mathbf{j}$; $\nabla f(1/2, \pi/3) = \sqrt{3}\mathbf{i} + \mathbf{j}$; $D_{\mathbf{u}}f(1/2, \pi/3) = -\sqrt{3}$

17. $\mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}$; $\nabla f = 2xy^2(2z + 1)^2\mathbf{i}2x^2y(2z + 1)^2\mathbf{j} + 4x^2y^2(2z + 1)\mathbf{k}$; $\nabla f(1, -1, 1) = 18\mathbf{i} - 18\mathbf{j} + 12\mathbf{k}$; $D_{\mathbf{u}}f(1, -1, 1) = -\frac{18}{\sqrt{2}} + \frac{12}{\sqrt{2}} = -\frac{6}{\sqrt{2}} = -3\sqrt{2}$

18. $\mathbf{u} = \frac{1}{\sqrt{6}}\mathbf{i} - \frac{2}{\sqrt{6}}\mathbf{j} + \frac{1}{\sqrt{6}}\mathbf{k}$; $\nabla f = \frac{2x}{z^2}\mathbf{i} - \frac{2y}{z^2}\mathbf{j} + \frac{2y^2 - 2x^2}{z^3}\mathbf{k}$; $\nabla f(2, 4, -1) = 4\mathbf{i} - 8\mathbf{j} - 24\mathbf{k}$;
$D_{\mathbf{u}}f(2, 4, -1) = \frac{4}{\sqrt{6}} - \frac{16}{\sqrt{6}} - \frac{24}{\sqrt{6}} = -6\sqrt{6}$

19. $\mathbf{u} = -\mathbf{k}$; $\nabla f = \frac{xy}{\sqrt{x^2y + 2y^2z}}\mathbf{i} + \frac{x^2 + 4z}{2\sqrt{x^2y + 2y^2z}}\mathbf{j} + \frac{y^2}{\sqrt{x^2y + 2y^2z}}\mathbf{k}$;
$\nabla f(-2, 2, 1) = -\mathbf{i} + \mathbf{j} + \mathbf{k}$; $D_{\mathbf{u}}f(-2, 2, 1) = -1$

20. $\mathbf{u} = -(4\mathbf{i} - 4\mathbf{j} + 2\mathbf{k})/\sqrt{36} = -\frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}$; $\nabla f = 2\mathbf{i} - 2y\mathbf{j} + 2z\mathbf{k}$; $\nabla f(4, -4, 2) = 2\mathbf{i} + 8\mathbf{j} + 4\mathbf{k}$;
$D_{\mathbf{u}}f(4, -4, 2) = -\frac{4}{3} + \frac{16}{3} - \frac{4}{3} = \frac{8}{3}$

21. $\mathbf{u} = (-4\mathbf{i} - \mathbf{j}/\sqrt{17}$; $\nabla f = 2(x - y)\mathbf{i} - 2(x - y)\mathbf{j}$; $\nabla f(4, 2) = 4\mathbf{i} - 4\mathbf{j}$;
$D_{\mathbf{u}}f(4, 2) = -\frac{16}{\sqrt{17}} + \frac{4}{\sqrt{17}} = -\frac{12}{\sqrt{17}}$

22. $\mathbf{u} = (-2\mathbf{i} + 5\mathbf{j}/\sqrt{29}$; $\nabla f = (3x^2 - 5y)\mathbf{i} - (5x - 2y)\mathbf{j}$; $\nabla f(1, 1) = -2\mathbf{i} - 3\mathbf{j}$;
$D_{\mathbf{u}}f(1, 1) = \frac{4}{\sqrt{29}} - \frac{15}{\sqrt{29}} = -\frac{11}{\sqrt{29}}$

23. $\nabla f = 2e^{2x}\sin y\mathbf{i} + e^{2x}\cos y\mathbf{j}$; $\nabla f(0, \pi/4) = \sqrt{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$
The maximum $D_{\mathbf{u}}$ is $\left[(\sqrt{2})^2 + (\sqrt{2}/2)^2\right]^{1/2} = \sqrt{5/2}$ in the direction $\sqrt{2}\mathbf{i} + (\sqrt{2}/2)\mathbf{j}$.

24. $\nabla f = (xye^{x-y} + ye^{x-y}\mathbf{i} + (-xye^{x-y} + xe^{x-y}\mathbf{j}$; $\nabla f(5,5) = 30\mathbf{i} - 20\mathbf{j}$
The maximum $D_{\mathbf{u}}$ is $\left[30^2 + (-20)^2\right]^{1/2} = 10\sqrt{13}$ in the direction $30\mathbf{i} - 20\mathbf{j}$.

25. $\nabla f = (2x + 4z)\mathbf{i} + 2z^2\mathbf{j} + (4x + 4yz)\mathbf{k}$; $\nabla f(1,2,-1) = -2\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$
The maximum $D_{\mathbf{u}}$ is $\left[(-2)^2 + (2)^2 + (-4)^2\right]^{1/2} = 2\sqrt{6}$ in the direction $-2\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$.

26. $\nabla f = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$; $\nabla f(3,1,-5) = -5\mathbf{i} - 15\mathbf{j} + 3\mathbf{k}$
The maximum $D_{\mathbf{u}}$ is $\left[(-5)^2 + (-15)^2 + (3)^2\right]^{1/2} = \sqrt{259}$ in the direction $-5\mathbf{i} - 15\mathbf{j} + 3\mathbf{k}$.

27. $\nabla f = 2x\sec^2(x^2 + y^2)\mathbf{i} + 2y\sec^2(x^2 + y^2)\mathbf{j}$;
$\nabla f(\sqrt{\pi/6}, \sqrt{\pi/6}) = 2\sqrt{\pi/6}\sec^2(\pi/3)(\mathbf{i} + \mathbf{j}) = 8\sqrt{\pi/6}(\mathbf{i} + \mathbf{j})$
The minimum $D_{\mathbf{u}}$ is $-8\sqrt{\pi/6}(1^2 + 1^2)^{1/2} = -8\sqrt{\pi/3}$ in the direction $-(\mathbf{i} + \mathbf{j})$.

28. $\nabla f = 3x^2\mathbf{i} - 3y^2\mathbf{j}$; $\nabla f(2,-2) = 12\mathbf{i} - 12\mathbf{j} = 12(\mathbf{i} - \mathbf{j})$
The minimum $D_{\mathbf{u}}$ is $-12\left[1^2 + (-1)^2\right]^{1/2} = -12\sqrt{2}$ in the direction $-(\mathbf{i} - \mathbf{j}) = -\mathbf{i} + \mathbf{j}$.

29. $\nabla f = \dfrac{\sqrt{z}e^y}{2\sqrt{x}}\mathbf{i} + \sqrt{xz}e^y\mathbf{j} + \dfrac{\sqrt{x}}{2\sqrt{z}}\mathbf{k}$; $\nabla f(16,0,9) = \dfrac{3}{8}\mathbf{i} + 12\mathbf{j} + \dfrac{2}{3}\mathbf{k}$. The minimum $D_{\mathbf{u}}$ is
$-\left[(3/8)^2 + 12^2 + (2/3)^2\right]^{1/2} = -\sqrt{83281}/24$ in the direction $-\dfrac{3}{8}\mathbf{i} - 12\mathbf{j} - \dfrac{2}{3}\mathbf{k}$.

30. $\nabla f = \dfrac{1}{x}\mathbf{i} + \dfrac{1}{y}\mathbf{j} - \dfrac{1}{z}\mathbf{k}$; $\nabla f(1/2, 1/6, 1/3) = 2\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$
The minimum $D_{\mathbf{u}}$ is $-\left[2^2 + 6^2(-3)^2\right]^{1/2} = -7$ in the direction $-2\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}$.

31. Using implicit differentiation on $2x^2 + y^2 = 9$ we find $y' = -2x/y$. At $(2,1)$ the slope of the tangent line is $-2(2)/1 = -4$. Thus, $\mathbf{u} = \pm(\mathbf{i} - 4\mathbf{j})/\sqrt{17}$. Now, $\nabla f = \mathbf{i} + 2y\mathbf{j}$ and $\nabla f(3,4) = \mathbf{i} + 8\mathbf{j}$. Thus, $D_{\mathbf{u}} = \pm(1/\sqrt{17} - 32\sqrt{17}) = \pm 31/\sqrt{17}$.

32. $\nabla f = (2x + y - 1)\mathbf{i} + (x + 2y)\mathbf{j}$; $D_{\mathbf{u}}f(x,y) = \dfrac{2x + y - 1}{\sqrt{2}} + \dfrac{x + 2y}{\sqrt{2}} = \dfrac{3x + 3y - 1}{\sqrt{2}}$ Solving $(3x + 3y - 1)/\sqrt{2} = 0$ we see that $D_{\mathbf{u}}$ is 0 for all points on the line $3x + 3y = 1$.

33. (a) Vectors perpendicular to $4\mathbf{i} + 3\mathbf{j}$ are $\pm(3\mathbf{i} - 4\mathbf{j})$. Take $\mathbf{u} = \pm\left(\dfrac{3}{5}\mathbf{i} - \dfrac{4}{5}\mathbf{j}\right)$.

(b) $\mathbf{u} = (4\mathbf{i} + 3\mathbf{j})/\sqrt{16 + 9} = \dfrac{4}{5}\mathbf{i} + \dfrac{3}{5}\mathbf{j}$

(c) $\mathbf{u} = -\dfrac{4}{5}\mathbf{i} - \dfrac{3}{5}\mathbf{j}$

34. $D_{-\mathbf{u}}f(a,b) = \nabla f(a,b) \cdot (-\mathbf{u}) = -\nabla f(a,b) \cdot \mathbf{u} = -D_{\mathbf{u}}f(a,b) = -6$

35. (a) $\nabla f = (3x^2 - 6xy^2)\mathbf{i} + (-6x^2y + 3y^2)\mathbf{j}$
$$D_u f(x,y) = \frac{3(3x^2 - 6xy^2)}{\sqrt{10}} + \frac{-6x^2y + 3y^2}{\sqrt{10}} = \frac{9x^2 - 18xy^2 - 6x^2y + 3y^2}{\sqrt{10}}$$

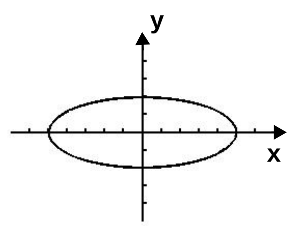
(b) $F(x,y) = \dfrac{3}{\sqrt{10}}(3x^2 - 3xy^2 - 2x^2y + y^2)$; $\nabla F = \dfrac{3}{\sqrt{10}}[(6x - 6y^2 - 4xy)\mathbf{i} + (-12xy - 2x^2 + 2y)\mathbf{j}]$

$$\begin{aligned} D_uF(x,y) &= \left(\frac{3}{\sqrt{10}}\right)\left(\frac{3}{\sqrt{10}}\right)(6x - 6y^2 - 4xy) + \left(\frac{1}{\sqrt{10}}\right)\left(\frac{3}{\sqrt{10}}\right)(-12xy - 2x^2 + 2y) \\ &= \frac{9}{5}(3x - 3y^2 - 2xy) + \frac{3}{5}(-6xy - x^2 + y) = \frac{1}{5}(27x - 27y^2 - 36xy - 3x^2 + 3y) \end{aligned}$$

36. Let $\nabla f(a,b) = \alpha\mathbf{i} + \beta\mathbf{j}$. Then $D_{\mathbf{u}}f(a,b) = \nabla f(a,b)\cdot\mathbf{u} = \dfrac{5}{13}\alpha - \dfrac{12}{13}\beta = 7$ and $D_{\mathbf{v}}f(a,b) = \nabla f(a,b)\cdot\mathbf{v} = \dfrac{5}{13}\alpha - \dfrac{12}{13}\beta = 3$. Solving for α and β, we obtain $\alpha = 13$ and $\beta = -13/6$. Thus, $\nabla f(a,b) = 13\mathbf{i} - (13/6)\mathbf{j}$.

37.

38. $\nabla f = \langle 2x, -5y\rangle$, $|\nabla f| = \sqrt{10x^2 + 25y^2} = 10$, $4x^2 + 25y^2 = 100$, $\dfrac{x^2}{25} + \dfrac{y^2}{4} = 1$



39. $\nabla T = 4x\mathbf{i} + 2y\mathbf{j}$; $\nabla T(4,2) = 16\mathbf{i} + 4\mathbf{j}$. The minimum change in temperature (that is, the maximum decrease in temperature) is in the direction $-\nabla T(4,3) = -16\mathbf{i} - 4\mathbf{j}$.

40. Let $x(t)\mathbf{i} + y(t)\mathbf{j}$ be the vector equation of the path. At (x,y) on this curve, the direction of a tangent vector is $x'(t)\mathbf{i} + y'(t)\mathbf{j}$. Since we want the direction of motion to be $-\nabla T(x,y)$, we have $x'(t)\mathbf{i} + y'(t)\mathbf{j} = -\nabla T(x,y) = 4x\mathbf{i} + 2y\mathbf{j}$. Separating variables in $dx/dt = 4x$, we obtain $dx/x = 4dt, \ln x = 4t + c_1$, and $x = C_1e^{4t}$. Separating variables in $dy/dt = 2y$, we obtain $dy/y = 2dt$, $\ln y = 2t + c_2$, and $y = C_2e^{2t}$. Since $x(0) = 4$ and $y(0) = 2$, we have $x = 4e^{4t}$ and $y = 2e^{2t}$. The equation of the path is $4e^{4t}\mathbf{i} + 2e^{2t}\mathbf{j}$ for $t \geq 0$, or eliminating the parameter, $x = y^2, y \geq 0$.

41. Let $x(t)\mathbf{i} + y(t)\mathbf{j}$ be the vector equation of the path. At (x,y) on this curve, the direction of a tangent vector is $x'(t)\mathbf{i} + y'(t)\mathbf{j}$. Since we want the direction of motion to be $\nabla T(x,y)$, we have $x'(t)\mathbf{i} + y'(t)\mathbf{j} = \nabla T(x,y) = -4x\mathbf{i} - 2y\mathbf{j}$. Separating variables in $dx/dt = -4x$, we obtain $dx/x = -4dt, \ln x = -4t + c_1$, and $x = C_1e^{-4t}$. Separating variables in $dy/dt = -2y$, we obtain $dy/y = -2dt, \ln y = -2t + c_2$, and $y = C_2e^{-2t}$. Since $x(0) = 3$ and $y(0) = 4$, we have $x = 3e^{-4t}$ and $y = 4e^{-2t}$. The equation of the path is $3e^{-4t}\mathbf{i} + 4e^{-2t}\mathbf{j}$, or eliminating the parameter, $16x = 3y^2$, $y \geq 0$.

42. Substituing $x = 0$, $y = 0$, $z = 1$, and $T = 500$ into $t = \dfrac{k}{x^2+y^2+z^2}$ we see that $k = 500$ and $T(x,y,z) = \dfrac{500}{x^2+y^2+z^2}$.

(a) $\mathbf{u} = \frac{1}{3}\langle 1,-2,-2\rangle = \frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$
$\nabla T = -\dfrac{1000x}{(x^2+y^2+z^2)^2}\mathbf{i} - \dfrac{1000y}{(x^2+y^2+z^2)^2}\mathbf{j} - \dfrac{1000z}{(x^2+y^2+z^2)^2}\mathbf{k}$
$\nabla T(2,3,3) = -\dfrac{500}{121}\mathbf{i} - \dfrac{750}{121}\mathbf{j} - \dfrac{750}{121}\mathbf{k}$
$D_u T(2,3,3) = \dfrac{1}{3}\left(-\dfrac{500}{121}\right) - \dfrac{2}{3}\left(-\dfrac{750}{121}\right) - \dfrac{2}{3}\left(-\dfrac{750}{121}\right) = \dfrac{2500}{363}$

(b) The direction of maximum increase is

$$\nabla T(2,3,3) = -\frac{500}{121}\mathbf{i} - \frac{750}{121}\mathbf{j} - \frac{750}{121}\mathbf{k} = \frac{252}{121}(-2\mathbf{i} - 3\mathbf{j} - 3\mathbf{k}).$$

(c) The maximum rate of change of T is $|\nabla T(2,3,3)| = \dfrac{250}{121}\sqrt{4+9+9} = \dfrac{250\sqrt{22}}{121}$.

43. $\nabla U = \dfrac{Gmx}{(x^2+y^2)^{3/2}}\mathbf{i} + \dfrac{Gmy}{(x^2+y^2)^{3/2}}\mathbf{j} = \dfrac{Gm}{(x^2+y^2)^{3/2}}(x\mathbf{i}+y\mathbf{j})$
The maximum and minimum values of $D_{\mathbf{u}}U(x,y)$ are obtained when $\mathbf{u}$ is in the directions ∇U and
$-\nabla U$, respectively. Thus, at a point (x,y), not (0,0), the directions of maximum and minimum increase in U are $x\mathbf{i}+y\mathbf{j}$ and $-x\mathbf{i}-y\mathbf{j}$, respectively. A vector at (x,y) in the direction $\pm(x\mathbf{i}+y\mathbf{j})$ lies on a line through the origin.

44. Since $\nabla f = f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j}$, we have $\partial f/\partial x = 3x^2 + y^3 + ye^{xy}$. Integrating, we obtain $f(x,y) = x^3 + xy^3 + e^{xy} + g(y)$. Then $f_y = 3xy^2 + xe^{xy} + g'(y) = -2y^2 + 3xy^2 + xe^{xy}$. Thus, $g'(y) = -2y^2$, $g(y) = -\frac{2}{3}y^3 + c$, and $f(x,y) = x^3 + xy^3 + e^{xy} - \frac{2}{3} + C$.

45. $\nabla(cf) = \dfrac{\partial}{\partial x}(cf)\mathbf{i} + \dfrac{\partial}{\partial y}\mathbf{j} = cf_x\mathbf{i} + cf_y\mathbf{j} = c(f_x\mathbf{i} + f_y\mathbf{j}) = c\nabla f$

46. $\nabla(f+g) = (f_x+g_x)\mathbf{i} + (f_y+g_y)\mathbf{j} = (f_x\mathbf{i} + f_y\mathbf{j}) + (g_x\mathbf{i} + g_y\mathbf{j}) = \nabla f + \nabla g$

47. $\nabla(fg) = (fg_x + f_xg)\mathbf{i} + (fg_y + f_yg)\mathbf{j} = f(g_x\mathbf{i} + g_y\mathbf{j}) + g(f_x\mathbf{i} + f_y\mathbf{j}) = f\nabla g + g\nabla f$

48. $\nabla(f/g) = \left[(gf_x - fg_x)/g^2\right]\mathbf{i} + \left[(gf_y - fg_y)/g^2\right]\mathbf{j} = g(f_x\mathbf{i} + f_y\mathbf{j})/g^2 - f(g_x\mathbf{i} + g_y\mathbf{j})/g^2$
$= g\nabla f/g^2 - f\nabla g/g^2 = (g\nabla f - f\nabla g)/g^2$

49. $r(x,y) = \sqrt{x^2+y^2}$ so $\dfrac{\partial r}{\partial x} = \dfrac{x}{\sqrt{x^2+y^2}} = \dfrac{x}{r}$ and $\dfrac{\partial r}{\partial y} = \dfrac{y}{\sqrt{x^2+y^2}} = \dfrac{y}{r}$
This gives $\nabla r = \left\langle \dfrac{x}{r}, \dfrac{y}{r}\right\rangle = \dfrac{1}{r}\langle x,y\rangle = \dfrac{\mathbf{r}}{r}$

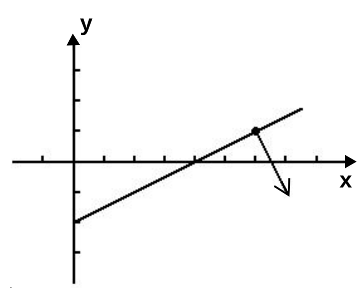
50. $\dfrac{\partial\,(f(r))}{\partial x} = \dfrac{df}{dr}\dfrac{\partial r}{\partial x}$ and $\dfrac{\partial\,(f(r))}{\partial y} = \dfrac{df}{dr}\dfrac{\partial r}{\partial y}$ so that $\nabla f(r) = \langle \dfrac{\partial\,(f(r))}{\partial x}, \dfrac{\partial\,(f(r))}{\partial y}\rangle$
$= \langle \dfrac{df}{dr}\dfrac{\partial r}{\partial x}, \dfrac{df}{dr}\dfrac{\partial r}{\partial y}\rangle = \dfrac{df}{dr}\langle \dfrac{\partial r}{\partial x}, \dfrac{\partial r}{\partial y}\rangle = f'(r)\nabla r = f'(r)\mathbf{r}/r$

51. Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$.
$D_v f = (f_x\mathbf{i} + f_y\mathbf{j}) \cdot \mathbf{v} = v_1 f_x + v_2 f_y$
$D_u D_v f = \left[\dfrac{\partial}{\partial x}(v_1 f_x + v_2 f_y)\mathbf{i} + \dfrac{\partial}{\partial y}(v_1 f_x + v_2 f_y)\mathbf{j}\right] \cdot \mathbf{u} = [(v_1 f_{xx} + v_2 f_{yz})\mathbf{i} + (v_1 f_{xy} + v_2 f_{yy})\mathbf{j}] \cdot \mathbf{u}$
$= u_1 v_1 f_{xx} + u_1 v_2 f_{yx} + u_2 v_1 f_{xy} + u_2 v_2 f_{yy}$
$D - uf = (f_x\mathbf{i} + f_y\mathbf{j}) \cdot \mathbf{u} = u_1 f_x + u_2 f_y$
$D_v D_u f = \left[\dfrac{\partial}{\partial x}(u_1 f_x + u_2 f_y)\mathbf{i} + \dfrac{\partial}{\partial y}(u_1 f_x + u_2 f_y)\mathbf{j}\right] \cdot \mathbf{v} = [(u_1 f_{xx} + u_2 f_{yx})\mathbf{i} + (u_1 f_{xy} + u_2 f_{yy})\mathbf{j}] \cdot \mathbf{v}$
$= u_1 v_1 f_{xx} + u_2 v_1 f_{yx} + u_1 v_2 f_{xy} + u_2 v_2 f_{yy}$
Since the second partial derivatives are continuous, $f_{xy} = f_{yx}$ and $D_u D_v f = D_v D_u f$. [Note that this result is a generalization $f_{xy} = f_{yx}$ since $D_i D_j f = f_{yx}$ and $D_j D_i f = f_{xy}$]

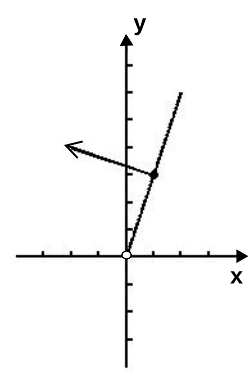
52. $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$
$= \left(\dfrac{\partial f_3}{\partial y} - \dfrac{\partial f_2}{\partial z}\right)\mathbf{i} - \left(\dfrac{\partial f_3}{\partial x} - \dfrac{\partial f_1}{\partial z}\right)\mathbf{j} + \left(\dfrac{\partial f_2}{\partial x} - \dfrac{\partial f_1}{\partial y}\right)\mathbf{k}$

13.7 Tangent Planes and Normal Lines

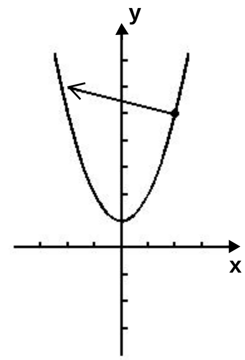
1. Since $f(6,1) = 4$, the level curve is $x - 2y = 4$. $\nabla f = \mathbf{i} - 2\mathbf{j}$; $\nabla f(6,1) = \mathbf{i} - 2\mathbf{j}$



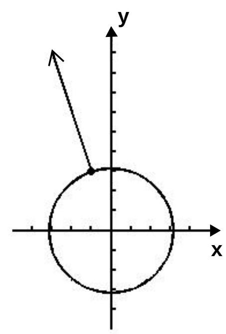
2. Since $f(1,3) = 5$, the level curve is $y + 2x = 5x$ or $y = 3x$, $x \neq 0$.
$\nabla f = -\dfrac{y}{x^2}\mathbf{i} + \dfrac{1}{x}\mathbf{j}$; $\nabla f(1,3) = -3\mathbf{i} + \mathbf{j}$



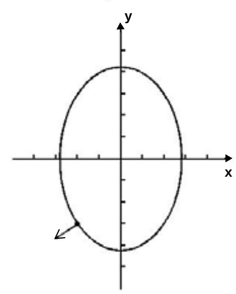
3. Since $f(2,5)=1$, the level curve is $y=x^2+1$. $\nabla f=-2x\mathbf{i}+\mathbf{j}$; $\nabla f(2,5)=-4\mathbf{i}+\mathbf{j}$



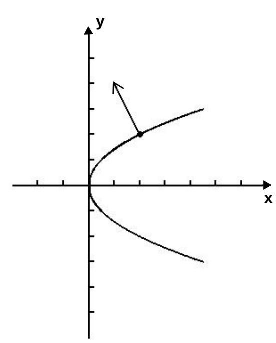
4. Since $f(-1,3)=10$, the level curve is $x^2+y^2=10$. $\nabla f=2x\mathbf{i}+2y\mathbf{j}$; $\nabla f(-1,3)=-2\mathbf{i}+6\mathbf{j}$



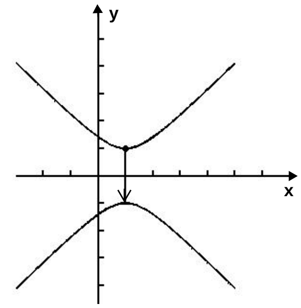
5. Since $f(-2,-3)=2$, the level curve is $x^2/4+y^2/0=2$ or $x^2/8+y^2/18=1$. $\nabla f=\dfrac{x}{2}\mathbf{i}+\dfrac{2y}{9}\mathbf{j}$; $\nabla f(-2,-3)=-\mathbf{i}-\dfrac{2}{3}\mathbf{j}$



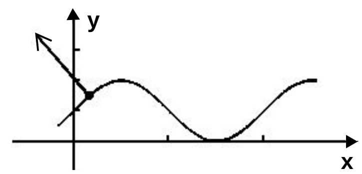
6. Since $f(2,2)=2$, the level curve is $y^2=2x,\ \ x\neq 0$. $\nabla f=-\dfrac{y^2}{x^2}\mathbf{i}+\dfrac{2y}{x}\mathbf{j}$; $\nabla f(2,2)=-\mathbf{i}+2\mathbf{j}$



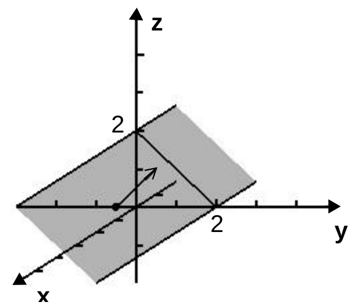
7. Since $f(1,1) = -1$, the level curve is $(x-1)^2 - y^2 = -1$ or $y^2 - (x-1)^2 = 1$. $\nabla f = 2(x-1)\mathbf{i} - 2y\mathbf{j}$; $\nabla f(1,1) = -2\mathbf{j}$



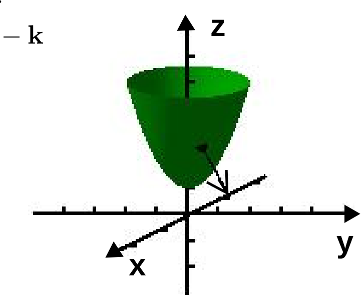
8. Since $f(\pi/6, 3/2) = 1$, the level curve is $y - 1 = \sin x$ or $y = 1 + \sin x$, $\sin x \neq 0$. $\nabla f = \dfrac{-(y-1)\cos x}{\sin^2 x}\mathbf{i} + \dfrac{1}{\sin x}\mathbf{j}$; $\nabla f(\pi/6, 3/2) = -\sqrt{3}\mathbf{i} + 2\mathbf{j}$



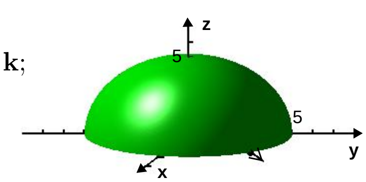
9. Since $f(3,1,1) = 2$, the level curve is $y + z = 2$ $\nabla f = \mathbf{j} + \mathbf{k}$; $\nabla f(3,1,1) = \mathbf{j} + \mathbf{k}$



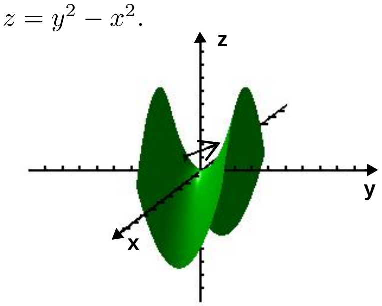
10. Since $f(1,1,3) = -1$, the level curve is $x^2 + y^2 - z = -1$ or $z = 1 + x^2 + y^2$. $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k}$; $\nabla f(1,1,3) = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$



11. Since $F(3,4,0) = 5$, the level curve is $x^2 + y^2 + z^2 = 25$.
$\nabla F = \dfrac{x}{\sqrt{x^2+y^2+z^2}}\mathbf{i} + \dfrac{y}{\sqrt{x^2+y^2+z^2}}\mathbf{j} + \dfrac{z}{\sqrt{x^2+y^2+z^2}}\mathbf{k}$;
$\nabla F(3,4,0) = \dfrac{3}{4}\mathbf{i} + \dfrac{4}{5}\mathbf{j}$



12. Since $F(0,-1,1)=0$, the level curve is $x^2-y^2+z=0$ or $z=y^2-x^2$.
$\nabla F = 2x\mathbf{i} - 2y\mathbf{j} + \mathbf{k}$; $\nabla F(0,-1,1) = 2\mathbf{i} + \mathbf{k}$



13. $F(x,y,z) = x^2+y^2-z$; $\nabla F = 2x\mathbf{i}+2y\mathbf{j}-\mathbf{k}$. We want $\nabla F = c\left(4\mathbf{i}+\mathbf{j}+\frac{1}{2}\mathbf{k}\right)$ or $2x = 4c$, $2y = c$, $-1 = c/2$. From the third equation $c=-2$. Thus, $x=-4$ and $y=-1$. Since $z = x^2+y^2 = 16+1 = 17$, the point on the surface is $(-4,-1,-17)$.

14. $F(x,y,z) = x^3+y^3+z$; $\nabla F = 3x^2\mathbf{i}+2y\mathbf{j}+\mathbf{k}$. We want $\nabla F = c(27\mathbf{i}+8\mathbf{j}+\mathbf{k})$ or $3x^2 = 27c$, $2y=8c$, $1=c$. From $c=1$ we obtain $x=\pm 3$ and $y=4$. Since $z = 15-x^3-y^2 = 15-(\pm 3)^3-16 = -1\mp 27$, the points on the surface are $(3,4,-28)$ and $(-3,4,26)$.

15. $F(x,y,z0 = x^2+y^2+z^2$; $\nabla F = 2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}$. $\nabla F(-2,2,1) = -4\mathbf{i}+4\mathbf{j}+2\mathbf{k}$. The equation of the tangent plane is $-4(x+2)+4(y-2)+2(z-1)=0$ or $-2x+2y+z=9$.

16. $F(x,y,z) = 5x^2-y^2+4z^2$; $\nabla F = 10x\mathbf{i}-2y\mathbf{j}+8z\mathbf{k}$; $\nabla F(2,4,1) = 20\mathbf{i}-8\mathbf{j}+8\mathbf{k}$.The equation of the tangent plane is $20(x-2)-8(y-4)+8(z-1)=0$ or $5x-2y+2z=4$.

17. $F(x,y,z) = x^2-y^2-3z^2; \nabla F = 2x\mathbf{i}-2y\mathbf{j}-6z\mathbf{k}$; $\nabla F(6,2,3) = 12\mathbf{i}-4\mathbf{j}-18\mathbf{k}$. The equation of the tangent plane is $12(x-6)-4(y-2)-18(z-3)=0$ or $6x-2y-9z=5$.

18. $F(x,y,z) = xy+yz+zx; \nabla F = (y+z)\mathbf{i}+(x+z)\mathbf{j}+(y+x)\mathbf{k}$; $\nabla F(1,-3,-5) = -8\mathbf{i}-4\mathbf{j}-2\mathbf{k}$. The equation of the tangent plane is $-8(x-1)-4(y+3)-2(z+5)=0$ or $4x+2y+z=-7$.

19. $F(x,y,z) = x^2+y^2+z$; $\nabla F = 2x\mathbf{i}+2y\mathbf{j}+\mathbf{k}$; $\nabla F(3,-4,0) = 6\mathbf{i}-8\mathbf{j}+\mathbf{k}$. The equation of the tangent plane is $6(x-3)-8(y+4)+z=0$ or $6x-8y+z=50$.

20. $F(x,y,z) = xz$; $\nabla F = z\mathbf{i}+x\mathbf{k}$; $\nabla F(2,0,3) = 3\mathbf{i}+2\mathbf{k}$. The equation of the tangent plane is $3(x-2)+2(z-3)=0$ or $3x+2z=12$.

21. $F(x,y,z) = \cos(2x+y)-z$; $\nabla F = -2\sin(2x+y)\mathbf{i}-\sin(2x+y)\mathbf{j}-\mathbf{k}$; $\nabla F(\pi/2,\pi/4,-1\sqrt{2}) = \sqrt{2}\mathbf{i}+\dfrac{\sqrt{2}}{2}\mathbf{j}-\mathbf{k}$. The equation of the tangent plane is $\sqrt{2}\left(x-\dfrac{\pi}{2}\right)+\dfrac{\sqrt{2}}{2}\left(y-\dfrac{\pi}{4}\right)-\left(z+\dfrac{1}{\sqrt{2}}\right) = 0$, $2\left(x-\dfrac{\pi}{2}\right)+\left(y-\dfrac{\pi}{4}\right)-\sqrt{2}\left(z+\dfrac{1}{\sqrt{2}}\right)=0$, or $2x+y-\sqrt{2}z = \dfrac{5\pi}{4}+1$.

22. $F(x,y,z) = x^2y^3+6z$; $\nabla F = 2xy^3\mathbf{i}+3x^2y^2\mathbf{j}+6\mathbf{k}$; $\nabla F(2,1,1) = 4\mathbf{i}+12\mathbf{j}+6\mathbf{k}$. The equation of the tangent plane is $4(x-2)+12(y-1)+6(z-1)=0$ or $2x+6y+3z=13$.

23. $F(x,y,z) = \ln(x^2+y^2)-z$; $\nabla F = \dfrac{2x}{x^2+y^2}\mathbf{i}+\dfrac{2y}{x^2+y^2}\mathbf{j}-\mathbf{k}$; $\nabla F(1/\sqrt{2},1/\sqrt{2},0) = \sqrt{2}\mathbf{i}+\sqrt{2}\mathbf{j}-\mathbf{k}$. The equation of the tangent plane is $\sqrt{2}\left(x-\dfrac{1}{\sqrt{2}}\right)+\sqrt{2}\left(y-\dfrac{1}{\sqrt{2}}\right)-(z-0) = 0$, $2\left(x-\dfrac{1}{\sqrt{2}}\right)+2\left(y-\dfrac{1}{\sqrt{2}}\right)-\sqrt{2}z=0$, or $2x+2y-\sqrt{2}z = 2\sqrt{2}$.

24. $F(x, y, z) = 8e^{-2y}\sin 4x - z$; $\nabla F = 32e^{-2y}\cos 4x\mathbf{i} - 16e^{-2y}\sin 4x\mathbf{j} - \mathbf{k}$; $\nabla F(\pi/24, 0, 4) = 16\sqrt{3}\mathbf{i} - 8\mathbf{j} - \mathbf{k}$. The equation of the tangent plane is

$$16\sqrt{3}(x - \pi/24) - 8(y - 0) - (z - 4) = 0 \text{ or } 16\sqrt{3}x - 8y - z = \frac{2\sqrt{3}\pi}{3} - 4.$$

25. The gradient of $F(x, y, z) = x^2 + y^2 + z^2$ is $\nabla F = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$, so the normal vector to the surface at (x_0, y_0, z_0) is $2x_0\mathbf{i} + 2y_0\mathbf{j} + 2z_0\mathbf{k}$. A normal vector to the plane $2x + 4y + 6z = 1$ is $2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$. Since we want the tangent plane to be parallel to the given plane, we find c so that $2x_0 = 2c$, $2y_0 = 4c$, $2z_0 = 6c$ or $x_0 = c$, $y_0 = 2c$, $z_0 = 3c$. Now, (x_0, y_0, z_0) is on the surface, so $c^2 + (2c)^2 + (3c)^2 = 14c^2 = 7$ and $c = \pm 1/\sqrt{2}$. Thus, the points on the surface are $(\sqrt{2}/2, \sqrt{2}, 3\sqrt{2}/2)$ and $-\sqrt{2}/2, -\sqrt{2}, -3\sqrt{2}/2)$.

26. The gradient of $F(x, y, z) = x^2 - 2y^2 - 3z^2$ is $\nabla F(x, y, z) = 2x\mathbf{i} - 4y\mathbf{j} - 6z\mathbf{k}$, so a normal vector to the surface at (x_0, y_0, z_0) is $\nabla F(x_0, y_0, z_0) = 2x_0\mathbf{i} - 4y_0\mathbf{j} - 6z_0\mathbf{k}$. A normal vector to the plane $8x+4y+6z = 5$ is $8\mathbf{i}+4\mathbf{j}+6\mathbf{k}$. Since we want the tangent plane to be parallel to the given plane, we find c so that $2x_0 = 8c$, $-4y_0 = 4c$, $-6z_0 = 6c$ or $x_0 = 4c$, $y_0 = -c$, $z_0 = -c$. Now (x_0, y_0, z_0) is on the surface, so $(4c)^2 - 2(-c)^2 - 3(-c)^2 = 11c^2 = 33$ and $c = \pm\sqrt{3}$. Thus, the points on the surface are $(4\sqrt{3}, -\sqrt{3}, -\sqrt{3})$ and $(-4\sqrt{3}, \sqrt{3}, \sqrt{3})$.

27. The gradient of $F(x, y, z) = x^2+4x+y^2+z^2-2z$ is $\nabla F = (2x+4)\mathbf{i}+2y\mathbf{j}+(2z-2)\mathbf{k}$, so a normal to the surface at (x_0, y_0, z_0) is $(2x_0+4)\mathbf{i}+2y_0\mathbf{j}+(2z_0-2)\mathbf{k}$. A horizontal plane has normal $c\mathbf{k}$ for $c \neq 0$. Thus, we want $2x_0 + 4 = 0$, $2y_0 = 0$, $2z_0 - 2 = c$ or $x_0 = -2$, $y_0 = 0$, $z_0 = c + 1$. Since (x_0, y_0, z_0) is on the surface, $(-2)^2 + 4(-2) + (c + 1)^2 - 2(c + 1) = c^2 - 5 = 11$ and $c = \pm 4$. The points on the surface are $(-2, 0, 5)$ and $(-2, 0, -3)$.

28. The gradient of $F(x, y, z) = x^2 + 3y^2 + 4z^2 - 2xy$ is $\nabla F = (2x - 2y)\mathbf{i} + (6y - 2x)\mathbf{j} + 8z\mathbf{k}$, so a normal to the surface at (x_0, y_0, z_0) is $2(x_0 - y_0)\mathbf{i} + 2(3y_0 - x_0)\mathbf{j} + 8z_0\mathbf{k}$.

    (a) A normal to the xz plane is $c\mathbf{j}$ for $c \neq 0$. Thus, we want $2(x_0 - y_0) = 0$, $2(3y_0 - x_0) = c$, $8z_0 = 0$ or $x_0 = y_0$, $3y_0 - x_0 = c/2$, $z_0 = 0$. Solving the first two equations, we obtain $x_0 = y_0 = c/4$. Since (x_0, y_0, z_0) is on the surface, $(c/4)^2 + 3(c/4)^2 + 4(0)^2 - 2(c/4)(c/4) = 2c^2/16 = 16$ and $c = \pm 16/\sqrt{2}$. Thus, the points on the surface are $(4/\sqrt{2}, 4/\sqrt{2}, 0)$ and $(-4\sqrt{2}, -4\sqrt{2}, 0)$.

    (b) A normal to the yz-plane is $c\mathbf{i}$ for $c \neq 0$. Thus, we want $2(x_0 - y_0) = c$, $2(3y_0 - x_0) = 0$, $8z_0 = 0$ or $x_0 - y_0 = c/2$, $x_0 = 3y_0$, $z_0 = 0$. Solving the first two equations, we obtain $x_0 = 3c/4$ and $y_0 = c/4$. Since (x_0, y_0, z_0) is on the surface, $(3c/4)^2 + 3(c/4)^2 + 4(0)^2 - 2(3c/4)(c/4) = 6c^2/16 = 16$ and $c = \pm 16\sqrt{6}$. Thus, the points on the surface are $(12/\sqrt{6}, 4/\sqrt{6}, 0)$ and $(-12/\sqrt{6}, -4/\sqrt{6}, 0)$.

    (c) A normal to the xy-plane is $c\mathbf{k}$for $c \neq 0$. Thus, we want $2(x_0 - y_0) = 0$, $2(3y_0 - x_0) = 0$, $8z_0 = c$ or $x_0 = y_0$, $3y_0 - x_0 = 0$, $z_0 = c/8$. Solving the first two equations, we obtain $x_0 = y_0 = 0$. Since (x_0, y_0, z_0) is on the surface, $0^2 + 3(0)^2 + 4(c/8)^2 - 2(0)(0) = c^2/16 = 16$ and $c = \pm 16$. Thus, the points on the surface are $(0, 0, 2)$ and $(0, 0, -2)$.

29. If (x_0, y_0, z_0) is on $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, then $x_0^2/a^2 + y_0^2/b^2 + z_0^2/c^2 = 1$ and $x_0, y_0, z_0)$ is on the plane $xx_0/a^2 + yy_0/b^2 + zz_0/c^2 = 1$. A normal to the surface at (x_0, y_0, z_0) is

$\nabla F(x_0, y_0, z_0) = (2x - 0/a^2)\mathbf{i} + (2y_0/b^2)\mathbf{j} + (2z_0/c^2)\mathbf{k}$. A normal to the plane is $(x_0/a^2)\mathbf{i} + (y_0/b^2)\mathbf{j} + (z_0/c^2)\mathbf{k}$. Since the normal to the surface is a multiple of the normal to the plane, the normal vectors are parallel and the plane is tangent to the surface.

30. If (x_0, y_0, z_0) is on $x^2/a^2 - y^2/b^2 + z^2/c^2 = 1$, then $x_0^2/b^2 - y_0^2/b^2 + z_0^2/c^2 = 1$ and (x_0, y_0, z_0) is on the plane $xx_0/a^2 - yy_0/b^2 + zz_0/c^2 = 1$. A normal to the surface at (x_0, y_0, z_0) is $\nabla F(x_0, y_0, z_0) = (2x_0/a^2)\mathbf{i} - (2y_0/b^2)\mathbf{j} + (2z_0/c^2)\mathbf{k}$. A normal to the plane is $(x_0/a^2)\mathbf{i} - (y_0/b^2)\mathbf{j} + (z_0/c^2)\mathbf{k}$. Since the normal to the surface is a multiple of the normal to the plane, the normal vectors are parallel, and the plane is tangent to the surface.

31. $F(x, y, z) = x^2 + 2y^2 + z^2$; $\nabla F = 2x\mathbf{i} + 4y\mathbf{j} + 2z\mathbf{k}$; $\nabla F(1, -1, 1) = 2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$. Parametric equations of the line are $x = 1 + 2t$, $y = -1 - 4t$, $z = 1 + 2t$.

32. $F(x, y, z) = 2x^2 - 4y^2 - z$; $\nabla F = 4x\mathbf{i} - 8y\mathbf{j} - \mathbf{k}$; $\nabla F(3, -2, 2) = 12\mathbf{i} + 16\mathbf{j} - \mathbf{k}$. Parametric equations of the line are $x = 3 + 12t$; $y = -2 + 16t$, $z = 2 - t$.

33. $F(x, y, z) = 4x^2 + 9y^2 - z$; $\nabla F = 8x\mathbf{i} + 18y\mathbf{j} - \mathbf{k}$; $\nabla F(1/2, 1/3, 3) = 4\mathbf{i} + 6\mathbf{j} - \mathbf{k}$. Symmetric equations of the line are $\dfrac{x - 1/2}{4} = \dfrac{y - 1/3}{6} = \dfrac{z - 3}{-1}$.

34. $F(x, y, z) = x^2 + y^2 - z^2$; $\nabla F = 2x\mathbf{i} + 2y\mathbf{j} - 2z\mathbf{k}; \nabla F(3, 4, 5) = 6\mathbf{i} + 8\mathbf{j} - 10\mathbf{k}$. Symmetric equations of the line are $\dfrac{x - 3}{6} = \dfrac{y - 4}{8} = \dfrac{z - 5}{-10}$.

35. Let $F(x, y, z) = x^2 + y^2 - z^2$. Then $\nabla F = 2x\mathbf{i} + 2y\mathbf{j} - 2z\mathbf{k}$ and a normal to the surface at (x_0, y_0, z_0) is $x_0\mathbf{i} + y_0\mathbf{j} - z_0\mathbf{k}$. An equation of the tangent plane at (x_0, y_0, z_0) is $x_0(x - x_0) + y_0(y - y_0) - z_0(z - z_0) = 0$ or $x_0x + y_0y - z_0z = x_0^2 + y_0^2 - z_0^2$. Since (x_0, y_0, z_0) is on the surface, $z_0^2 = x_0^2 + y_0^2$ and $x_0^2 + y_0^2 - z_0^2 = 0$. Thus, the equation of the tangent plane is $x_0x + y_0y - z_0z = 0$, which passes through the origin.

36. Let $F(x, y, z) = \sqrt{x} + \sqrt{y} + \sqrt{z}$. Then $\nabla F = \dfrac{1}{2\sqrt{x}}\mathbf{i} + \dfrac{1}{2\sqrt{y}}\mathbf{j} + \dfrac{1}{2\sqrt{z}}\mathbf{k}$ and a normal to the surface at (x_0, y_0, z_0) is $\dfrac{1}{2\sqrt{x_0}}\mathbf{i} + \dfrac{1}{2\sqrt{y_0}}\mathbf{j} + \dfrac{1}{2\sqrt{z_0}}\mathbf{k}$. An equation of the tangent plane at (x_0, y_0, z_0) is $\dfrac{1}{2\sqrt{x_0}}(x - x_0) + \dfrac{1}{2\sqrt{y_0}}(y - y_0) + \dfrac{1}{2\sqrt{z_0}}(z - z_0) = 0$ or $\dfrac{1}{\sqrt{x_0}}x + \dfrac{1}{\sqrt{y_0}}y + \dfrac{1}{\sqrt{z_0}}z = \sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0} = \sqrt{a}$. The sum of the intercepts is $\sqrt{x_0}\sqrt{a} + \sqrt{y_0}\sqrt{a} + \sqrt{z_0}\sqrt{a} = (\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0})\sqrt{a} = \sqrt{a} \cdot \sqrt{a} = a$.

37. A normal to the surface at (x_0, y_0, z_0) is $\nabla F(x_0, y_0, z_0) = 2x_0\mathbf{i} + 2y_0\mathbf{j} + 2z_0\mathbf{k}$. Parametric equations of the normal line are $x = x_0 + 2x_0t$, $y = y_0 + 2y_0t$, $z = z_0 + 2z_0t$. Letting $t = -1/2$, we see that the normal line passes through the origin.

38. The normal lines to $F(x, y, z) = 0$ and $G(x, y, z) = 0$ are $F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$ and $G_x\mathbf{i} + G_y\mathbf{j} + G_z\mathbf{k}$, respectively. These vectors are orthogonal if and only if their dot product is 0. Thus, the surfaces are orthogonal at P if and only if $F_xG_x + F_yG_y + F_zG_z = 0$.

39. We have $F(x, y, z) = x^2 + y^2 + z^2$ and $G(x, y, z) = x^2 + y^2 - z^2$.
$\nabla F = \langle 2x, 2y, 2z \rangle \neq 0$ except at the origin

$\nabla G = \langle 2x, 2y, -2z\rangle \neq 0$ except at the origin
Therefore, the gradient vectors are nonzero at each of the intersection points. Now

$$\begin{aligned} F_xG_x + F_yG_y + F_xG_z &= (2x)(2x) + (2y)(2y) + (2z)(-2z) \\ &= 4x^2 + 4y^2 - 4z^2 \\ &= 4(x^2 + y^2 + z^2) = 4(0) = 0 \end{aligned}$$

The second to last equality follows from the fact that the intersection points lie on both surfaces and hence satisfy the second equation $x^2 + y^2 - z^2 = 0$.

40. Let $F(x, y, z) = x^2 - y^2 + z^2 - 4$ and $G(x, y, z) = 1/xy^2 - z$. Then

$$\begin{aligned} F_xG_x + F_yG_y + F_zG_z &= (2x)(-1/x^2y^2) + (-2y)(-2/xy^3) + (2z)(-1) \\ &= -2/xy^2 + 4/xy^2 - 2z = 2(1/xy^2 - z). \end{aligned}$$

For (x, y, z) on both surfaces, $F(x, y, z) = G(x, y, z) = 0$. Thus, $F_xG_x + F_yG_y + F_zG_z = 2(0)$ and the surfaces are orthogonal at points of intersection.

13.8 Extrema of Multivariable Functions

1. $f_x = 2x$; $f_{xx} = 2$; $f_{xy} = 0$; $f_y = 2y$; $f_{yy} = 2$; $D = 4$. Solving $f_x = 0$ and $f_y = 0$, we obtain the critical point $(0, 0)$. Since $D(0, 0) = 4 > 0$ and $f_{xx}(0, 0) = 2 > 0$, $f(0, 0) = 5$ is a relative minimum.

2. $f_x = 8x$; $f_{xx} = 8$; $f_{xy} = 0$; $f_y = 16y$; $f_{yy} = 16$; $D = 128$. Solving $f_x = 0$ and $f_y = 0$, we obtain the critical point $(0, 0)$. Since $D(0, 0) = 128 > 0$ and $f_{xx}(0, 0) = 8 > 0$, $f(0, 0) = 0$ is a relative minimum.

3. $f_x = -2x + 8$; $f_{xx} = -2$; $fxy = 0$; $f_y = -2y + 6$; $f_{yy} = -2$; $D = 4$. Solving $f_x = 0$ and $f_y = 0$ we obtain the critical point $(4, 3)$. Since $D(4, 3) = 4 > 0$ and $f_{xx}(4, 3) = -2 < 0$, $f(4, 3,) = 25$ is a relative maximum.

4. $f_x = 6x - 6$; $fxx = 6$; $fxy = 0$; $f_y = 4y + 8$; $f_{yy} = 4$; $D = 24$. Solving $f_x = 0$ and $f_y = 0$, we obtain the critical point $(1, -2)$. Since $D(1, -2) = 24 > 0$ and $f_{xx}(1, -2) = 6 > 0$, $f(1, -2) = -11$ is a relative minimum.

5. $f_x = 10x + 20$; $f_{xx} = 10$; $f_{xy} = 0$; $f_y = 10y - 10$; $f_{yy} = 10$; $D = 100$. Solving $f_x = 0$ and $f_y = 0$, we obtain the critical point $(-2, 1)$. Since $D(-2, 1) = 100 > 0$ and $f_{xx}(-2, 1) = 10 > 0$, $f(-2, 1) = 15$ is a relative minimum.

6. $f_x = -8x - 8$; $f_{xx} = -8$; $f_{xy} = 0$; $f_y = -4y + 12$; $f_{yy} = -4$; $D = 32$. Solving $f_x = 0$ and $f_y = 0$, we obtain the critical point $(-1, 3)$. Since $D(-1, 3) = 32 > 0$ and $f_{xx}(-1, 3) = -8 < 0$, $f(-1, 3) = 27$ is a relative maximum.

7. $f_x = 12x^2 - 12$; $f_{xx} = 24x$; $f_{xy} = 0$; $f_y = 3y^2 - 3$; $f_{yy} = 6y$; $D = 144xy$. Solving $f_x = 0$ and $f_y = 0$, we obtain the critical points $(-1, -1)$, $(-1, 1)$, $(1, -1)$, and $(1, 1)$. Since $D(-1, 1) = -144 < 0$ and $D(1, -1) = -144 < 0$, these points do not give relative extrema. Since $D(-1, -1) = 144 > 0$ and $f_{xx}(-1, -1) = -24 < 0$, $f(-1, -1) = 10$ is a

relative maximum. Since $D(1,1) = 144 > 0$ and $f_{xx}(1,1) = 24 > 0$, $f(1,1) = -10$ is a relative minimum.

8. $f_x = -3x^2 + 27$; $f_{xx} = -6x$; $f_{xy} = 0$; $f_y = 6y^2 - 24$; $f_{yy} = 12y$; $D = -72xy$. Solving $f_x = 0$, $f_y = 0$, we obtain the critical points $(-3,-2)$, $(-3,2)$, $(3,-2)$, and $(3,2)$. Since $D(-3,-2) = -432 < 0$ and $D(3,2) = -432 < 0$, these points do no give relative extrema. Since $D(-3,2) = 432 > 0$ and $f_{xx}(-3,2) = 18 > 0$, $f(-3,2) = 432 > 0$ and $f_{xx}(3,-2) = -18 < 0, f(3,-2) = 89$ is a relative maximum.

9. $f_x = 4x - 2y - 10$; $f_{xx} = 4$; $f_{xy} = -2$; $f_y = 8y - 2x - 2$; $f_{yy} = 8$; $D = 32 - (-2)^2 = 28$. Setting $f_x = 0$ and $f_y = 0$, we obtain $4x - 2y = 10$ and $8y - 2x = 2$ or $2x - y = 5$ and $4y - x = 1$. Solving, we obtain the critical point $(3,1)$. Since $D(3,1) = 28 > 0$ and $f_{xx}(3,1) = 4 > 0$, $f(3,1) = -14$ is a relative minimum.

10. $f_x = 10x + 5y - 10$; $f_{xx} = 10$; $f_{xy} = 5$; $f_y = 10y + 5x - 5$; $f_{yy} = 10$; $D = 100 - (5)^2 = 75$. Setting $f_x = 0$ and $f_y = 0$, we obtain $10x + 5y = 10$ and $10y + 5x = 5$ or $2x + y = 2$ and $2y + x = 1$. Solving, we obtain the critical point $(1,0)$. Since $D(1,0) = 75 > 0$ and $f_{xx}(1,0) = 10 > 0$, $f(1,0) = 13$ is a relative minimum.

11. $f_x = 2t - 8$; $f_{xx} = 0$; $f_{xy} = 2$; $f_y = 2x - 5$; $f_{yy} = 0$; $D = 0 - 2^2 = -4$. Since $D(x,y) = -4 < 0$ for all (x,y), there are no relative extrema.

12. $f_x = 2y + 6$; $f_{xx} = 0$; $f_{xy} = 2$; $f_y = 2x + 10$; $f_{yy} = 0$; $D = 0 - 2^2 = -4$. Since $D(x,y) = -4 < 0$ for all (x,y), there are no relative extrema.

13. $f_x = -6x^2 + 6y$; $f_{xx} = -12x$; $f_{xy} = 6$; $f_y = -6y^2 + 6x$; $f_{yy} = -12y$; $D = 144xy - 36$. Setting $f_x = 0$ and $f_y = 0$, we obtain $-6x^2 + 6y = 0$ and $-6y^2 + 6x = 0$ or $y = x^2$ and $x = y^2$. Substituting $x = y^2$ into $y = x^2$, we obtain $y = y^4$ or $y(y^3 - 1) = 0$. Thus, $y = 0$ and $y = 1$. The critical points are $(0,0)$ and $(1,1)$. Since $D(0,0) = -36 < 0$, $(0,0)$ does not give a relative extremum. Since $D(1,1) = 108 > 0$ and $f_{xx}(1,1) = -12 < 0$, $f(1,1) = 12$ is a relative maximum.

14. $f_x = 3x^2 - 6y$; $f_{xx} = 6x$; $f_xy = -6$; $f_y = 3y^2 - 6x$; $f_{yy} = 6y$; $D = 36xy - 36$. Setting $f_x = 0$ and $f_y = 0$, we obtain $3x^2 - 6y = 0$ and $3y^2 - 6x = 0$ or $x^2 = 2y$ and $y^2 = 2x$. Substituting $y = x^2/2$ into $y^2 = 2x$ we obtain $x^4 = 8x$ or $x(x^3 - 8) = 0$. Thus, $x = 0$ and $x = 2$. The critical points $(0,0)$ and $(2,2)$. Since $D(0,0) = -36 < 0$, $f(0,0)$ is not an extremum. Since $D(2,2) = 108 > 0$ and $f_{xx}(2,2) = 12 > 0$, $f(2,2) = 19$ is a relative minimum.

15. $f_x = y + 2/x^2$; $f_{xx} = -4/x^3$; $f_{xy} = 1$; $f_y = x + 4/y^2$; $f_{yy} = -8/y^3; D = 32/x^3Y^3 - 1$. Setting $f_x = 0$ and $f_y = 0$ we obtain $y + 2/x^2 = 0$ and $x + 4/y^2 = 0$. Substituting $y = -2/x^2$ into $x + 4/y^2 = 0$ we obtain $x + x^4 = x(1 + x^3) = 0$. Since $x = 0$ is not in the domain of f, the only critical point is $(-1,-2)$. Since $D(-1,-2) = 3 > 0$ and $f_{xx}(-1,-2) = 4 > 0$, $f(-1,-2) = 14$ is a relative minimum.

16. $f_x = -6xy - 3y^2 + 36y$; $f_{xx} = -6y$; $f_{xy} = -6x - 6y + 36 = 6(6 - x - y)$; $f_y = -3x^2 - 6xy + 36x$; $f_{yy} = -6x$; $D = 36xy - 36(6 - x - y)^2$. Setting $f_x = 0$ and $f_y = 0$ we obtain $-6xy - 3y^2 + 36y = 0$ and $-3x^2 - 6xy + 36x = 0$ or $-3y(2x + y - 12) = 0$ and $-3x(x + 2y - 12) = 0$. Letting $y = 0$, the first equation is satisfied and the second

equation becomes $-3x(x-12)=0$. Thus, $(0,0)$ and $(12,0)$ are critical points. Similarly, letting $x=0$ we obtain the critical point $(0,12)$. Finally solving $2x+y=12$ and $x+2y=12$ we obtain the critical point $(4,4)$. Since $D(0,0)=-36^2<0$, $D(0,12)=-36^2<0$, and $D(12,0)=-36^2<0$, none of these points give relative extrema. Since $D(4,4)=432>0$ and $f_{xx}(4,4)=-24<0$, $f(4,4)=192$ is a relative maximum.

17. $f_x=(xe^x+e^x)\sin y$; $f_{xx}=(xe^x+2e^x)\sin y$; $f_{xy}=(xe^x+e^x)\cos y$; $f_y=xe^x\cos y$; $f_{yy}=-xe^x\sin y$; $D=-xe^{2x}(x+2)\sin^2 y-e^{2x}(x+1)^2\cos^2 y$. Setting $f_x(x,y)=0$ and $f_y(x,y)=0$ we obtain $(xe^x+e^x)\sin y=0$ and $xe^x\cos y=0$. Since $e^x>0$ for all x, we have $(x+1)\sin y=0$ and $x\cos y=0$. When $x=-1$, we must have $\cos y=0$ or $y=\pi/2+k\pi$, k an integer. When $x=0$, we must have $\sin y=0$ or $y=k\pi$, k an integer. Thus, the critical points are $(0,k\pi)$ and $(-1,\pi/2+k\pi)$, k an integer. Since $D(0,k\pi)=0-\cos^2 k\pi<0$, $(0,k\pi)$ does not give a relative extrema. Now, $D(-1,\pi/2+k\pi)=e^{-2}\sin^2(\pi/2+k\pi)-0>0$ and $f_{xx}(-1,\pi/2+k\pi)=e^{-1}\sin(\pi/2+k\pi)$. Since $f_{xx}(-1,\pi/2+k\pi)$ is positive for k even and negative for k odd, $f(-1,\pi/2+k\pi)=-e^{-1}$ are relative minima for k even, and $f(-1,\pi/2+k\pi)=e^{-1}$ are relative maxima for k odd.

18. $f_x=(2x+4)e^{y^2-3y+x^2+4x}$; $f_{xx}=[(2x+4)^2+2]e^{y^2-3y+x^2+4x}$; $f_{xy}=(2x+4)(2y-3)e^{y^2-3y+x^2+4x}$; $f_y=(2y-3)e^{y^2-3y+x^2+4x}$; $f_{yy}=[(2y-3)^2+2]e^{y^2-3y+x^2+4x}$; $D=[(2x+4)^2+2][(2y-3)^2+2]\cdot e^{2(y^2-3y+x^2+4x)}-[(2x+4)(2y-3)]^2e^{2(y^2-3y+x^2+4x)}$. Setting $f_x=0$ and $f_y=0$ and using the fact that an exponential function is always positive, we obtain $2x+4=0$ and $2y-3=0$. Thus, a critical point is $(-2,3/2)$. Since $D(-2,3/2)=4e^{2(9/4-9/2+4-8)}>0$ and $f_{xx}(-2,3/2)=2e^{9/4-9/2+4-8}>0$, $f(-2,3/2)=e^{9/4-9/2+4-8}=e^{-25/4}$ is a relative minimum.

19. $f_x=\cos x$; $f_{xx}=-\sin x$; $f_{xy}=0$; $f_y=\cos y$; $f_{yy}=-\sin y$; $D=\sin x\sin y$. Solving $f_x=0$ and $f_y=0$, we obtain the critical points $(\pi/2+m\pi,\pi/2+n\pi)$ for m and n integers. For m even and n odd or m odd and n even, $D<0$ and no relative extrema result. For m and n both even, $D>0$ and $f_{xx}<0$ and $f(\pi/2+m\pi,\pi/2+n\pi)=2$ are relative maxima. For m and n both odd, $D>0$ and $fxx>0$ and $f(\pi/2+m\pi,\pi/2+n\pi)=-2$ are relative minima.

20. $f_x=y\cos xy$; $f_{xx}=-y^2\sin xy$; $f_{xy}=-xy\sin xy+\cos xy$; $f_y=x\cos xy$; $f_{yy}=-x^2\sin xy$; $D=x^2y^2\sin^2 xy-(-xy\sin xy+\cos xy)^2=2xy\sin xy\cos xy-\cos^2 xy$. Setting $f_x=0$ and $f_y=0$ we see that $(0,0)$ is a critical point. Also, solving $\cos xy=0$ we obtain $xy=\pi/2+k\pi$ or $y=\pi(1+2k)/2x$ for k an integer. Since $D(0,0)=-1<0$, $(0,0)$ does not give a relative extrema. For any of the critical points $(x,\pi(1+2k)/2x)$, $D=0$ and no conclusion can be drawn from the second partials test. Since $-1\le\sin xy\le 1$ for all $(x,y)f(x,\pi(1+2k)/2x)=-1$ for k odd are relative minima and $f(x,\pi(1+2k)/2x)=1$ for k even are relative maxima.

21. Let the numbers be x,y, and $21-x-y$. We want to maximize $P(x,y)=xy(21-x-y)=21xy-x^2y-xy^2$. Now $P_x=21y-2xy-y^2$; $P_{xx}=-2y$; $P_{xy}=21-2x-2y$; $P_y=21x-x^2-2xy$; $P_{yy}=-2x$; $D=4xy-(21-2x-2y)^2$. Setting $P_x=0$ and $P_y=0$, we obtain $y(21-2x-y)=0$ and $x(21-x-2y)=0$. Letting $x=0$ and $y=0$, we obtain the critical points $(0,0)$, $(0,21)$, and $(21,0)$. Each of these results in $P=0$ which is clearly not a maximum. Solving $21-2x-y=0$ and $21-x-2y=0$, we obtain the critical point $(7,7)$.

Since $D(7,7) = 147 > 0$ and $P_{xx}(7,7) = -14 < 0$, $P(7,7) = 343$ is a maximum. The three numbers are 7,7, and 7.

22. Let the sides of the base of the box by x and y. Then, since the volume of the box is 1, its height is $1/xy$ and $S = 2xy + 2x(1/xy) + 2y(1/xy) = 2xy + 2/y + 2/x$, $x > 0$, $y > 0$. Now $S_x = 2y - 2/x^2$; $S_{xx} = 4/x^3$; $S_{xy} = 2$; $S_y = 2x - 2/y^2$; $S_{yy} = 4/y^3$; $D = 16/x^3y^3 - 4$. Setting $S_x = 0$ and $S_y = 0$ we obtain $y = 1/x^2$ and $x = 1/y^2$. The critical point is $(1,1)$. Since $D(1,1) = 12 > 0$ and $S_{xx}(1,1) = 4 > 0$, $S(1,1) = 6$ is a minimum. The box is 1 foot on each side.

23. Let $(x, y, 1 - x - 2y)$ be a point on the plane $x + 2y + z = 1$. We want to minimize $f(x,y) = x^2 + y^2 + (1 - x - 2y)^2$. Now $f_x = 2x - 2(1 - x - 2y)$; $f_{xx} = 4$; $f_{xy} = 4$; $f_y = 2y - 4(1 - x - 2y)$; $f_{yy} = 10$; $D = 40 - 4^2 = 24$. Setting $f_x = 0$ and $f_y = 0$ we obtain $2x-2(1-x-2y) = 0$ and $2y-4(1-x-2y) = 0$ or $2x+2y = 1$ and $2x+5y = 2$. Thus, $(1/6, 1/3)$ is a critical point. Since $D = 24 > 0$ and $f_{xx} = 4 > 0$ for all (x,y), $f(1/6, 1/3) = 1/6$ is a minimum. Thus, the point on the plane closest to the origin is $(1/6, 1/3, 1/6)$.

24. Let $(x, y, 1 - x - y)$ be a point on the plane $x + y + z = 1$. We want to minimize the square of the distance between the point and the plane. This is given by

$$f(x,y) = (x-2)^2 + (y-3)^2 + (-x-y)^2 = 2x^2 + 2y^2 - 4x - 6y + 2xy + 13.$$

$f_x = 4x - 4 + 2y$; $f_{xx} = 4$; $f_{xy} = 2$; $f_y = 4y - 6 + 2x$; $f_{yy} = 4$; $D = 16 - 2^2 = 12$. Setting $f_x = 0$ and $f_y = 0$ we obtain $4x - 4 + 2y = 0$ and $4y - 6 + 2x = 0$ or $2x + y = 2$ and $x + 2y = 3$. Thus, $(1/3, 4/3)$ is a critical point. Since $D = 12 > 0$ and $f_{xx} = 4 > 0$ for all (x,y), $f(1/3, 4/3) = 25/3$ is a minimum. Thus, the least distance between the point and the plane is $\sqrt{25/3} = 5/\sqrt{3}$.

25. Let $(x, y, 8/xy)$ be a point on the surface. We want to minimize the square of the distance to the origin or $f(x,y) = x^2 + y^2 + 64/x^2y^2$. Now $f_x = 2x - 128/x^3y^2$; $f_{xx} = 2 + 384/x^4y^2$; $f_{xy} = 256/x^3y^3$; $f_y = 2y - 128/x^2y^3$; $f_{yy} = 2 + 384/x^2y^4$; $D = (2 + 384/x^4y^2)(2 + 384/x^2y^4) - (256)^2/x^6y^6$. Setting $f_x = 0$ and $f_y = 0$ we obtain $2x - 128/x^3y^2 = 0$ and $2y - 128/x^2y^3 = 0$ or $x^4y^2 = 64$ and $x^2y^4 = 64$; $x \neq 0$, $y \neq 0$. This gives $x^4y^2 = x^2y^4$, $x^2y^2(x^2 - y^2) = 0$ or $x^2 = y^2$. Thus, $x^6 = 64$ and $x = \pm 2$. Similarly, $y = \pm 2$ and the critical points are $(-2,-2)$, $(-2,2)$, $(2,-2)$, and $(2,2)$. Since $D(\pm 2, \pm 2) = 48 > 0$ and $f_{xx}(\pm 2, \pm 2) = 8 > 0$, $f(\pm 2, \pm 2) = 12$ are minima. The points closest to the origin are $(-2,-2,2)$, $(-2,2,-2)$, $(2,-2,-2)$, and $(2,2,2)$. The minimum distance is $\sqrt{12} = 2\sqrt{3}$.

26. We will minimize the square of the distance between the lines. This is given by

$$\begin{aligned} f(s,t) &= [(3+2s) - t]^2 + [(6+2s) - (4-2t)]^2 + [(8-2s) - (1+t)]^2 \\ &= (2s - t + 3)^2 + (2s + 2t + 2)^2 + (-2s - t - 7)^2 = 12s^2 + 6t^2 + 8st - 8s + 12t + 62. \end{aligned}$$

$f_s = 24s+8t-8$; $f_{ss}24$; $f_{st} = 8$; $f_t = 12t+8s-12$; $f_{tt} = 12$; $D = 24(12)-64 = 224$. Solving $24s+8t-8 = 0$ and $12t+8s-12 = 0$ we obtain the critical point $(0,1)$. Since $D(0,1) = 224 > 0$ and $f_{ss}(0,1) = 24 > 0$, we see that $f(0,1)$ is a minimum. The corresponding points on the lines are $(3,6,8)$ on $\mathcal{L}_2$ and $(1,2,2)$ on $\mathcal{L}_1$. The minimum distance is $\sqrt{f(0,1)} = \sqrt{56} = 2\sqrt{14}$.

27. We will maximize the square of the volume of the box in the first octant,

$$V(x,y) = x^2Y^2z^2 = x^2y^2(c^2 - c^2x^2/a^2 - c^2y^2/b^2).$$

$V_x = 2c^2xy^2 - 4c^2x^3y^2/a^2 - 2c^2xy^4/b^2$; $V_{xx} = 2c^2y^2 - 12c^2x^2y^2/a^2 - 2c^2y^4/b^2$; $V_{xy} = 4c^2xy - 8c^2x^3y/a^2 - 8c^2xy^3/b^2$; $V_y = 2c^2x^2y - 2c^2x^4/a^2 - 4c^2x^2y^3/b^2$; $V_{yy} = 2c^2x^2 - 2c^2x^4/a^2 - 12c^2x^2y^2/b^2$; $D = V_{xx}V_{yy} - V_{xy}^2$. Setting $V_x = 0$ and $V_y = 0$ we obtain $xy^2 - 2x^3y^2/a^2 - xy^4/b^2 = 0$, $x^2y - x^4y/a^2 - 2x^2y^3/b^2 = 0$, or, assuming $x > 0$ and $y > 0$, $2b^2x^2 + a^2y^2 = a^2b^2$. Solving, we obtain $x^2 = a^2/3$ and $y^2 = b^2/3$. Thus, $(a/\sqrt{3}, b/\sqrt{3})$ is a critical point. Since

$$D(a/\sqrt{3}, b/\sqrt{3}) = (-\frac{14}{9}b^2c^2)(-\frac{14}{9}a^2c^2) - (-\frac{4}{9}abc^2)^2 = \frac{20}{9}a^2b^2c^4 > 0$$

and $v_{xx} = -\frac{14}{9}b^2c^2 < 0$, $V(a/\sqrt{3}, b/\sqrt{3}) = a^2b^2c^2/27$. The maximum volume is

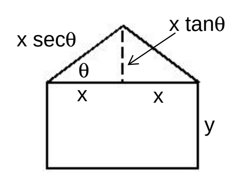
$$8\sqrt{V(a/\sqrt{3}, b/\sqrt{3})} = 8\sqrt{3}abc/9.$$

28. Let $a + b + c = k$. Then $c = k - a - b$ and we want to maximize

$$V(a,b) = 4\pi ab(k - a - b)/3 = 4\pi(kab - a^2b - ab^2)/3.$$

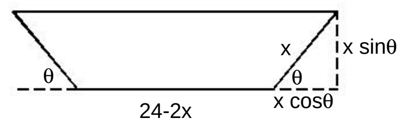
$V_a = \frac{4\pi}{3}(kb - 2ab - b^2)$; $V_{aa} = -\frac{8\pi}{3}$; $V_{ab} = \frac{4\pi}{3}(k - 2a - 2b)$; $V_b = \frac{4\pi}{3}(ka - a^2 - 2ab)$; $V_{bb} = -\frac{8\pi}{3}a$; $D = \frac{64x^2}{9}ab - \frac{16x^2}{9}(k - 2a - 2b)^2$. Setting $V_a = 0$ and $V_b = 0$ we obtain $kb - 2ab - b^2 = 0$ or $ka - a^2 - 2ab = 0$, $a \neq 0$, $b \neq 0$, or $2a + b = k$ and $a + 2b = k$. Solving, we get $a = b = k/3$. Since $D(k/3, k/3) = 16\pi^2k^2/27 > 0$ and $V_{aa}(k/3, k/3) = -8\pi k/9 < 0$, the volume is maximized when $a = b = k/3$. Since $c = k - a - b = k/3$, $a = b = c$ and the ellipsoid is a sphere.

29. The perimeter is given by $P = 2x + 2y + 2x\sec\theta$ and the area is $a = 2xy + x^2\tan\theta$. Solving P for $2y$ and substituting in A, we obtain $A = Px - 2x^2(1 + \sec\theta) + x^2\tan\theta$. Now $A_x(x,\theta) = P - 4x(1 + \sec\theta)2x\tan\theta$; $A_{xx}(x\theta) = -4(1 + \sec\theta) + 2\tan\theta$; $A_{x\theta}(x,\theta) = -4x\sec\theta\tan\theta + 2x\sec^2\theta$; $A_\theta(x\theta) = x^2\sec\theta(\sec\theta - 2\tan\theta)$; $A_{\theta\theta}(x,\theta) = 2x^2\sec\theta(\tan\theta - 2\sec^2\theta + 1)$. We assume that $x > 0$ and $0 \leq \theta \leq \pi/2$.



Setting $A_x = 0$ and $A_\theta = 0$, we obtain $P - 4x(1 + \sec\theta) + 2x\tan\theta = 0$ and $x^2\sec\theta(\sec\theta - 2\tan\theta) = 0$. We note from the second equation and the fact that $\sec\theta \neq 0$ for all θ that $\sec\theta - 2\tan\theta = 0$. Solving for θ, we obtain $\theta = 30°$ and solving $A_x = 0$ for x, we obtain $x_) = P/(4 + 2\sqrt{3})$. Since $D(x_0, 30°) = (-2\sqrt{3} + 2)(4x_0^2(\sqrt{3} - 5)/3\sqrt{3}) - 0^2 > 9$ and $A_{xx} = 2 - 2\sqrt{3} < 0$, $A(x_0, 30°)$ is a maximum. Letting $x = P/(4 + 2\sqrt{3})$ and $\theta = 30°$ in $P = 2x + 2y + 2x\sec\theta$, we obtain $P = 2y + P?\sqrt{3}$. Thus, the area is maximized for $x = P/(4 + 2\sqrt{3})$, $y = P(\sqrt{3} - 1)/2\sqrt{3}$, and $\theta = 30°$

30. We want to maximize $A(x,\theta) = (x\sin\theta)(24 - 2x + x\cos\theta) = 24x\sin\theta - 2x^2\sin\theta + \frac{1}{2}x^2\sin 2\theta$. Now $A_x = 24\sin\theta - 4x\sin\theta + x\sin 2\theta$; $A_{xx} = -4\sin\theta + \sin 2\theta$; $A_{x\theta} = 24\cos\theta - 4x\cos\theta + 2x\cos 2\theta$; $A_\theta = 24x\cos\theta - 2x^2\cos\theta + x^2\cos 2\theta$; $A_{\theta\theta} = -24x\sin\theta + 2x^2\sin\theta - 2x^2\sin 2\theta$.



We assume $0 < x < 12$ and $=< \theta < \pi/2$. Setting $A_x = 0$ and $A_\theta = 0$ we obtain $24\sin\theta - 4x\sin\theta + 2x\sin\theta\cos\theta = 0$ and $24x\cos\theta - 2x^2\cos\theta + x^2(2\cos^2\theta - 1) = 0$ or $12 - 2x + x\cos\theta = 0$ and $2x\cos^2\theta - 2x\cos\theta + 2\cos\theta - x = 0$. Solving the first equation for $\cos\theta$ and substituting into the second equation, we obtain $2x(2-12/x)^2 - 2x(12-12/x) + 24(2-12/x) - x = 0$. Simplifying, we find $x = 8$ and $\cos\theta = 1/2$ or $\theta = 60°$. Since $D(8,60°) = (-3\sqrt{3}/2)(-96\sqrt{3}) - (-12)^2 = 288 > 0$ and $A_{xx} = -3\sqrt{3}/2 < 0$, $A(8,60° = 48\sqrt{3})$ square inches is the maximum area.

31. $f_x = -\dfrac{2}{3}x^{-1/3}$, $f_y = -\dfrac{2}{3}y^{-1/3}$. Since $f_x = 0$ and $f_y = 0$ have no solutions, $f(x,y)$ has no critical points and Theorem 13.8.2 does not apply. However, for all (x,y),
$f(0,0) = 16 \geq 16 - (x^{1/3})^2 - (y^{1/3})^2 = f(x,y)$, and $f(0,0) = 16$ is an absolute maximum.

32. $f_x = -4x^3y^2$; $f_{xx} = -12x^2y^2$; $f_{xy} = -8x^3y$; $f_y = -2x^4y$; $f_{yy} = -2x^4$;
$D = 24x^6y^2 - 64x^6y^2 = -40x^6y^2$. Setting $f_x = 0$ and $f_y = 0$ we see that $(0,y)$ and $(x,0)$ are critical points for any x and y. Since, for any critical point, $D = 0$, Theorem 13.8.2 does not apply. However, for all $(x,y), f(0,0) = 1 \geq 1 - (x^2y)^2 = f(x,y)$, and $f(0,0) = 1$ is an absolute maximum.

33. $f_x = 10x$; $f_{xx} = 10$; $f_{xy} = 0$; $f_y = 4y^3$; $f_{yy} = 12y^2$; $D = 120y^2$. Solving $f_x = 0$ and $f_y = 0$ we obtain the critical point $(0,0)$. Since $D(0,0) = 0$, Theorem 13.8.2 does not apply. However, for any (x,y), $f(0,0) = -8 \leq 5x^2 + y^4 - 8 = f(x,y)$ and $f(0,0) = -8$ is an absolute minimum.

34. $f_x = \dfrac{x}{\sqrt{x^2+y^2}}$; $f_{xx} = \dfrac{y^2}{(x^2+y^2)^{3/2}}$; $f_{xy} = -\dfrac{xy}{(x^2+y^2)^{3/2}}$; $f_y = \dfrac{y}{\sqrt{x^2+y^2}}$;
$f_{yy} = \dfrac{x^2}{(x^2+y^2)^{3/2}}$; $D = \dfrac{x^2y^2}{(x^2+y^2)^3} - (\dfrac{-xy}{(x^2+y^2)^{3/2}})^2 = 0$. Since $D = 0$ for all (x,y), Theorem 13.8.2 does not apply. However, for all (x,y), $f(0,0) = 0 \leq \sqrt{x^2+y^2} = f(x,y)$, so $f(0,0) = 0$ is an absolute minimum.

In Problems 35-38 we parameterize the boundary of R by letting $x = \cos t$ and $y = \sin t$; $0 \leq t \leq 2\pi$. Then, for $(x(t), y(t))$ on the boundary, we maximize or minimize $F(t) = f(\cos t, \sin t)$ on $[0, 2\pi]$.

35. $f_x = 1$; $f_y = \sqrt{3}$. There are no critical points on the interior of R. On the boundary we consider $F(t) = \cos t + \sqrt{3}\sin t$. Solving $F'(t) = -\sin t + \sqrt{3}\cos t = 0$, we obtain critical points at $t = \pi/3$ and $t = 4\pi/3$. Comparing $F(0) = 1, F(\pi/3) = 2$, and $F(4\pi/3) = -2$, we see that $f(1/2, \sqrt{3}/2) = 2$ is an absolute maximum and $f(-1/2, -\sqrt{3}/2) = -2$ is an absolute minimum.

36. $f_x = y$; $f_y = x$. Solving $f_x = 0$ and $f_y = 0$ we obtain the critical point $(0,0)$ with corresponding function value $f(0,0) = 0$. On the boundary we consider $F(t) = \cos t \sin t = \dfrac{1}{2}\sin 2t$. Solving $F'(t) = \cos 2t = 0$, we obtain critical points at $\pi/4$, $3\pi/4, 5\pi/4$, and $7\pi/4$. Comparing $f(0,0,) = 0$, $F(0) = 0$, $F(\pi/4) = 1/2$, $F(3\pi/4) = -1/2$, $F(5\pi/4) = 1/2$, and $F(7\pi/4) = -1/2$, we see that $f(\sqrt{2}/2, \sqrt{2}/2) = f(-\sqrt{2}/2, -\sqrt{2}/2) = 1/2$ are absolute maxima and $f(-\sqrt{2}/2, \sqrt{2}/2) = f(\sqrt{2}/2, -\sqrt{2}/2) = -1/2$ are absolute minima.

37. $f_x = 2x + y$; $f_y = x + 2y$. Solving $f_x = 0$ and $f_y = 0$ we obtain the critical point $(0,0)$ with corresponding function value $f(0,0) = 0$. On the boundary we consider $F(t) = \cos^2 t + \cos t \sin t + \sin^2 t = 1 + \dfrac{1}{2}\sin 2t$. Solving $F'(t) = \cos 2t = 0$ we obtain critical points at $\pi/4$, $3\pi/4$, $5\pi/4$, and $7\pi/4$. Comparing $f(0,0) = 0$, $F(0) = 1$, $F(\pi/4) = 3/2$;
$F(3\pi/4) = 1/2$, $F(5\pi/4) = 3/2$, and $F(7\pi/4) = 1/2$, we see that $f(\sqrt{2}/2, \sqrt{w}/2) = f(-\sqrt{2}/2, -\sqrt{2}/2) = 3/2$ are absolute maxima and $f(0,0) = 0$ is an absolute minimum.

38. $f_x = -2x$; $f_y = -6y + 4$. Solving $f_x = 0$ and $f_y = 0$, we obtain the critical point $(0, 2/3)$, which is inside R, with corresponding function value $f(0, 2/3) = 5$. On the boundary we consider $F(t) = -\cos^2 t - 3\sin^2 t + 4\sin t + 1$. Solving $F'(t) = 2\cos t \sin t - 6\sin t \cos t + 4\cos t = 4\cos t - 4\sin t \cos t = 0$, we obtain critical points at $\pi/2$ and $3\pi/2$. Comparing $f(0, 2/3) = 5$, $F(0) = 0$, $F(\pi/2) = 2$, and $F(3\pi/2) = -6$, we see that $f(0, -1) = -6$ is an absolute minimum and $f(0, 2/3) = 5$ is an absolute maximum.

39. $f_x = 4$; $f_y = -6$. There are no critical points over the region R, so absolute extrema must occur on the boundary. We parameterize the boundary by $x = 2\cos t$ and $y = \sin t$ for $0 \le t \le 2\pi$. Considering $F(t) = 8\cos t - 6\sin t$ we obtain $F'(t) = -8\sin t - 6\cos t$. Solving $F'(t) = 0$ we find $\tan t = -3/4$. Using $1 + \tan^2 t = \sec^2 t$ we see that $\sec^2 t = 25/16$ and $\cos t = -4/5$, t is in the second quadrant and $\sin t = 3/5$. The corresponding points on the boundary of R are $(8/5, -3/5)$ and $(-8/5, 3/5)$. Comparing $f(0) = F(2\pi) = f(2,0) = 8$, $f(8/5, -3/5) = 10$, and $f(-8/5, 3/5) = -10$ we see that the absolute minimum is $f(-8/5, 3/5) = -10$ and the absolute maximum is $f(8/5, -3/5) = 10$.

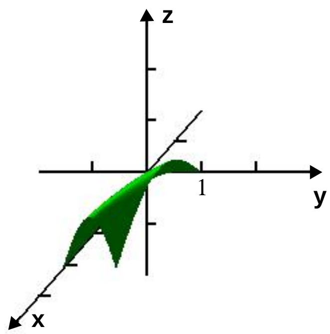
40. $f_x = y - 2$; $f_y = x - 1$. Solving $f_x = 0$ and $f_y = 0$ we obtain the critical point $(1,2)$ in the region. On $x = 0$, $F(y) = f(0,y) = -y + 6$, which has no critical points for $0 \le y \le 8$. The endpoints of the interval are $(0,0)$ and $(0,8)$. On $y = 0$, $G(x) = f(x,0) = -2x + 6$, which has no critical points for $0 \le x \le 4$. The endpoints of the interval are $(0,0)$ and $(4,0)$. On $y = -2x + 8$, $H(x) = f(x, -2x + 8) = x(-2x + 8) = x(-2x + 8) - 2x - (-2x + 8) + 6 = -2x^2 + 8x - 2$. Solving $H'(x) = -4x + 8 = 0$ we obtain $x = 2$. The corresponding point on the triangle is $(2,4)$. Comparing $f(0,0) = 6$, $f(0,8) = -2$; $f(4,0) = -2$; $f(2,4) = 6$, and $f(1,2) = 4$ we see that absolute maxima are $f(0,0) = f(2,4) = 6$ and absolute minima are $f(0,8) = f(4,0) = -2$.

41. (a) $f_x = y\cos xy$; $f_y = x\cos xy$. Setting $f_x = 0$ and $f_y = 0$ we obtain $y\cos xy = 0$ and $x\cos xy = 0$. If $y = 0$ from the first equation, then necessarily $x = 0$ from the second equation. Thus, $(0,0)$ is a critical point. For $x \ne 0$ and $y \ne 0$ we have $\cos xy = 0$ or $xy = \pi/2$. Thus, all points $(x, \pi/2x)$ for $0 \le x \le \pi$ are also critical points.

(b) Since $0 \le \sin xy \le 1$ for $0 \le x \le \pi$ and $0 \le y \le 1$, $f(x,y) = \sin xy$ has absolute minima at any points for which $\sin xy = 0$ and absolute maxima at any points for which

$\sin xy = 1$. Thus, $f(x,y)$ has absolute minima when $xy = 0$ or $xy = \pi$, that is, at the points $(0,y)$, $(x,0)$, and $(\pi,1)$ which are in the region. Absolute maxima occur when $xy = \pi/2$ or along the curve $y = \pi/2x$ in the region

(c)



42. We want to maximize $P(x,y) = R(x,y) - C(x,y) = 108x - 8x^2 + 192y - 6y^2 - 4xy - 20$. Now $P_x = 108 - 16x - 4y$; $P_{xx} = -16$; $P_{xy} = -4$; $P_y = 192 - 4x$; $P_{yy} = -12$; $D = 192 - 16 = 176$. Setting $P_x = 0$ and $P_y = 0$ we obtain $108 - 16x - 4y = 0$ and $192 - 12y - 4x = 0$ or $4x + y = 27$ and $x + 3y = 48$. Solving, we see that $(3,15)$ is a critical point. Since $D(3,15) = 175 > 0$ and $P_{xx}(3,15) = -16 < 0$, $P(3,15) = 1582$ is the maximum profit

43. Since the volume of the box is 60, the height is $60/xy$. Then

$$C(x,y) = 10xy + 20xy + 2[2x60/xy + 2y60/xy] = 30xy + 240/y + 240/x.$$

$C_x = 30y - 240/x^2$; $C_{xx} = 480/x^3$, $C_{xy} = 30$; $C_y = 30x - 240/y^2$; $c_{yy} = 480/y^3$; $D = 480^2/x^3y^3 - 900$. Setting $C_x = 0$ and $C_y = 0$ we obtain $30y - 240/x^2 = 0$ and $30x - 240/y^2 = 0$ or $y = 8/x^2$ and $x = 8/y^2$. Substituting the first equation into the second, we have $x = x^4/8$ or $x(x^3 - 8) = 0$. Thus, $(2,2)$ is a critical point. Since $D(2,2) = 2700 > 0$ and $C_{xx}(2,2) = 60 > 0$, $C(2,2)$ is a minimum. Thus, the cost is minimized when the base of the box is 2 feet square and the height is 15 feet.

13.9 Method of Least Squares

1. $\sum_{i=1}^{4} x_i = 14$, $\sum_{i=1}^{4} y_i = 8$, $\sum_{i=1}^{4} x_i y_i = 30$, $\sum_{i=1}^{4} x_i^2 = 54$, $m = \dfrac{4(30) - 14(8)}{4(54) - (14)^2} = 0.4$, $b = \dfrac{54(8) - 30(14)}{4(54) - (14)^2} = 0.6$, $y = 0.4x + 0.6$

2. $\sum_{i=1}^{4} x_i = 6$, $\sum_{i=1}^{4} y_i = 14$, $\sum_{i=1}^{4} x_i y_i = 34$, $\sum_{i=1}^{4} x_i^2 = 14$, $m = \dfrac{4(34) - 6(14)}{4(14) - (6)^2} = 2.6$, $b = \dfrac{14(14) - 34(6)}{4(14) - (6)^2} = -0.4$, $y = 2.6x - 0.4$

3. $\displaystyle\sum_{i=1}^{5} x_i = 15,\ \sum_{i=1}^{5} y_i = 15,\ \sum_{i=1}^{5} x_i y_i = 56,\ \sum_{i=1}^{5} x_i^2 = 55,\ m = \frac{5(56) - 15(15)}{5(55) - (15)^2} = 1.1,$
$\displaystyle b = \frac{55(15) - 56(15)}{5(55) - (15)^2} = -0.3,\quad y = 1.1x - 0.3$

4. $\displaystyle\sum_{i=1}^{5} x_i = 14,\ \sum_{i=5}^{4} y_i = 14,\ \sum_{i=1}^{5} x_i y_i = 55,\ \sum_{i=1}^{5} x_i^2 = 54,\ m = \frac{5(55) - 14(14)}{5(54) - (14)^2} \approx 1.06757,$
$\displaystyle b = \frac{54(14) - 55(14)}{5(54) - (14)^2} \approx -0.189189,\quad y \approx 1.06757x - 0.189189$

5. $\displaystyle\sum_{i=1}^{7} x_i = 21,\ \sum_{i=1}^{7} y_i = 42,\ \sum_{i=1}^{7} x_i y_i = 164,\ \sum_{i=1}^{7} x_i^2 = 91,\ m = \frac{7(164) - 21(42)}{7(91) - (21)^2} \approx 1.35714,$
$\displaystyle b = \frac{91(42) - 164(21)}{7(91) - (21)^2} \approx 1.92857,\quad y \approx 1.35714x + 1.92857$

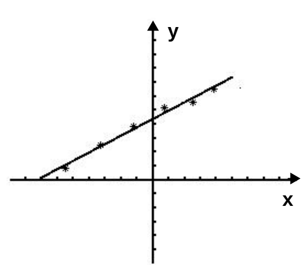
6. $\displaystyle\sum_{i=1}^{7} x_i = 28,\ \sum_{i=1}^{7} y_i = 17.2,\ \sum_{i=1}^{7} x_i y_i = 80.2,\ \sum_{i=1}^{7} x_i^2 = 140,\ m = \frac{7(80.2) - 28(17.2)}{7(140) - (28)^2} \approx 0.407143,$
$\displaystyle b = \frac{140(17.2) - 80.2(28)}{7(140) - (28)^2} \approx 0.828571,\quad y \approx 0.407143x + 0.828571$

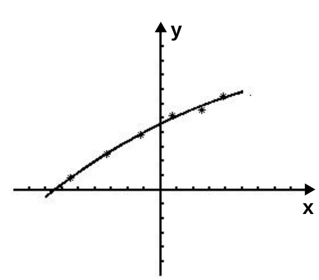
7. $\displaystyle\sum_{i=1}^{6} T_i = 420,\ \sum_{i=1}^{6} v_i = 1055,\ \sum_{i=1}^{6} T_i v_i = 68,000,\ \sum_{i=1}^{6} T_i^2 = 36,400,\ m = \frac{6(68,000) - 420(1055)}{6(36,400) - (420)^2} \approx$
$\displaystyle -0.835714,\quad b = \frac{36,400(1055) - 68,000(420)}{6(36,400) - (420)^2} \approx 234.333,\quad v \approx -0.835714T + 234.333.$

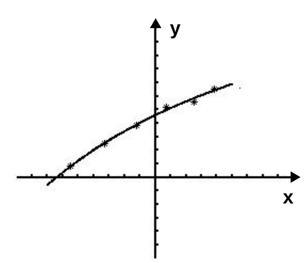
8. $\displaystyle\sum_{i=1}^{6} T_i = 3150,\ \sum_{i=1}^{6} R_i = 29.57,\ \sum_{i=1}^{6} T_i R_i = 17,878,\ \sum_{i=1}^{6} T_i^2 = 1,697,500,\ m = \frac{6(17,878) - 3150(29.57)}{6(1,697,500) - (3150)^2} \approx$
$\displaystyle 0.05,\quad b = \frac{1,697,500(29.57) - 17,878(3150)}{6(1,697,500) - (3150)^2} \approx -23.32,\quad R \approx 0.05T - 23.32 - 23.32.$ When $T = 700,\quad R \approx 14.34.$

9. (a) least-squares line: $y = 0.5966x + 4.3665$
least-squares quadratic: $y = -0.0232x^2 + 0.5618x + 4.5942$
least-squares cubic: $y = 0.00079x^3 - 0.0212x^2 + 0.5498x + 4.5840$

(b) least-squares line | least-squares quadratic | least-squares cubic



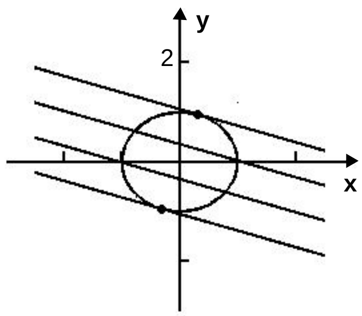




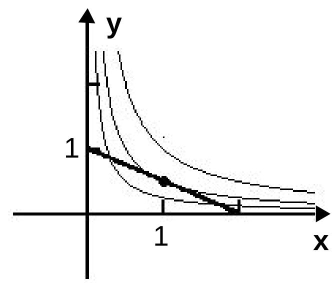
10. The least-squares line is given by $y = 2.0533x - 3837.115$. Plugging 2020 in for x, we predict that the population will be 310.551 million.

13.10 Lagrange Multipliers

1. f has constrained extrema where the level lines intersect the circle.



2. f has constrained extrema where the level curve intersect the line.



3. $f_x = 1$; $f_y = 3$; $g_x = 2x$; $g_y = 2y$. We need to solve $1 = 2\lambda x$, $3 = 2\lambda y$, $x^2 + y^2 - 1 = 0$. Dividing the second equation by the first, we obtain $3 = y/x$ or $y = 3x$. Substituting into the third equation, we have $x^2 + 9x^2 = 1$ or $x = \pm 1/\sqrt{10}$. For $x = 1/\sqrt{10}$, $y = 3/\sqrt{10}$ and for $x = -1/\sqrt{10}$, $y = -3/\sqrt{10}$. A constrained maximum is $f(1/\sqrt{10}, 3/\sqrt{10}) + \sqrt{10}$ and a constrained minimum is $f(-1/\sqrt{10}, -3/\sqrt{10}) = -\sqrt{10}$.

4. $f_x = y$; $f_y = x$; $g_x = 1/2$; $g_y = 1$. We need to solve $y = \lambda/2$, $x = \lambda$, $x/2 + y - 1 = 0$. From the first two equations $y = x/2$. Substituting into the second equation, we have $x = 1$. Thus, $f(1, 1/2) = 1/2$ is a constrained extremum. Since $(0, 1)$ satisfies the constraint and $f(0, 1) = 0 < 1/2$, $f(1, 1/2) = 1/2$ is a constrained maximum.

5. $f_x = y$; $f_y = x$; $g_x = 2x$; $g_y = 2y$. We need to solve $y = 2\lambda x$, $x = 2\lambda y$, $x^2 + y^2 - 2 = 0$. Substituting the second equation into the first, we obtain $y = 4\lambda^2 y$ or $y(4\lambda^2 - 1) = 0$. If $y = 0$, then from the second equation $x = 0$. Since $g(0, 0) = -2 \neq 0$, $(0, 0)$ does not satisfy the constraint. Thus, $\lambda = \pm 1/2$ and $y = \pm x$. Substituting into the third equation, we have $2x^2 = 2$ or $x = \pm 1$. Solutions of the system are $x = 1$, $y = 1$, $\lambda = 1/2$, $x = -1$, $y = -1$; $\lambda = 1/2$, $x = 1$, $y = -1$, $\lambda = -1/2$, and $x = -1$, $y = 1$, $\lambda = -1/2$. Thus, $f(1, 1) = f(-1, -1) = 1$ are constrained maxima and $f(1, -1) = f(-1, 1) = -1$ are constrained minima.

6. $f_x = 2x$; $f_y = 2y$; $g_x = 2$; $g_y = 1$. We need to solve $2x = 2\lambda$, $2y = \lambda$, $2x + y - 5 = 0$. Substituting the second equation into the first, we find $x = 2y$. Substituting into the third equation, we have $4y + y - 5 = 0$ or $y = 1$. A constrained extremum is $f(2, 1) = 5$. Since $(0, 5)$ satisfies the constraint and $f(0, 5) = 25 > 5$, $f(2, 1) = 5$ is a constrained minimum.

7. $f_x = 6x$; $f_y = 6y$; $g_x = 1$; $g_y = -1$. We need to solve $6x = \lambda$, $6y = -\lambda$; $x - y - 1 = 0$. From the first two equations, we obtain $x + y = 0$. Solving this with the third equation, we

obtain $x = 1/2, \quad y = -1/2$. Thus, $f(1/2, -1/2) = 13/2$ is a constrained extremum. Since $(1, 0)$ satisfies the constraint and $f(1, 0) = 8 > 13/2, \quad f(1/2 - 1/2) = 13/2$ is a constrained minimum.

8. $f_x = 8x; \;\; f_y = 4y; \;\; g_x = 8x; \;\; g_y = 2y$. We need to solve $8x = 8\lambda x, \;\; 4y = 2\lambda y, \;\; 4x^2 + y^2 - 4 = 0$ or $x(\lambda - 1) = 0, \quad y(\lambda - 2) = 0, \quad 4x^2 + y^2 = 4$. If $x = 0$, then from the third equation $y = \pm 2$. If $y = 0$, then from the third equation $x = \pm 1$. The cases $\lambda = 1$ and $\lambda = 2$ lead also to $y = 0$ and $x = 0$, respectively. Thus, $f(0, 2) = f(0, -2) = 18$ are constrained maxima and $f(1, 0) = f(-1, 0) = 14$ are constrained minima.

9. $f_x = 2x; \;\; f_y = 2y; \;\; g_x = 4x^3; \;\; g_y = 4y^3$. We need to solve $2x = 4\lambda x^3, \;\; 2y = 4\lambda y^3, \;\; x^4 + y^4 - 1 = 0$ or $x(2\lambda x^2 - 1) = 0, \quad y(2\lambda y^2 - 1) = 0, \quad x^4 + y^4 = 1$. If $x = 0$, the from the third equation $y = \pm 1$. If $y = 0$, then $x = \pm 1$. From $2\lambda x^2 = 1 = 2\lambda y^2$ we have $x^2 = y^2$. Substituting into the third equation, we obtain $x = \pm 1/\sqrt[4]{2}$ and $y = \pm 1/\sqrt[4]{2}$. Solutions of the system are $(0, \pm 1), \quad (\pm 1, 0)$, and $(\pm 1/\sqrt[4]{2}, \pm 1/[4]2)$. Thus, $f(0, \pm 1) = f(\pm 1, 0) = 1$ are constrained minima and $f(\pm 1/\sqrt[4]{2}, \pm 1/\sqrt[4]{2}) = \sqrt{2}$ are constrained maxima.

10. $f_x = 16x - 8y; \;\; f_y = 4y - 8x; \;\; g_x = 2x; \;\; g_y = 2y$. We need to solve $16x - 8y = 2\lambda x, \;\; 4y - 8x = 2\lambda y, x^2 + y^2 - 10 = 0$ or $8 - 47/x = \lambda, \quad x^2 + y^2 = 10$. From the first two equations, we obtain $6 - 4y/x = -4x/y, \quad 6(y/x) - 4(y/x)^2 = -4$, and $2(y/x)^2 - 3(y/x) - 2 = 0$. Factoring, we have $(2y/x + 1)(y/x - 2) = 0$. Then $y = -x/2$ and $y = 2xs$. Substituting $y = -x/2$ into the third equation, we have $x^2 + x^2/4 = 10$ and $x = \pm 2\sqrt{2}$. Substituting $y = 2x$ into the third equation, we have $x^2 + 4x^2 = 10$ and $x = \pm\sqrt{2}$. Solutions of the system are $(2\sqrt{2}, -\sqrt{2}), \quad (-2\sqrt{2}, \sqrt{2}), (\sqrt{2}, 2\sqrt{2})$, and $(-\sqrt{2}, -2\sqrt{2})$. Thus, $f(2\sqrt{2}, -\sqrt{2}) = f(-2\sqrt{2}, \sqrt{2}) = 100$ are constrained maxima and $f(\sqrt{2}, 2\sqrt{2}) = f(-\sqrt{2}, -2\sqrt{2}) = 0$ are constrained minima.

11. $f_x = 3x^2 y; \;\; f_y = x^3; \;\; g_x = 1/2\sqrt{x}; \;\; g_y = 1/2\sqrt{y}$. We need to solve $3x^2 y = \lambda/2\sqrt{x}, \;\; x^3 = \lambda/2\sqrt{y}, \;\; \sqrt{x} + \sqrt{y} - 1 = 0$ or $6x^{5/2} y = \lambda, \;\; 2x^3 y^{1/2} = \lambda, \;\; \sqrt{x} + \sqrt{y} = 1$. From the first two equations, we obtain $3x^{5/2} y = x^3 y^{1/2}$ and $3\sqrt{y} = \sqrt{x}$. Substituting into the third equation, we have $3\sqrt{y} + \sqrt{y} = 4\sqrt{y} = 1$. Then, $y = 1/16$ and $x = 9/16$. Since $(1/4, 1/4)$ satisfies the constraint and $f(1/4, 1/4) = 1/256, \;\; f(9/16, 1/16) + 729/65,536$ is a constrained maximum. We also consider $x = 0$, which requires $y = 1$; and $y = 0$, which requires $x = 1$. Since $x \geq 0$ and $y \geq 0, \;\; f(0, 1) = f(1, 0) = 0 \leq x^3 y = f(x, y)$ for all (x, y) which satisfy $\sqrt{x} + \sqrt{y} = 1$. Thus, $f(0, 1) = 0$ and $f(1, 0) = 0$ are constrained minima.

12. $f_x = y^2; \;\; f_y = 2xy; \;\; g_x = 2x; \;\; g_y = 2y$. We need to solve $y^2 = 2\lambda x, \;\; 2xy = 2\lambda y, \;\; x^2 + y^2 - 27 = 0$ or $y^2 = 2\lambda x, \;\; y(x - \lambda) = 0, \;\; x^2 + y^2 = 27$. When $y = 0$ in the third equation, we obtain $x^2 = 27$ or $x = \pm 3\sqrt{3}$, and $\lambda = 0$. When $x = \lambda$ in the second equation, we obtain $y^2 = 2x^2$ from the first equation and $x^2 + 2x^2 = 3x^2 = 27$ from the third equation. This gives $x = \pm 3$ and $y = \pm 3\sqrt{2}$. Since $f(\pm 3\sqrt{3}, 0) = 0$, we see that $f(-3, 3\sqrt{2}) = f(-3, -3\sqrt{2}) = -54$ are constrained minima and $f(3, 3\sqrt{2}) = f(3, -3\sqrt{2}) = 54$ are constrained maxima.

13. $F_x = 1; \;\; F_y = 2; \;\; F_z = 1; \;\; g_x = 2x; \;\; g_y = 2y; \;\; g_z = 2z$. We need to solve $1 = 2\lambda x, \;\; 2 = 2\lambda y; \;\; 1 = 2\lambda z, \;\; x^2 + y^2 + z^2 - 30 = 0$. From the first and second equations, we obtain $y = 2x$. From the first and third equations, we obtain $z = x$. Substituting into the fourth equation, we have $x^2 + 4x^2 + x^2 = 6x^2 = 30$. Thus, $x = \pm\sqrt{5}, \;\; y = \pm 2\sqrt{5}, \;\; z = \pm\sqrt{5}$. Then, $F(\sqrt{5}, 2\sqrt{5}, \sqrt{5}) = 6\sqrt{5}$ is a constrained maximum and $F(-\sqrt{5}, -2\sqrt{5}, -\sqrt{5}) = -6\sqrt{5}$ is a constrained minimum.

14. $F_x = 2x$; $F_y = 2y$; $F_x = 2z$; $g_x = 1$; $f_y = 2$; $g_z = 3$. We need to solve $2x = \lambda$, $2y = 2\lambda$, $2z = 3\lambda, x + 2y + 3z - 4 = 0$. From the first and second equations, $y = 2x$. From the first and third equations, $z = 3x$. Substituting into the fourth equation, we have $x+4x+9x = 14x = 4$. Thus, $x = 2/7$, $y = 4/7$, $andz = 6/7$. Then, $F(2/7, 4/7, 6/7) = 56/49$ is a constrained extremum. Since $(4, 0, 0)$ satisfies the constraint and $F(4, 0, 0) = 16 > 56/49$, $F(2/7, 4/7, 6/7) = 56/49$ is a constrained minimum.

15. $F_x = yz$; $F_y = xz$; $F_z = xy$; $g_x = 2x$; $g_y = y/2$; $g_z = 2z/9$. We need to solve $yz = 2\lambda x$, $xz = \lambda y/2$, $xy = 2\lambda z/9$ or $xyz/2 = \lambda x^2$, $xyz/2 = \lambda y^2/4$, $xyz/2 = \lambda z^2/9$ along with $x^2 + y^2/4 + z^2/9 - 1 = 0$ or $x^2 + Y^2/4 + z^2/9 = 1$ for $x > 0$, $y > 0$, $z > 0$. From the first three equations and the fact that $\lambda \neq 0$, $x^2 = y^2/4 = z^2/9$. Substituting into the third equation, we obtain $x^2 + x^2 + x^2 = 3x^2 = 1$, so $x^2 = 1/3$, $y^2 = 4/3$, and $z^2 = 3$. Thus, $F(\sqrt{3}/3, 2\sqrt{3}/3, \sqrt{3}) = 2\sqrt{3}/3$ is a constrained extremum. Since $(\sqrt{23}/6, 1, 1)$ satisfies the constraint and $F(\sqrt{23}/6, 1, 1) = \sqrt{23}/6 < 2\sqrt{3}/3$, $F(\sqrt{3}/3, 2\sqrt{3}/3, \sqrt{3}) = 2\sqrt{3}/3$ is a constrained maximum.

16. $F_x = yz$; $F_y = xz$; $F_z = xy$; $g_x = 3x^2$; $g_y = 3y^2$; $g_z = 3z^2$. We need to solve $yz + 3\lambda x^2$, $xz = 3\lambda y^2$, $xy = 3\lambda z^2$ or $xyz = 3\lambda x^3$, $syz = 3\lambda y^3$, $xyz = 3\lambda z^3$ along with $x^3 + y^3 + z^3 - 24 = 0$ or $x^3 + y^3 + z^3 = 24$. Taking $\lambda = 0$ we see that $(\sqrt[3]{24}, 0, 0)$, $(0, \sqrt[3]{24}, 0)$, and $(0, 0, \sqrt[3]{24})$ satisfy the system. If $\lambda \neq 0$, the the first three equations imply $x^3 = y^3 = z^3$. Substituting into the fourth equation, we obtain $x^3+x^3+x^3 = 3x^3 = 24$ or $x = 2$. Then $(2, 2, 2)$ satisfies the system. Since $\sqrt[3]{24} = 2\sqrt[3]{3}$, $F(2\sqrt[3]{3}, 0, 0) = F(0, 2\sqrt[3]{3}, 0) = F(0, 0, 2\sqrt[3]{3}) = 5$ is a constrained minimum and $F(2, 2, 2) = 13$ is a constrained maximum.

17. $F_x = 3x^2$; $F_y = 3y^2$; $F_z = 3z^2$; $g_x = 1$; $g_y = 1$; $g_z = 1$. We need to solve $3x^2 = \lambda$, $3y^2 = \lambda$, $3z^2 = \lambda$, $x + y + z - 1 = 0$ for $x > 0$, $y > 0$, $z > 0$, and hence $\lambda > 0$. From the first three equations $x^2 = y^2 = z^2$, and since x, y, and z are positive, $x = y = z$. Then, from the fourth equation, $x = y = z = 1/3$ and $F(1/3, 1/3, 1/3) = 1/9$ is a constrained extremum. Since $(1/2, 1/4, 1/4)$ satisfies the constraint and $F(1/2, 1/4, 1/4) = 5/32 > 1/9$, $F(1/3, 1/3, 1/3) = 1/9$ is a constrained minimum.

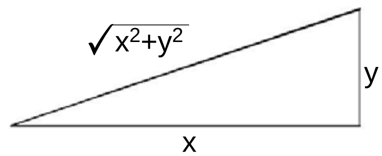
18. $F_x = 8xy^2z^2$; $F_y = 8x^2yz^2$; $F_z = 8x^2Y^2z$; $g_x = 2x$; $g_y = 2y$; $g_z = 2z$. We need to solve $8xy^2z^2 = 2\lambda x$, $8x^2yz^2 = 2\lambda y$, $8x^2y^2z = 2\lambda z$ or $4x^2y^2z^2 = \lambda x^2$, $4x^2y^2z^2 = \lambda y^2$, $4x^2y^2z^2 = \lambda z^2$ along with $x^2 + y^2 + z^2 - 9 = 0$ or $x^2 + y^2 + z^2 = 9$ for $x > 0$, $y > 0$, $z > 0$, and hence $lambda > 0$. From the first three equations, we see $x^2 = y^2 = z^2$. Substituting into the third equation, we obtain $x^2 + x^2 + x^2 = 3x^2 = 9$. Thus, since x, y, and z are positive, $x = y = z = \sqrt{3}$ and $F(\sqrt{3}, \sqrt{3}, \sqrt{3}) = 108$ is a constrained extremum. Since $(1, 2, 2)$ satisfies the constraint and $F(1, 2, 2) = 64 < 108$, $F(\sqrt{3}, \sqrt{3}, \sqrt{3}) = 108$ is a constrained maximum.

19. $F_x = 2x$; $F_y = 2y$; $F_z = 2z$; $g_x = 2$; $g_y = 1$; $g_z = 1$; $h_x = -1$; $h_y = 2$; $h_z = -3$. We need to solve $2x = 2\lambda - \mu$, $2y = \lambda + 2\mu$, $2z = \lambda - 3\mu$ subject to $2x+y+z = 1$, $-x+2y-3z = 4$. Solving the first three equations for x, y, and z, respectively, and substituting into the constraint equations, we obtain $2\lambda - \mu + \lambda/2 - 3\mu/2 = 1$, $-\lambda + \mu/2 + \lambda + 2\mu - 3\lambda/2 + 9\mu/2 = 4$ or $6\lambda - 3\mu = 2$, $-3\lambda + 14\mu = 8$. From this, we obtain $\lambda = 52/75$ and $\mu = 54/75$. Then $x = 1/3$, $y = 16/15$, and $z = -11/15$. Thus, $F(1/3, 16/15, -11/15) = 134/75$ is a constrained minimum.

20. $F_x = 2x$; $F_y = 2y$; $F_z = 2z$; $g_x = 4$; $g_y = 0$; $g_z = 1$; $h_x = 2x$; $h_y = 2y$; $h_z = -2z$. We need to solve $2x = 4\lambda + 2x\mu$, $2y = 2y\mu$, $2z = \lambda - 2z\mu$ subject to $4x + z = 7$, $z^2 = x^2 + y^2$.

Consider the second equation. If $y = 0$, then the constraint equations become $4x + z = 7$ and $z^2 = x^2$. The solutions of these equations are $x = z = 7/5$ and $x = -z = 7/3$. In either case, the first and third equations can be solved for λ and μ. Thus, $(7/5, 0, 7/5)$ and $(7/3, 0, -7/3)$ are candidates for constrained extrema. Now, if $y \neq 0$, then from the second equations $\mu = 0$. In this case, $2x = 4\lambda$ and $2z = \lambda$ or $x = 4z$. Then the first constraint equation becomes $16z + z = 17z = 7$, so $z = 7/17$. Then, $x = 28/17$ and $y^2 = z^2 - x^2 < 0$. Hence, the system has no solution when $y \neq 0$. Thus, $F(7/5, 0, 7/5) = 98/25$ is a constrained minimum and $F(7/3, 0, -7/3) = 98/9$ is a constrained maximum.

21. We want to maximize $A(x, y_x y/2$ subject to $P(x, y) = x + y + \sqrt{x^2 + y^2} - 4 = 0$. $A_x = y/2$; $A_y = x/2$; $P_x = 1 + x/\sqrt{x^2 + y^2}$; $P_y = 1 + y/\sqrt{x^2 + y^2}$. We need to solve $y/2 = \lambda + \lambda x/\sqrt{x^2 + y^2}$, $x/2 = \lambda + \lambda y/\sqrt{x^2 + y^2}$, $x + y + \sqrt{x^2 + y^2} - 4 = 0$ for $x > 0$, $y > 0$, and hence $\lambda > 0$. Subtracting the second equation from the first, we have $(y - x)/2 = \lambda(x - y)/\sqrt{x^2 + y^2}$ or $(y - x) = (y - x)(-2\lambda/\sqrt{x^2 + y^2})$. Since $-2\lambda/\sqrt{x^2 + y^2}$ is negative, it cannot equal 1, and hence $y - x = 0$ or $y = x$. Substituting in the third equation gives $2x + \sqrt{2x^2} = (2 + \sqrt{2})x = 4$. Thus, $x = y = 4/(2 + \sqrt{2})$ and this maximum area is $A(4/(2 + \sqrt{2}), 4/(2 + \sqrt{2})) = 4/(3 + 2\sqrt{2})$.



22. Let the base of the box have dimensions x and y and let the height be z. We want to maximize $V(x, y, z) = xyz$ subject to $S(x, y, z) = xy + 2yz + 2xz - 75 = 0$. Now $V_x = yz$; $V_y = xz$; $V_z = xy$; $S_x = y + 2z$; $S_y = x + 2z$; $S_z = 2y + 2x$. We need to solve $yz = \lambda(y + 2z)$, $xz = \lambda(x + 2z)$, $xy = \lambda(2y + 2x)$, $xy + 2yz + 2xz - 75 = 0$ or $xyz = \lambda(xy + 2xz)$, $xyz = \lambda(xy + 2yz)$, $xyz = \lambda(2yz + 2xz)$, $xy + 2yz + 2xz = 75$, for $x > 0$, $y > 0$, $z > 0$, and thus $\lambda > 0$. From the first three equations, we have $xy + 2xz = xy + 2yz$, which gives $xz = yz$ or $x = y$; and $xy + 2yz = 2yz + 2xz$, which gives $xy = 2xz$ or $y = 2z$. Substituting $x = y = 2z$ into the fourth equation, we obtain $4z^2 + 4z^2 + 4z^2 = 12z^2 = 75$. Thus, $z = 5/2cm$ and $x = y = 5cm$. When the box is closed, $S(x, y, z) = 2(xy + yz + xz) - 75$, $S_x = 2(y + z)$, $S_y = 2(x + z)$, $S_z = 2(x + y)$, and we need to solve $xyz = 2\lambda(xy + xz)$, $xyz = 2\lambda(xy + yz)$, $xyz = 2\lambda(yz + xz)$, $2(xy + yz + xz) = 75$ for $x > 0$, $y > 0$, $z > 0$, and thus $\lambda > 0$. From the first three equations, we have $xy + xz = xy + yz$, which gives $xz = yz$ or $x = y$; and $xy + yz = yz + xz$, which gives $xy = xz$ or $y = z$. Substituting $x = y = z$ into the fourth equation, we obtain $2(x^2 + x^2 + x^2) = 6x^2 = 75$. Thus, $x = y = z = 5/\sqrt{2}$. The box is a cube with each side $5/\sqrt{2}cm$.

23. We want to maximize $V(x, y) = 9\pi x + 3\pi y$ subject to $S(x, y) = 9\pi + 6\pi x + 3\pi\sqrt{9 + y^2} - 81\pi$. Now $V_x = 9\pi$; $V_y = 3\pi$; $S_x = 6\pi$; $S_y = 3\pi y/\sqrt{9 + y^2}$. We need to solve $9\pi = 6\pi\lambda$, $3\pi = 3\pi\lambda y/\sqrt{9 + y^2}$, $9\pi + 6\pi x + 3\sqrt{9 + y^2} - 81\pi = 0$ for $x > 0$ and $y > 0$. From the first equation, $\lambda = 3/2$. Using $\lambda = 3/2$ in the second equation gives $y = 6/\sqrt{5}$. From the third equation, we have $6x + 3\sqrt{9 + 36/5} = 72$ or $x = 12 - 9/2\sqrt{5}$. The volume is maximum when $x = 12 - 9/2\sqrt{5}m$ and $y = 6/\sqrt{5}m$.

24. $U_x = \dfrac{1}{3}x^{-2/3}y^{2/3}$; $U_y = \dfrac{2}{3}x^{1/3}y^{-1/3}$; $g_x = 1$; $g_y = 6$. We need to solve $\dfrac{1}{3}x^{-2/3}y^{2/3} = \lambda$, $\dfrac{2}{3}x^{1/3}y{-}1/3 = 6\lambda$, $x + 6y - 18 = 0$ or $y = 3\lambda x^{2/3}y^{1/3}$, $\dfrac{1}{3}x = 3\lambda x^{2/3}y^{1/3}$, $x + 6y = 18$. From the first two equations, $y = x/3$. Substituting into the third equation, we

have $x + 2x = 3x = 18$. Thus, $x = 6$ and $y = 2$. Since $(12, 1)$ satisfies the constraint and $U(12, 1) = 12^{1/3}$, $U(6, 2) = 6^{1/3}2^{2/3} = 24^{1/3}$ is a constrained maximum.

25. We want to maximize $z(x, y) = P - x - y$ subject to $z^2/xy^3 = k$ or $(P - x - y)^2 - kxy^3 = 0$. Now $z_x = -1$; $z_y = -1$; $g_x = -2(P - x - y) - ky^3$; $g_y = -2(P - x - y) - 3kxy^2$. We need to solve $-1 = -2\lambda(P - x - y) - \lambda ky^3$, $-1 = -2\lambda(P - x - y) - 3\lambda kxy^2$, $(P - x - y)^2 = kxy^3$ for $x > 0$, $y > 0$, and $z > 0$. From the first two equations, we have $y = 3x$. Substituting into the third equation, we obtain $(P - 4x)^2 = 27kx^4$ or $\sqrt{27k}x^2 = P - 4x$. (Since $z > 0$, $z = P - x - y = P - 4x > 0$.) Using the quadratic formula and the fact that $x > 0$, we find

$$x = \frac{-4 + \sqrt{16 + 4P\sqrt{27k}}}{2\sqrt{27k}} = \frac{-2 + \sqrt{4 + P\sqrt{27k}}}{27k}.$$

Then the maximum value of z is $P - 4x = P + 4(2 - \sqrt{4 + P\sqrt{27k}}/\sqrt{27k}$.

26. (a) See part **(b).**

    (b) Maximizing $1/(x_1^2 \cdots x_n^2)$ is equivalent to minimizing the denominator $F(x_1, \ldots x_n) = x_1^2 + \cdots + x_n^2$. The constraint is still $x_1 + \cdots x_n = 1$, which we can write as $g(x_1, \ldots x_n) = x_1 + \cdots + x_n - 1 = 0$. Since $\partial F/\partial x_i = 2x_i$ and $\partial g/\partial x_i = 1$, we can get the equations $2x_1 = \lambda,$, $2x_2 = \lambda, , \ldots, 2x_n = \lambda$, $x_1 + \cdots x_n = 1$. The solution is $x_1 = \ldots = 1/2$ (and $\lambda = 1/2n$).

27. $f(x, y)$ is the square of the distance from a point on the graph of $x^4 + y^4 = 1$ to the origin. The points $(0, \pm 1)$ and $(\pm 1, 0)$ are closest to the origin, while $(\pm 1/\sqrt[4]{2}, \pm 1/\sqrt[4]{2})$ are farthest from the origin.

28. $F(x, y, z)$ is the square of the distance from a point on the plane $x + 2y + 3z = 4$ to the origin. The point $(2/7, 4/7, 6/7)$ is closest to the origin.

29. F is the square of the distance of points on the intersection of the planes $2x + y + z = 1$ and $-x + 2y - 3z = 4$ from the origin. The point $(1/3, 16/15, -11/15)$ is closest to the origin.

30. F is the square of the distance of points on the intersection of the plane $4x + z = 7$ and the circular cone $z^2 = x^2 = y^2$. The point $(7/5, 0, 7/5)$ is closest to the origin and the point $(7/3, 0, -7/3)$ is farthest from the origin.

31. We want to minimize $f(x,) = x^2 + y^2$ subject to $xy^2 = 1$. Now $f_x = 2x$; $f_y = 2y$; $g_x = y^2$; $g_y = 2xy$. We need to solve $2x = \lambda y^2$, $2y = 2\lambda xy$, $xy^2 - 1 = 0$ or $2xy = \lambda y^3$, $2xy = 2\lambda x^2 y$, $xy^2 = 1$ for $x > 0, y > 0$, and hence $\lambda > 0$. From the first two equations, we have $y^3 = 2x^2 y$ or $y^2 = 2x^2$. Substituting into the third equation gives $2x^3 = 1$ or $x = 2^{-1/3}$. Again, from the third equation we have $y = 1/(2^{-1/3})^{1/2} = 2^{1/6}$. Thus, the point closest to the origin is $(2^{-1/3}, 2^{1/6})$. Since the surface is $F(x, y, z) = xy^2 - 1 = 0$, $\nabla F = y^2\mathbf{i} + 2xy\mathbf{j}$ is normal to the surface at (x, y, z). Thus, a normal to the surface at $(2^{-1/3}, 2^{1/6}, 0)$ is $\nabla F(2^{-1/3}, 2^{1/6}, 0)$ is $\nabla F(2^{-1/3}, 21/6, 0) = 2^{1/3}\mathbf{i} + 2(2^{-1/3})(2^{1/6})\mathbf{j} = 2^{1/3}\mathbf{i} + 2^{5/6}\mathbf{j} = 2^{2/3}(2^{-1/3}\mathbf{i} + 2^{1/6}\mathbf{j})$. Since $\nabla F(2^{-1/3}, 2^{1/6}, 0)$ is a multiple of the vector from the origin to $P(2^{-1/3}, 2^{1/6}, 0)$, this vector is perpendicular to the surface.

32. $F_x = \frac{1}{3}x^{-2/3}y^{1/3}z^{1/3}$; $F_y = \frac{1}{3}x^{1/3}y^{-2/3}z^{1/3}$; $F_z = \frac{1}{3}x^{1/3}y^{1/3}z^{-2/3}$; $g_x = 1$; $g_y = 1$; $g_z = 1$. We need to solve $\frac{1}{3}x^{-2/3}y^{1/3}z^{1/3} = \lambda$, $\frac{1}{3}x^{1/3}y^{-2/3}z^{1/3} = \lambda$, $\frac{1}{3}x^{1/3}y^{1/3}z^{-2/3} = \lambda$, $x + y + z - k = 0$ or $x^{1/3}y^{1/3}Z1/3 = 3\lambda x$, $x^{1/3}y^{1/3}z^{1/3} = 3\lambda y$, $x^{1/3}y^{1/3}z^{1/3} = \lambda z$, $x + y + z = k$. From the first three equations, $x = y = z$. Substituting into the fourth equation, $x + x + x = 3x = k$. Thus, $x = y = z = k/3$ and $F(k/3, k/3, k/3) = k/3$ is a constrained maximum.

33. For any $x + y + z = k$, by Problem 32, $\sqrt[3]{xyz} \le \frac{k}{3} = \frac{x+y+z}{3}$.

34. Distance from the xz-plane is measured by $|y|$. Alternatively, we will find the extreme values of $F(x, y, z) = y^2$ subject to $g(x, y, z) = x^2 + z^2 - 1 = 0$ and $h(x, y, z) = x + y + 2z - 4 = 0$. Now $F_x = 0$; $F_y = 2y$; $F_z = 0$; $g_x = 2x$; $g_y = 0$; $g_z = 2z$; $h_x = 1$; $h_y = 1$; $h_z = 2$. We need to solve $0 = 2\lambda x + \mu$, $2y = \mu$, $0 = 2\lambda z + 2\mu$, $x^2 + z^2 = 1$, $x + y + 2z = 4$. By inspection we see that if $\lambda = 0$, then $\mu = 0$ and $y = 0$. Similarly, if $\mu = 0$, then $\lambda = 0$ and $y = 0$. Substituting $y = 0$ into the fourth and fifth equations, we obtain the system $x^2 + z^2 = 1$, $x + 2z = 4$, which is inconsistent. Thus, $\mu \ne 0$ and $\lambda \ne 0$. Now, solving the first three equations for x, y, and z and substituting into the fourth and fifth equations, we obtain the system

$$\frac{\mu^2}{4\lambda^2} + \frac{\mu^2}{\lambda^2} = 1, \quad -\frac{\mu}{2\lambda} + \frac{\mu}{2} - \frac{2\mu}{\lambda} = 4 \text{ or } 5\mu^2 = 4\lambda^2, \quad (\lambda - 5)\mu = 8\lambda.$$

Solving the second equation for μ and substituting into the first, we obtain $5(\frac{8\lambda}{\lambda - 5})^2 = 4\lambda^2$ or $80 = (\lambda - 5)^2$. Thus, $\lambda = 5 \pm 4\sqrt{5}$. From $\mu = 8\lambda/(\lambda - 5)$ we find that corresponding values of μ are $8 \pm 2\sqrt{5}$. Since $2y = \mu$ we see that the objective function $F(x, y, z) = y^2$ is minimized when $\mu = 8 - 2\sqrt{5}$ and maximized when $\mu = 8 + 2\sqrt{5}$. Corresponding values of x, y, and z are $x = -\frac{\mu}{2\lambda} = -\frac{8 \pm 2\sqrt{5}}{10 \pm 8\sqrt{5}} = \mp\frac{1}{\sqrt{5}} \approx |mp0.45$, $y - \mu/2 = 4 \pm \sqrt{5} \approx 6.24$ and 1.76, $z = -\mu/\lambda = 2x = \mp 2/\sqrt{5} \approx \mp 0.89$. The closest point is $(-1/\sqrt{5}, 4 - \sqrt{5}, -2\sqrt{5})$ or about $(-4.5, 6.24, -0.89)$.

Chapter 13 in Review

A. True/False

1. False; see Example 3 in Section 13.2 in the text.

2. False; (0,4.1) is in the domain of g but not in the domain of f.

3. True

4. True

5. False; consider $z = y^2$.

6. False; consider $f(x, y) = xy$ at $(0, 0)$.

7. False; ∇f is perpendicular to the level curve $f(x, y) = c$.

8. True

9. True

10. False; at a saddle point $f_x = f_y = 0$, but there is no extremum.

B. Fill in the Blanks

1. $\lim\limits_{(x,y)\to(1,1)} \dfrac{3x^2 + xy^2 - 3xy - 2y^3}{5x^2 - y^2} = \dfrac{3 + 1 - 3 - 2}{5 - 1} = -\dfrac{1}{4}$

2. where $x - y + 1 = 0$

3. $3x^2 + y^2 = 3(2)^2 + (-4)^2 = 28$

4. $\dfrac{\partial}{\partial \xi} T(p, q) = \dfrac{\partial T}{\partial p}\dfrac{\partial p}{\partial \xi} + \dfrac{\partial T}{\partial q}\dfrac{\partial q}{\partial \xi} = T_p g_\xi + T_q h_\xi$

5. $\dfrac{d}{dw} F(r, s) = \dfrac{\partial F}{\partial r}\dfrac{dr}{dw} + \dfrac{\partial F}{\partial s}\dfrac{ds}{dw} = F_r g'(w) + F_s h'(w)$

6. $dg = g_s \Delta s + g_t \Delta t = \dfrac{2}{t^2}\Delta s - \dfrac{4s}{t^3}\Delta t$

7. f_{yyzx}

8. $\dfrac{\partial^3 f}{\partial y^2 \partial x}$

9. Using the Fundamental Theorem of Calculus, we have $\dfrac{\partial f}{\partial y}(x, y) = \dfrac{\partial}{\partial y}\left[\int_x^y F(t)dt\right] = F(y)$
$\dfrac{\partial f}{\partial x}(x, y) = \dfrac{\partial}{\partial x}\left[\int_x^y F(t)dt\right] = \dfrac{\partial}{\partial x}\left[-\int_y^x F(t)dt\right] = -\dfrac{\partial}{\partial x}\left[\int_y^x F(t)dt\right] = -F(x)$

10. $\nabla F(x_0, y_0, z_0) = \mathbf{i} + \mathbf{j} + \mathbf{k}$

11. $F_{x,y,z} = \dfrac{\partial^2}{\partial z \partial y} f_x(x, y)g(y)h(z) = \dfrac{\partial}{\partial z}[f_x(x, y)g'(y)h(z) + f_{xy}(x, y)g(y)h(z)]$
$= f_x(x, y)g'(y)h'(z) + f_{xy}(x, y)g(y)h'(z)$

12. The distinct fourth-order partial derivatives are $f_{xxxx}, f_{xxxy}, f_{xxyy}, f_{xyyy}$, and f_{yyyy}.

C. Exercises

1. $z_y = -x^3ye^{-x^3y} + e^{-x^3y}$

2. $z_u = -\dfrac{v\sin uv}{\cos uv} = -v\tan uv$

3. $f_r = \dfrac{3}{2}r^2(r^3+\theta^2)^{-1/2};\quad f_{r\theta} = -\dfrac{3}{2}r^2\theta(r^3+\theta^2)^{-3/2}$

4. $\dfrac{\partial f}{\partial x} = 2(2x+xy^2)(2+y^2) = 2(2+y^2)^2x;\quad \dfrac{\partial^2 f}{\partial x^2} = 2(2+y^2)^2$

5. $\dfrac{\partial z}{\partial y} = 3x^2y^2\sinh x^2y^3;\quad \dfrac{\partial^2 z}{\partial y^2} = 9x^4y^4\cosh x^2y^3 + 6x^2y\sinh x^2y^3$

6. $\dfrac{\partial z}{\partial y} = -4y(e^{x^2}+e^{-y^2});\quad \dfrac{\partial^2 z}{\partial x\partial y} = -8xye^{x^2};\quad \dfrac{\partial^3 z}{\partial^2 x\partial y} = -16x^2ye^{x^2} - 8ye^{x^2}$

7. $F_s = 3s^2t^5v^{-4};\quad F_{st} = 15s^2t^4v^{-4};\quad F_{stv} = -60s^2t^4v^{-5}$

8. $\dfrac{\partial w}{\partial z} = \dfrac{xy}{z^2} + \dfrac{x}{y} + \dfrac{y}{x};\quad \dfrac{\partial^2 w}{\partial y\partial z} = -\dfrac{x}{z^2} - \dfrac{x}{y^2} + \dfrac{1}{x};\quad \dfrac{\partial^3 w}{\partial^2 y\partial z} = \dfrac{2x}{y^3};\quad \dfrac{\partial^4 w}{\partial x\partial^2 y\partial z} = \dfrac{2}{y^3}$

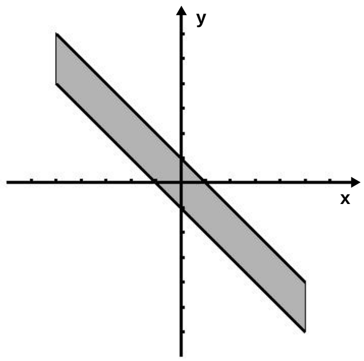
9. $\nabla f = -\dfrac{y}{x^2}\dfrac{1}{1+y^2/x^2}\mathbf{i} + \dfrac{1}{x}\dfrac{1}{1+y^2/x^2}\mathbf{j} = -\dfrac{y}{x^2+y^2}\mathbf{i} + \dfrac{x}{x^2+y^2}\mathbf{j};\quad \nabla f(1,-1) = \dfrac{1}{2}\mathbf{i} + \dfrac{1}{2}\mathbf{j}$

10. $\nabla F = \dfrac{2x}{z^4}\mathbf{i} - \dfrac{9y^2}{z^4}\mathbf{j} - \dfrac{4(x^2-3y^3)}{z^5}\mathbf{k};\quad \nabla F(1,2,1) - 2\mathbf{i} - 36\mathbf{j} + 92\mathbf{k}$

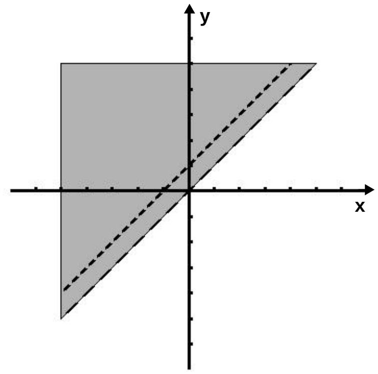
11. $\nabla f = (2xy-y^2)\mathbf{i} + (x^2-2xy)\mathbf{j};\quad \mathbf{u} = \dfrac{2}{\sqrt{40}}\mathbf{i} + \dfrac{6}{\sqrt{40}}\mathbf{j} = \dfrac{1}{\sqrt{10}}(\mathbf{i}+3\mathbf{j});$
$D_uf = \dfrac{1}{\sqrt{10}}(2xy-y^2+3x^2-6xy) = \dfrac{1}{\sqrt{10}}(3x^2-4xy-y^2)$

12. $\nabla F = \dfrac{2x}{x^2+y^2+z^2}\mathbf{i} + \dfrac{2y}{x^2+y^2+z^2}\mathbf{j} + \dfrac{2z}{x^2+y^2+z^2}\mathbf{k};\quad \mathbf{u} = -\dfrac{2}{3}\mathbf{i} + \dfrac{1}{3}\mathbf{j} + \dfrac{2}{3}\mathbf{k};$
$D_uF = \dfrac{-4x+2y+4z}{3(x^2+y^2+z^2)}$

13. $\{(x,y)|(x+y)^2 \leq 1\} = \{(x,y)|\ |x+y| \leq 1\}$



14. $\{(x,y)|y > x,\ \ y \neq x+1\}$



15. $\Delta z = 2(x+\Delta x)(y+\Delta y) - (y+\Delta y)^2 - (2xy - y^2) = 2x\Delta y + 2y\Delta x + 2\Delta x\Delta y - 2y\Delta y - (\Delta y)^2$

16. $\Delta z = (x+\Delta x)^2 - 4(y+\Delta y)^2 + 7(x+\Delta x) - 9(y+\Delta y) + 10 - (x^2 - 4y^2 + 7x - 9y + 10)$
$= 2\Delta x + (\Delta x)^2 - 8y\Delta y - 4(\Delta y)^2 + 7\Delta x - 9\Delta y$

17. $z_x = \dfrac{4x+3y-(x-2y)4}{(4x+3y)^2} = \dfrac{11y}{(4x+3y)^2};\ \ z_y = \dfrac{(4x+3y)(-2)-(x-2y)3}{(4x+3y)^2} = \dfrac{-11x}{(4x+3y)^2};$
$dz = \dfrac{11y}{(4x+3y)^2}dx - \dfrac{11x}{(4x+3y)^2}dy$

18. $A_x = 2y + 2z;\ \ A_y = 2x + 2z;\ \ A_z = 2y + 2x;\ \ dA = 2(y+z)dx + 2(x+z)dy + 2(x+y)dz$

19. $z_y = 4y/\sqrt{x^2+4y^2},\ \ z_y(-\sqrt{5},1) = 4/3,\ \ z(-\sqrt{5},1) = 3.$ The line is given by $x = -\sqrt{5}$ and $z - 3 = \dfrac{4}{3}(y-1)$. Symmetric equations of the line are $x = -\sqrt{5},\ \ \dfrac{z-3}{4} = \dfrac{y-1}{3}$.

20. The direction vector is $\overrightarrow{PQ} = 2\mathbf{i} + 2\mathbf{j}$. $\nabla z = (y+2x)\mathbf{i} + x\mathbf{j}$. $\mathbf{u} = \dfrac{1}{\sqrt{2}}\mathbf{i} + \dfrac{1}{\sqrt{2}}\mathbf{j};\ \ D_u = \nabla z \cdot \mathbf{u} = (y+2x+x)/\sqrt{2} = (y+3x)/\sqrt{2};\ \ D_u(2,3) = 9/\sqrt{2}.$ The slope of the tangent line is $9/\sqrt{2}$.

21. $f_x = 2xy^4,\ \ f_y = 4x^2y^3.$

(a) $\mathbf{u} = \mathbf{i},\ \ D_u(1,1) = f_x(1,1) = 2$

(b) $\mathbf{u} = (\mathbf{i} - \mathbf{j}/\sqrt{2},\ \ D_u(1,1) = (2-4)/\sqrt{2} = -2/\sqrt{2}$

(c) $\mathbf{u} = \mathbf{j},\ \ D_u(1,1) = f_y(1,1) = 4$

22. (a)
$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dxy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}\\ &= \frac{x}{\sqrt{x^2+y^2+z^2}}6\cos 2t + \frac{y}{\sqrt{x^2+y^2+z^2}}(-8\sin 2t) + \frac{z}{\sqrt{x^2+y^2+z^2}}15t^2\\ &= \frac{(6x\cos 2t - 8y\sin 2t + 15zt^2)}{\sqrt{x^2+y^2+z^2}}\end{aligned}$$

(b) $$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dxy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} \\ &= \frac{x}{\sqrt{x^2+y^2+z^2}}\frac{6}{r}\cos\frac{2t}{r} + \frac{y}{\sqrt{x^2+y^2+z^2}}\left(\frac{8r}{t^2}\sin\frac{2r}{t}\right) + \frac{z}{\sqrt{x^2+y^2+z^2}}15t^2r^3 \\ &= \frac{\left(\dfrac{6x}{r}\cos\dfrac{2t}{r} + \dfrac{8yr}{t^2}\sin\dfrac{2r}{t} + 15zt^2r^3\right)}{\sqrt{x^2+y^2+z^2}}\end{aligned}$$

23. $F(x,y,z) = \sin xy - z$; $\nabla F = y\cos xy\mathbf{i} + x\cos xy\mathbf{j} - \mathbf{k}$; $\nabla F(1/2, 2\pi/3, \sqrt{3}/2) = \frac{\pi}{3}\mathbf{i} + \frac{1}{4}\mathbf{j} - \mathbf{k}$. The equation of the tangent plane is $\frac{\pi}{3}(x-\frac{1}{2}) + \frac{1}{4}(y-\frac{2\pi}{3}) - (z-\frac{\sqrt{3}}{2}) = 0$ or $4\pi x + 3y - 12z = 4\pi - 6\sqrt{3}$.

24. We want to find a normal to the surface that is parallel to $\mathbf{k}$. $\nabla F = (y-2)\mathbf{i} + (x-2y)\mathbf{j} + 2z\mathbf{k}$. We need $y-2=0$ and $x-2y=0$. The tangent plane is parallel to $z=2$ when $y=2$ and $x=4$. In this case $z^2=5$. The points are $(4,2,\sqrt{5})$ and $(4,2,-\sqrt{5})$.

25. $\nabla F = 2x\mathbf{i} + 2y\mathbf{j}$; The equation of the tangent plane is $6(x-3) + 8(y-4) = 0$ or $3x + 4y = 25$.

26. We want to minimize

$$D_u f = \mathbf{u}\cdot\nabla f = \frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j})\cdot[(3x^2+3y-6x)\mathbf{i} + (3x+3y^2)\mathbf{j}] = \frac{3}{\sqrt{2}}(x^2+y-2x+x+y^2)$$

or equivalently, we want to minimize $F(x,y) = x^2 - x + y^2 + y$. Now $F_x = 2x-1$; $F_{xx} = 2$; $F_{xy} = 0$;
$F_y = 2y+1$; $F_{yy} = 2$; $D = 4$. Solving $F_x = 0$ and $F_y = 0$ we obtain $x = 1/2$ and $y = -1/2$. Since $D = 4 > 0$ and $F_{xx} = 2 > 0$, F, and hence $D_u f$, has a minimum at $(1/2, -1/2)$.

27. We want to maximize $v(x,y,z) = xyz$ subject to $x + 2y + z = 6$. Now $V_x = yz$; $V_y = xz$; $V_Z = xy$; $g_x = 1$; $g_y = 2$; $g_z = 1$. We need to solve $yz = \lambda$, $xz = 2\lambda$, $xy = \lambda$ or $xyz = \lambda x$, $xyz = 2\lambda y$, $xyz = \lambda z$ along with $x + 2y + z - 6 = 0$ or $x + 2y + z = 6$. From the first three equations, we have $x = 2y = z$. Substituting into the fourth equation gives $x + x + x = 3x = 6$ or $x = 2$. Then $y = 1$ and $z = 2$ and $V(2,1,2) = 4$ is the maximum volume.

28. (a) $M = \dfrac{c^2}{G}D\theta^2$

(b) $dM = \dfrac{c^2}{G}(\theta^2 dD + 2D\theta d\theta)$

(c) We have $\dfrac{dM}{M} = \dfrac{c^2}{G}\left(\dfrac{\theta^2}{M}dD + \dfrac{2D\theta}{M}d\theta\right) = \dfrac{\theta^2}{D\theta^2}dD + \dfrac{2D\theta}{D\theta^2}d\theta = \dfrac{dD}{D} + 2\dfrac{d\theta}{\theta}$, so

$$\left|\frac{dM}{M}\right| = \left|\frac{dD}{D} + 2\frac{d\theta}{\theta}\right| \le \left|\frac{dD}{D}\right| + 2\left|\frac{d\theta}{\theta}\right| \le 0.10 + 2(0.02) = 0.14 = 14\%.$$

29. We are given $v = 14\sqrt{5}ry^{-1/2}$, $dr = -1$, $dy = 1$, $r = 20$, and $y = 25$. Now, $dv = 14\sqrt{5}y^{-1/2}dr - 7\sqrt{5}ry^{-3/2}dy$ and the approximate change in volume is

$$\Delta v \approx 14\sqrt{5}(25)^{-1/2}(-1) - 7\sqrt{5}(20)(25)^{-3/2}(1) = -98\sqrt{5}/25 \approx -8.77\text{cm/s}.$$

30. $\Delta f = 2x\mathbf{i} + 2y\mathbf{j}$, $\Delta f(3,4) = 6\mathbf{i} = 8\mathbf{j}$

(a) $\Delta f(1,-2)2\mathbf{i} - 4\mathbf{j}$; $\mathbf{u} = (2\mathbf{i} - 4\mathbf{j})\sqrt{20} = (\mathbf{i} - 2\mathbf{j})/\sqrt{5}$;
$D_u f(3,4) = 6\sqrt{5} - 16\sqrt{5} = -10\sqrt{5} = -2\sqrt{5}$

(b) $\mathbf{v} = (6\mathbf{i} + 8\mathbf{j})/\sqrt{100} = (3\mathbf{i} + 4\mathbf{j})/5$; $D_v f(3,4) = 18/5 + 32/5 = 10$

31. Let $g(r,\theta) = \dfrac{R^2 - r^2}{R^2 - 2rR\cos(\theta - \phi) + r^2}$. Then, after a straightforward but lengthy computaiton, we find

$$g_r = \frac{(2r^2R + 2R^3)\cos(\theta - \phi) - 4rR^2}{(R^2 - 2rR\cos(\theta - \phi) + r^2)^2,}$$

$$g_{rr} = \frac{8R^4\cos^2(\theta - \phi) + (-12rR^3 - 4r^3R)\cos(\theta - \phi) - 4R^4 + 12r^2R^2}{[R^2 - 2rR\cos(\theta - \phi) + r^2]^3},$$

$$g_{\theta\theta} = \frac{(4r^4R^2 - 4r^2R^4)\cos^2(\theta - \phi) + (2r^5R - 2rR^5)\cos(\theta - \phi) - 8r^4R^2 + 8r^2R^4}{(R^2 - 2rR\cos\theta + r^2)^3}$$

and $r^2g_{rr} + rg_r + g_{\theta\theta}$. Then

$$\begin{aligned} r^2U_{rr} + rU_r + U_{\theta\theta} &= \frac{r^2}{2\pi}\int_{\pi}^{\pi} g(r,\theta)f(\phi)d\phi + \frac{r}{2\pi}\int_{\pi}^{\pi} g(r,\theta)f(\phi)d\phi + \frac{1}{2\pi}\int_{\pi}^{\pi} g(r,\theta)f(\phi)d\phi \\ &= \frac{1}{2\pi}\int_{\pi}^{\pi}(r^2g_{rr} + rg_r + g_{\theta\theta})d\phi = 0. \end{aligned}$$

32. $f_x = A\alpha x^{\alpha-1}y^{\beta} = \dfrac{\alpha Ax^{\alpha}y^{\beta}}{x} = \dfrac{\alpha z}{x}$; $f_y = A\beta x^{\alpha}y^{\beta-1} = \dfrac{\beta Ax^{\alpha}y^{\beta}}{y} = \dfrac{\beta y}{y}$;
$f_{xx} = \dfrac{x\alpha z_x - \alpha z}{x^2} = \dfrac{x\alpha(\alpha z/x) - \alpha z}{x^2} = \dfrac{\alpha^2 z - \alpha z}{x^2} = \dfrac{\alpha(\alpha - 1)}{x^2}$;
$f_{yy} = \dfrac{y\beta z_y - \beta z}{y^2} = \dfrac{y\beta(\beta z - \beta z)}{y^2} = \dfrac{\beta^2 z - \beta z}{y^2} = \dfrac{\beta(\beta - 1)z}{y^2}$;
$f_{xy} = f_{yz} = \dfrac{\alpha(\beta z)/y}{x} = \dfrac{\alpha\beta z}{xy}$

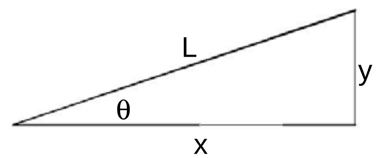
33. Since $D = 4(6) - 5^2 = -1 < 0$, $f(a,b)$ is not a relative extremum.

34. Since $D = 2(7) - 0^2 = 14 > 0$ and $f_{xx} = 2 > 9$, $f(a,b)$ is a relative minimum.

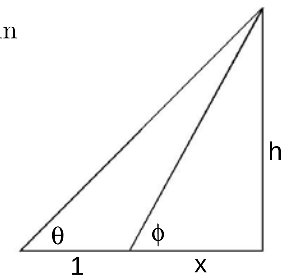
35. Since $D = (-5)(-9) - 6^2 = 9 > 0$ and $f_{xx} = -5 < 0$, $f(a,b)$ is a relative maximum.

36. Since $D = (-2)(-8) - 4^2 = 0$, no determination is possible.

37. Since $x = L\cos\theta$ and $y = L\sin\theta$,
$A = \dfrac{1}{2}xy = \dfrac{1}{2}L^2\sin\theta\cos\theta = \dfrac{1}{4}l^2\sin 2\theta.$



38. Substituting $x = h\cot\phi$ into $\tan\theta = \dfrac{h}{1+x}$ and solving, we obtain
$h = \dfrac{\tan\theta}{1-\tan\theta\cos\phi}.$



39. $A = xy - (y-2z)(x-2z) - z^2 = 2(x+y)z - 5z^2$

40. We are given $V(x,y,z) = xyz$, $x = 30$, $y = 40$, $z = 25$, and $dx = dy = -1$ and $dz = -1/2$. Then $dV = yzdx + dzdy + dydz$, so the approximate volume of plastic is

$$|dV| = 40(25)(1) + 30(25)(1) + 30(40)(1/2) = 2350\text{cm}^3$$

.

41. $V = (2x)(2y)z = 4xy\left(4 - \sqrt{x^2-y^2}\right) = 16xy - 4xy\sqrt{x^2+y^2}$

42. $C(x,y,z) = 1.5(2xy + 2xz + 2yz + xz + 5yz) = \dfrac{3}{2}(2xy + 3xz + 7yz)$

Chapter 14

Multiple Integrals

14.1 The Double Integral

1. With $f(x,y) = x + 3y + 1$ and $\Delta A_k = 1$,

$$\begin{aligned}\int\int_R (x+3y+1)\mathrm{d}A &\approx f(1/2,1/2) + f(3/2,1/2) + f(5/2,1/2) + f(1/2,3/2)\\ &\quad + f(3/2,3/2) + f(5/2,3/2) + f(1/2,5/2) + f(3/2,5/2)\\ &= 3+4+5+6+7+8+9+10 = 52.\end{aligned}$$

2. With $f(x,y) = 2x + 4y$ and $\Delta A_k = 1/4$,

$$\begin{aligned}\int\int_R (2x+4y)\mathrm{d}A &\approx \frac{1}{4}[2(3/2) + 4(1/2) + 2(2) + 4(1/2) + 2(5/2) + 4(1/2) + 2(1/2) + 4(1)\\ &\quad + 2(3/2) + 4(1) + 2(1) + 4(1) + 2(1/2) + 4(3/2) + 2(1) + 4(3/2) + 2(3/2)\\ &\quad + 4(3/2) + 2(1) + 4(2) + 2(1/2) + 4(2) + 2(1/2) + 4(5/2)]\\ &= \frac{1}{4}(3+2+4+2+5+2+4+4+3+4+2+4+1+6+2+6+3\\ &\quad + 6+2+8+1+8+1+10) = \frac{93}{4}.\end{aligned}$$

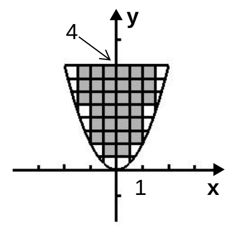
3. (a) With $f(x,y) = x + y$, and $\Delta A_k = 1$,

$$\begin{aligned}\int\int_R (x+y)\ \mathrm{d}A &\approx (-3/2+1/2) + (-1/2+1/2) + (1/2+1/2) + (3/2+1/2)\\ &\quad + (-3/2+3/2) + (-1/2+3/2) + (1/2+3/2) + (3/2+3/2)\\ &= \frac{1}{2}(-2+0+2+4+0+2+4+6) = \frac{16}{2} = 8.\end{aligned}$$

(b) With $f(x,y) + y + 4$ and $\Delta A_k = 1$,

$$\int\int_R (x+y)\ \mathrm{d}A \approx (-2+1)+(-1+1)+(0+1)+(1+1)+(-2+2)+(-1+2)+(0+2)+(1+2)$$
$$= 8.$$

4. With $f(x,y) = xy$ and $\Delta A_k = 1/4$,



$$\int\int_R xy dA \approx \frac{1}{4}[0(1/2)+(1/2)(1/2)+(-1/2)(1)+(0)(1)+(1/2)(1)$$
$$+(1)(1)+(-1/2)(3/2)+(0)(3/2)+(1/2)(3/2)$$
$$+(1)(3/2)+(-1/2)(2)+(0)(2)+(1/2)(2)+(1)(2)$$
$$+(-1)(5/2)+(-1/2)(5/2)+(0)(5/2)+(1/2)(5/2)$$
$$+(1)(5/2)+(3/2)(5/2)+(-1)(3)+(-1/2)(3)+(0)(3)$$
$$+(1/2)(3)+(1)(3)+(3/2)(3)+(-1)(7/2)+(-1/2)(7/2)$$
$$+(0)(7/2)+(1/2)(7/2)+(1)(7/2)+(3/2)(7/2)] = 73/16$$

5. $\int\int_R 10\mathrm{d}A = 10\int\int_R \mathrm{d}A = 10(6) = 60$

6. $\int\int_R 10\mathrm{d}A = 10\int\int_R \mathrm{d}A = 10(12) = 120$

7. $\int\int_R 10\mathrm{d}A = 10\int\int_R \mathrm{d}A = 10\left[\frac{1}{4}\pi(2)^2\right] = 10\pi$

8. $\int\int_R 10\mathrm{d}A = 10\int\int_R \mathrm{d}A = 10\left[\frac{1}{2}(5)\left(\frac{5}{2}\right)\right] = \frac{125}{2}$

9. No, since $x+5y$ is negative at $(3,-1)$ which is in R.

10. Yes, since x^2+y^2 is nonnegative on R.

11. $\int\int_R 10\mathrm{d}A = 10\int\int_R \mathrm{d}A = 10(8) = 80$

12. $\int\int_R -5x\mathrm{d}A = -5\int\int_R x\mathrm{d}A = -5(3) = -15$

13. $\int\int_R (2x+4y)\mathrm{d}A = 2\int\int_R x\mathrm{d}A + 4\int\int_R y\mathrm{d}A = 2(3)+4(7) = 34$

14. $\int\int_R (2x+4y)\mathrm{d}A = \int\int_R x\mathrm{d}A - \int\int_R y\mathrm{d}A = 3-7 = -4$

15. $\int\int_R (3x+7y+1)\mathrm{d}A = 3\int\int_R x\mathrm{d}A + 7\int\int_R y\mathrm{d}A + \int\int_R \mathrm{d}A = 3(3)+7(7)+8 = 66$

16. $\displaystyle\int\int_R y^2\mathrm{d}A - \int\int_R (2+y)^2\mathrm{d}A = \int\int_R y^2\mathrm{d}A - 4\int\int_R \mathrm{d}A - 4\int\int_R y\mathrm{d}A - \int\int_R y^2\mathrm{d}A \quad = -4(8)-4(7) = -6$

17. $\int\int_R f(x,y)\mathrm{d}A = \int\int_{R_1} f(x,y)\mathrm{d}A + \int\int_{R_2} f(x,y)\mathrm{d}A = 4+14 = 18$

18. Since $\int\int_R f(x,y)\mathrm{d}A = \int\int_{R_1} f(x,y)\mathrm{d}A + \int\int_{R_2} f(x,y)\mathrm{d}A$, $25 = 30 + \int\int_{R_2} f(x,y)\mathrm{d}A$ and $\int\int_{R_2} f(x,y)\mathrm{d}A = -5$.

14.2 Iterated Integrals

1. $\int dy = y + c_1(x)$

2. By holding y fixed,
$\int (1-2y)dy = x - 2yx + c_2(y)$

3. By holding y fixed,
$$\int (6x^2y - 3x\sqrt{y})dx = 6\left(\frac{x^3}{3}\right)y - 3\left(\frac{x^2}{2}\right)\sqrt{y} + c_2(y)$$
$$= 2x^3y - \frac{3}{2}x^2\sqrt{y} + c_2(y)$$

4. By holding x fixed,
$$\int (6x^2y - 3x\sqrt{y})dy = 6x^2\left(\frac{y^2}{2}\right) - 3x\frac{y^{3/2}}{(3/2)} + c_1(x)$$
$$= 3x^2y^2 - 2xy^{3/2} + c_1(x)$$

5. By holding x fixed,
$\int \frac{1}{x(y+1)}dy = \frac{\ln|y+1|}{x} + c_1(x)$

6. By holding x fixed,
$$\int (1 + 10x - 5y^4)dx = x + 10\left(\frac{x^2}{2}\right) - 5xy^4 + c_2(y)$$
$$= x + 5x^2 - 5xy^4 + c_2(y)$$

7. By holding y fixed,
$$\int (12y\cos 4x - 3\sin y)dx = 12y\left(\frac{\sin 4x}{4}\right) - 3x\sin y + c_2(y)$$
$$= 3y\sin 4x - 3x\sin y + c_2(y)$$

8. By holding x fixed,
$$\int \sec^2 3xy dy = \frac{\tan 3xy}{3x} + c_1(x)$$

9. By holding y fixed,
$$\int \frac{y}{\sqrt{2x+3y}}dy = y\sqrt{2x+3y} + c_2(y)$$

10. By holding x fixed,
$$\int (2x+5y)^6 dy = \left(\frac{1}{5}\right)\frac{(2x+5y)^7}{7} + c_1(x)$$
$$= \frac{(2x+5y)^7}{35} + c_1(x)$$

11. $\int_{-1}^{3}(6xy - 5e^y)\ \mathrm{dx} = (3x^2y - 5xe^y)\Big|_{-1}^{3} = (27y - 15e^y) - (3y + 5e^y) = 24y - 20e^y$

12. $\displaystyle\int_1^2 \tan xy \;\; \mathrm{d}y = \frac{1}{x} \ln|\sec xy|\Big|_1^2 = \frac{1}{x}\ln|\sec 2x - \sec x|$

13. $\displaystyle\int_1^{3x} x^3 e^{xy} = x^2 e^{xy}\big|_1^{3x} = x^2(e^{3x^2} - e^x)$

14. $\displaystyle\int_{\sqrt{y}}^{y^3} (8x^3y - 4xy^2) \;\; \mathrm{d}x = (2x^4y - 2x^2y^2)\big|_{\sqrt{y}}^{y^3} = (2y^{13} - 2y^8) - (2y^3 - 2y^3) = 2y^{13} - 2y^8$

15. $\displaystyle\int_0^{2x} \frac{xy}{x^2+y^2} \;\; \mathrm{d}y = \frac{x}{2}\ln(x^2+y^2)\Big|_0^{2x} = \frac{x}{2}[\ln(x^2+4x^2) - \ln x^2] = \frac{x}{2}\ln 5$

16. $\displaystyle\int_{x^3}^{x} e^{2y/x} \;\; \mathrm{d}y = \frac{x}{2}e^{2y/x}\Big|_{x^3}^{x} = \frac{x}{2}(e^{2y/x} - e^{2x^3/x}) = \frac{x}{2}(e^2 - e^{2x^2})$

17. $\displaystyle\int_{\tan y}^{\sec y} (2x + \cos y) \;\; \mathrm{d}x = (x^2 + x\cos y)\big|_{\tan y}^{\sec y} = \sec^2 y + \sec y \cos y - \tan^2 y - \tan y \cos y$

$$= \sec^2 y + 1 - \tan^2 y - \sin y = 2 - \sin y$$

18. $\displaystyle\int_{\sqrt{y}}^{1} y \ln x \;\; \mathrm{d}x$ $\boxed{\text{Integration by parts}}$

$$= y(x\ln x - x)\big|_{\sqrt{y}}^{1} = y(0-1) - y(\sqrt{y}\ln\sqrt{y} - \sqrt{y}) = -y - y\sqrt{y}\left(\frac{1}{2}\ln y - 1\right)$$

19. $\displaystyle\int_x^{\pi/2} \cos x \sin^3 y dy = \cos x \cos y \left(\frac{-\sin^2 y}{3} - \frac{2}{3}\right)\Bigg|_x^{\pi/2} = 0 - \cos^2 x\left(\frac{-\sin^2 x}{3} - \frac{2}{3}\right)$

$$= \frac{\cos^2 x \sin^3 x}{3} + \frac{2\cos^2 x}{3} = \frac{\cos^2(1-\cos^2 x)}{3} + \frac{2\cos^2 x}{3}$$
$$= \frac{\cos^2 x}{3} - \frac{\cos^4 x}{3} + \frac{2\cos^2 x}{3} = \cos^2 x - \frac{1}{3}\cos^4 x$$

20. $\displaystyle\int_{1/2}^{1} y\cos^2 xy dx = y\left(\frac{\sin xy \cos xy}{2y} + \frac{x}{2}\right)\Bigg|_{1/2}^{1} = \frac{\sin xy \cos xy}{2} + \frac{xy}{2}\Bigg|_{1/2}^{1}$

$$= \left(\frac{\sin x \cos x}{2} + \frac{x}{2}\right) - \left(\frac{\sin\frac{x}{2}\cos\frac{x}{2}}{2} + \frac{x}{4}\right) = \frac{\sin x\cos x - \sin\frac{x}{2}\cos\frac{x}{2}}{2} + \frac{x}{4}$$

21. $\displaystyle\int_1^2 \int_{-x}^{x^2} (8x - 10y + 2) \;\; \mathrm{d}y \;\; \mathrm{d}x = \int_1^2 (8xy - 5y^2 + 2y)\Big|_{-x}^{x^2} \;\; \mathrm{d}x$

$$= \int_1^2 [(8x^3 - 5x^4 + 2x^2) - (-8x^2 - 5x^2 - 2x)] \;\; \mathrm{d}x$$
$$= \int_1^2 (8x^3 - 5x^4 + 15x^2 + 2x) \;\; \mathrm{d}x = (2x^4 - x^5 + 5x^3 + x^2)\big|_1^2$$
$$= 44 - 7 = 37$$

22. $$\int_{-1}^{1}\int_{0}^{y}(x+y)^2\ \mathrm{d}x\ \mathrm{d}y = \int_{-1}^{1}\left[\int_{0}^{y}(x+y)^2\ \mathrm{d}x\right]\ \mathrm{d}y = \int_{-1}^{1}\frac{1}{3}(x+y)^3\Big|_0^y\ \mathrm{d}y$$
$$= \frac{1}{3}\int_{-1}^{1}[(y+y)^3-(0+y)^3]\ \mathrm{d}y = \frac{1}{3}\int_{-1}^{1}7y^3\ \mathrm{d}y = \frac{1}{3}\left(\frac{7}{4}y^4\right)\Big|_{-1}^{1} = 0$$

23. $$\int_0^{\sqrt{2}}\int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}}(2x-y)\ \mathrm{d}x\ \mathrm{d}y = \int_0^{\sqrt{2}}(x^2-xy)\Big|_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}}$$
$$= \int_0^{\sqrt{2}}\left[(2-y^2-y\sqrt{2-y^2})-(2-y^2+y\sqrt{2-y^2})\right]\ \mathrm{d}y$$
$$= \int_0^{\sqrt{2}}(-2\sqrt{2-y^2})\ \mathrm{d}y = \frac{2}{3}(2-y^2)^{3-2}\Big|_0^{\sqrt{2}} = \frac{2}{3}(0)-\frac{2}{3}2^{3/2} = -\frac{4}{3}\sqrt{2}$$

24. $$\int_0^{\pi/4}\int_0^{\cos x}(1+4y\tan^2 x)\ \mathrm{d}y\ \mathrm{d}x = \int_0^{\pi/4}(y+2y^2\tan^2 x)\Big|_0^{\cos x} = \int_0^{\pi/4}(\cos x+2\cos^2 x\tan^2 x)\ \mathrm{d}x$$
$$= \int_0^{\pi/4}(\cos x+2\sin^2 x)\ \mathrm{d}x = \left(\sin x + x - \frac{1}{2}\sin 2x\right)\Big|_0^{\pi/4}$$
$$= \frac{\sqrt{2}}{2}+\frac{\pi}{4}-\frac{1}{2} = \frac{2\sqrt{2}+\pi-2}{4}$$

25. $$\int_0^{\pi}\int_y^{3y}\cos(2x+y)\ \mathrm{d}x\ \mathrm{d}y\int_0^{\pi}\frac{1}{2}\sin(2x+y)\Big|_y^{3y}\ \mathrm{d}y\frac{1}{2}\int_0^{\pi}(\sin 7y-\sin 3y)\ \mathrm{d}y$$
$$= \frac{1}{2}\left(-\frac{1}{7}\cos 7y+\frac{1}{3}\cos 3y\right)\Big|_0^{\pi}$$
$$= \frac{1}{2}\left[-\frac{1}{7}(-1)+\frac{1}{3}(-1)-\left(-\frac{1}{7}+\frac{1}{3}\right)\right] = -\frac{4}{21}$$

26. $$\int_1^2\int_0^{\sqrt{x}}2y\sin\pi x^2\ \mathrm{d}y\ \mathrm{d}x = \int_1^2\left[\int_0^{\sqrt{x}}2y\sin\pi x^2\ \mathrm{d}y\right]\ \mathrm{d}x = \int_1^2 y^2\sin\pi x^2\Big|_0^{\sqrt{x}}\ \mathrm{d}x = \int_1^2 x\sin\pi x^2\ \mathrm{d}x$$
$$= -\frac{1}{2\pi}\cos\pi x^2\Big|_1^2 = -\frac{1}{2\pi}(\cos 4\pi-\cos\pi) = -\frac{1}{2\pi}(1-(-1)) = -\frac{1}{\pi}$$

27. $$\int_1^{\ln 3}\int_0^x 6e^{x+2y}\ \mathrm{d}y\ \mathrm{d}x = \int_1^{\ln 3}e^{x+2y}\Big|_0^x\ \mathrm{d}x = \int_1^{\ln 3}(3e^{3x}-3e^x)\ \mathrm{d}x = (e^{3x}-3e^x)\big|_1^{\ln 3}$$
$$= (27-9)-(e^3-3e) = 18-e^3+3e$$

28. $\int_0^1\int_0^{2y}e^{-y^2}\ \mathrm{d}x\ \mathrm{d}y = \int_0^1 xe^{-y^2}\Big|_0^{2y}\ \mathrm{d}y = \int_0^1 2ye^{-y^2}\ \mathrm{d}y = -e^{-y^2}\Big|_0^1 = -e^{-1}-(-1) = 1-e^{-1}$

29. $$\int_0^3 \int_{x+1}^{2x+1} \frac{1}{\sqrt{y-x}}\ dy\ dx = \int_0^3 2\sqrt{y-x}\Big|_{x+1}^{2x+1}\ dx = 2\int_0^3 (\sqrt{x+1}-1)\ dx$$
$$= 2\left[\frac{2}{3}(x+1)^{3/2}\right]\Big|_0^3 = 2\left[\left(\frac{16}{3}-3\right)-\left(\frac{2}{3}\right)\right] = \frac{10}{3}$$

30. $$\int_0^1 \int_0^y x(y^2-x^2)^{3/2}\ dx\ dy = \int_0^1 -\frac{1}{5}(y^2-x^2)^{5/2}\Big|_0^y = -\frac{1}{5}\int_0^1 (-y^5)\ dy = -\frac{1}{30}y^6\Big|_0^1 = \frac{1}{30}$$

31. $$\int_1^9 \int_0^x \frac{1}{x+y^2}\ dy\ dx = \int_1^9 \frac{1}{x}\tan^{-1}\frac{y}{x}\Big|_0^x = \int_1^9 \frac{\pi}{4x}\ dx = \frac{\pi}{4}\ln|x|\Big|_1^9 = \frac{\pi}{4}\ln 9$$

32. $$\int_0^{1/2} \int_0^y \frac{1}{\sqrt{1-x^2}}\ dx\ dy = \int_0^{1/2} \sin^{-1}x\Big|_0^y\ dy = \int_0^{1/2}\sin^{-1}y\ dy \quad \boxed{\text{Integration by parts}}$$
$$= \left(y\sin^{-1}y+\sqrt{1-y^2}\right)\Big|_0^{1/2} = \frac{1}{2}\left(\frac{\pi}{6}\right)+\frac{\sqrt{3}}{2}-1 = \frac{\pi+6\sqrt{3}-12}{12}$$

33. $$\int_1^e \int_1^y \frac{y}{x}\ dx\ dy = \int_1^e y\ln x\Big|_1^y\ dy = \int_1^e y\ln y\ dy \quad \boxed{\text{Integration by parts}}$$
$$= \left(\frac{1}{2}y^2\ln y - \frac{1}{4}y^2\right)\Big|_1^e = \frac{1}{2}e^2-\frac{1}{4}e^2-\left(-\frac{1}{4}\right) = \frac{1}{4}(e^2+1)$$

34. $$\int_1^4 \int_1^{\sqrt{x}} 2ye^{-x}\ dy\ dx = \int_1^4 y^2e^{-x}\Big|_1^{\sqrt{x}}\ dx = \int_1^4 (xe^{-x}-e^{-x})\ dx \quad \boxed{\text{Integration by parts}}$$
$$= (-xe^{-x}-e^{-x}+e^{-x})\big|_1^4 = -4e^{-4}+e^{-1}$$

35. $$\int_0^6 \int_0^{\sqrt{25-y^2}/2} \frac{1}{\sqrt{(25-y^2)-x^2}}\ dx\ dy = \int_0^6 \left(\sin^{-1}\frac{x}{\sqrt{25-y^2}}\right)\Bigg|_0^{\sqrt{25-y^2}/2}\ dy$$
$$= \int_0^6 \sin^{-1}\frac{1}{2}\ dy = \int_0^6 \frac{\pi}{6}\ dy = \pi$$

36. $$\int_0^2 \int_{y^2}^{\sqrt{20-y^2}} y\ dx\ dy = \int_0^2 xy\Big|_{y^2}^{\sqrt{20-y^2}}\ dy = \int_0^2 (y\sqrt{20-y^2}-y^3)\ dy = \left[-\frac{1}{3}(20-y^2)^{3/2}-\frac{1}{4}y^4\right]\Big|_0^2$$
$$= \left[-\frac{1}{3}(64)-4\right]-\left[-\frac{1}{3}(40\sqrt{5})-0\right] = \frac{40\sqrt{5}-76}{3}$$

37. $$\int_{\pi/2}^{\pi} \int_{\cos y}^0 e^x\sin y\ dx\ dy = \int_{\pi/2}^{\pi} e^x\sin y\Big|_{\cos y}^0\ dy = \int_{\pi/2}^{\pi}(\sin y - e^{\cos y}\sin y)\ \mathrm{d}$$
$$= (-\cos y + e^{\cos y})\big|_{\pi/2}^{\pi} = (1+e^{-1})-(0+1) = e^{-1}$$

38. $$\int_0^1 \int_0^{y^{1/3}} 6x^2 \ln(y+1) \ \mathrm{d}x \ \mathrm{d}y = \int_0^1 2x^3 \ln(y+1)\Big|_0^{y^{1/3}} \ \mathrm{d}y = 2\int_0^1 y\ln(y+1) \ \mathrm{d}y$$

$$\boxed{\text{Integration by parts}}$$

$$= \left[y^2\ln(y+1) - \frac{1}{2}y^2 + y - \ln(y+1)\right]\Big|_0^1$$

$$= (\ln 2\frac{1}{2} + 1 - \ln 2) - (0 - 0 + 0 - \ln 1) = \frac{1}{2}$$

39. $$\int_\pi^{2\pi} \int_0^x (\cos x - \sin y) \ \mathrm{d}y \ \mathrm{d}x = \int_\pi^{2\pi} (y\cos x + \cos y)\Big|_0^x \ \mathrm{d}x = \int_\pi^{2\pi} (x\cos x + \cos x - 1) \ \mathrm{d}x$$

$$\boxed{\text{Integration by parts}}$$

$$= (\cos x + x\sin x + \sin x - x)|_\pi^{2\pi} = (1 - 2\pi) - (-1 - \pi) = 2 - \pi$$

40. $$\int_1^3 \int_0^{1/x} \frac{1}{x+1} \ \mathrm{d}y \ \mathrm{d}x = \int_1^3 \frac{y}{x+1}\Big|_0^{1/x} \ \mathrm{d}x = \int_1^3 \frac{1}{x(x+1)} \ \mathrm{d}x = \int_1^3 \left(\frac{1}{x} - \frac{1}{x+1}\right) \ \mathrm{d}x$$

$$= [\ln x - \ln(x+1)]|_1^3 = (\ln 3 - \ln 4) - (0 - \ln 2) = \ln 3/2$$

41. $$\int_{\pi/12}^{5\pi/12} \int_1^{\sqrt{2\sin 2\theta}} r \ \mathrm{d}r \ \mathrm{d}\theta = \int_{\pi/12}^{5\pi/12} \frac{1}{2}r^2\Big|_1^{\sqrt{2\sin 2\theta}} \ \mathrm{d}\theta = \int_{\pi/12}^{5\pi/12} \left(\sin 2\theta - \frac{1}{2}\right) \ \mathrm{d}\theta = -\frac{1}{2}(\cos 2\theta + \theta)\Big|_{\pi/12}^{5\pi/12}$$

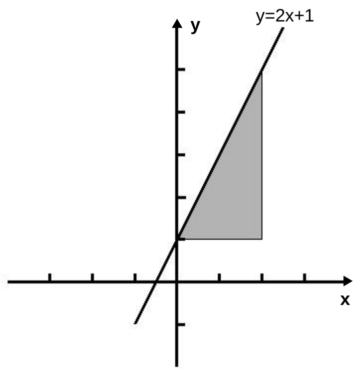
$$= -\frac{1}{2}\left[\left(-\frac{\sqrt{3}}{2} + \frac{5\pi}{12}\right) - \left(\frac{\sqrt{3}}{2} + \frac{5\pi}{12}\right)\right] = \frac{\sqrt{3}}{2} - \frac{\pi}{6}$$

42. $$\int_0^{\pi/3} \int_{3\cos\theta}^{1+\cos\theta} r \ \mathrm{d}r \ \mathrm{d}\theta = \int_0^{\pi/3} \frac{1}{2}r^2\Big|_{3\cos\theta}^{1+\cos\theta} \ \mathrm{d}\theta$$

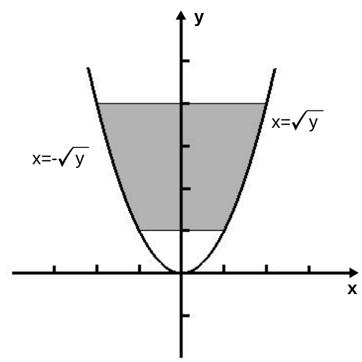
$$= \frac{1}{2}\int_0^{\pi/3} (1 + 2\cos\theta - \cos^2\theta) \ \mathrm{d}\theta = \frac{1}{2}(\theta + 2\sin\theta - 4\theta - 2\sin 2\theta)\Big|_0^{\pi/3}$$

$$= \frac{1}{2}(-\pi + \sqrt{3} - \sqrt{3}) = -\frac{\pi}{2}$$

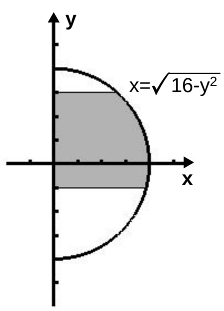
43.



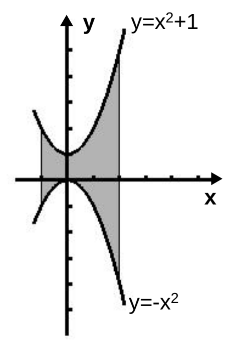
44.



45.



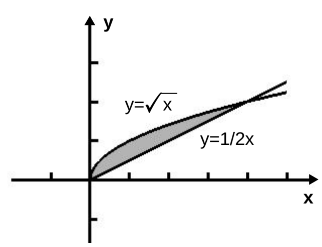
46.



47. $$\int_0^4 \int_{x/2}^{\sqrt{x}} x^2 y\,dy\,dx = \int_0^4 \frac{1}{2}x^2 y\Big|_{x/2}^{\sqrt{x}} dx = \int_0^4 \frac{1}{2}x^2\left(x - \frac{x^2}{4}\right)dx$$

$$= \int_0^4 \left(\frac{1}{2}x^2 - \frac{1}{8}x^4\right)dx = \left(\frac{1}{8}x^4 - \frac{1}{40}x^5\right)\Big|_0^4$$

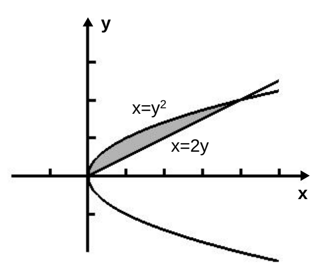
$$= 32 - \frac{128}{5} = \frac{32}{5}$$



$$\int_0^2 \int_{y^2}^{2y} x^2 y\,dx\,dy = \int_0^2 \frac{1}{3}x^3 y\Big|_{y^2}^{2y} dy = \int_0^2 \frac{1}{3}y(8y^3 - y^6)dy$$

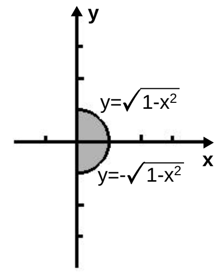
$$= \int_0^2 \left(\frac{8}{3}y^4 - \frac{1}{3}y^7\right)dy = \left(\frac{8}{15}y^5 - \frac{1}{24}y^8\right)\Big|_0^2$$

$$= \frac{256}{15} - \frac{32}{3} = \frac{32}{5}$$

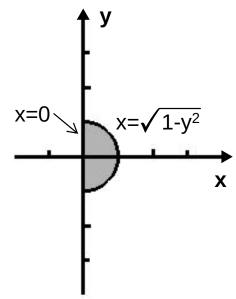


Therefore $\displaystyle\int_0^4 \int_{x/2}^{\sqrt{x}} x^2 y\,dy\,dx = \int_0^2 \int_{y^2}^{2y} x^2 y\,dx\,dy$

48. $$\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2xdydx = \int_0^1 2xy\Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx$$
$$= \int_0^1 \left(2x\sqrt{1-x^2} + 2\sqrt{1-x^2}\right) dx$$
$$= \int_0^1 4x\sqrt{1-x^2}dx = -\frac{4}{3}(1-x^2)^{3/2}\Big|_0^1$$
$$= \frac{4}{3}$$



$$\int_{-1}^1 \int_0^{\sqrt{1-y^2}} 2xdxdy = \int_{-1}^1 x^2\Big|_0^{\sqrt{1-y^2}} dy = \int_{-1}^1 (1-y^2)dy$$
$$= \left(y - \frac{1}{3}y^3\right)\Big|_{-1}^1$$
$$= \left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right) = \frac{4}{3}$$



Therefore, $\displaystyle\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2xdydx = \int_{-1}^1 \int_0^{\sqrt{1-y^2}} 2xdxdy$

49.
$$\int_{-1}^2 \int_0^3 x^2 \;\; \mathrm{d}y \;\; \mathrm{d}x = \int_{-1}^2 x^2y\Big|_0^3 \;\; \mathrm{d}x = \int_{-1}^2 3x^2 \;\; \mathrm{d}x = x^3\big|_{-1}^2 = 8-(-1) = 9$$
$$\int_0^3 \int_{-1}^2 x^2 \;\; \mathrm{d}x \;\; \mathrm{d}y = \int_0^3 \frac{1}{3}x^3\Big|_{-1}^2 \;\; \mathrm{d}y = \int_0^3 \left(\frac{8}{3} + \frac{1}{3}\right) \;\; \mathrm{d}y = \int_0^3 3 \;\; \mathrm{d}y = 3y|_0^3 = 9$$

50. $$\int_{-2}^2 \int_2^4 (2x+4y) \;\; \mathrm{d}x \;\; \mathrm{d}y = \int_{-2}^2 (x^2+4xy)\Big|_2^4 \;\; \mathrm{d}y = \int_{-2}^2 [(16+16y) - (4+8y)] \;\; \mathrm{d}y$$
$$= \int_{-2}^2 (12+8y) \;\; \mathrm{d}y = (12y+4y^2)\big|_{-2}^2 = (24+16) - (-24+16) = 48$$
$$\int_2^4 \int_{-2}^2 (2x+4y) \;\; \mathrm{d}y \;\; \mathrm{d}x = \int_2^4 (2xy+2y^2)\Big|_{-2}^2 \;\; \mathrm{d}x = \int_2^4 [(4x+8) - (-4+8)] \;\; \mathrm{d}x$$
$$= \int_2^4 8x \;\; \mathrm{d}x = 4x^2\big|_2^4 = 64 - 16 = 48$$

51. $$\int_1^3 \int_0^\pi (3x^2y - 4\sin y) \;\; \mathrm{d}y \;\; \mathrm{d}x = \int_1^3 \left(\frac{3}{2}x^2y^2 - 4\cos y\right)\Big|_0^\pi \;\; \mathrm{d}x = \int_1^3 \left[\left(\frac{3\pi^2}{2}x^2 - 4\right) - (4)\right] \;\; \mathrm{d}x$$
$$= \int_1^3 \left(\frac{3\pi^2}{2}x^2 - 8\right) \;\; \mathrm{d}x = \left(\frac{\pi^2}{2}x^3 - 8x\right)\Big|_1^3$$
$$= \left(\frac{27\pi^2}{2} - 24\right) - \left(\frac{\pi^2}{2} - 8\right) = 13\pi^2 - 16$$

$$\int_0^{\pi}\int_1^3 (3x^2y - 4\sin y)\ \text{d}y\ \text{d}x = \int_0^{\pi} (x^3y - 4x\sin y)\Big|_1^3 dy = \int_0^{\pi} [(27y - 12\sin y) - (y - 4\sin y)]dy$$
$$= \int_0^{\pi} (26y - 8\sin y)dy = (13y^2 + 8\cos y)\Big|_0^{\pi}$$
$$= (13\pi^2 - 8) - (8) = 13\pi^2 - 16$$

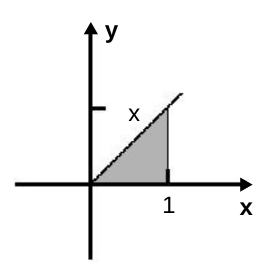
52. $$\int_0^1\int_0^2 \left(\frac{8y}{x+1} - \frac{2x}{y^2+1}\right) dxdy = \int_0^1 \left(8y\ln|x+1| - \frac{x^2}{y^2+1}\right)\Bigg|_0^2 dy = \int_0^1 \left(8y\ln 3 - \frac{4}{y^2+1}\right) dy$$
$$= (4y^2\ln 3 - 4\tan^{-1} y)\Big|_0^1 = 4\ln 3 - \pi$$
$$\int_0^2\int_0^1 \left(\frac{8y}{x+1} - \frac{2x}{y^2+1}\right) dydx = \int_0^2 \left(\frac{4y^2}{x+1} - 2x\tan^{-1} y\right)\Bigg|_0^1 = \int_0^2 \left(\frac{4}{x+1} - \frac{\pi}{2}\right) dx$$
$$= \left(4\ln|x+1| - \frac{\pi}{4}x^2\right)\Bigg|_0^2 = 4\ln 3 - \pi$$

53. We use the fact that $\int_{\alpha}^{\beta} kF(t)dt = k\int_{\alpha}^{\beta} F(t)dt$. Then
$$\int_c^d\int_a^b f(x)g(y)dxdy = \int_c^d g(y)\left[\int_a^b f(x)dx\right] dy = \left[\int_a^b f(x)dx\right]\left[\int_c^d g(y)dy\right]$$

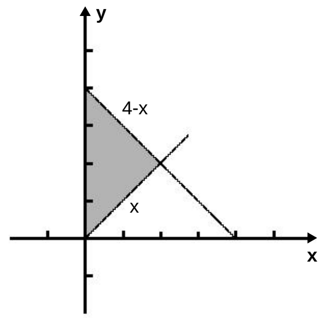
54. $$\int_0^{\infty}\int_0^{\infty} xye^{-(2x^2+3y^2)}dxdy = \int_0^{\infty}\int_0^{\infty} \left(xe^{-2x^2}\right)\left(ye^{-3y^2}\right) dxdy$$
$$= \left(\int_0^{\infty} xe^{-2x^2}dx\right)\cdot\left(\int_0^{\infty} ye^{-3y^2}dy\right)$$
$$= \left(\lim_{a\to\infty}\int_0^a xe^{-2x^2}dx\right)\cdot\left(\lim_{b\to\infty}\int_0^b ye^{-3y^2}dy\right)$$
$$= \left(\lim_{a\to\infty} -\frac{e^{-2x^2}}{4}\Bigg|_0^a\right)\cdot\left(\lim_{b\to\infty} -\frac{e^{-3y^2}}{6}\Bigg|\right)$$
$$= \left(\lim_{a\to\infty}\left[\frac{-e^{-2a^2}}{4} + \frac{1}{4}\right]\right)\cdot\left(\lim_{b\to\infty}\left[\frac{-e^{-3b^2}}{6} + \frac{1}{6}\right]\right)$$
$$= \left(\frac{1}{4}\right)\cdot\left(\frac{1}{6}\right) = \frac{1}{24}$$

14.3 Evaluation of Double Integrals

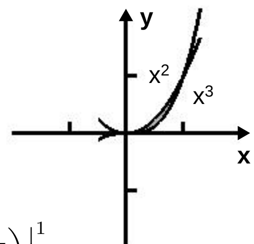
1. $$\int\int_R x^3y^2dA = \int_0^1\int_0^x x^3y^2dydx = \int_0^1 \frac{1}{3}x^3y^3\Bigg|_0^x dx = \frac{1}{3}\int_0^1 x^6dx$$
$$= \frac{1}{21}x^7\Bigg|_0^1 = \frac{1}{21}$$



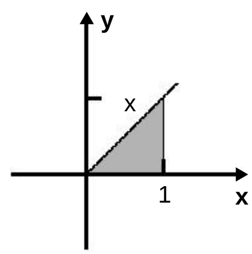
2. $$\int\int_R (x+1)dA = \int_0^2 \int_x^{4-x} (x+1)dydx = \int_0^2 (xy+y)\Big|_x^{4-x} dx$$
$$= \int_0^2 [(4x - x^2 + 4 - x) - (x^2 + x)]dx$$
$$= \int_0^2 (2x - 2x^2 + 4)dx$$
$$= \left(x^2 - \frac{2}{3}x^3 + 4x\right)\Big|_0^2 = \frac{20}{3}$$



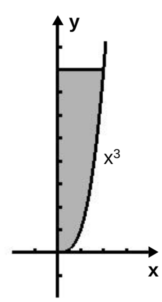
3. $$\int\int_R (2x+4y+1)dA = \int_0^1 \int_{x^3}^{x^2} (2x+4y+1)dydx$$
$$= \int_0^1 (2xy + 2y^2 + y)\Big|_{x^3}^{x^2} dx$$
$$= \int_0^1 [(2x^3 + 2x^4 + x^2) - (2x^4 + 2x^6 + x^3)]dx$$
$$= \int_0^1 (x^3 + x^2 - 2x^6)dx = \left(\frac{1}{4}x^4 + \frac{1}{3}x^3 - \frac{2}{7}x^7\right)\Big|_0^1$$
$$= \frac{1}{4} + \frac{1}{3} - \frac{2}{7} = \frac{25}{84}$$



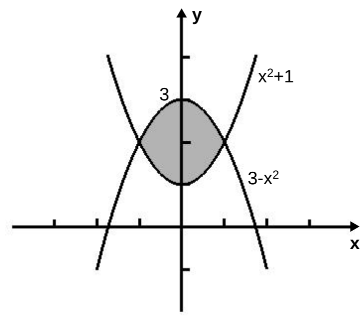
4. $$\int\int_R xe^y dA = \int_0^1 \int_0^x xe^y dydx = \int_0^1 xe^y\Big|_0^x dx$$
$$= \int_0^1 (xe^x - x)dx \quad \boxed{\text{Integration by parts}}$$
$$= \left(xe^x - e^x - \frac{1}{2}x^2\right)\Big|_0^1 = \left(e - e - \frac{1}{2}\right) - (-1) = \frac{1}{2}$$



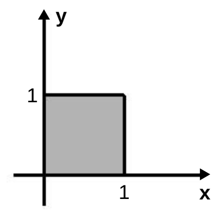
5. $$\int\int_R 2xydA = \int_0^2 \int_{x^3}^8 2xydydx = \int_0^2 xy^2\Big|_{x^3}^8 dx$$
$$= \int_0^2 (64x - x^7)dx = \left(32x^2 - \frac{1}{8}x^8\right)\Big|_0^2 = 96$$



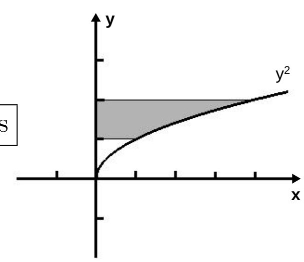
6. $$\int\int_R \frac{x}{\sqrt{y}}dA = \int_{-1}^{1}\int_{x^2+1}^{3-x^2} xy^{-1/2}dydx = \int_{-1}^{1} 2x\sqrt{y}\Big|_{x^2+1}^{3-x^2} dx$$
$$= 2\int_{-1}^{1}(x\sqrt{3-x^2} - x\sqrt{x^2+1})dx$$
$$= 2[-\frac{1}{3}(3-x^2)^{3/2} - \frac{1}{3}(x^2+1)^{3/2}]\Big|_{-1}^{1}$$
$$= -\frac{2}{3}[(2^{3/2}+2^{3/2}) - (2^{3/2}+2^{3/2})] = 0$$



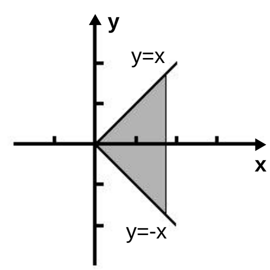
7. $$\int\int_R \frac{y}{1+xy}dA = \int_0^1\int_0^1 \frac{y}{1+xy}dxdy = \int_0^1 \ln(1+xy)\Big|_0^1 dy$$
$$= \int_0 1\ln(1+y)dy = [(1+y)\ln(1+y) - (1+y)]|_0^1$$
$$= (2\ln 2 - 2) - (-1) = 2\ln 2 - 1$$



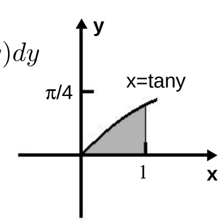
8. $$\int\int_R \sin\frac{\pi x}{y}dA = \int_1^2\int_0^{y^2} \sin\frac{\pi x}{y}dxdy = \int_1^2 \left(-\frac{y}{x}\cos\frac{\pi x}{y}\right)\Big|_0^{y^2} dy$$
$$= \int_1^2 \left(-\frac{y}{\pi}\cos\pi y + \frac{y}{\pi}\right) dy \quad \boxed{\text{Integration by parts}}$$
$$= \left(-\frac{y}{\pi^2}\sin\pi y - \frac{1}{\pi^3}\cos\pi y + \frac{y^2}{2\pi}\right)\Big|_1^2$$
$$= \left(-\frac{1}{\pi^3} + \frac{2}{\pi}\right) - \left(\frac{1}{\pi^3} + \frac{1}{2\pi}\right) = \frac{3\pi^2-4}{2\pi^3}$$



9. $$\int\int_R \sqrt{x^2+1}dA = \int_0^{\sqrt{3}}\int_{-x}^{x} \sqrt{x^2+1}dydx = \int_0^{\sqrt{3}} y\sqrt{x^2+1}\Big|_{-x}^{x} dx$$
$$= \int_0^{\sqrt{3}}(x\sqrt{x^2+1} + x\sqrt{x^2+1})dx$$
$$= \int_0^{\sqrt{3}} 2x\sqrt{x^2+1}dx = \frac{2}{3}(x^2+1)^{3/2}\Big|_0^{\sqrt{3}}$$
$$= \frac{2}{3}(4^{3/2} - 1^{3/2}) = \frac{14}{3}$$



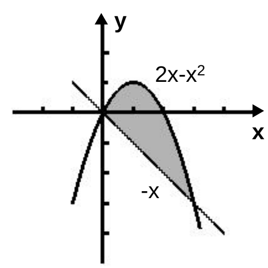
10. $$\int\int_R xdA = \int_0^{\pi/4}\int_{\tan y}^{1} xdxdy = \int_0^{\pi/4} \frac{1}{2}x^2\Big|_{\tan y}^{1} dy = \frac{1}{2}\int_0^{\pi 4}(1-\tan^2 y)dy$$
$$= \frac{1}{2}\int_0^{\pi 4}(2-\sec^2 y)dy = \frac{1}{2}(2y - \tan y)\Big|_0^{\pi/4}$$
$$= \frac{1}{2}\left(\frac{\pi}{2} - 1\right) = \frac{\pi}{4} - \frac{1}{2}$$



11. $$\int\int_R (x+y)dA = \int_0^4 \int_0^2 (x+y)dxdy + \int_0^2 \int_2^4 (x+y)dxdy$$
$$= \int_0^4 \left(\frac{1}{2}x^2 + xy\right)\Big|_0^2 dy + \int_0^2 \left(\frac{1}{2}x^2 + xy\right)\Big|_2^4 dy$$
$$= \int_0^4 (2+2y)dy + \int_0^2 [(8+4y)-(2+2y)]dy = (2y+y^2)\big|_0^4 + (6y+y^2)\big|_0^2$$
$$= 24 + 16 = 40$$

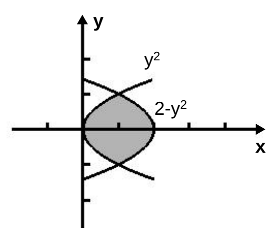
12. $$\int\int_R (x+y)dA = \int_0^4 \int_0^4 (x+y)dxdy - \int_1^3 \int_1^3 (x+y)dxdy$$
$$= \int_0^4 \left(\frac{1}{2}x^2 + xy\right)\Big|_0^4 dy - \int_1^3 \left(\frac{1}{2}x^2 + xy\right)\Big|_1^3 dy$$
$$= \int_0^4 (8+4y)dy - \int_1^3 \left[\left(\frac{9}{2}+3y\right) - \left(\frac{1}{2}+y\right)\right] dy$$
$$= (8y+2y^2)\big|_0^4 - (4y+y^2)\big|_1^3 = 64 - (21-5) = 48$$

13. $$A = \int_0^3 \int_{-x}^{2x-x^2} dydx = \int_0^3 (2x - x^2 + x)dx$$
$$= \left(\frac{3}{2}x^2 - \frac{1}{3}x^3\right)\Big|_0^3 = \frac{9}{2}$$

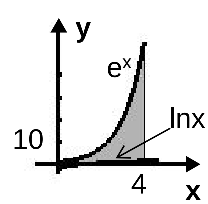


14. Using symmetry,

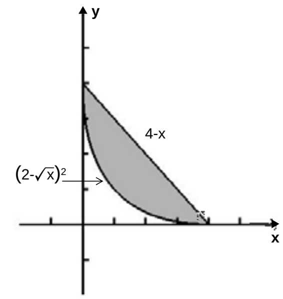
$$A = 2\int_0^1 \int_{y^2}^{2-y^2} dxdy = 2\int_0^1 (2-y^2-y^2)dy = 2\left(2y - \frac{2}{3}y^3\right)\Big|_0^1 = \frac{8}{3}.$$



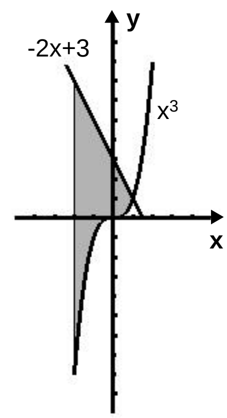
15. $$A = \int_1^4 \int_{\ln x}^{e^x} dydx = \int_1^4 (e^x - \ln x)dx = (e^x - x\ln x + x)\big|_1^4$$
$$= (e^4 - 4\ln 4 + 4) - (e+1) = e^4 - e - 4\ln 4 + 3$$



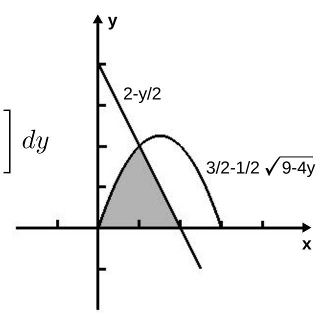
16. $A = \int_0^4 \int_{(2-\sqrt{x})^2}^{4-x} dydx = \int_0^4 [4 - x - (2 - \sqrt{x})^2]dx$
$= \int_0^4 (4\sqrt{x} - 2x)dx = \left(\frac{8}{3}x^{3/2} - x^2\right)\Big|_0^4 = \frac{16}{3}$



17. $A = \int_{-2}^1 \int_{x^3}^{-2x+3} dydx = \int_{-2}^1 (-2x + 3 - x^3)dx$
$= \left(-x^2 + 3x - \frac{1}{4}x^4\right)\Big|_{-2}^1 = \frac{7}{4} - (-14) = \frac{63}{4}$



18. Expressing $y = -x^2 + 3x$ and $y = -2x + 4$ as functions of y, we have $x = \frac{3}{2} - \frac{1}{2}\sqrt{9 - 4y}$ and $x = 2 - \frac{1}{2}y$.
$A = \int_0^2 \int_{3/2-\sqrt{9-4y}/2}^{2-y/2} dxdy = \int_0^2 \left[\left(2 - \frac{y}{2}\right) - \left(\frac{3}{2} - \frac{1}{2}\sqrt{9 - 4y}\right)\right] dy$
$= \left[\frac{1}{2}y - \frac{1}{4}y^2 - \frac{1}{12}(9 - 4y)^{3/2}\right]\Big|_0^2 = -\frac{1}{12} - \left(-\frac{27}{12}\right) = \frac{13}{6}$



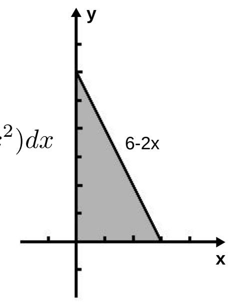
19. The correct integral is (**c**).
$V = 2\int_{-2}^2 \int_0^{\sqrt{4-y^2}} (4 - y)dxdy = 2\int_{-2}^2 (4 - y)x\Big|_0^{\sqrt{4-y^2}} dy = 2\int_{-2}^2 (4 - y)\sqrt{4 - y^2}dy$
$= 2\left[2y\sqrt{4 - y^2} + 8\sin^{-1}\frac{y}{2} + \frac{1}{3}(4 - y^2)^{3/2}\right]\Big|_{-2}^2 = 2(4\pi - (-4\pi)] = 16\pi$

20. The correct integral is **(b).**

$$V = 8\int_0^r \int_0^{\sqrt{r^2-y^2}} (r^2-y^2)^{1/2}dxdy = 8\int_0^r (r^2-y^2)^{1/2}x\Big|_0^{\sqrt{r^2-y^2}} dy$$
$$= 8\int_0^r (r^2-y^2)dy = 8\left[r^y - \frac{y^3}{3}\right]\Big|_0^r$$
$$= 8\left(r^3 - \frac{r^3}{3}\right) = 8\left(\frac{2r^3}{3}\right) = \frac{16}{3}r^3$$

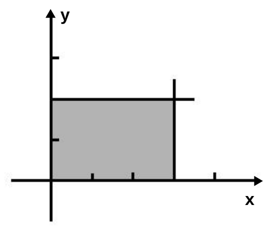
21. Setting $z = 0$ we have $y = 6 - 2x$.

$$V = \int_0^3 \int_0^{6-2x} (6-2x-y)dydx = \int_0^3 \left(6y - 2xy - \frac{1}{2}y^2\right)\Big|_0^{6-2x} dx$$
$$= \int_0^3 \left[6(6-2x) - 2x(6-2x) - \frac{1}{2}(6-2x)^2\right] dx = \int_0^3 (18 - 12x + 2x^2)dx$$
$$= \left(18x - 6x^2 + \frac{2}{3}x^3\right)\Big|_0^3 = 18$$



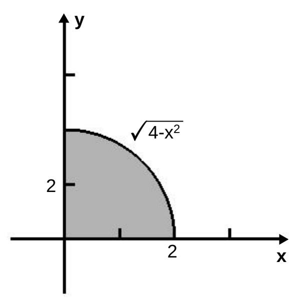
22. Setting $z = 0$ we have $y \pm 2$.

$V = \int_0^3 \int_0^2 (4 - y^2)dydx = \int_0^3 \left(4y - \frac{1}{3}y^3\right)\Big|_0^2 dx = \int_0^3 \frac{16}{3}dx = 16$



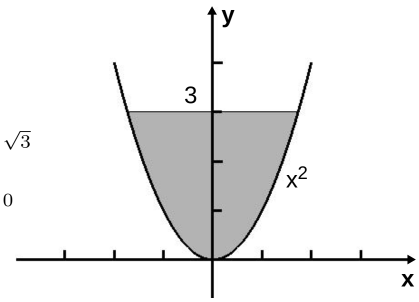
23. Solving for z, we have $x = 2 - \frac{1}{2}x + \frac{1}{2}y$. Setting $z = 0$, we see that this surface (plane) intersects the xy-plane in the line $y = x - 4$. Since $z(0,0) = 2 > 0$, the surface lies above the xy-plane over the quarter-circular region.

$$V = \int_0^2 \int_0^{\sqrt{4-x^2}} \left(2 - \frac{1}{2}x + \frac{1}{2}y\right) dydx$$
$$= \int_0^2 \left(2y - \frac{1}{2}xy + \frac{1}{4}y^2\right)\Big|_0^{\sqrt{4-x^2}} dx$$
$$= \int_0^2 \left(2\sqrt{4-x^2} - \frac{1}{2}x\sqrt{4-x^2} + 1 - \frac{1}{4}x^2\right) dx$$
$$= \left[x\sqrt{4-x^2} + 4\sin^{-1}\frac{x}{2} + \frac{1}{6}(4-x^2)^{3/2} + x - \frac{1}{12}x^3\right]\Big|_0^2$$
$$= \left(2\pi + 2 - \frac{2}{3}\right) - \frac{4}{3} = 2\pi$$



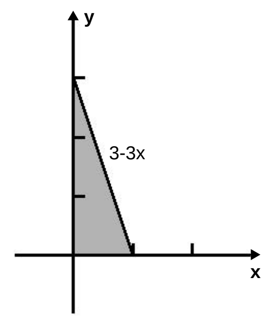
24. Setting $z = 0$ we have $y = 3$. Using symmetry,

$$V = 2\int_0^{\sqrt{3}}\int_{x^2}^{3}(3-y)dydx = 2\int_0^{\sqrt{3}}(3y - \frac{1}{2}y^2)\bigg|_{x^2}^{3} dx$$
$$= 2\int_0^{\sqrt{3}}(\frac{9}{2} - 3x^2 + \frac{1}{2}x^4)dx = 2\left(\frac{9}{2}x - x^3 + \frac{1}{10}x^5\right)\bigg|_0^{\sqrt{3}}$$
$$= 2\left(\frac{9}{2}\sqrt{3} - 3\sqrt{3} + \frac{9}{10}\sqrt{3}\right) = \frac{24\sqrt{3}}{5}.$$



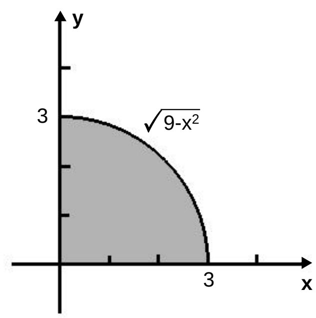
25. Note that $z = 1 + x^2 + y^2$ is always positive. Then

$$V = \int_0^1\int_0^{3-3x}(1 + x^2 + y^2)dydx = \int_0^1\left(y + x^2y + \frac{1}{3}y^3\right)\bigg|_0^{3-3x} dx$$
$$= \int_0^1 [(3-3x) + x^2(3-3x) + 9(1-x)^3]dx$$
$$= \int_0^1 (12 - 30x + 30x^2 - 12x^3)dx$$
$$= (12x - 15x^2 + 10x^3 - 3x^4)\big|_0^1 = 4.$$



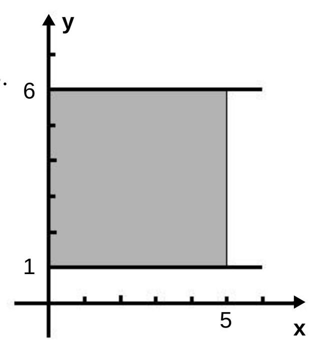
26. In the first octant, $z = x + y$ is nonnegative. Then

$$V = \int_0^3\int_0^{\sqrt{9-x^2}}(x+y)dydx = \int_0^3\left(xy + \frac{1}{2}y^2\right)\bigg|_0^{\sqrt{9-x^2}} dx$$
$$= \int_0^3\left(x\sqrt{9-x^2} + \frac{9}{2} - \frac{1}{2}x^2\right)dx$$
$$= \left[-\frac{1}{3}(9-x^2)^{3/2} + \frac{9}{2}x - \frac{1}{6}x^3\right]\bigg|_0^3$$
$$= \left(\frac{27}{2} - \frac{9}{2}\right) - (-9) = 18.$$

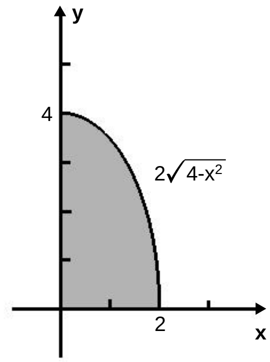


27. In the first octant $z = 6/y$ is positive. Then

$$V = \int_1^6\int_0^5 \frac{6}{y}\, dxdy = \int_1^6 \frac{6x}{y}\bigg|_0^5 dy = 30\int_1^6 \frac{dy}{y} = 30\ln y\big|_1^6 = 30\ln 6.$$

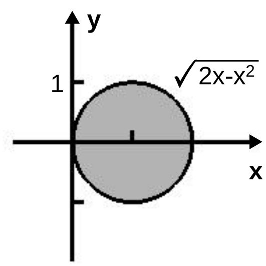


28. Setting $z = 0$, we have $x^2/4 + y^2/16 = 1$. Using symmetry,



$$
\begin{aligned}
V &= 4\int_0^2 \int_0^{2\sqrt{4-x^2}} (4 - x^2 - \frac{1}{4}y^2)dydx \\
&= 4\int_0^2 (4y - x^2y - \frac{1}{12}y^3)\Big|_0^{2\sqrt{4-x^2}} dx \\
&= 4\int_0^2 [8\sqrt{4-x^2} - 2x^2\sqrt{4-x^2} - \frac{2}{3}(4-x^2)^{3/2}]dx \\
&\boxed{\text{Trig substitution}} \\
&= 4\left[4x\sqrt{4-x^2} + 16\sin^{-1}\frac{x}{2} - \frac{1}{4}x(2x^2-4)\sqrt{4-x^2} - 4\sin^{-1}\frac{x}{2}\right. \\
&\quad \left. + \frac{1}{12}x(2x^2-20)\sqrt{4-x^2} \ -4\sin\frac{x}{2}\right]\Big|_0^2 \\
&= 4\left(\frac{16\pi}{2} - \frac{4\pi}{2} - \frac{4\pi}{2}\right) - (0) = 16\pi.
\end{aligned}
$$

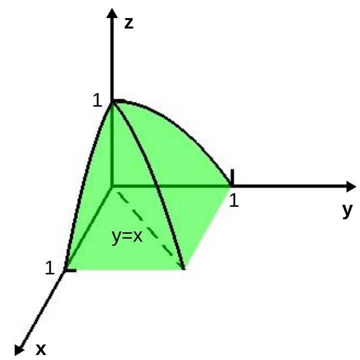
29. Note that $z = 4 - y^2$ is positive for $|y| \le 1$. Using symmetry,



$$
\begin{aligned}
V &= 2\int_0^2 \int_0^{\sqrt{2x-x^2}} (4-y^2)dydx = 2\int_0^2 (4y - \frac{1}{3}y^3)\Big|_0^{\sqrt{2x-x^2}} dx \\
&= 2\int 0^2 \left[4\sqrt{2x-x^2} - \frac{1}{3}(2x-x^2)\sqrt{2x-x^2}\right] dx \\
&= 2\int_0^2 (4\sqrt{1-(x-1)^2} - \frac{1}{3}[1-(x-1)^2]\sqrt{1-(x-1)^2})dx \\
&\boxed{u = x-1, \ \ du = dx} \\
&= 2\int_{-1}^1 [4\sqrt{1-u^2} - \frac{1}{3}(1-u^2)\sqrt{1-u^2}]du = 2\int_{-1}^1 \left(\frac{11}{3}\sqrt{1-u^2} + \frac{1}{3}u^2\sqrt{1-u^2}\right) du \\
&\boxed{\text{Trig substitution}} \\
&= 2\left[\frac{11}{6}u\sqrt{1-u^2} + \frac{11}{6}\sin u + \frac{1}{24}x(2x^2-1)\sqrt{1-u^2} + \frac{1}{24}\sin^{-1}u\right]\Big|_{-1}^1 \\
&= 2\left[\left(\frac{11}{6}\frac{\pi}{2} + \frac{1}{24}\frac{\pi}{2}\right) - \left(-\frac{11}{6}\frac{\pi}{2} - \frac{1}{24}\frac{\pi}{2}\right)\right] = \frac{15}{4}.
\end{aligned}
$$

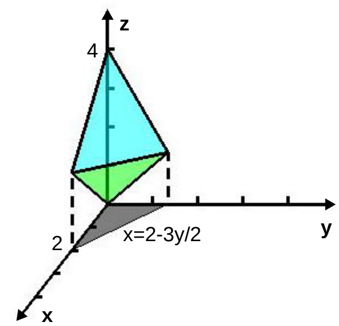
30. From $z = 1 - x^2$ and $z = 1 - y^2$ we have $1 - x^2 = 1 - y^2$ or $y = x$ (in the first octant). Thus, the surfaces intersect in the plane $y = x$. Using symmetry,

$$V = 2\int_0^1 \int_x^1 (1 - y^2)\, dydx = 2\int_0^1 \left(y - \frac{1}{3}y^3\right)\Big|_x^1 dx$$
$$= 2\int_0^1 \left(\frac{2}{3} - x + \frac{1}{3}x^3\right) dx$$
$$= 2\left(\frac{2}{3}x - \frac{1}{2}x^2 + \frac{1}{12}x^4\right)\Big|_0^1 = \frac{1}{2}.$$



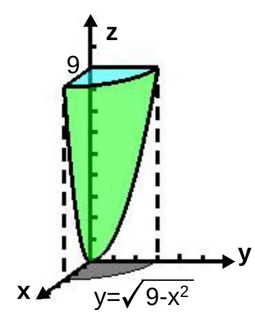
31. From $z = 4 - x - 2y$ and $z = x + y$, we have $4 - x - 2y = x + y$ or $x = 2 - \dfrac{3}{2}y$.

$$V = \int_0^{4/3}\int_0^{2-3y/2} [4 - x - 2y) - (x + y)]dxdy$$
$$= \int_0^{4/3} (4x - x^2 - 3xy)\Big|_0^{2-3y/2} dy$$
$$= \int_0^{4/3} \left[4(2 - \frac{3}{2}y) - \left(2 - \frac{3}{2}y\right)^2 - 3\left(2 - \frac{3}{2}y\right)y\right] dy$$
$$= \int_0^{4/3} \left(4 - 6y + \frac{9}{4}y^2\right) dy$$
$$= \left(4y - 3y^2 + \frac{3}{4}y^3\right)\Big|_0^{4/3} = \frac{16}{9}$$



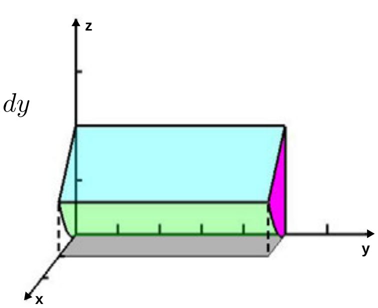
32. Using symmetry,

$$V = 4\int_0^3 \int_0^{\sqrt{9-x^2}} (9 - x^2 - y^2)dydx = 4\int_0^3 [(9 - x^2)y - \frac{1}{3}y^3]\Big|_0^{\sqrt{9-x^2}}$$
$$= \frac{8}{3}\int 0^3 (9 - x^2)^{3/2}dx \quad \boxed{\text{Trig substitution}}$$
$$= \frac{8}{3}[-\frac{x}{8}(2x^2 - 45)\sqrt{9 - x^2} + \frac{243}{8}\sin^{-1}\frac{x}{3}]\Big|_0^3 = \frac{8}{3}(\frac{243}{8}\frac{\pi}{2}) = \frac{81\pi}{2}.$$

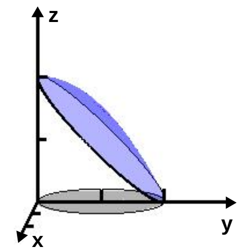


33. From $z = x^2$ and $z = -x + 2$ we have $x^2 = -x + 2$ or $x = 1$ (in the first octant). Then

$$V = \int_0^5 \int_0^1 (-x + 2 - x^2)dxdy = \int_0^5 (-\frac{1}{2}x^2 + 2x - \frac{1}{3}x^3)\Big|_0^1 dy$$
$$= \int_0^5 \frac{7}{6}dy = \frac{35}{6}.$$

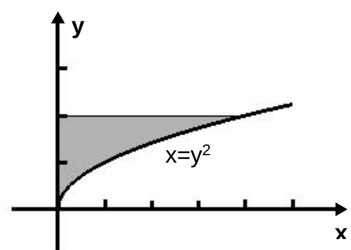


34. From $2z = 4 - x^2 - y^2$ and $z = 2 - y$ we have $4 - x^2 - y^2 = 4 - 2y$ or $x^2 + (y-1)^2 = 1$. We find the volume in the first octant and use symmetry.



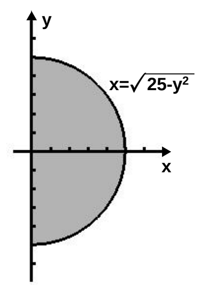
$$V = 2\int_0^2 \int_0^{\sqrt{1-(y-1)^2}} \left[\left(2 - \frac{1}{2}x^2 - \frac{1}{2}y^2\right) - (2-y)\right] dxdy$$
$$= 2\int_0^2 \left(-\frac{1}{6}x^3 - \frac{1}{2}xy^2 + xy\right)\Bigg|_0^{\sqrt{1-(y-1)^2}} dy$$
$$= 2\int_0^2 \left(-\frac{1}{6}\left[1-(y-1)^2\right]^{3/2} - \frac{1}{2}y^2\sqrt{1-(y-1)^2} + y\sqrt{1-(y-1)^2}\right) dy$$
$$= 2\int_0^2 \left(-\frac{1}{6}[1-(y-1)^2]^{3/2} + \frac{1}{2}(2y-y^2)\sqrt{1-(y-1)^2}\right) dy$$
$$= 2\int_0^2 \left(-\frac{1}{6}[1-(y-1)^2]^{3/2} + \frac{1}{2}[1-(y-1)^2]^{3/2}\right) dy$$
$$= \frac{2}{3}\int_0^2 [1-(y-1)^2]^{3/2} dy \quad \boxed{\text{Trig substitution}}$$
$$= \frac{2}{3}\left[-\frac{y-1}{8}[2(y-1)^2-5]\sqrt{1-(y-1)^2} + \frac{3}{8}\sin^{-1}(y-1)\right]\Bigg|_0^2 = \frac{2}{3}\left[\frac{3}{8}\frac{\pi}{2} - \frac{3}{8}\left(-\frac{\pi}{2}\right)\right] = \frac{\pi}{4}$$

35. Solving $x = y^2$ for y, we obtain $y = \sqrt{x}$. Thus,



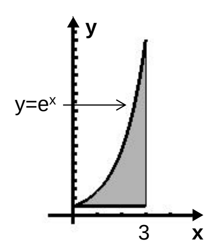
$$\int_0^2 \int_0^{y^2} f(x,y)dxdy = \int_0^4 \int_{\sqrt{x}}^2 f(x,y)dydx.$$

36. Solving $x = \sqrt{25-y^2}$ for y, we obtain $y = \pm\sqrt{25-x^2}$. Thus,



$$\int_{-5}^5 \int_0^{\sqrt{25-y^2}} f(x,y)dxdy = \int_0^5 \int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} f(x,y)dydx.$$

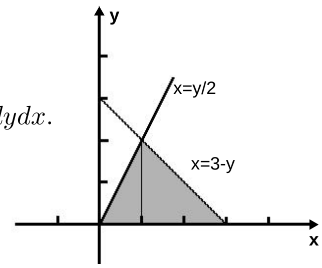
37. Solving $y = e^x$ for x, we obtain $x = \ln y$. Thus,



$$\int_0^3 \int_1^{e^x} f(x,y)dydx = \int_1^{e^3} \int_{\ln y}^3 f(x,y)dxdy.$$

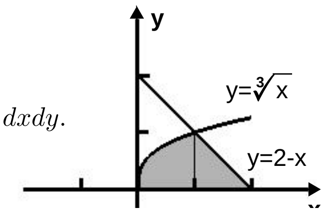
38. Solving $x = 3 - y$ and $x = y/2$ for y, we obtain $y = 3 - x$ and $y = 2x$. Thus,

$$\int_0^2 \int_{y/2}^{3-y} f(x,y)\,dx\,dy = \int_0^1 \int_0^{2x} f(x,y)\,dy\,dx + \int_1^3 \int_0^{3-x} f(x,y)\,dy\,dx.$$



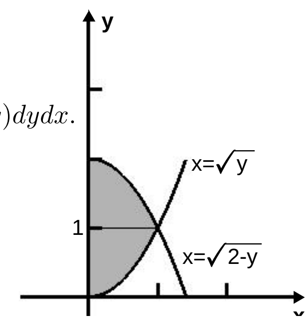
39. Solving $y = \sqrt[3]{x}$ and $y = 2 - x$ for x, we obtain $x = y^3$ and $x = 2 - y$. Thus,

$$\int_0^1 \int_0^{\sqrt[3]{x}} f(x,y)\,dy\,dx + \int_1^2 \int_0^{2-x} f(x,y)\,dy\,dx = \int_0^1 \int_{y^3}^{2-y} f(x,y)\,dx\,dy.$$

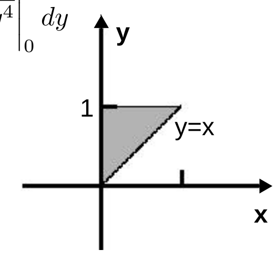


40. Solving $x = \sqrt{y}$ and $x = \sqrt{2-y}$ for y, we obtain $y = x^2$ and $y = 2 - x^2$. Thus,

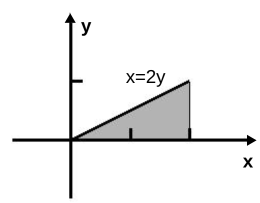
$$\int_0^1 \int_0^{\sqrt{y}} f(x,y)\,dx\,dy + \int_1^2 \int_0^{\sqrt{2-y}} f(x,y)\,dx\,dy = \int_0^1 \int_{x^2}^{2-x^2} f(x,y)\,dy\,dx.$$



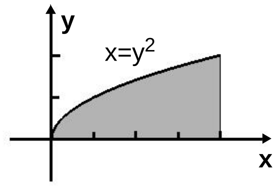
41. $$\int_0^1 \int_x^1 x^2\sqrt{1+y^4}\,dy\,dx = \int_0^1 \int_0^y x^2\sqrt{1+y^4}\,dx\,dy = \int_0^1 \frac{1}{3}x^3\sqrt{1+y^4}\Big|_0^y dy$$
$$= \frac{1}{3}\int_0^1 y^3\sqrt{1+y^4}\,dy = \frac{1}{3}\left[\frac{1}{6}(1+y^4)^{3/2}\right]\Big|_0^1$$
$$= \frac{1}{18}(2\sqrt{2} - 1)$$



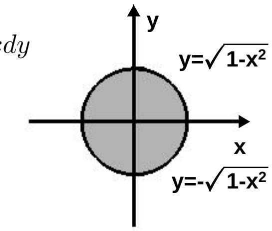
42. $$\int_0^1 \int_{2y}^2 e^{-y/x}\,dx\,dy = \int_0^2 \int_0^{x/2} e^{-y/x}\,dy\,dx = \int_0^2 -xe^{-y/x}\Big|_0^{x/2}$$
$$= \int_0^2 (-xe^{-1/2} + x)\,dx = \int_0^2 (1 - e^{-1/2})x\,dx$$
$$= \frac{1}{2}(1 - e^{-1/2})x^2\Big|_0^2 = 2(1 - e^{-1/2})$$



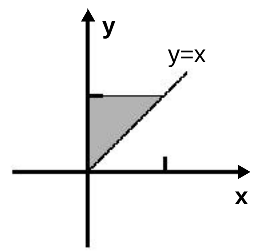
43. $$\int_0^2 \int_{y^2}^4 \cos x^{3/2} dx dy = \int_0^4 \int_0^{\sqrt{x}} \cos x^{3/2} dy dx = \int_0^4 y \cos x^{3/2} \Big|_0^{\sqrt{x}} dx$$
$$= \int_0^4 \sqrt{x} \cos x^{3/2} dx = \frac{2}{3} \sin x^{3/2} \Big|_0^4 = \frac{2}{3} \sin 8$$



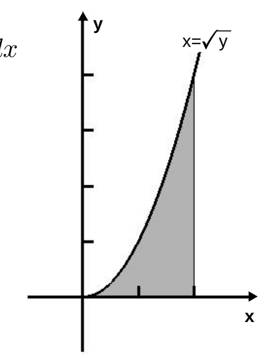
44. $$\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x\sqrt{1-x^2-y^2} dy dx = \int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x\sqrt{1-x^2-y^2} dx dy$$
$$= \int_{-1}^1 [-\frac{1}{3}(1-x^2-y^2)^{3/2}] \Big|_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dy$$
$$= -\frac{1}{3} \int_{-1}^1 (0-0) dy = 0$$



45. $$\int_0^1 \int_x^1 \frac{1}{1+y^4} dy dx = \int_0^1 \int_0^y \frac{1}{1+y^4} dx dy = \int_0^1 \frac{x}{1+y^4} \Big|_0^y dy$$
$$= \int_0^1 \frac{y}{1+y^4} dy = \frac{1}{2} \tan^{-1} y^2 \Big|_0^1 = \frac{\pi}{8}$$



46. $$\int_0^4 \int_{\sqrt{y}}^2 \sqrt{x^3+1} dx dy = \int_0^2 \int_0^{x^2} \sqrt{x^3+1} dy dx = \int_0^2 y\sqrt{x^3+1} \Big|_0^{x^2} dx$$
$$= \int_0^2 x^2\sqrt{x^3+1} dx = \frac{2}{9}(x^3+1)^{3/2} \Big|_0^2$$
$$= \frac{2}{9}(9^{3/2} - 1^{3/2}) = \frac{52}{9}$$



47. $$f_{ave} = \frac{1}{A} \int_c^d \int_a^b xy dx dy = \frac{1}{A} \int_c^d \frac{x^2 y}{2} \Big|_a^b dy$$
$$= \frac{1}{A} \int_c^d \frac{(b^2-a^2)y}{2} dy = \frac{1}{A} \left[\frac{(b^2-a^2)y^2}{4} \right] \Big|_c^d$$
$$= \frac{1}{A} \frac{(b^2-a^2)(d^2-c^2)}{4}$$

But $A = (b-a)(d-c)$, so

$$f_{ave} = \frac{(b^2-a^2)(d^2-c^2)}{4(b-a)(d-c)} = \frac{(b+a)(d+c)}{4}$$

48. $$f_{ave} = \frac{1}{A}\int_{-\sqrt{3}}^{\sqrt{3}}\int_{-\sqrt{9-3y^2}}^{\sqrt{9-3y^2}} 9 - x^2 - 3y^2 dxdy$$
$$= \frac{1}{A}\int_{-\sqrt{3}}^{\sqrt{3}} 9x - -\frac{x^3}{3} - 3y^2x\Big|_{-\sqrt{9-3y^2}}^{\sqrt{9-3y^2}} dy$$
$$= \frac{1}{A}\int_{-\sqrt{3}}^{\sqrt{3}} \left(9\sqrt{9-3y^2} - \frac{(9-3y^2)^{3/2}}{3} - 3y^2\sqrt{9-3y^2}\right)$$
$$- \left(-9\sqrt{9-3y^2} + \frac{(9-3y^2)^{3/2}}{3} + 3y^2\sqrt{9-3y^2}\right) dy$$
$$= \frac{1}{A}\int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{9-3y^2}\left(9 - \frac{9-3y^2}{3} - 3y^2 + 9 - \frac{9-3y^2}{3} - 3y^2\right) dy$$
$$= \frac{1}{A}\int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{9-3y^2}(12-4y^2)dy$$
$$= \frac{1}{A}\left[\frac{12}{\sqrt{3}}\left(\frac{\sqrt{3}y}{2}\sqrt{9-3y^2} + \frac{9}{2}\sin^{-1}\frac{\sqrt{3}y}{3}\right) - \frac{4}{3\sqrt{3}}\left(\frac{\sqrt{3}y}{3}(6y^2-9)\sqrt{9-3y^2} + \frac{81}{8}\sin^{-1}\frac{\sqrt{3}y}{9}\right)\right]\Bigg|_{-\sqrt{3}}^{\sqrt{3}}$$
$$= \frac{27\pi\sqrt{3}}{2A}$$

49. Let S be the solid with base R and height described by the function $f(x, y)$. The volume of S is equal to the volume of the solid with base R and constant height f_{ave}.

50. (a) $$\int\int_R \cos 2\pi(x+y)dA = \int_c^d\int_a^b \cos 2\pi(x+y)dxdy = \frac{1}{2\pi}\int_c^d [\sin 2\pi(b+y) - \sin 2\pi(a+b)]dy$$
$$= \frac{1}{2\pi}\int_c^d [(\sin 2\pi b\cos 2\pi y + \cos 2\pi b\sin 2\pi y) - (\sin 2\pi a\cos 2\pi y + \cos 2\pi a\sin 2\pi y)]\, dy$$
$$= \frac{1}{2\pi}\int_c^d (S_1\cos 2\pi y + C_1\sin 2\pi y)dy$$
$$= \frac{1}{4\pi^2}[S_1(\sin 2\pi d - \sin 2\pi c) - C_1(\cos 2\pi d - \cos 2\pi c)] = \frac{1}{4\pi^2}(S_1S_2 - C_1C_2)$$
$$\int\int_R \sin 2\pi(x+y)dA = \int_c^d\int_a^b \sin 2\pi(x+y)dxdy - \frac{1}{2\pi}\int_c^d [\cos 2\pi(b+y) - \cos 2\pi(a+y)]dy$$
$$= -\frac{1}{2\pi}\int_c^d [(\cos 2\pi b\cos 2\pi y - \sin 2\pi b\sin 2\pi y) - (\cos 2\pi a\cos 2\pi y - \sin 2\pi a\sin 2\pi y)]dy$$
$$= -\frac{1}{2\pi}\int_c^d (C_1\cos 2\pi y - S_1\sin 2\pi y)dy$$
$$= -\frac{1}{4\pi^2}[C_1(\sin 2\pi d - \sin 2\pi c) + S_1(\cos 2\pi d - \cos 2\pi c)] = -\frac{1}{4\pi^2}(C_1S_2 + S_1C_2)$$

(b) If $b - a = n$ is an integer, then $b = a + n$ and
$$\sin 2\pi b = \sin 2\pi(a+n) = \sin 2\pi a\cos 2\pi n + \cos 2\pi a\sin 2\pi n = \sin 2\pi a$$
$$\cos 2\pi b = \cos 2\pi(a+n) = \cos 2\pi a\cos 2\pi n - \sin 2\pi a\sin 2\pi n = \cos 2\pi a.$$

In this case, $S_1 = 0$ and $C_1 = 0$, so $\int\int_R \cos 2\pi(x+y)dA = 0$ and $\int \sin_R \sin 2\pi(x+y)dA = 0$. Similarly, is $d-c$ is an integer, the double integrals are zero.

(c) If both integrals are 0, then

$$\begin{aligned} 0 &= (S_1S_2 - C_1C_2)^2 + (C_1S_2 + S_1C_2)^2 = S_1^2S_2^2 + C_1^2C_2^2 + C_1^2S_2^2 + S_1^2C_2^2 \\ &= (S_1^2 + C_1^2)(S_2^2 + C_2^2). \end{aligned}$$

Thus, either $S_1^2 + C_1^2 = 0$, in which case $S_1 = C_1 = 0$, or $S_2^2 + C_2^2 = 0$, in which case $S_2 = C_2 = 0$. Suppose $S_1 = C_1 = 0$, and $b - a = k$ or $b = a + k$. We want to show that k is an integer. Consider

$$\begin{aligned} S_1 &= \sin 2\pi b - \sin 2\pi a = \sin 2\pi(a+k) - \sin 2\pi a \\ &= \sin 2\pi a \cos 2\pi k + \cos 2\pi a \sin 2\pi k - \sin 2\pi a \\ C_1 &= \cos 2\pi b - \cos 2\pi a = \cos 2\pi(a+k) - \cos 2\pi a \\ &= \cos 2\pi a \cos 2\pi k - \sin 2\pi a \sin 2\pi k - \cos 2\pi a \end{aligned}$$

$$\begin{aligned} S_1 - C_1 &= (\sin 2\pi a - \cos 2\pi a)\cos 2\pi k + (\sin 2\pi a + \cos 2\pi a)\sin 2\pi k - (\sin 2\pi a - \cos 2\pi a) \\ &= (\sin 2\pi a - \cos 2\pi a)(\cos 2\pi k - 1) + (\sin 2\pi a + \cos 2\pi a)\sin 2\pi k. \end{aligned}$$

Since a is arbitrary we must have $\cos 2\pi k - 1 = 0$ and $\sin 2\pi k = 0$, which implies k is an integer. Similarly, if $S_2 = C_2 = 0$, $d - c$ must be an integer.

51. By Problem 50 **(a)** we have $\int\int_{R_k} \cos 2\pi(x+y)dA = \int\int_{R_k} \sin 2\pi(x+y)dA = 0$ for $k = 1,\ 2,\ \cdots,\ n$. Then

$$\int\int_R \cos 2\pi(x+y)dA = \int\int_{R_1} \cos 2\pi(x+y)dA + \cdots + \int\int_{R_n} \cos 2\pi(x+y)dA = 0 + \cdots + 0 = 0$$

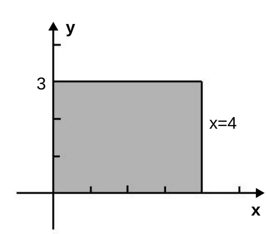
and

$$\int\int_R \sin 2\pi(x+y)dA = \int\int_{R_1} \sin 2\pi(x+y)dA + \cdots + \int\int_{R_n} \sin 2\pi(x+y)dA = 0 + \cdots + 0 = 0.$$

Therefore by Problem 45 **(c)**, at least one of the two sides of R must have integer length.

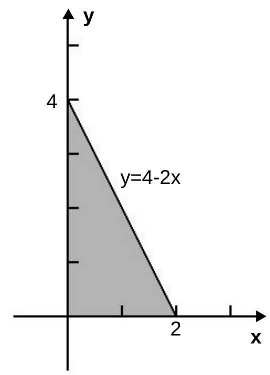
14.4 Center of Mass and Moments

1. $$\begin{aligned} m &= \int_0^3 \int_0^4 xydxdy = \int_0^3 \frac{1}{2}x^2y\Big|_0^4 dy = \int_0^3 8ydy \\ &= 4y^2\Big|_0^3 = 36 \\ M_y &= \int_0^3 \int_0^4 x^2ydxdy = \int_0^3 \frac{1}{3}x^3y\Big|_0^4 dy \\ &= \int_0^3 \frac{64}{3}ydy = \frac{32}{3}y^2\Big|_0^3 = 96 \end{aligned}$$



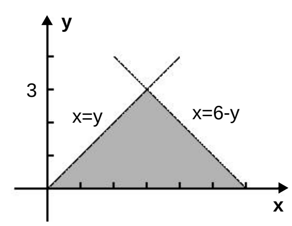
$$M_x = \int_0^3 \int_0^4 xy^2 dxdy = \int_0^3 \frac{1}{2}x^2y^2\Big|_0^4$$
$$= \int_0^3 8y^2 dy = \frac{8}{3}y^3\Big|_0^3 = 72$$
$\overline{x} = M_y/m = 96/36 = 8/3$; $\overline{y} = M_x/m = 72/36 = 2$. The center of mass is $(8/3, 2)$.

2. $$m = \int_0^2 \int_0^{4-2x} x^2 dydx = \int_0^2 x^2y\big|_0^{4-2x} dx = \int_0^2 x^2(4-2x)dx$$



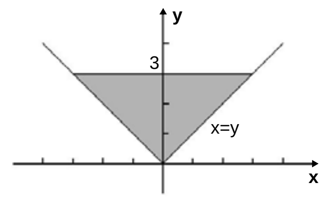
$$= \int_0^2 (4-2^2-2x^3)dx = \left(\frac{4}{3}x^3 - \frac{1}{2}x^4\right)\Big|_0^2 = \frac{32}{3} - 8 = \frac{8}{3}$$
$$M_y = \int_0^2 \int_0^{4-2x} x^3 dydx = \int_0^3 x^3y\Big|_0^{4-2x} dx = \int_0^2 x^3(4-2x)dx$$
$$= \int_0^2 (4x^3 - 2x^4)dx = \left(x^4 - \frac{2}{5}x^5\right)\Big|_0^2$$
$$= 16 - \frac{64}{5} = \frac{16}{5}$$
$$M_x = \int_0^2 \int_0^{4-2x} x^2 ydydx = \int_0^2 \frac{1}{2}x^2y^2\Big|_0^{4-2x} dx = \frac{1}{2}\int_0^2 x^2(4-2x)^2 dx$$
$$= \frac{1}{2}\int_0^2 (16x^2 - 16x^3 + 4x^4)dx = 2\int_0^2 (4x^2 - 4x^3x^4)dx = 2\left(\frac{4}{3}x^3 - x^4 + \frac{1}{5}x^5\right)\Big|_0^2$$
$$= 2\left(\frac{32}{15} - 16 + \frac{32}{15}\right) = \frac{32}{15}$$
$\overline{x} = M_y/m = 16/5 = 6/5$; $\overline{y} = M_x/m = \dfrac{32/15}{8/3} = 4/5$.
The center of mass is $(6/5, 4/5)$.

3. Since both the region and ρ are symmetric with respect to the line $x = 3$, $\overline{x} = 3$.



$$m = \int_0^3 \int_y^{6-y} 2ydxdy = \int_0^3 2xy\Big|_y^{6-y}$$
$$= \int_0^3 2y(6-y-y)dy = \int_0^3 (12y - 4y^2)dy$$
$$= \left(6y^2 - \frac{4}{3}y^3\right)\Big|_0^3 = 18$$
$$M_x = \int_0^3 \int_y^{6-y} 2y^2 dxdy = \int_0^3 2xy^2\Big|_y^{6-y} dxdy = \int_0^3 2y^2(6-y-y)dy = \int_0^3 (12y^2 - 4y^3)dy$$
$$= (4y^3 - y^4)\big|_0^3 = 27$$
$\overline{y} = M_x/m = 27/18 = 3/2$. The center of mass is $(3, 3/2)$.

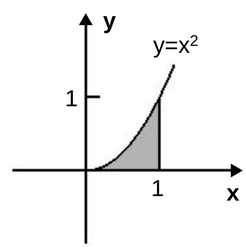
4. Since both the region and ρ are symmetric with respect to the y-axis, $\overline{x} = 0$. Using symmetry,



$$m = \int_0^3 \int_0^y (x^2 + y^2) dx dy = \int_0^3 \left(\frac{1}{3}x^3 + xy^2 \right) \Big|_0^y dy$$
$$= \int_0^3 \left(\frac{1}{3}y^3 + y^3 \right) dy = \frac{4}{3} \int_0^3 y^3 dy = \frac{1}{3}y^4 \Big|_0^3 = 27$$
$$M_x = \int_0^3 \int_0^y (x^2 y + y^3) dx dy = \int_0^3 \left(\frac{1}{3}x^3 y + xy^3 \right) \Big|_0^y dy = \int_0^3 \left(\frac{1}{3}y^4 + y^4 \right) dy = \frac{4}{3} \int_0^3 y^4 dy$$
$$= \frac{4}{15} y^5 \Big|_0^3 = \frac{324}{5}$$

$\overline{y} = M_x/m = \dfrac{324/5}{27} = 12/5$. The center of mass is $(0, 12/5)$.

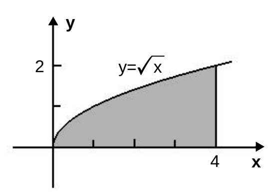
5. $$m = \int_0^1 \int_0^{x^2} (x + y) dy dx = \int_0^1 \left(xy + \frac{1}{2}y^2 \right) \Big|_0^{x^2} dx$$



$$= \int_0^1 \left(x^3 + \frac{1}{2}x^4 \right) dx = \left(\frac{1}{4}x^4 + \frac{1}{10}x^5 \right) \Big|_0^1 = \frac{7}{20}$$
$$M_y = \int_0^1 \int_0^{x^2} (x^2 + xy) dy dx = \int_0^1 \left(x^2 y + \frac{1}{2}xy^2 \right) \Big|_0^{x^2} dx$$
$$= \int_0^1 \left(x^4 + \frac{1}{2}x^5 \right) dx = \left(\frac{1}{5}x^5 + \frac{1}{12}x^6 \right) \Big|_0^1 = \frac{17}{60}$$
$$M_x = \int_0^1 \int_0^{x^2} (xy + y^2) dy dx = \int_0^1 \left(\frac{1}{2}xy^2 + \frac{1}{3}y^3 \right) \Big|_0^{x^2} = \int_0^1 \left(\frac{1}{2}x^5 + \frac{1}{3}x^6 \right) dx$$
$$= \left(\frac{1}{12}x^6 + \frac{1}{21}x^7 \right) \Big|_0^1 = \frac{11}{84}$$

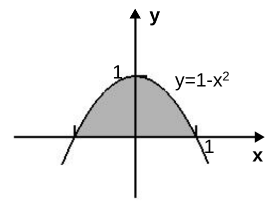
$\overline{x} = M_y = m = \dfrac{17/60}{7/20} = 17/21$; $\overline{y} + M_x/m = \dfrac{11/84}{7/20} = 55/147$.
The center of mass is $(17/21, 55/147)$.

6. $$m = \int_0^4 \int_0^{\sqrt{x}} (y + 5) dy dx = \int_0^4 (\frac{1}{2}y^2 + 5y) \Big|_0^{\sqrt{x}} dx$$

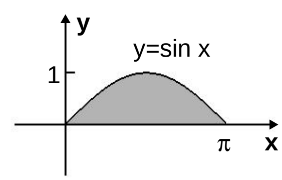


$$= \int_0^4 \left(\frac{1}{2}x + 5\sqrt{x} \right) dx = \left(\frac{1}{4}x^2 + \frac{10}{3}x^{3/2} \right) \Big|_0^4 = \frac{92}{3}$$
$$M_y = \int_0^4 \int_0^{\sqrt{x}} (xy + 5x) dy dx = \int_0^4 \left(\frac{1}{2}xy^2 + 5xy \right) \Big|_0^{\sqrt{x}} dx$$
$$= \left(\frac{1}{6}x^3 + 2x^{5/2} \right) \Big|_0^4 = \frac{224}{3}$$

$$M_x = \int_0^4 \int_0^{\sqrt{x}} (y^2+5y)dydx = \int_0^4 \left(\frac{1}{3}y^3 + \frac{5}{2}y^2\right)\Big|_0^{\sqrt{x}} dx = \int_0^4 \left(\frac{1}{3}x^{3/2} + \frac{5}{2}x\right)dx$$
$$= \left(\frac{2}{15}x^{5/2} + \frac{5}{4}x^2\right)\Big|_0^4 = \frac{364}{15}$$
$\overline{x} = M_y/m = \dfrac{224/3}{92/3} = 56/23$; $\overline{y} = M_x/m = \dfrac{364/15}{92/3} = 91/115$.
The center of mass is (56/23, 91/115).



7. The density is $\rho = ky$. Since both the region and ρ are symmetric with respect to the y-axis, $\overline{x} = 0$. Using symmetry,
$$m = 2\int_0^1 \int_0^{1-x^2} kydydx = 2k\int_0^1 \frac{1}{2}y^2\Big|_0^{1-x^2} dx = k\int 0^1 (1-x^2)^2 dx$$
$$= k\int_0^1 (1-2x^2+x^4)dx = k(x - \frac{2}{3}x^3 + \frac{1}{5}x^5)\Big|_0^1$$
$$= k\left(1 - \frac{2}{3} + \frac{1}{5}\right) = \frac{8}{15}k$$
$$M_x = 2\int_0^1 \int_0^{1-x^2} ky^2dydx = 2k\int_0^1 \frac{1}{3}y^3\Big|_0^{1-x^2} dx = \frac{2}{3}k\int_0^1 (1-x^2)^3 dx$$
$$= \frac{2}{3}k\int_0^1 (1-3x^2+3x^4-x^6)dx = \frac{2}{3}k\left(x - x^3 + \frac{3}{5}x^5 - \frac{1}{7}x^7\right)\Big|_0^1$$
$$= \frac{2}{3}k\left(1 - 1 + \frac{3}{5} - \frac{1}{7}\right) = \frac{32}{105}k$$
$\overline{y} = M_x/m = \dfrac{32k/105}{8k/15} = 4/7$. The center of mass is (0,4/7).



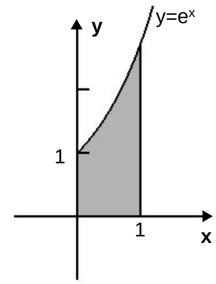
8. The density is $\rho = kx$.
$$m = \int_0^\pi \int_0^{\sin x} kxdydx = \int_0^\pi kxy\Big|_0^{\sin x} dx = \int_0^\pi kx\sin xdx$$
$\boxed{\text{Integration by parts}}$
$$= k(\sin x - x\cos x)|_0^\pi = k\pi$$
$$M_y = \int_0^\pi \int_0^{\sin x} kx^2dydx = \int_0^\pi kx^2y\Big|_0^{\sin x} dx = \int_0^\pi kx^2\sin xdx \quad \boxed{\text{Integration by parts}}$$
$$= k(-x^2\cos x + 2\cos x + 2x\sin x)|_0^\pi = k[(\pi^2-2)-2] = k(\pi^2-4)$$
$$M_x = \int_0^\pi \int_0^{\sin x} kxydydx = \int_0^\pi \frac{1}{2}kxy^2\Big|_0^{\sin x} dx$$
$$= \int_0^\pi \frac{1}{2}kx\sin^2 xdx = \int_0^\pi \frac{1}{4}kx(1-\cos 2x)dx$$
$$= \frac{1}{4}k\left[\int_0^\pi xdx - \int_0^\pi x\cos 2xdx\right] \quad \boxed{\text{Integration by parts}}$$

$= \frac{1}{4}k[\frac{1}{2}x^2\Big|_0^\pi - \frac{1}{4}(\cos 2x + 2x\sin 2x)\Big|_0^\pi] = \frac{1}{4}k(\frac{1}{2}\pi^2) = \frac{1}{8}k\pi^2$

$\overline{x} = M_y/m = \frac{k(\pi^2-4)}{k\pi} = \pi - 4/\pi$; $\overline{y} = M_x/m = \frac{k\pi^2/8}{k\pi} = \pi/8$.

The center of mass is $(\pi - 4/\pi, \pi/8)$.

9. $m = \int_0^1 \int_0^{e^x} y^3 dydx = \int_0^1 \frac{1}{4}y^4\Big|_0^{e^x} dx = \int_0^1 \frac{1}{4}e^{4x}dx$



$= \frac{1}{16}e^{4x}\Big|_0^1 = \frac{1}{16}(e^4-1)$

$M_y = \int_0^1 \int_0^{e^x} xy^3 dydx = \int_0^1 \frac{1}{4}xy^4\Big|_0^{e^x} dx$

$= \int_0^1 \frac{1}{4}xe^{4x}dx$ $\boxed{\text{Integration by parts}}$

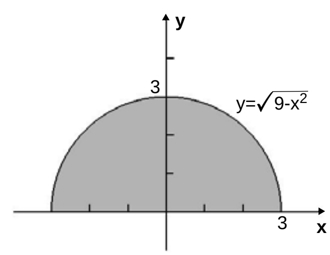
$= \frac{1}{4}\left(\frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x}\right)\Big|_0^1 = \frac{1}{4}\left(\frac{3}{16}e^4 + \frac{1}{16}\right) = \frac{1}{64}(3e^4+1)$

$M_x = \int_0^1 \int_0^{e^x} y^4 dydx = \int_0^1 \frac{1}{5}y^5\Big|_0^{e^x} dx = \int_0^1 \frac{1}{5}e^{5x}dx = \frac{1}{25}e^{5x}\Big|_0^1 = \frac{1}{25}(e^5-1)$

$\overline{x} = M_y/m = \frac{(3e^4+1)/64}{(e^4-1)/16} = \frac{3e^4+1}{4(e^4-1)}$; $\overline{y} = M_x/m = \frac{(e^5-1)/25}{(e^4-1)/16} = \frac{16(e^5-1)}{25(e^4-1)}$

The center of mass is $(\frac{3e^4+1}{4(e^4-1)}, \frac{16(e^5-1)}{25(e^4-1)} \approx (0.77, 1.76)$.

10. Since both the region and ρ are symmetric with respect to the y-axis, $\overline{x} = 0$. Using symmetry,



$m = 2\int_0^3 \int_0^{\sqrt{9-x^2}} x^2 dydx = 2\int_0^3 x^2 y\Big|_0^{\sqrt{9-x^2}} dx$

$= 2\int_0^3 x^2\sqrt{9-x^2}dx$ $\boxed{\text{Trig substitution}}$

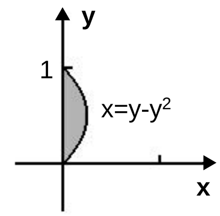
$= 2\left[\frac{x}{8}(2x^2-9)\sqrt{9-x^2} + \frac{81}{8}\sin^{-1}\frac{x}{3}\right]\Big|_0^3 = \frac{81}{4}\cdot\frac{\pi}{2} = \frac{81\pi}{8}$.

$M_x = 2\int_0^3 \int_0^{\sqrt{9-x^2}} x^2 ydydx = 2\int_0^3 \frac{1}{2}x^2y^2\Big|_0^{\sqrt{9-x^2}} dydx = \int_0^3 x^2(9-x^2)dx$

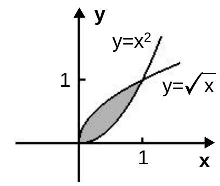
$= (3x^2 - \frac{1}{5}x^5\Big|_0^3 = \frac{162}{5}$

$\overline{y} = M_x/m = \frac{162/5}{81\pi/8} = 16/5\pi$. The center of mass is $(0,\ 16/5\pi)$.

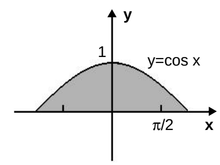
11. $I_x = \int_0^1 \int_0^{y-y^2} 2xy^2 dxdy = \int_0^1 x^2y^2\Big|_0^{y-y^2} dy = \int_0^1 (y-y^2)^2 y^2 dy$
$= \int_0^1 (y^4 - 2y^5 + y^6)dy = \left(\frac{1}{5}y^5 - \frac{1}{3}y^6 + \frac{1}{7}y^7\right)\Big|_0^1 = \frac{1}{105}$



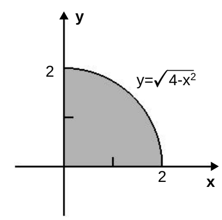
12. $I_x = \int_0^1 \int_{x^2}^{\sqrt{x}} x^2y^2 dydx = \int_0^1 \frac{1}{3}x^2y^3\Big|_{x^2}^{\sqrt{x}} dx = \frac{1}{3}\int_0^1 (x^{7/2} - x^8)dx$
$= \frac{1}{3}(\frac{2}{9}x^{9/2} - \frac{1}{9}x^9)\Big|_0^1 = \frac{1}{27}$



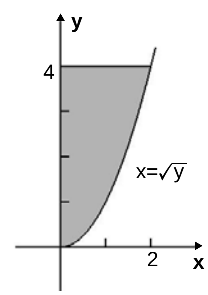
13. Using symmetry,
$I_x = 2\int_0^{\pi/2} \int_0^{\cos x} ky^2 dydx = 2k\int_0^{\pi/2} \frac{1}{3}y^3\Big|_0^{\cos x} dx = \frac{2}{3}k\int_0^{\pi/2} \cos^3 x dx$
$= \frac{2}{3}k\int_0^{\pi/2} \cos x(1 - \sin^2 x)dx = \frac{2}{3}k(\sin x - \frac{1}{3}\sin^3 x)\Big|_0^{\pi/2} = \frac{4}{9}k.$



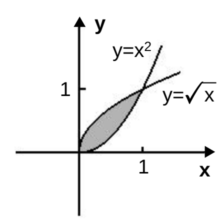
14. $I_x = \int_0^2 \int_0^{\sqrt{4-x^2}} y^3 dydx = \int_0^2 \frac{1}{4}y^4\Big|_0^{\sqrt{4-x^2}} dx = \frac{1}{4}\int_0^2 (4-x^2)^2 dx$
$= \frac{1}{4}\int_0^2 (16 - 8x^2 + x^4)dx = \frac{1}{4}\left(16x - \frac{8}{3}x^3 + \frac{1}{5}x^5\right)\Big|_0^2$
$= \frac{1}{4}\left(32 - \frac{64}{3} + \frac{32}{5}\right) = 8\left(1 - \frac{2}{3} + \frac{1}{5}\right) = \frac{64}{15}$

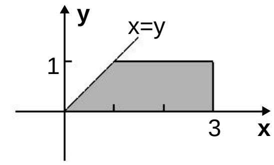


15. $I_y = \int_0^4 \int_0^{\sqrt{y}} x^2y dxdy = \int_0^4 \frac{1}{3}x^3y\Big|_0^{\sqrt{y}} dy = \frac{1}{3}\int_0^4 y^{3/2}ydy = \frac{1}{3}\int_0^4 y^{5/2}dy$
$= \frac{1}{3}\left(\frac{2}{7}y^{7/2}\right)\Big|_0^4 = \frac{2}{21}(4^{7/2}) = \frac{256}{21}$

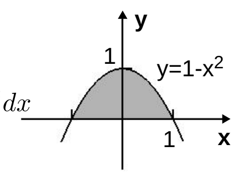


16. $I_y = \int_0^1 \int_{x^2}^{\sqrt{x}} x^4 dydx = \int_0^1 x^4y\Big|_{x^2}^{\sqrt{x}} dx = \int_0^1 (x^{9/2} - x^6)dx$
$= (\frac{2}{11}x^{11/2} - \frac{1}{7}x^7)\Big|_0^1 = \frac{3}{77}$

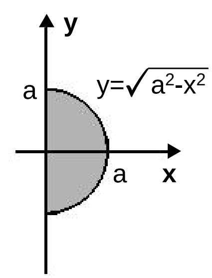




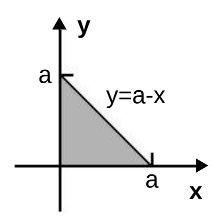
17. $I_y = \int_0^1 \int_y^3 (4x^3 + 3x^2 y)dxdy = \int_0^1 (x^4 + x^3 y)\Big|_y^3 dy$
$= \int_0^1 (81 + 27y - 2y^4)dy$
$= \left(81y + \frac{27}{2}y^2 - \frac{2}{5}y^5\right)\Big|_0^1 = \frac{941}{10}$



18. The density is $\rho = ky$. Using symmetry,
$I_y = 2\int_0^1 \int_0^{1-x^2} kx^2 ydydx = 2\int_0^1 \frac{1}{2}kx^2 y^2\Big|_0^{1-x^2} dx = k\int_0^1 x^2(1-x^2)^2 dx$
$= k\int_0^1 (x^2 - 2x^4 + x^6)dx = k(\frac{1}{3}x^3 - \frac{2}{5}x^5 + \frac{1}{7}x^7)\Big|_0^1 = \frac{8k}{105}.$



19. Using symmetry,
$m = 2\int_0^a \int_0^{\sqrt{a^2-y^2}} xdxdy = 2\int_0^a \frac{1}{2}x^2\Big|_0^{\sqrt{a^2-y^2}} dy = \int_0^a (a^2 - y^2)dy$
$= (a^2 y - \frac{1}{3}y^3)\Big|_0^a = \frac{2}{3}a^3.$
$I_y = 2\int_0^a \int_0^{\sqrt{a^2y^2}} x^3 dxdy = 2\int_0^a \frac{1}{4}x^4\Big|_0^{\sqrt{a^2-y^2}} dy = \frac{1}{2}\int_0^a (a^2 - y^2)^2 dy$
$= \frac{1}{2}\int_0^a (a^4 - 2a^2y^2 + y^4)dy = \frac{1}{2}(a^4 y - \frac{2}{3}a^2 y^3 + \frac{1}{5}y^5)\Big|_0^a = \frac{4}{15}a^5$
$R_g = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{4a^5/15}{2a^3/3}} = \sqrt{\frac{2}{5}}a$



20. $m = \int_0^a \int_0^{a-x} kdydx = \int_0^a ky\big|_0^{a-x} dx = k\int_0^a (a-x)dx =$
$k(ax - \frac{1}{2}x^2)\Big|_0^a = \frac{1}{2}ka^2$
$I_x = \int_0^a \int_0^{a-x} ky^2 dydx = \int_0^a \frac{1}{3}ky^3\Big|_0^{a-x} dx = \frac{1}{3}k\int_0^a (a-x)^3 dx$
$= \frac{1}{3}k\int_0^a (a^3 - 3a^2 x - x^3)dx = \frac{1}{3}k(a^3 x - \frac{3}{2}a^2 x^2 + ax^3 - \frac{1}{4}x^4)\Big|_0^a = \frac{1}{12}ka^4$
$R_g = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{ka^4/12}{ka^2/2}} = \sqrt{\frac{1}{6}}a$

21. (a) Using symmetry,

$$I_x = 4\int 0^a \int_0^{b\sqrt{a^2-x^2}/a} y^2 dydx = \frac{4b^3}{3a^3}\int_0^a (a^2-x^2)^{3/2}dx \quad \boxed{x = a\sin\theta, \ \ dx = a\cos\theta d\theta}$$
$$= \frac{4}{3}ab^3\int_0^{\pi/2}\cos^4\theta d\theta = \frac{4}{3}ab^3\int_0^{\pi/2}\frac{1}{4}(1+\cos 2\theta)^2 d\theta$$
$$= \frac{1}{3}ab^3\int_0^{\pi/2}(1+\cos 2\theta+\frac{1}{2}+\frac{1}{2}\cos 4\theta)d\theta = \frac{1}{3}ab^3(\frac{3}{2}\theta+\frac{1}{2}\sin 2\theta+\frac{1}{8}\sin 4\theta)\Big|_0^{\pi/2}$$
$$= \frac{ab^3\pi}{4}.$$

(b) Using symmetry,

$$I_y = 4\int_0^a\int_0^{b\sqrt{a^2-x^2}/a} x^2 dydx = \frac{4b}{a}\int_0^a x^2\sqrt{a^2-x^2}dx \quad \boxed{x = a\sin\theta, \ \ dx = a\cos\theta d\theta}$$
$$= 4a^3b\int_0^{\pi/2}\sin^2\theta\cos^2\theta d\theta = 4a^3b\int_0^{\pi/2}\frac{1}{4}(1-\cos^2 2\theta)d\theta$$
$$= a^3b\int_0^{\pi/2}(1-\frac{1}{2}-\frac{1}{2}\cos 4\theta)d\theta = a^3b(\frac{1}{2}\theta-\frac{1}{8}\sin 4\theta)\Big|_0^{\pi/2} = \frac{a^3b\pi}{4}.$$

(c) Using $m = \pi ab$, $R_g = \sqrt{I_x/m} = \dfrac{1}{2}\sqrt{ab^3\pi/\pi ab} = \dfrac{1}{2}b.$

(d) $R_g = \sqrt{I_y/m} = \dfrac{1}{2}\sqrt{a^3b\pi/\pi ab} = \dfrac{1}{2}a$

22. The equation of the ellipse is $9x^2/a^2 + 4y^2/b^2 = 1$ and the equation of the parabola is $y = \pm(9bx^2/8a^2 - b/2)$. Letting I_e and I_p represent the moments of inertia of the ellipse and parabola, respectively, about the x-axis, we have

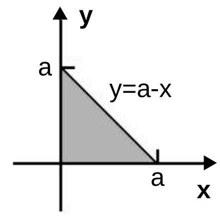
$$I_e = 2\int_{-a/3}^0\int_0^{b\sqrt{a^2-9x^2}2a} y^2 dydx = \frac{b^3}{12a^3}\int_{-a/3}^0 (a^2-9x^2)^{3/2}dx \quad \boxed{x = \frac{a}{3}\sin\theta, \ \ dx = \frac{a}{3}\cos\theta d\theta}$$
$$= \frac{b^3}{12a^3}\frac{a^4}{3}\int_{-\pi/3}^0\cos^4\theta d\theta = \frac{b^3a}{36}\frac{3\pi}{16} = \frac{ab^3\pi}{192}$$

and

$$I_p = 2\int_0^{2a/3}\int_0^{b/2-9bx^2/8a^2} y^2 dydx = \frac{2}{3}\int_0^{2a/3}\left(\frac{b}{2}-\frac{9b}{8a^2}x^2\right)^3 dx$$
$$= \frac{2}{3}\frac{b^3}{8}\int_0^{2a/3}\left(1-\frac{9}{4a^2}x^2\right)^3 dx = \frac{b^3}{12}\int_0^{2a/3}\left(1-\frac{27}{4a^2}x^2+\frac{243}{16a^4}x^4-\frac{729}{64a^6}x^6\right)dx$$
$$= \frac{b^3}{12}\left(x-\frac{9}{4a^2}x^3+\frac{243}{80a^4}x^5-\frac{729}{64a^6}x^7\right)\Bigg|_0^{2a/3} = \frac{b^3}{12}\frac{32a}{105} = \frac{8ab^3}{315}.$$

Then $I_x = I_e + I_p = \dfrac{ab^3\pi}{192} + \dfrac{8ab^3}{315}.$

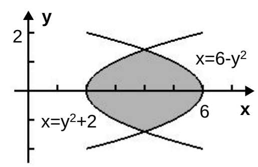
23. From Problem 20, $m = \frac{1}{2}ka^2$ and $I_x = \frac{1}{12}ka^4$.



$$I_y = \int_0^a \int_0^{a-x} kx^2 dydx = \int_0^a kx^2 y\Big|_0^{a-x} dx = k\int_0^a x^2(a-x)dx$$
$$= k\left(\frac{1}{3}ax^3 - \frac{1}{4}x^4\right)\Big|_0^4 = \frac{1}{12}ka^4$$
$$I_0 = I_x + I_y = \frac{1}{12}ka^4 + \frac{1}{12}ka^4 = \frac{1}{6}ka^4$$

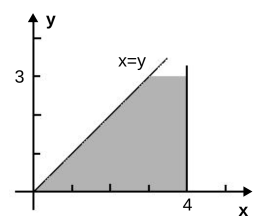
24. From Problem 12, $I_x = \frac{1}{27}$, and from Problem 16, $I_y = \frac{3}{77}$. Thus, $I_0 = I_x + I_y = \frac{1}{27} + \frac{3}{77} = \frac{158}{2079}$.

25. The density is $\rho = k/(x^2+y^2)$. Using symmetry,



$$I_0 = 2\int_0^{\sqrt{2}} \int_{y^2+2}^{6-y^2} (x^2+y^2)\frac{k}{x^2+y^2} dxdy = 2\int_0^{\sqrt{2}} kx\Big|_{y^2+2}^{6-y^2} dy$$
$$= 2k\int_0^{\sqrt{2}} (6-y^2-y^2-2)dy = 2k\left(4y - \frac{2}{3}y^3\right)\Big|_0^{\sqrt{2}}$$
$$= 2k\left(\frac{8}{3}\sqrt{2}\right) = \frac{16\sqrt{2}}{3}k.$$

26. $$I_0 = \int_0^3 \int_y^4 k(x^2+y^2)dxdy = k\int_0^3 \left(\frac{1}{3}x^3 + xy^2\right)\Big|_y^4 dy$$



$$= k\int_0^3 \left(\frac{64}{3} + 4y^2 - \frac{1}{3}y^3 - y^3\right) dy$$
$$= k\left(\frac{64}{3}y + \frac{4}{3}y^3 - \frac{1}{3}y^4\right)\Big|_0^3 = 73k$$

27. From Problem 20, $m = \frac{1}{2}ka^2$, and from Problem 21, $I_0 = \frac{1}{6}ka^4$.

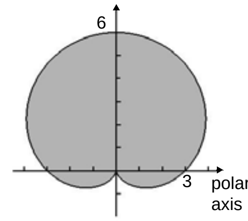
Then $R_g = \sqrt{I_0/m} = \sqrt{\dfrac{ka^4/6}{ka^2/2}} = \sqrt{\dfrac{1}{3}}a$.

28. Since the plate is homogeneous, the density is $\rho = m/lw$. Using symmetry,

$$I_0 = 4\int_0^{l/2} \int_0^{w/2} \frac{m}{lw}(x^2+y^2)dydx = \frac{4m}{lw}\int_0^{l/2} \left(x^2 y + \frac{1}{3}y^3\right)\Big|_0^{w/2} dx$$
$$= \frac{4m}{lw}\int_0^{l/2} \left(\frac{w}{2}x^2 + \frac{w^3}{24}\right) dx = \frac{4m}{lw}\left(\frac{w}{6}x^3 + \frac{w^3}{24}x\right)\Big|_0^{l/2} = \frac{4m}{lw}\left(\frac{wl^3}{48} + \frac{lw^3}{48}\right) = m\frac{l^2+w^2}{12}.$$

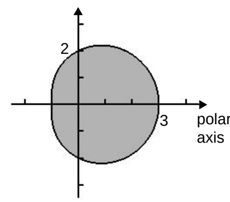
14.5 Double Integrals in Polar Coordinates

1. Using symmetry,



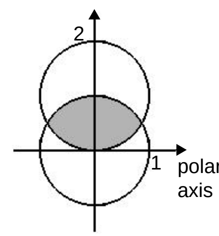
$$A = 2\int_{-\pi/2}^{\pi/2}\int_0^{3+3\sin\theta} r\,dr\,d\theta = 2\int_{-\pi/2}^{\pi/2} \frac{1}{2}r^2\Big|_0^{3+3\sin\theta} d\theta$$
$$= \int_{-\pi/2}^{\pi/2} 9(1+\sin\theta)^2 d\theta = 9\int_{-\pi/2}^{\pi/x}(1+2\sin\theta+\sin^2\theta)d\theta$$
$$= 9\left(\theta - 2\cos\theta + \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right)\Big|_{-\pi/2}^{\pi/2}$$
$$= 9\left[\frac{3}{2}\frac{\pi}{2} - \frac{3}{2}\left(-\frac{\pi}{2}\right)\right] = \frac{27\pi}{2}$$

2. Using symmetry,



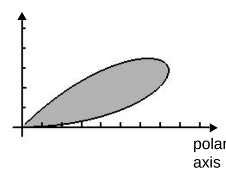
$$A = 2\int_0^{\pi}\int_0^{2+\cos\theta} r\,dr\,d\theta = 2\int_0^{\pi}\frac{1}{2}r^2\Big|_0^{2+\cos\theta} d\theta = \int_0^{\pi}(2+\cos\theta)^2 d\theta$$
$$= \int_0^{\pi}(4+4\cos\theta+\cos^2\theta)d\theta = \left(4\theta + 4\sin\theta + \frac{1}{2}\theta + \frac{1}{4}\cos 2\theta\right)\Big|_0^{\pi}$$
$$= \left(4\pi + \frac{\pi}{2} + \frac{1}{4}\right) - \left(\frac{1}{4}\right) = \frac{9\pi}{2}.$$

3. Solving $r = 2\sin\theta$ and $r = 1$, we obtain $\sin\theta = 1/2$ or $\theta = \pi/6$. Using symmetry,



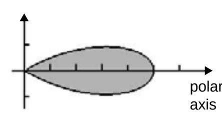
$$A = 2\int_0^{\pi/6}\int_0^{2\sin\theta} r\,dr\,d\theta + 2\int_{\pi/6}^{\pi/2}\int_0^1 r\,dr\,d\theta$$
$$= 2\int_0^{\pi/6}\frac{1}{2}r^2\Big|_0^{2\sin\theta} d\theta + 2\int_{\pi/6}^{\pi/2}\frac{1}{2}r^2\Big|_0^1 d\theta = \int_0^{\pi/6} 4\sin^2\theta\,d\theta + \int_{\pi/6}^{\pi/2} d\theta$$
$$= (2\theta - \sin 2\theta)\big|_0^{\pi/6} + \left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \frac{\pi}{3} - \frac{\sqrt{3}}{2} + \frac{\pi}{3} = \frac{4\pi - 3\sqrt{3}}{6}$$

4. $$A = \int_0^{\pi/4}\int_0^{8\sin 4\theta} r\,dr\,d\theta = \int_0^{\pi/4}\frac{1}{2}r^2\Big|_0^{8\sin 4\theta} d\theta = \frac{1}{2}\int_0^{\pi/4} 64\sin^2 4\theta\,d\theta$$



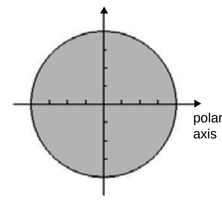
$$= 32\left(\frac{1}{2}\theta - \frac{1}{16}\sin 8\theta\right)\Big|_0^{\pi/4} = 4\pi$$

5. Using symmetry,

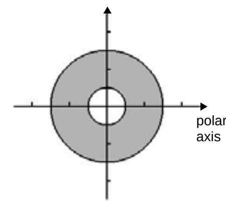


$$V = 2\int_0^{\pi/6}\int_0^{5\cos 3\theta} 4r\,dr\,d\theta = 4\int_0^{\pi/6} r^2\Big|_0^{5\cos 3\theta} d\theta = 4\int_0^{\pi/6} 25\cos^2 3\theta\,d\theta$$
$$= 100\left(\frac{1}{2}\theta + \frac{1}{12}\sin 6\theta\right)\Big|_0^{\pi/6} = \frac{25\pi}{3}$$

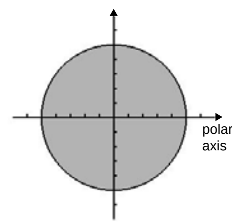
6. $V = \int_0^{2\pi}\int_0^2 \sqrt{9-r^2}\, r dr d\theta = \int_0^{2\pi} -\frac{1}{3}(9-r^2)^{3/2}\Big|_0^2 d\theta$
$= -\frac{1}{3}\int_0^{2\pi}(5^{3/2}-27)d\theta = \frac{1}{3}(27-5^{3/2})2\pi = \frac{2\pi(27-5\sqrt{5})}{3}$



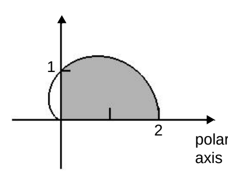
7. $V = \int_0^{2\pi}\int_1^3 \sqrt{16-r^2}\, r dr d\theta = \int_0^{2\pi} -\frac{1}{3}(16-r^2)^{3/2}\Big|_1^3 d\theta$
$= -\frac{1}{3}\int_0^{2\pi}(7^{3/2}-15^{3/2})d\theta = \frac{1}{3}(15^{3/2}-7^{3/2})2\pi = \frac{2\pi(15\sqrt{15}-7\sqrt{7})}{3}$



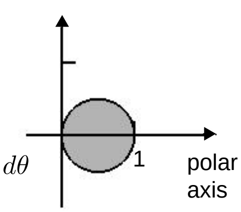
8. $V = \int_0 {}^2\pi \int_0^5 \sqrt{r^2}\, r dr d\theta = \int_0^{2\pi} \frac{1}{3}r^3\Big|_0^5 d\theta$
$= \int_0^{2\pi}\frac{125}{3}d\theta = \frac{250\pi}{3}$

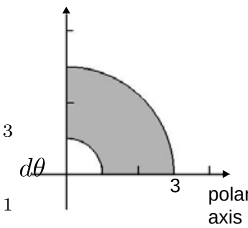


9. $V = \int_0^{\pi/2}\int_0^{1+\cos\theta}(r\sin\theta) r dr d\theta = \int_0^{\pi/2}\frac{1}{3}r^3\sin\theta\Big|_0^{1+\cos\theta} d\theta$
$= \frac{1}{3}\int_0^{\pi/2}(1+\cos\theta)^3\sin\theta d\theta = \frac{1}{3}\left[-\frac{1}{4}(1+\cos\theta)^4\right]\Big|_0^{\pi/2}$
$= -\frac{1}{12}(1-2^4) = \frac{5}{4}$



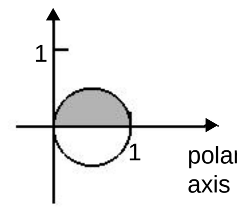
10. Using symmetry,
$V = 2\int_0^{\pi/2}\int_0^{\cos\theta}(2+r^2) r dr d\theta = \int_0^{\pi/2}\left(r^2+\frac{1}{4}r^4\right)\Big|_0^{\cos\theta} d\theta$
$= 2\int_0^{\pi/2}\left(\cos^2\theta+\frac{1}{4}\cos^4\theta\right)d\theta = 2\int_0^{\pi/2}\left[\cos^2\theta+\frac{1}{4}(\frac{1+\cos 2\theta}{2})^2\right]d\theta$
$= \int_0^{\pi/2}\left(2\cos^2\theta+\frac{1}{8}+\frac{1}{4}\cos 2\theta+\frac{1}{8}\cos^2 2\theta\right)d\theta$
$= \left(\theta+\frac{1}{2}\sin 2\theta+\frac{1}{8}\theta+\frac{1}{8}\sin 2\theta+\frac{1}{16}\theta+\frac{1}{64}\sin 4\theta\right)\Big|_0^{\pi/2} = \frac{19\pi}{32}.$





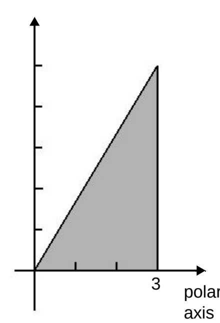
11. $m = \int_0^{\pi/2} \int_1^3 krdrd\theta = k \int_0^{\pi/2} \frac{1}{2}r^2\Big|_1^3 d\theta = \frac{1}{2}k \int_0^{\pi/2} 8d\theta = 2k\pi$

$$M_y = \int_0^{\pi/2} \int_1^3 kxrdrd\theta = k \int_0^{\pi/2} \int_1^3 r^2 \cos\theta drd\theta = k \int_0^{\pi/2} \frac{1}{3}r^3 \cos\theta\Big|_1^3 d\theta$$
$$= \frac{1}{3}k \int_0^{\pi/2} 26\cos\theta d\theta = \frac{26}{3}k\sin\theta\Big|_0^{\pi/2} = \frac{26}{3}k$$
$\overline{x} = M_y/m = \dfrac{26k/3}{2k\pi} = \dfrac{13}{3\pi}$.
Since the region and density function are symmetric about the ray $\theta = \pi/4$, $\overline{y} = \overline{x} = 13/3\pi$ and the center of mass is $(13/3\pi,\ 13/3\pi)$.



12. The interior of the upper-half circle is traced from $\theta = 0$ to $\pi/2$. The density is kr. Since both the region and the density are symmetric about the polar axis, $\overline{y} = 0$.

$$m = \int_0^{\pi/2} \int_0^{\cos\theta} kr^2drd\theta = k \int_0^{\pi/2} \frac{1}{3}r^3\Big|_0^{\cos\theta} d\theta = \frac{k}{3} \int_0^{\pi/2} \cos^3\theta d\theta$$
$$= \frac{k}{3}\left(\frac{2}{3} + \frac{1}{3}\cos^2\theta\right)\sin\theta\Big|_0^{\pi/2} = \frac{2k}{9}$$
$$M_y = k \int_0^{\pi/2} \int_0^{\cos\theta} (r\cos\theta)(r)(rdrd\theta) = k \int_0^{\pi/2} \int_0^{\cos\theta} r^3\cos\theta drd\theta = k \int_0^{\pi/2} \frac{1}{4}r^4\cos\theta\Big|_0^{\cos\theta} d\theta$$
$$= \frac{k}{4} \int_0^{\pi/2} \cos^5\theta d\theta = \frac{k}{4}\left(\sin\theta - \frac{2}{3}\sin^3\theta + \frac{1}{5}\sin^5\theta\right)\Big|_0^{\pi/2} = \frac{2k}{15}$$
Thus, $\overline{x} = \dfrac{2k/15}{2k/9} = 3/5$ and the center of mass is $(3/5, 0)$.

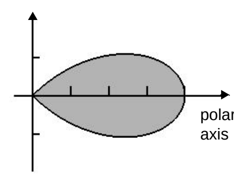


13. In polar coordinates the line $x = 3$ becomes $r\cos\theta = 3$ or $r = 3\sec\theta$. The angle of inclination of the line $y = \sqrt{3}x$ is $\pi/3$.

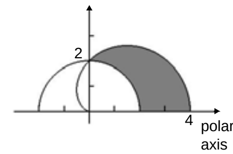
$$m = \int_0^{\pi/3} \int 0^{3\sec\theta} r^2rdrd\theta = \int_0^{\pi/3} \frac{1}{4}r^4\Big|_0^{3\sec\theta} d\theta$$
$$= \frac{81}{4} \int_0^{\pi/3} \sec^4\theta d\theta = \frac{81}{4} \int_0^{\pi/3} (1 + \tan^2\theta)\sec^2\theta d\theta$$
$$= \frac{81}{4}(\tan\theta + \frac{1}{3}\tan^3\theta)\Big|_0^{\pi/3} = \frac{81}{4}(\sqrt{3} + \sqrt{3}) = \frac{81}{2}\sqrt{3}$$

$$M_y = \int_0^{\pi/3}\int_0^{3\sec\theta} xr^2 rdrd\theta = \int_0^{\pi/3}\int_0^{3\sec\theta} r^4\cos\theta drd\theta$$
$$= \int_0^{\pi/3} \frac{1}{5}r^5\cos\theta\Big|_0^{3\sec\theta} d\theta = \frac{243}{5}\int_0^{\pi/3}\sec^5\theta\cos\theta d\theta$$
$$= \frac{243}{5}\int_0^{\pi/3}\sec^4\theta d\theta = \frac{243}{5}(2\sqrt{3}) = \frac{486}{5}\sqrt{3}$$
$$M_x = \int_0^{\pi/3}\int_0^{3\sec\theta} yr^2 rdrd\theta = \int_0^{\pi/3}\int_0^{3\sec\theta} r^4\sin\theta d\theta = \int_0^{\pi/3}\frac{1}{5}r^5\sin\theta\Big|_0^{3\sec\theta}$$
$$= \frac{243}{5}\int_0^{\pi/3}\sec^5\theta\sin\theta d\theta = \frac{243}{5}\int 0^{\pi/3}\tan\theta\sec^4\theta d\theta = \frac{243}{5}\int_0^{\pi/3}\tan\theta(1+\tan^2\theta)\sec^2\theta d\theta$$
$$= \frac{243}{5}\int_0^{\pi/3}(\tan\theta+\tan^3\theta)\sec^2\theta d\theta = \frac{243}{5}\left(\frac{1}{2}\tan^2\theta + \frac{1}{4}\tan^4\theta\right)\Big|_0^{\pi/3} = \frac{243}{5}(\frac{3}{2}+\frac{9}{4}) = \frac{729}{4}$$
$\overline{x} = M_y/m = \dfrac{486\sqrt{3}/5}{81\sqrt{3}/2} = 12/5;\ \ \overline{y} = M_x/m = \dfrac{729/4}{81\sqrt{3}/2} =$
$3\sqrt{3}/2$. The center of mass is $(12/5, 3\sqrt{3}/2)$.

14. Since both the region and the density are symmetric about the x-axis, $\overline{y} = 0$. Using symmetry,



$$m = 2\int_0^{\pi/4}\int_0^{4\cos 2\theta} krdrd\theta = 2k\int_0^{\pi/4}\frac{1}{2}r^2\Big|_0^{4\cos 2\theta} d\theta$$
$$= 16k\int_0^{\pi/4}\cos^2 2\theta d\theta = 16k(\frac{1}{2}\theta + \frac{1}{8}\sin 4\theta)\Big|_0^{\pi/4} = 2k\pi$$
$$M_y = 2\int_0^{\pi/4}\int_0^{4\cos 2\theta} kxrdrd\theta = 2k\int_0^{\pi/4}\int_0^{4\cos 2\theta} r^2\cos\theta drd\theta = 2k\int_0^{\pi/4}\frac{1}{3}r^3\cos\theta\Big|_0^{4\cos 2\theta} d\theta$$
$$= \frac{128}{3}k\int_0^{\pi/4}\cos^3 2\theta\cos\theta d\theta = \frac{128}{3}k\int_0^{\pi/4}(1-2\sin^2\theta)^3\cos\theta d\theta$$
$$= \frac{128}{3}k\int_0^{\pi/4}(1-6\sin^2\theta+12\sin^4\theta-8\sin^6\theta)\cos\theta d\theta$$
$$= \frac{128}{3}k(\sin\theta - 2\sin^3\theta + \frac{12}{5}\sin^5\theta - \frac{8}{7}\sin^7\theta)\Big|_0^{\pi/4}$$
$$= \frac{128}{3}k\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} + \frac{3\sqrt{2}}{10} - \frac{\sqrt{2}}{14}\right) = \frac{1024}{105}\sqrt{2}k$$
$\overline{x} = M_y/m = \dfrac{1024\sqrt{2}/105}{2k\pi} = \dfrac{512\sqrt{2}}{105\pi}.$
The center of mass is $(512\sqrt{2}/105\pi, 0)$ or approximately $(2.20, 0)$.



15. The density is $\rho = k/r$.

$$m = \int_0^{\pi/2}\int_2^{2+2\cos\theta} \frac{k}{r} r\,dr\,d\theta = k\int_0^{\pi/2}\int_2^{2+2\cos\theta} dr\,d\theta$$

$$= k\int_0^{\pi/2} 2\cos\theta\,d\theta = 2k(\sin\theta)\Big|_0^{\pi/2} = 2k$$

$$M_y = \int_0^{\pi/2}\int_2^{2+2\cos\theta} x\frac{k}{r} r\,dr\,d\theta = k\int_0^{\pi/2}\int_2^{2+2\cos\theta} r\cos\theta\,dr\,d\theta = k\int 0^{\pi/2}\frac{1}{2}r^2\Big|_2^{2+2\cos\theta}\cos\theta\,d\theta$$

$$= \frac{1}{2}k\int_0^{\pi/2}(8\cos\theta + 4\cos^2\theta)\cos\theta\,d\theta = 2k\int_0^{\pi/2}(2\cos^2\theta + \cos\theta - \sin^2\theta\cos\theta)d\theta$$

$$= 2k(\theta + \frac{1}{2}\sin 2\theta + \sin\theta - \frac{1}{3}\sin^3\theta)\Big|_0^{\pi/2} = 2k(\frac{\pi}{2} + \frac{2}{3}) = \frac{3\pi+4}{3}k$$

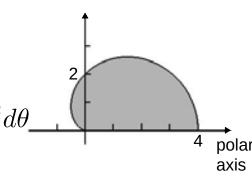
$$M_x = \int_0^{\pi/2}\int_2^{2+2\cos\theta} y\frac{k}{r} r\,dr\,d\theta = k\int_{\pi/2}\int_2^{2+2\cos\theta} r\sin\theta\,dr\,d\theta = k\int_0^{\pi/2}\frac{1}{2}r^2\Big|_2^{2+2\cos\theta}\sin\theta\,d\theta$$

$$= \frac{1}{2}k\int_0^{\pi/2}(8\cos\theta + 4\cos^2\theta)\sin\theta\,d\theta = \frac{1}{2}k\left(-4\cos^2\theta - \frac{4}{3}\cos^3\theta\right)\Big|_0^{\pi/2}$$

$$= \frac{1}{2}k\left[-\left(-4-\frac{4}{3}\right)\right] = \frac{8}{3}k$$

$$\overline{x} = M_y/m = \frac{(3\pi+4)k/3}{2k} = \frac{3\pi+4}{6};\ \ \overline{y} = M_x/m = \frac{8k/3}{2k} = \frac{4}{3}$$

The center of mass is $((3\pi+4)/6, 4/3)$.



16. $$m = \int_0^{\pi}\int_0^{2+2\cos\theta} kr\,dr\,d\theta = k\int_0^{\pi}\frac{1}{2}r^2\Big|_0^{2+2\cos\theta} d\theta = 2k\int_0^{\pi}(1+\cos\theta)^2 d\theta$$

$$= 2k\int_0^{\pi}(1 + 2\cos\theta + \cos^2\theta)d\theta$$

$$= 2k\left(\theta + 2\sin\theta + \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right)\Big|_0^{\pi} = 3\pi k$$

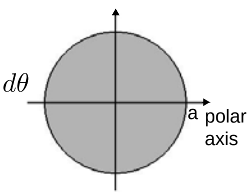
$$M_y = \int_0^{\pi}\int_0^{2+2\cos\theta} kxr\,dr\,d\theta = k\int_0^{\pi}\int_0^{2+2\cos\theta} r^2\cos\theta\,dr\,d\theta = k\int_0^{\pi}\frac{1}{3}r^3\Big|_0^{2+2\cos\theta}\cos\theta\,d\theta$$

$$= \frac{8}{3}k\int_0^{\pi}(1+\cos\theta)^3\cos\theta\,d\theta = \frac{8}{3}k\int_0^{\pi}(\cos\theta + 3\cos^2\theta + 3\cos^3\theta + \cos^4\theta)d\theta$$

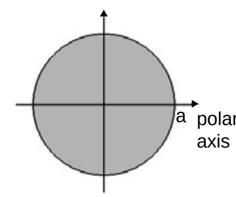
$$= \frac{8}{3}k\left[\sin\theta + \left(\frac{3}{2}\theta + \frac{3}{4}\sin 2\theta\right) + (3\sin\theta - \sin^3\theta) + \left(\frac{3}{8}\theta + \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta\right)\right]\Bigg|_0^{\pi}$$

$$= \frac{8}{3}k\left(\frac{15}{8}\pi\right) = 5\pi k$$

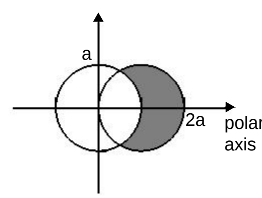
$$M_x = \int_0^{\pi}\int_0^{2+2\cos\theta} kyr\,dr\,d\theta = k\int_0^{\pi}\int_0^{2+2\cos\theta} r^2\sin\theta\,dr\,d\theta = k\int_0^{\pi}\frac{1}{3}r^3\Big|_0^{2+2\cos\theta}\sin\theta\,d\theta$$
$$= \frac{8}{3}k\int_0^{\pi}(1+\cos\theta)^3\sin\theta\,d\theta = \frac{8}{3}k\int_0^{\pi}(1+3\cos\theta+3\cos^2\theta+\cos^3\theta)\sin\theta\,d\theta$$
$$= \frac{8}{3}k\left(-\cos\theta-32\cos^2\theta-\cos^3\theta-\frac{1}{4}\cos^4\theta\right)\Big|_0^{\pi} = \frac{8}{3}k\left[\frac{1}{4}-\left(-\frac{15}{4}\right)\right] = \frac{32}{3}k$$
$\overline{x} = M_y/m = \dfrac{5\pi k}{3\pi k} = 5/3$; $\overline{y} = M_x/m = \dfrac{32k/3}{3\pi k} = 32/9\pi$. The center of mass is $(5/3, 32/9\pi)$.



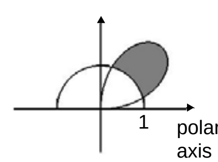
17. $I_x = \displaystyle\int_0^{2\pi}\int_0^a y^2kr\,dr\,d\theta = k\int_0^{2\pi}\int_0^a r^3\sin^2\theta\,dr\,d\theta = k\int_0^{2\pi}\frac{1}{4}r^4\sin^2\theta\Big|_0^a d\theta$
$$= \frac{ka^4}{4}\int_0^{2\pi}\sin^2\theta\,d\theta = \frac{ka^4}{4}\left(\frac{1}{2}\theta-\frac{1}{4}\sin 2\theta\right)\Big|_0^{2\pi} = \frac{k\pi a^4}{4}$$



18. $I_x = \displaystyle\int_0^{2\pi}\int_0^a y^2\frac{1}{1+r^4}r\,dr\,d\theta = \int_0^{2\pi}\int_0^a \frac{r^3}{1+r^4}\sin^2\theta\,dr\,d\theta$
$$= \int_0^{2\pi}\frac{1}{4}\ln(1+r^4)\Big|_0^a\sin^2\theta\,d\theta = \frac{1}{4}\ln(1+a^4)\left(\frac{1}{2}\theta-\frac{1}{4}\sin 2\theta\right)\Big|_0^{2\pi}$$
$$= \frac{\pi}{4}\ln(1+a^4)$$



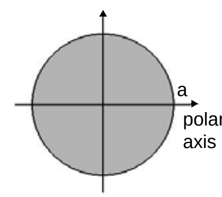
19. Solving $a = 2a\cos\theta$, $\cos\theta = 1/2$ or $\theta = \pi/3$. The density is k/r^3. Using symmetry,
$$I_y = 2\int_0^{\pi/3}\int_a^{2a\cos\theta} x^2\frac{k}{r^3}r\,dr\,d\theta = 2k\int_0^{\pi/3}\int_a^{2a\cos\theta}\cos^2\theta\,dr\,d\theta$$
$$= 2k\int_0^{\pi/3}(2a\cos^3\theta - a\cos^2\theta)\,d\theta$$
$$= 2ak\left(2\sin\theta-\frac{2}{3}\sin^3\theta-\frac{1}{2}\theta-\frac{1}{4}\sin 2\theta\right)\Big|_0^{\pi/3}$$
$$= 2ak\left(\sqrt{3}-\frac{\sqrt{3}}{4}-\frac{\pi}{6}-\frac{\sqrt{3}}{8}\right) = \frac{5ak\sqrt{3}}{4}-\frac{ak\pi}{3}$$



20. Solving $1 = 2\sin 2\theta$, we obtain $\sin 2\theta = 1/2$ or $\theta = \pi/12$ and $\theta = 5\pi/12$.

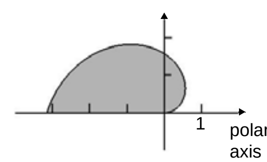
$$
\begin{aligned}
I_y &= \int_{\pi/12}^{5\pi/12}\int_1^{2\sin 2\theta} x^2\sec^2\theta r dr d\theta = \int_{\pi/12}^{5\pi/12}\int_1^{2\sin 2\theta} r^3 dr d\theta \\
&= \int \pi/12^{5\pi/12}\frac{1}{4}r^4\Big|_1^{2\sin 2\theta} d\theta = 4\int_{\pi/12}^{5\pi 12}\sin^4 2\theta d\theta \\
&= 2\left(\frac{3}{4}\theta - \frac{1}{4}\sin 4\theta + \frac{1}{32}\sin 8\theta\right)\Big|_{\pi/12}^{5\pi/12} \\
&= 2\left[\left(\frac{5\pi}{16} + \frac{\sqrt{3}}{8} - \frac{\sqrt{3}}{64}\right) - \left(\frac{\pi}{16} - \frac{\sqrt{3}}{8} + \frac{\sqrt{3}}{64}\right)\right] = \frac{8\pi + 7\sqrt{3}}{16}
\end{aligned}
$$

21. From Problem 17, $I_x = k\pi a^4/4$. By symmetry, $I_y = I_x$. Thus $I_0 = k\pi a^4/2$.



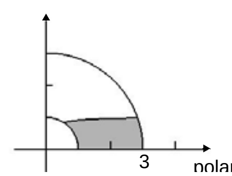
22. The density is $\rho = kr$. $I_0 = \int_0^{\pi}\int_0^{\theta} r^2(kr) r dr d\theta = k\int_0^{\pi}\frac{1}{5}r^5\Big|_0^{\theta} d\theta$

$$= \frac{1}{5}k\int_0^{\pi}\theta^5 d\theta = \frac{1}{5}k\left(\frac{1}{6}\theta^6\right)\Big|_0^{\pi} = \frac{k\pi^6}{30}$$



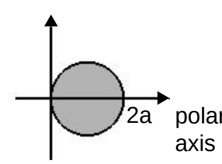
23. The density is $\rho = k/r$. $I_0 = \int_1^3\int_0^{1/r} r^2\frac{k}{r} r d\theta dr = k\int_1^3\int_0^{1/r} f^2 d\theta dr$

$$= k\int_1^3 r^2\left(\frac{1}{r}\right) dr = k\left(\frac{1}{2}r^2\right)\Big|_1^3 = 4k$$



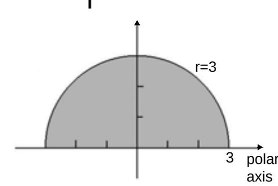
24. $I_0 = \int_0^{\pi}\int_0^{2a\cos\theta} r^2 kr dr d\theta = k\int_0^{\pi}\frac{1}{4}r^4\Big|_0^{2a\cos\theta} d\theta = 4ka^4\int_0^{\pi}\cos^4\theta d\theta$

$$= 4ka^4\left(\frac{3}{8}\theta + \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta\right)\Big|_0^{\pi} = 4ka^4\left(\frac{3\pi}{8}\right) = \frac{3k\pi a^4}{2}$$

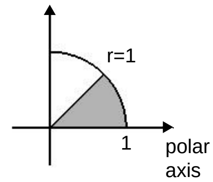


25. $\int_{-3}^{3}\int_0^{\sqrt{9-x^2}}\sqrt{x^2+y^2} dy dx = \int_0^{\pi}\int 0^3|r| r dr d\theta$

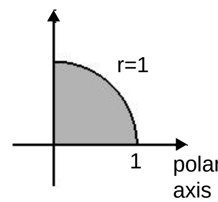
$$= \int_0^{\pi}\frac{1}{3}r^3\Big|_0^3 d\theta = 9\int_0^{\pi} d\theta = 9\pi$$



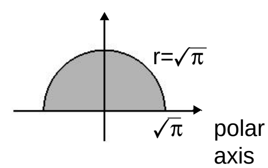
26. $$\int_0^{\sqrt{2}/2}\int_y^{\sqrt{1-y^2}} \frac{y^2}{\sqrt{x^2+y^2}}dxdy = \int_0^{\pi/4}\int_0^1 \frac{r^2\sin^2\theta}{|r|}rdrd\theta$$
$$= \int_0^{\pi/4}\int_0^1 r^2\sin^2\theta drd\theta$$
$$= \int_0^{\pi/4} \frac{1}{3}r^3\sin^2\theta\bigg|_0^1 d\theta = \frac{1}{3}\int_0^{\pi/4}\sin^2\theta d\theta$$
$$= \frac{1}{3}(\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta)\bigg|_0^{\pi/4} = \frac{\pi-2}{24}$$



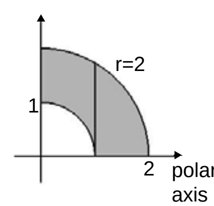
27. $$\int_0^1\int_0^{\sqrt{1-y^2}} e^{x^2+y^2}dxdy = \int_0^{\pi/2}\int_0^1 e^{r^2}rdrd\theta = \int_0^{\pi/2}\frac{1}{2}e^{r^2}\bigg|_0^1 d\theta$$
$$= \frac{1}{2}\int_0^{\pi/2}(e-1)d\theta = \frac{\pi(e-1)}{4}$$

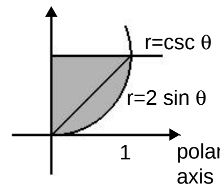


28. $$\int_{-\sqrt{x}}^{\sqrt{x}}\int_0^{\sqrt{\pi-x^2}} \sin(x^2+y^2)dydx = \int_0^{\pi}\int_0^{\sqrt{x}}(\sin r^2)rdrd\theta$$
$$= \int_0^{\pi} -\frac{1}{2}\cos r^2\bigg|_0^{\sqrt{x}} d\theta$$
$$= -\frac{1}{2}\int_0^{\pi}(-1-1)d\theta = \pi$$



29. $$\int_0^1\int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} \frac{x^2}{x^2+y^2}dydx = \int_1^2\int_0^{\sqrt{4-x^2}} \frac{x^2}{x^2+y^2}dydx$$
$$= \int_0^{\pi/2}\int_1^2 \frac{r^2\cos^2\theta}{r^2}rdrd\theta$$
$$= \int_0^{\pi/2}\int_1^2 r\cos^2\theta drd\theta$$
$$= \int_0^{\pi/2}\frac{1}{2}r^2\bigg|_1^2\cos^2\theta d\theta = \frac{3}{2}\int_0^{\pi/2}\cos^2\theta d\theta$$
$$= \frac{3}{2}\left(\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right)\bigg|_0^{\pi/2} = \frac{3\pi}{8}$$





30. $\displaystyle\int_0^1\int_0^{\sqrt{2y-y^2}}(1-x^2-y^2)dxdy$

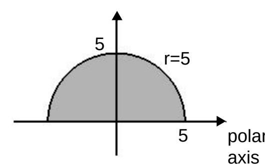
$$= \int_0^{\pi/4}\int_0^{2\sin\theta}(1-r^2)rdrd\theta + \int_{\pi/4}^{\pi/2}\int_0^{\csc\theta}(1-r^2)rdrd\theta$$

$$= \int_0^{\pi/4}\left(\frac{1}{2}r^2-\frac{1}{4}r^4\right)\Big|_0^{2\sin\theta}d\theta + \int_{\pi/4}^{\pi/2}\left(\frac{1}{2}r^2-\frac{1}{4}r^4\right)\Big|_0^{\csc\theta}d\theta$$

$$= \int_0^{\pi/4}(2\sin^2\theta-4\sin^4\theta)d\theta + \int_{\pi/4}^{\pi/2}\left(\frac{1}{2}\csc^2\theta-\frac{1}{4}\csc^4\theta\right)d\theta$$

$$= \left[\theta-\frac{1}{2}\sin 2\theta-\left(\frac{3}{2}\theta-\sin 2\theta+\frac{1}{8}\sin 4\theta\right)\right] + \left[-\frac{1}{2}\cot\theta-\frac{1}{4}(-\cot\theta-\frac{1}{3}\cot^3\theta)\right]\Big|_{\pi/4}^{\pi/2}$$

$$= \left(-\frac{\pi}{8}+\frac{1}{2}\right)+\left[0-\left(-\frac{1}{4}+\frac{1}{12}\right)\right] = \frac{16-3\pi}{24}$$

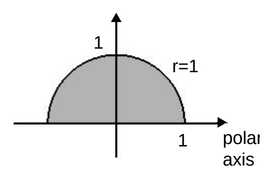


31. $\displaystyle\int_{-5}^5\int_0^{\sqrt{25-x^2}}(4x+3y)dydx = \int_0^{\pi}\int_0^5(4r\cos\theta+3r\sin\theta)rdrd\theta$

$$= \int_0^{\pi}\int_0^5(4r^2\cos\theta+3r^2\sin\theta)drd\theta$$

$$= \int_0^{\pi}\left(\frac{4}{3}r^3\cos\theta+r^3\sin\theta\right)\Big|_0^5 d\theta$$

$$= \int_0^{\pi}\left(\frac{500}{3}\cos\theta+125\sin\theta\right)d\theta = \left(\frac{500}{3}\sin\theta-125\cos\theta\right)\Big|_0^{\pi} = 250$$



32. $\displaystyle\int_0^1\int_0^{\sqrt{1-y^2}}\frac{1}{1+\sqrt{x^2+y^2}}dxdy = \int_0^{\pi/2}\int_0^1\frac{1}{1+r}rdrd\theta$

$$= \int_0^{\pi/2}\int_0^1(1-\frac{1}{1+r})drd\theta = \int_0^{\pi/2}[r-\ln(1+r)]\Big|_0^1 d\theta$$

$$= \int_0^{\pi/2}(1-\ln 2)d\theta = \frac{\pi}{2}(1-\ln 2)$$

33. $\displaystyle I^2 = \int_0^{\infty}\int_0^{\infty}e^{-(x^2+y^2)}dxdy = \int_0^{\pi/2}\int_0^{\infty}e^{-r^2}rdrd\theta = \int_0^{\pi/2}\lim_{t\to\infty}-\frac{1}{2}e^{-r^2}\Big|_0^t d\theta$

$$= \int_0^{\pi/2}\lim_{t\to\infty}\left(-\frac{1}{2}e^{-t^2}+\frac{1}{2}\right)d\theta = \int_0^{\pi/2}\frac{1}{2}d\theta = \frac{\pi}{4};\quad I = \frac{\sqrt{\pi}}{2}$$

34. $$\int\int_R (x+y)dA = \int_0^{\pi/2}\int_{2\sin\theta}^{2}(r\cos\theta + r\sin\theta)rdrd\theta = \int_0^{\pi/2}\int_{2\sin\theta}^{2} r^2(\cos\theta+\sin\theta)drd\theta$$
$$= \int_0^{\pi/2}\frac{1}{3}r^3(\cos\theta+\sin\theta)\Big|_{2\sin\theta}^{2} d\theta = \frac{8}{3}\int_0^{\pi/2}(\cos\theta+\sin\theta-\sin^3\theta\cos\theta-\sin^4\theta)d\theta$$
$$= \frac{8}{3}]\left(\sin\theta-\cos\theta-\frac{1}{4}\sin^4\theta+\frac{1}{4}\sin^3\theta\cos\theta-\frac{3}{8}\theta+\frac{3}{16}\sin 2\theta\right)\Big|_0^{\pi/2}$$
$$= \frac{8}{3}\left[\left(1-\frac{1}{4}-\frac{3\pi}{16}\right)-(-1)\right] = \frac{28-3\pi}{6}$$

35. The volume of the cylindrical portion of the tank is $V_c = \pi(4.2)^2 19.3 \approx 1069.56m^3$. We take the equation of the ellipsoid to be
$$\frac{x^2}{(4.2)^2}+\frac{x^2}{(5.15)^2} = 1 \text{ or } z = \pm\frac{5.15}{4.2}\sqrt{(4.2)^2-x^2-y^2}.$$
The volume of the ellipsoid is
$$V_e = 2\left(\frac{5.15}{4.2}\right)\int\int_R\sqrt{(4.2)^2-x^2-y^2}dxdy = \frac{10.3}{4.2}\int_0^{2\pi}\int_0^{4.2}[(4.2)^2-r^2]^{1/2}rdrd\theta$$
$$= \frac{10.3}{4.2}\int_0^{2\pi}\left[\left(-\frac{1}{2}\right)\frac{2}{3}[(4.2)^2-r^2]^{3/2}\Big|_0^{4.2}\right]d\theta = \frac{10.3}{4.2}\frac{1}{3}\int_0^{2\pi}(4.2)^3 d\theta$$
$$= \frac{2\pi}{3}\frac{10.3}{4.2}(4.2)^3 \approx 380.53.$$
The volume of the tank is approximately $1069.56 + 380.53 = 1450.09m^3$.

36. (a) With $b > 2$ we have
$$\int\int_C IdA = \frac{1}{2}\int_0^{2\pi}\int_0^R\frac{r}{(r+c)^b}dr \quad \boxed{u = r+c, \;\; du = dr}$$
$$= \pi a\int_c^{R+c}\frac{u-c}{u^b}du = \pi a\int_c^{R+c}(u^{1-b}-cu^{-b})du = \pi a\left(\frac{r^2-b}{2-b}-c\frac{r^{1-b}}{1-b}\right)\Big|_c^{R+c}$$
$$= \pi a\left(\frac{c^{2-b}}{b-2}-\frac{c^2-b}{b-1}\right)-\pi a\left[\frac{(R+c)^{2-b}}{b-2}-\frac{c(R+c)^{1-b}}{b-1}\right]$$
$$= \frac{\pi a}{(b-1)(b-2)c^{b-2}}-\pi a\left[\frac{1}{(b-2)(R+c)^{b-2}}-\frac{c}{(b-1)(R+c)^{b-1}}\right].$$
(b) $$\lim_{R\to\infty}\int\int_C I(r)dA = \frac{\pi a}{(b-1)(b-2)c^{b-2}}$$

(c) Identifying $a = 68.585$, $b = 2.351$, and $c = 0.248$ in part **b** we find that the total number of infections in the plane is approximately 741.25.

37. (a) $$P = \int\int_C D(r)dA = \int_0^{2\pi}\int_0^R D_0 e^{-r/d}rdrd\theta = 2\pi D_0\int_0^R re^{-r/d}dr$$
$$= 2\pi D_0(-dre^{-r/d}-d^2e^{r/d})\Big|_0^R = 2\pi dD_0[d-(R+d)e^{-R/d}]$$

(b) Using

$$\int\int_C rD(r)dA = \int_0^{2\pi}\int_0^R rD_0e^{-r/d}rdrd\theta = 2\pi D_0\int_0^R r^2e^{-r/d}dr$$
$$= 2\pi D_0(-2d^3e^{-r/d} - 2d^2re^{-r/d} - dr^2e^{-r/d})\Big|_0^R$$
$$= 2\pi dD_0\left[2d^2 - (R^2+2dR+2d^2)e^{-R/d}\right]$$

we have

$$\frac{\int\int_C rD(r)dA}{\int\int_C D(r)dA} = \frac{2d^2 - (R^2+2dR+2d^2)e^{-R/d}}{d-(R+d)e^{-R/d}}$$

(c) Letting $R \longrightarrow \infty$ in the result of parts **(a)** and **(b)** we find that the total population is $2\pi d^2D_0$ and the average commute for the total population is $2d^2/d = 2d$.

38. In the first case, let the circle centered at $(D/2, 0)$ be described by the equation $r = D\cos\theta$ for $-\pi/2 \le \theta \le \pi/2$ and assume that the snow is plowed to the origin. Then

$$\int\int_R rdA = \int_{-\pi/2}^{\pi/2}\int_0^{D\cos\theta} r^2drd\theta = \frac{D^3}{3}\int_0^{\pi/2}(1-\sin^2\theta)\cos\theta d\theta$$
$$= \frac{2D^3}{3}\left(\sin\theta - \frac{1}{3}\sin^3\theta\right)\Bigg|_0^{\pi/2} = \frac{4D^3}{9}.$$

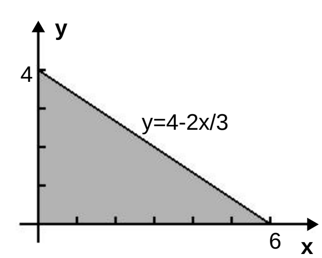
In the second case, let the circle centered at the origin be described by the equation $r = D/2$ for $0 \le \theta \le 2\pi$, and assume the snow is plowed to the origin. Then

$$\int\int_R rdA = \int_0^{2\pi}\int_0^{D/2} r^2drd\theta = \frac{2\pi}{3}r^3\Big|_0^{D/2} = \frac{\pi D^3}{12}.$$

The ratio of these integrals is $\dfrac{4D^3/9}{\pi D^3/12} = \dfrac{16}{3\pi} \approx 1.698$, which means that plowing snow to one point on the perimeter is approximately 69.8% more costly than plowing to the center.

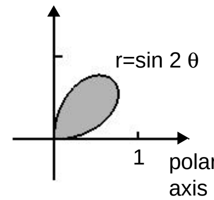
14.6 Surface Area

1. Letting $z = 0$, we have $2x + 3y = 12$. Using $f(x, y) = z = 3 - \frac{1}{2}x - \frac{3}{4}y$ we have $f_x = -\frac{1}{2}$, $f_y = -\frac{3}{4}$, $1 + f_x^2 + f_y^2 = \frac{29}{16}$. Then



$$A = \int_0^6\int_0^{4-2x/3}\sqrt{29/16}dydx = \frac{\sqrt{29}}{4}\int_0^6(4-\frac{2}{3}x)dx$$
$$= \frac{\sqrt{29}}{4}(4x - \frac{1}{3}x^2)\Big|_0^6 = \frac{\sqrt{29}}{4}(24-12) = 3\sqrt{29}.$$

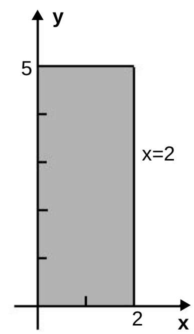
2. We see from the graph in Problem 1 that the plane is entirely above the region bounded by $r = \sin 2\theta$ in the first octant. Using $f(x,y) = z = 3 - \frac{1}{2}x - \frac{3}{4}y$ we have $f_x = -\frac{1}{2}$, $f_y = -\frac{3}{4}$, $1 + f_x^2 + f_y^2 = \frac{29}{16}$. Then



$$A = \int_0^{\pi/2}\int_0^{\sin 2\theta} \sqrt{29/16}\, r\,dr\,d\theta = \frac{\sqrt{29}}{4}\int_0^{\pi/2} \frac{1}{2}r^2\Big|_0^{\sin 2\theta} d\theta$$

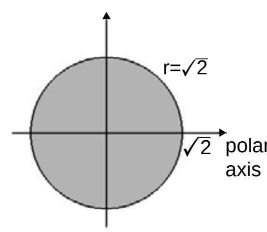
$$= \frac{\sqrt{29}}{8}\int_0^{\pi/2} \sin^2 2\theta\, d\theta = \frac{\sqrt{29}}{8}\left(\frac{1}{2}\theta - \frac{1}{8}\sin 4\theta\right)\Big|_0^{\pi/2} = \frac{\sqrt{29}\pi}{32}.$$

3. Using $f(x,y) = z = \sqrt{16 - x^2}$ we see that for $0 \le x \le 2$ and $0 \le y \le 5$, $z.0$. Thus, the surface is entirely above the region. Now $f_x = -\frac{x}{\sqrt{16 - x^2}}$, $f_y = 0$, $1 + f_x^2 + f_y^2 = 1 + \frac{x^2}{16 - x^2} = \frac{16}{16 - x^2}$ and



$$A = \int_0^5\int_0^2 \frac{4}{\sqrt{16 - x^2}}\,dx\,dy = 4\int_0^5 \sin^{-1}\frac{x}{4}\Big|_0^2 dy = 4\int_0^5 \frac{\pi}{6}\,dy = \frac{10\pi}{3}.$$

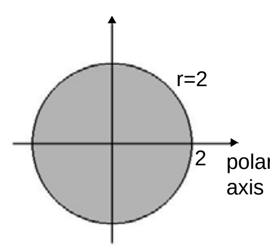
4. The region in the xy-plane beneath the surface is bounded by the graph of $x^2 + y^2 = 2$. Using $f(x,y) = z = x^2 + y^2$ we have $f_x = 2x$, $f_y = 2y$, $1 + f_x^2 + f_y^2 = 1 + 4(x^2 + y^2)$. Then,



$$A = \int_0^{2\pi}\int_0^{\sqrt{2}} \sqrt{1 + 4r^2}\, r\,dr\,d\theta = \int_0^{2\pi} \frac{1}{12}(1 + 4r^2)^{3/2}\Big|_0^{\sqrt{2}} d\theta$$

$$= \frac{1}{12}\int_0^{2\pi}(27 - 1)\,d\theta = \frac{13\pi}{3}.$$

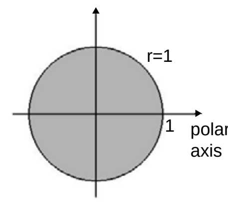
5. Letting $z = 0$ we have $x^2 + y^2 = 4$. Using $f(x,y) = z = 4 - (x^2 + y^2)$ we have $f_x = -2x$, $f_y = -2y$, $1 + f_x^2 + f_y^2 = 1 + 4(x^2 + y^2)$. Then



$$A = \int_0^{2\pi}\int_0^2 \sqrt{1 + 4r^2}\, r\,dr\,d\theta = \int_0^{2\pi} \frac{1}{3}(1 + 4r^2)^{3/2}\Big|_0^2 d\theta$$

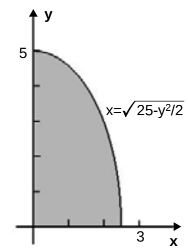
$$= \frac{1}{12}\int_0^{2\pi}(17^{3/2} - 1)\,d\theta = \frac{\pi}{6}(17^{3/2} - 1).$$

6. The surfaces $x^2 + y^2 + z^2 = 2$ and $z^2 = x^2 + y^2$ intersect on the cylinder $2x^2 + 2y^2 = 2$ or $x^2 + y^2 = 1$. There are portions of the sphere within the cone both above and below the xy-plane. Using $f(x, y) = \sqrt{2 - x^2 - y^2}$ we have $f_x = -\dfrac{x}{\sqrt{2 - x^2 - y^2}}$, $f_y = -\dfrac{y}{\sqrt{2 - x^2 - y^2}}$, $1 + f_x^2 + f_y^2 = \dfrac{2}{2 - x^2 - y^2}$. Then



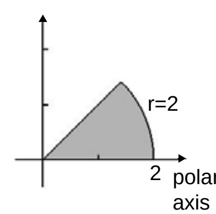
$$A = 2\left[\int_0^{2\pi}\int_0^1 \frac{\sqrt{2}}{\sqrt{2 - r^2}}r\,dr\,d\theta\right] = 2\sqrt{2}\int_0^{2\pi} -\sqrt{2 - r^2}\Big|_0^1 d\theta$$
$$= 2\sqrt{2}\int_0^{2\pi}(\sqrt{2} - 1)d\theta = 4\pi\sqrt{2}(\sqrt{2} - 1).$$

7. Using $f(x, y) = z = \sqrt{25 - x^2 - y^2}$ we have $f_x = -\dfrac{x}{\sqrt{25 - x^2 - y^2}}$, $f_y = -\dfrac{y}{\sqrt{25 - x^2 - y^2}}$, $1 + f_x^2 + f_y^2 = \dfrac{25}{25 - x^2 - y^2}$. Then



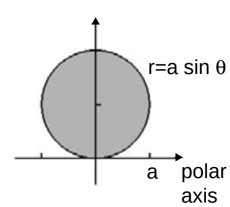
$$A = \int_0^5\int_0^{\sqrt{25-y^2}/2} \frac{5}{\sqrt{25 - x^2 - y^2}}dx\,dy$$
$$= 5\int_0^5 \sin^{-1}\frac{x}{\sqrt{25 - y^2}}\Bigg|_0^{\sqrt{25-y^2}/2} dy = 5\int_0^5 \frac{\pi}{6}dy = \frac{25\pi}{6}.$$

8. In the first octant, the graph of $z = x^2 - y^2$ intersects the xy-plane in the line $y = x$. The surface is in the first octant for $x > y$. Using $f(x, y) = z = x^2 - y^2$ we have $f_x = 2x$, $f_y = -2y$, $1 + f_x^2 + f_y^2 = 1 + 4x^2 + 4y^2$. Then



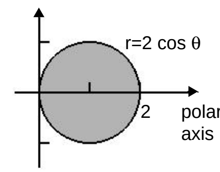
$$A = \int_0^{\pi/4}\int_0^2 \sqrt{1 + 4r^2}\,r\,dr\,d\theta = \int_0^{\pi/4} \frac{1}{12}(1 + 4r^2)^{3/2}\Big|_0^2 d\theta$$
$$= \frac{1}{12}\int_0^{\pi/4}(17^{3/2} - 1)d\theta = \frac{\pi}{48}(17^{3/2} - 1).$$

9. There are portions of the sphere within the cylinder both above and below the xy-plane. Using $f(x, y) = z = \sqrt{a^2 - x^2 - y^2}$ we have $f_x = -\dfrac{x}{\sqrt{1^2 - x^2 - y^2}}$, $f_y = -\dfrac{y}{\sqrt{a^2 - x^2 - y^2}}$, $1 + f_x^2 + f_y^2 = \dfrac{a^2}{a^2 - x^2 - y^2}$. Then, using symmetry,



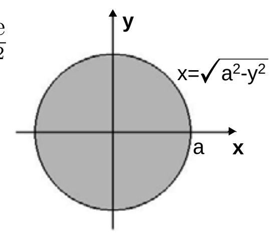
$$A = 2\left[2\int_0^{\pi/2}\int_0^{a\sin\theta}\frac{a}{\sqrt{a^2-r^2}}rdrd\theta\right] = 4a\int_0^{\pi/2} -\sqrt{a^2-r^2}\Big|_0^{a\sin\theta}d\theta$$
$$= 4a\int_0^{\pi/2}(a - a\sqrt{1-\sin^2\theta})d\theta = 4a^2\int_0^{\pi/2}(1-\cos\theta)d\theta$$
$$= 4a^2(\theta - \sin\theta)\Big|_0^{\pi/2} = 4a^2(\frac{\pi}{2}-1) = 2a^2(\pi-2).$$

10. There are portions of the cone within the cylinder both above and below the xy-plane. Using $f(x,y) = \frac{1}{2}\sqrt{x^2+y^2}$, we have $f_x = \frac{x}{2\sqrt{x^2+y^2}}$, $f_y = \frac{y}{2\sqrt{x^2+y^2}}$, $1+f_x^2+f_y^2 = \frac{5}{4}$. Then, using symmetry,



$$A = 2\left[2\int_0^{\pi/2}\int_0^{2\cos\theta}\sqrt{\frac{5}{4}}rdrd\theta\right] = 2\sqrt{5}\int_0^{\pi/2}\frac{1}{2}r^2\Big|_0^{2\cos\theta}d\theta$$
$$= 4\sqrt{5}\int_0^{\pi/2}\cos^2\theta d\theta = 4\sqrt{5}\left(\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta\right)\Big|_0^{\pi/2} = \sqrt{5}\pi.$$

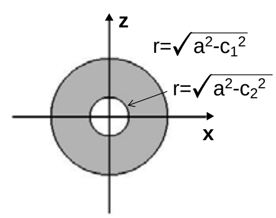
11. There are portions of the surface in each octant with areas equal to the area of the portion in the first octant. Using $f(x,y) = z = \sqrt{a^2-y^2}$ we have $f_x = 0$, $f_y = \frac{y}{\sqrt{a^2-y^2}}$, $1+f_x^2+f_y^2 = \frac{a^2}{a^2-y^2}$. Then



$$A = 8\int_0^a\int_0^{\sqrt{a^2-y^2}}\frac{a}{\sqrt{a^2-y^2}}dxdy$$
$$= 8a\int_0^a\frac{x}{\sqrt{a^2-y^2}}\Big|_0^{\sqrt{a^2-y^2}}dy = 8a\int_0^a dy = 8a^2.$$

12. From Example 1, the area of the portion of the hemisphere with $x^2+y^2=b^2$ is $2\pi a(a-\sqrt{a^2-b^2})$. Thus, the area of the sphere is $A = 2\lim_{b\to a}2\pi a(a-\sqrt{a^2-b^2}) = 2(2\pi a^2) = 4\pi a$.

13. The projection of the surface onto the xy-plane is shown in the graph. Using $f(x,z) = y = \sqrt{a^2-x^2-z^2}$ we have $f_x = -\frac{x}{\sqrt{a^2-x^2-z^2}}$, $f_z = -\frac{z}{\sqrt{a^2-x^2-z^2}}$, $1+f_x^2+f_z^2 = \frac{a^2}{a^2-x^2-z^2}$. Then



$$A = \int_0^{2\pi}\int_{\sqrt{a^2-c_2^2}}^{\sqrt{a^2-c_1^2}}\frac{a}{\sqrt{a^2-r^2}}rdrd\theta = a\int_0^{2\pi} -\sqrt{a^2-r^2}\Big|_{\sqrt{a^2-c_2^2}}^{\sqrt{a^2-C_1^2}}d\theta$$
$$= a\int_0^{2\pi}(c_2-c_1)d\theta = 2\pi a(c_2-c_1).$$

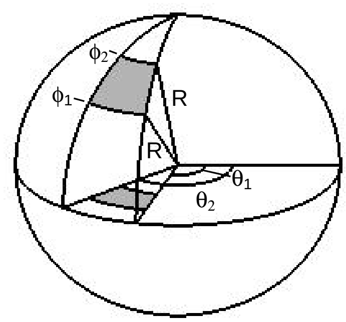
14. The surface area of the cylinder $x^2 + z^2 = a^2$ from $y = c_1$ to $y = c_2$ is the area of a cylinder of radius a and height $c_2 - c_1$. This is $2\pi a(c_2 - c_1)$.

15. The equations of the spheres are $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 + (z + a)^2 = 1$. Subtracting these equations, we obtain$(z - a)^2 - z^2 = 1 - a^2$ or $-2az + a^2 = 1 - a^2$. Thus, the spheres intersect on the plane $z = a - 1/2a$. The region of integration is $x^2 + y^2 + (a - 1/2a)^2 = a^2$ or $r^2 = 1 - 1/4a^2$. The area is

$$A = a\int_0^{2\pi}\int_0^{\sqrt{1-1/4a^2}} (a^2 - r^2)^{-1/r} r\,dr\,d\theta = 2\pi a[-(a^2 - r^2)^{1/2}]\Big|_0^{\sqrt{1-1/4a^2}}$$
$$= 2\pi a\left(a - \left[a^2 - \left(1 - \frac{1}{4a^2}\right)\right]^{1/2}\right) = 2\pi a\left(a - \left[\left(a - \frac{1}{2a}\right)^2\right]^{1/2}\right) = \pi.$$

16. (a) Both states span 7 degrees of longitude and 4 degrees of latitude, but Colorado is larger because it lies to the south of Wyoming. Lines of longitude converge as they go north, so the east-west dimensions of Wyoming are shorter than those of Colorado.

(b) We use the function $f(x, y) = \sqrt{R^2 - x^2 - y^2}$ to describe the northern hemisphere, where $R \approx 3960$ miles is the radius of the Earth. We need to compute the surface area over a polar rectangle P of the form $\theta_1 \le \theta \le \theta_2, R\cos\phi_2 \le r \le R\cos\phi_1$. We have $f_x = \dfrac{-x}{\sqrt{R^2 - x^2 - y^2}}$ and $f_y = \dfrac{-y}{\sqrt{R^2 - x^2 - y^2}}$ so that $\sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + \dfrac{x^2 + y^2}{R^2 - x^2 - y^2}} = \dfrac{R}{\sqrt{R^2 - r^2}}$.
Thus



$$A = \int\int_P \sqrt{1 + f_x^2 + f_y^2}\,dA = \int_{\theta_1}^{\theta_2}\int_{R\cos\phi_2}^{R\cos\phi_1} \frac{R}{\sqrt{R^2 - r^2}} r\,dr\,d\theta$$
$$= (\theta_2 - \theta_1)R\sqrt{R^2 - r^2}\Big|_{R\cos\phi_1}^{R\cos\phi_2} = (\theta_2 - \theta_1)R^2(\sin\phi_2 - \sin\phi_1).$$

The ratio of Wyoming to Colorado is then $\dfrac{\sin 45^\circ - \sin 41^\circ}{\sin 41^\circ - \sin 37^\circ} \approx 0.941$. Thus Wyoming is about 6% smaller than Colorado.

(c) 97,914/104,247 ≈ 0.939, which is close to the theoretical value of 0.941. (Our formula for the area says that the area of Colorado is approximately 103,924 square miles, while the area of Wyoming is approximately 97,801 square miles.

14.7 The Triple Integral

1. $\int_2^4 \int_{-2}^2 \int_{-1}^1 (x+y+z)dxdydz = \int_2^4 \int_{-2}^2 (\frac{1}{2}x^2 + xy + xz)\Big|_{-1}^{1} dydz$
$= \int_2^4 \int_{-2}^2 (2y+2z)dydz = \int_2^4 (y^2+2yz)\Big|_{-2}^{2} dz = \int_2^4 8zdz = 4z^2\big|_2^4 = 48$

2. $$\int_1^3 \int_1^x \int_2^{xy} 24xydzdydx = \int_1^3 \int_1^x 24xyz\Big|_2^{xy} dydx = \int 1^3 \int_1^x (24x^2y^2 - 48xy)dydx$$
$$= \int_1^3 (8x^2y^3 - 24xy^2)\Big|_1^x dx = \int_1^3 (8x^5 - 24x^3 - 8x^2 + 24x)dx$$
$$= \left(\frac{4}{3}x^6 - 6x^4 - \frac{8}{3}x^3 + 12x^2\right)\Big|_1^3 = 522 - \frac{14}{3} = \frac{1552}{3}$$

3. $$\int_0^6 \int_0^{6-x} \int_0^{6-x-z} dydzdx = \int_0^6 \int_0^{6-x} (6-x-z)dzdx = \int_0^6 (6z - xz - \frac{1}{2}z^2)\Big|_0^{6-x} dx$$
$$= \int_0^6 \left[6(6-x) - x(6-x) - \frac{1}{2}(6-x)^2\right] dx = \int_0^6 (18 - 6x + \frac{1}{2}x^2)dx$$
$$= \left(18x - 3x^2 + \frac{1}{6}x^3\right)\Big|_0^6 = 36$$

4. $$\int_0^1 \int_0^{1-x} \int 0^{\sqrt{y}} 4x^2z^3dzdydx = \int_0^1 \int_0^{1-x} x^2z^4\Big|_0^{\sqrt{y}} dydx = \int_0^1 \int_0^{1-x} x^2y^2dydx$$
$$= \int_0^1 \frac{1}{3}x^2y^3\Big|_0^{1-x} dx = \frac{1}{3}\int_0^1 x^2(1-x)^3dx = \frac{1}{3}\int_0^1 (x^2 - 3x^3 + 3x^4 - x^5)dx$$
$$= \frac{1}{3}\left(\frac{1}{3}x^3 - \frac{3}{4}x^4 + \frac{3}{5}x^5 - \frac{1}{6}x^6\right)\Big|_0^1 = \frac{1}{180}$$

5. $$\int_0^{\pi/2} \int_0^{y^2} \int_0^y \cos\frac{x}{y}dzdxdy = \int_0^{\pi/2} \int_0^{y^2} y\cos\frac{x}{y}dxdy = \int_0^{\pi/2} y^2\sin\frac{x}{y}\Big|_0^{y^2} dy$$
$$= \int_0^{\pi/2} y^2 \sin ydy \quad \boxed{\text{Integration by parts}}$$
$$= (-y^2\cos y + 2\cos y + 2y\sin y)\big|_0^{\pi/2} = \pi - 2$$

6. $$\int_0^{\sqrt{2}} \int_{\sqrt{y}}^2 \int_0^{e^{x^2}} xdzdxdy = \int_0^{\sqrt{2}} \int_{\sqrt{y}}^2 xe^{x^2}dxdy = \int_0^{\sqrt{2}} \frac{1}{2}e^{x^2}\Big|_{\sqrt{y}}^2 (e^4 - e^y)dy$$
$$= \frac{1}{2}(ye^4 - e^y)\Big|_0^{\sqrt{2}} = \frac{1}{2}[(e^4\sqrt{2} - e^{\sqrt{2}}) - (-1)] = \frac{1}{2}(1 + e^4\sqrt{2} - e^{\sqrt{2}})$$

7. $\displaystyle\int_0^1\int_0^1\int_0^{2-x^2-y^2} xye^z\,dzdxdy = \int_0^1\int_0^1 xye^z\Big|_0^{2-x^2y^2} dxdy = \int_0^1\int_0^1 (xye^{2-x^2-y^2}-xy)dxdy$

$\displaystyle = \int_0^1\left(-\frac{1}{2}ye^{2-x^2-y^2}-\frac{1}{2}x^2y\right)\Big|_0^1 dy = \int_0^1\left(-\frac{1}{2}ye^{1-y^2}-\frac{1}{2}y+\frac{1}{2}ye^{2-y^2}\right)dy$

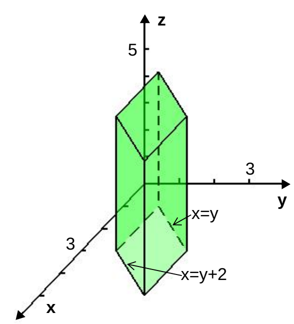
$\displaystyle = \left(\frac{1}{4}e^{1-y^2}-\frac{1}{4}y^2-\frac{1}{4}e^{2-y^2}\right)\Big|_0^1 = \left(\frac{1}{4}-\frac{1}{4}-\frac{1}{4}e)-(\frac{1}{4}e-\frac{1}{4}e^2\right) = \frac{1}{4}e^2-\frac{1}{2}e$

8. $\int_0^4\int_0^{1/2}\int_0^{x^2}\dfrac{1}{\sqrt{x^2-y^2}}dydxdz = \int_0^4\int_0^{1/2}\sin^{-1}\dfrac{y}{x}\Big|_0^{x^2} dxdz = \int_0^4\int_0^{1/2}\sin^{-1}x\,dxdz$ $\boxed{\text{Integration by parts}}$

$= \int_0^4\left(x\sin^{-1}x+\sqrt{1-x^2}\right)\Big|_0^{1/2} dz = \int_0^4\left(\dfrac{1}{2}\dfrac{\pi}{6}+\dfrac{\sqrt{3}}{2}-1\right)dz = \dfrac{\pi}{3}+2\sqrt{3}-4$

9. $\displaystyle\iiint_D z\,dV = \int_0^5\int_1^3\int_y^{y+2} z\,dxdydz = \int_0^5\int_1^3 2x\,dydz$

$\displaystyle = \int_0^5 2yz\Big|_1^3 dz = \int_0^5 4z\,dz = 2z^2\Big|_0^5 = 50$



10. Using symmetry,

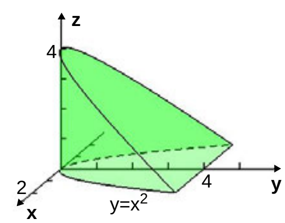
$\displaystyle\iiint_D (x^2+y^2)dV = 2\int_0^2\int_{x^2}^4\int_0^{4-y}(x^2+y^2)dzdydx$

$\displaystyle = 2\int_0^2\int_{x^2}^4 (x^2+y^2)z\Big|_0^{4-y} dydx$

$\displaystyle = 2\int_0^2\int_{x^2}^4 (4x^2-x^2y+4y^2-y^3)dydx$

$\displaystyle = 2\int_0^2 (4x^2y-\frac{1}{2}x^2y^2+\frac{4}{3}y^3-\frac{1}{4}y^4)\Big|_{x^2}^4 dx$

$\displaystyle = 2(\frac{8}{3}x^3+\frac{64}{3}x-\frac{4}{5}x^5-\frac{5}{42}x^7+\frac{1}{36}x^9)\Big|_0^2 = \frac{23,552}{315}.$

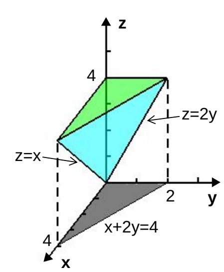


11. The other five integrals are

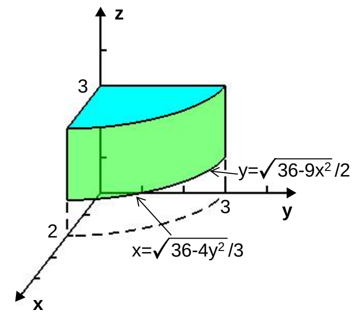
$\int_0^4\int_0^{2-x/2}\int_{x+2y}^4 f(x,y,z)dzdydx,\ \int_0^4\int_0^z\int_0^{(z-x)/2} f(x,y,z)dydxdz,$

$\int_0^4\int_x^4\int_0^{(z-x)/2} f(x,y,z)dydzdx,\ \int_0^4\int_0^{z/2}\int_0^{z-2y} f(x,y,z)dxdydz,$

$\int_0^2\int_{2y}^4\int_0^{z-2y} f(x,y,z)dxdzdy.$



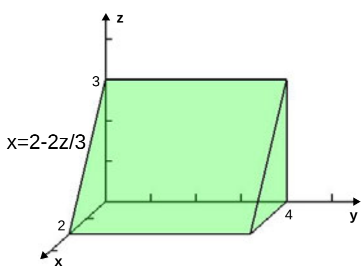
12. The other five integrals are
$\int_0^3 \int_0^{\sqrt{36-4y^2}/3} \int_1^3 f(x,y,z)dzdxdy$,
$\int_1^3 \int_0^2 \int_0^{\sqrt{36-9x^2}/2} f(x,y,z)dydxdz$,
$\int_1^3 \int_0^3 \int_0^{\sqrt{36-4y^2}/3} f(x,y,z)dxdydz$,
$\int_0^3 \int_1^3 \int_0^{\sqrt{36-4y^2}/3} f(x,y,z)dxdzdy$,
$\int_0^2 \int_1^3 \int_0^{\sqrt{36-9x^2}/2} f(x,y,z)dydzdx$.



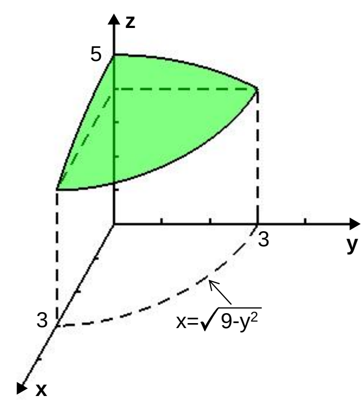
13. (a) $V = \int_0^2 \int_{x^3}^8 \int_0^4 dzdydx$ (b) $V = \int_0^8 \int_0^4 \int_0^{y^{1/3}} dxdz$
(c) $V = \int_0^4 \int_0^2 \int_{x^2}^8 dydxdz$

14. Solving $z = \sqrt{x}$ and $x + z = 2$, we obtain $x = 1, \ \ z = 1$.
(a) $V = \int_0^3 \int_0^1 \int_{z^2}^{2-z} dxdzdy$ **(b)** $V = \int_0^1 \int_{z^2}^{2-z} \int_0^3 dydxdz$
(c)$V = \int_0^3 \int_0^1 \int_0^{\sqrt{x}} dzdxdy + \int_0^3 \int_1^2 \int_0^{2-x} dzdxdy$

15.

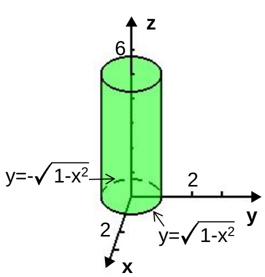


16.

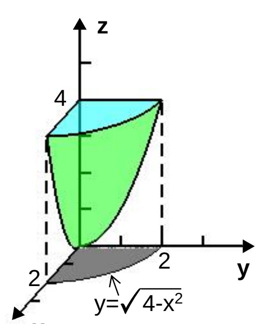


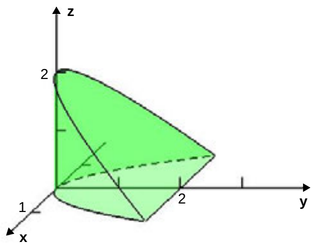
The region in the first octant is shown.

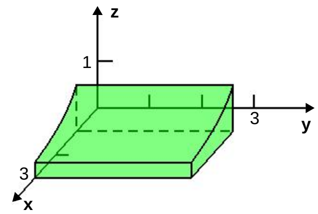
17.



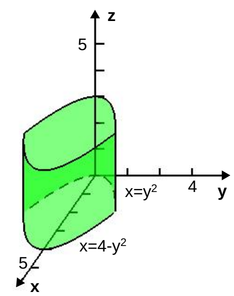
18.



19. 

20. 

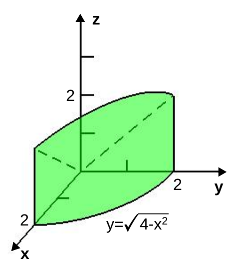
21. Solving $x = y^2$ and $4 - x = y^2$, we obtain $x = 2, \quad y = \pm\sqrt{2}$.
Using symmetry,



$$V = 2\int_0^3 \int_0^{\sqrt{2}} \int_{y^2}^{4-y^2} dxdydz = 2\int_0^3 \int_0^{\sqrt{2}} (4 - 2y^2)dydz$$

$$= 2\int_0^3 \left(4y - \frac{2}{3}y^3\right)\Bigg|_0^{\sqrt{2}} dz = 2\int_0^3 \frac{8\sqrt{2}}{3}dz = 16\sqrt{2}.$$

22. $$V = \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{x+y} dzdydx = \int_0^2 \int_0^{\sqrt{4-x^2}} z\Bigg|_0^{x+y} dydx$$

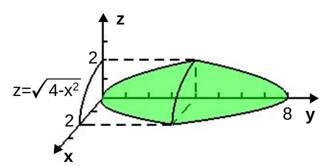


$$= \int_0^2 \int_0^{\sqrt{4-x^2}} (x+y)dydx = \int_0^2 \left(xy + \frac{1}{2}y^2\right)\Bigg|_0^{\sqrt{4-x^2}} dx$$

$$= \int_0^2 \left[x\sqrt{4-x^2} + \frac{1}{2}(4-x^2)\right] dx = \left[-\frac{1}{3}(4-x^2)^{3/2} + 2x - \frac{1}{6}x^3\right]\Bigg|_0^2$$

$$= \left(4 - \frac{4}{3}\right) - \left(-\frac{8}{3}\right) = \frac{16}{3}$$

23. Adding the two equations, we obtain $2y = 8$. Thus, the paraboloids intersect in the plane $y = 4$. Their intersection is a circle of radius 2. Using symmetry,



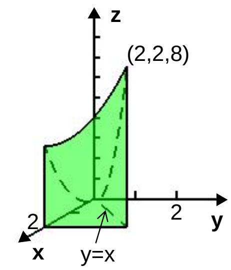
$$V = 4\int_0^2 \int_0^{\sqrt{4-x^2}} \int_{x^2+z^2}^{8-x^2-z^2} dydzdx$$

$$= 4\int_0^2 \int_0^{\sqrt{4-x^2}} (8 - 2x^2 - 2x^2)dzdx = 4\int_0^2 \left[2(4-x^2)z - \frac{2}{3}z^3\right]\Bigg|_0^{\sqrt{4-x^2}} dx$$

$$= 4\int_0^2 \frac{4}{3}(4-x^2)^{3/2}dx \quad \boxed{\text{Trig substitution}}$$

$$= \frac{16}{3}\left[-\frac{x}{8}(2x^2 - 20)\sqrt{4-x^2} + 6\sin^{-1}\frac{x}{2}\right]\Bigg|_0^2 = 16\pi.$$

24. Solving $x = 2$, $y = x$, and $z = x^2 + y^2$, we obtain the point (2,2,8).



$$V = \int_0^2 \int_0^x \int_0^{x^2+y^2} dzdydx = \int_0^2 \int_0^x (x^2 + y^2)dydx$$
$$= \int_0^2 (x^2y + \frac{1}{3}y^3)\Big|_0^x dx = \int_0^2 \frac{4}{3}x^3 dx = \frac{1}{3}x^4\Big|_0^2 = \frac{16}{3}.$$

25. We are given $\rho(x, y, z) = kz$.

$$m = \int_0^8 \int_0^4 \int_0^{y^{1/3}} kzdxdzdy = k \int_0^8 \int_0^4 xz\Big|_0^{y^{1/3}} dzdy = k \int_0^8 \int_0^4 y^{1/3}zdzdy$$
$$= k \int_0^8 \frac{1}{2}y^{1/3}z^2\Big|_0^4 dy = 8k \int_0^8 y^{1/3}dy = 8k\left(\frac{3}{4}y^{4/3}\right)\Big|_0^8 = 96k$$
$$M_{xy} = \int_0^8 \int_0^4 \int_0^{y^{1/3}} kz^2dxdzdy = k \int_0^8 \int_0^4 xz^2\Big|_0^{y^{1/3}} dzdy = k \int_0^8 \int_0^4 y^{1/3}z^2dzdy$$
$$= k \int_0^8 \frac{1}{3}y^{1/3}z^3\Big|_0^4 dy = \frac{64}{3}k \int_0^8 y^{1/3}dy = \frac{64}{3}k\left(\frac{3}{4}y^{4/3}\right)\Big|_0^8 = 256k$$
$$M_{xz} = \int_0^8 \int_0^4 \int_0^{y^{1/3}} kyzdxdzdy = k \int_0^8 \int_0^4 xyz\Big|_0^{y^{1/3}} dzdy = i \int_0^8 \int_0^4 y^{4/3}zdzdy$$
$$= k \int_0^8 \frac{1}{2}y^{4/3}z^2\Big|_0^4 dy = 8k \int_0^8 y^{4/3}dy = 8k\left(\frac{3}{7}y^{7/3}\right)\Big|_0^8 = \frac{3072}{7}k$$
$$M_{yz} = \int_0^8 \int_0^4 \int_0^{y^{1/3}} kxzdxdzdy = k \int_0^8 \int_0^4 \frac{1}{2}x^2z\Big|_0^{y^{1/3}} dzdy = \frac{1}{2}k \int_0^8 \int_0^4 y^{2/3}zdzdy$$
$$= \frac{1}{2}k \int_0^8 \frac{1}{2}y^{2/3}z^2\Big|_0^4 dy = 4k \int_0^8 y^{2/3}dy = 4k\left(\frac{3}{5}y^{5/3}\right)\Big|_0^8 = \frac{384}{5}k$$
$$\overline{x} = M_{yz}/m = \frac{384k/5}{96k} = 4/5; \quad \overline{y} = M_{xz}/m = \frac{3072k/7}{96k} = 32/7; \quad \overline{z} = M_{xy}/m = \frac{256k}{96k} = 8/3$$

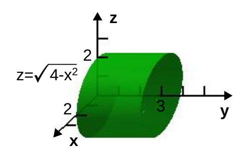
The center of mass is $(4/5, 32/7, 8/3)$.

26. We use the form of the integral in Problem 14(**b**) of this section. Without loss of generality, we take $\rho = 1$.

$$m = \int_0^1 \int_{z^2}^{2-z} \int_0^3 dydxdz = \int_0^1 \int_{z^2}^{2-z} 3dxdz = 3\int_0^1 (2 - z - z^2)dz$$
$$= 3(2z - \frac{1}{2}z^2 - \frac{1}{3}z^3)\Big|_0^1 = \frac{7}{2}$$
$$M_{xy} = \int_0^1 \int_{z^2}^{2-z} \int_0^3 zdydxdz = \int_0^1 \int_{z^2}^{2-z} yz\Big|_0^3 dxdz = \int_0^1 \int_{z^2}^{2-z} 3zdxdz$$
$$= 3\int_0^1 xz\Big|_{z^2}^{2-z} dz = 3\int_0^1 (2z - z^2 - z^3)dz = 3\left(z^2 - \frac{1}{3}z^3 - \frac{1}{4}z^4\right)\Big|_0^1 = \frac{5}{4}$$

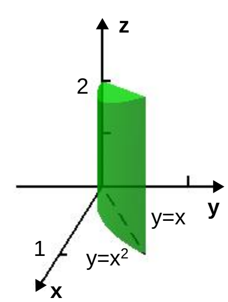
$$M_{xz} = \int_0^1 \int_{z^2}^{2-z} \int_0^3 y\,dydxdz = \int_0^1 \int_{z^2}^{2-z} \frac{1}{2}y^2\Big|_0^3 dxdz = \frac{9}{2}\int_0^1 \int_{z^2}^{2-z} dxdz$$
$$= \frac{9}{2}\int_0^1 (2-z-z^2)dz = \frac{9}{2}(2z - \frac{1}{2}z^2 - \frac{1}{3}z^3)\Big|_0^1 = \frac{21}{4}$$
$$M_{yz} = \int_0^1 \int_{z^2}^{2-z} \int_0^3 x\,dydxdz = \int_0^1 \int_{z^2}^{2-z} xy\Big|_0^3 dxdz = \int_0^1 \int_{z^2}^{2-z} 3x\,dxdz$$
$$= 3\int_0^1 \frac{1}{2}x^2\Big|_{z^2}^{2-z} dz = \frac{3}{2}\int_0^1 (4-4z+z^2-z^4)dz = \frac{3}{2}(4z - 2z^2 + \frac{1}{3}z^3 - \frac{1}{5}z^5)\Big|_0^1 = \frac{16}{5}$$
$\overline{x} = M_{yz}/m = \dfrac{16/5}{7/2} = 32/35,\ \ \overline{y} = M_{xz}/m = \dfrac{21/4}{7/2} = 3/2,\ \ \overline{z} = M_{xy}/m = \dfrac{5/4}{7/2} = 5/14.$
The centroid is $(32/35, 3/2, 5/14)$.

27. The density is $\rho(x,y,z) = ky$. Since both the region and the density function are symmetric with respect to the xy- and yz-planes, $\overline{x} = \overline{z} = 0$. Using symmetry,



$$m = 4\int_0^3 \int_0^2 \int_0^{\sqrt{4-x^2}} ky\,dzdxdy = 4k\int_0^3 \int_0^2 yz\Big|_0^{\sqrt{4-k^2}} dxdy$$
$$= 4k\int_0^3 \int_0^2 y\sqrt{4-x^2}dxdy = 4k\int_0^3 y\left(\frac{x}{2}\sqrt{4-x^2} + 2\sin^{-1}\frac{x}{2}\right)\Big|_0^2 dy$$
$$= 4k\int_0^3 \pi y\,dy = 4\pi k\left(\frac{1}{2}y^2\right)\Big|_0^3 = 18\pi k$$
$$M_{xz} = 4\int_0^3 \int_0^2 \int_0^{\sqrt{4-x^2}} ky^2\,dzdxdy = 4k\int_0^3 \int_0^2 y^2 z\Big|_0^{\sqrt{4-x^2}} dxdy = 4k\int_0^3 \int_0^2 y^2\sqrt{4-x^2}dxdy$$
$$= 4k\int_0^3 y^2\left(\frac{x}{2}\sqrt{4-x^2} + 2\sin^{-1}\frac{x}{2}\right)\Big|_0^2 dy = 4k\int_0^3 \pi y^2 dy = 4\pi k\left(\frac{1}{3}y^3\right)\Big|_0^3 = 36\pi k.$$
$\overline{y} = M_{xz}/m = \dfrac{36\pi k}{18\pi k} = 2$. The center of mass is (0,2,0).

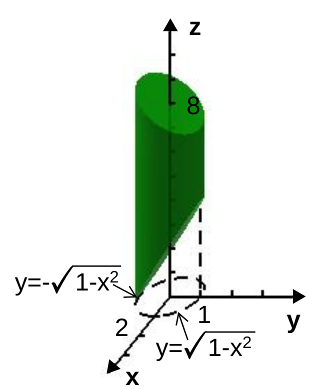
28. The density is $\rho(x,y,z) = kz$.



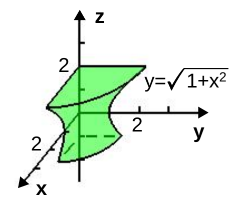
$$m = \int_0^1 \int_{x^2}^x \int_0^{y+2} kz\,dzdydx = k\int_0^1 \int_{x^2}^x \frac{1}{2}z^2\Big|_0^{y+2} dydx$$
$$= \frac{1}{2}k\int_0^1 \int_{x^2}^x (y+2)^2 dydx$$
$$= \frac{1}{2}k\int_0^1 \frac{1}{3}(y+2)^3\Big|_{x^2}^x dx$$
$$= \frac{1}{6}k\int_0^1 [(x+2)^3 - (x^2+2)^3]dx = \frac{1}{6}k\int_0^1 [(x+2)^3 - (x^6 + 6x^4 + 12x^2 + 8)]dx$$
$$= \frac{1}{6}k\left[\frac{1}{4}(x+2)^4 - \frac{1}{7}x^7 - \frac{6}{5}x^5 - 4x^3 - 8x\right]\Big|_0^1 = \frac{407}{840}k$$

$$M_{xy} = \int_0^1 \int_{x^2}^x \int_0^{y+2} kz^2 dzdydx = k\int_0^1 \int_{x^2}^x \frac{1}{3}z^3\Big|_0^{y+2} dydx = \frac{1}{3}k\int_0^1 \int_{x^2}^x (y+2)^3 dydx$$
$$= \frac{1}{3}k\int_0^1 \frac{1}{4}(y+2)^4\Big|_{x^2}^x dx = \frac{1}{12}k\int_0^1 [(x+2)^4 - (x^2+2)^4]dx$$
$$= \frac{1}{12}k\int_0^1 [(x+2)^4 - (x^8 + 8x^6 + 24x^4 + 32x^2 + 16)]dx$$
$$= \frac{1}{12}k\left[\frac{1}{5}(x+2)^5 - \frac{1}{9}x^9 - \frac{8}{7}x^7 - \frac{24}{5} - \frac{32}{3}x^3 - 16x\right]\Bigg|_0^1 = \frac{1493}{1890}k$$
$$M_{xz} = \int_0^1 \int_{x^2}^x \int_0^{y+2} kyz dzdydx = k\int_0^1 \int_{x^2}^x \frac{1}{2}yz^2\Big|_0^{y+2} dydx = \frac{1}{2}k\int_0^1 \int_{x^2}^x y(y+2)^2 dydx$$
$$= \frac{1}{2}k\int_0^1 \int_{x^2}^x (y^3 + 4y^2 + 4y)dydx = \frac{1}{2}k\int_0^1 \left(\frac{1}{4}y^4 + \frac{4}{3}y^3 + 2y^2\right)\Bigg|_{x^2}^x dx$$
$$= \frac{1}{2}k\int_0^1 \left(-\frac{1}{4}x^8 - \frac{4}{3}x^6 - 74x^4 + \frac{4}{3}x^3 + 2x^2\right) dx$$
$$= \frac{1}{2}k\left(-\frac{1}{36}x^9 - \frac{4}{21}x^7 - \frac{7}{20}x^5 + \frac{1}{3}x^4 + \frac{2}{3}x^3\right)\Bigg|_0^1 = \frac{68}{315}k$$
$$M_{yz} = \int_0^1 \int_{x^2}^x \int_0^{y+2} kxz dzdydx = k\int_0^1 \int_{x^2}^x \frac{1}{2}xz^2\Big|_0^{y+2} dydx = \frac{1}{2}k\int_0^1 \int_{x^2}^x x(y+2)^2 dydx$$
$$= \frac{1}{2}k\int_0^1 \frac{1}{3}x(y+2)^3\Big|_{x^2}^x dx = \frac{1}{6}k\int_0^1 [x(x+2)^3 - x(x^2+2)^3]dx$$
$$= \frac{1}{6}k\int_0^1 [x^4 + 6x^3 + 12x^2 + 8x - x(x^2+2)^3]dx$$
$$= \frac{1}{6}k\left[\frac{1}{5}x^5 + \frac{3}{2}x^4 + 4x^3 + 4x^2 - \frac{1}{8}(x^2+2)\right]\Bigg|_0^1 = \frac{21}{80}k$$
$\overline{x} = M_{yz}/m = \dfrac{21k/80}{407k/840} = 441/814$, $\overline{y} = M_{xz}/m = \dfrac{68k/315}{407k/840} = 544/1221$,
$\overline{z} = M_{xy}/m = \dfrac{1493k/1890}{407k/840} = 5972/3663$. The center of mass is (441/814,544/1221,5972/3663).

29. $m = \displaystyle\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{2+2y}^{8-y} (x+y+4)dzdydx$



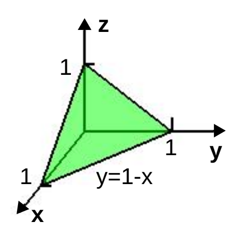
30. Both the region and the density function are symmetric with respect to the xy- and yz-planes. Thus,
$$m = 4\int_{-1}^{2}\int_{0}^{\sqrt{1+z^2}}\int_{0}^{\sqrt{1+z^2-y^2}} z^2\,dxdydz.$$



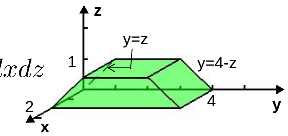
31. We are given $\rho(x,y,z) = kz$.
$$I_y = \int_0^8\int_0^4\int_0^{y^{1/3}} kz(x^2+z^2)dxdzdy = k\int_0^8\int_0^4 (\frac{1}{3}x^3z + xz^3)\Big|_0^{y^{1/3}} dzdy$$
$$= k\int_0^8\int_0^4\left(\frac{1}{3}yz + y^{1/3}z^3\right)dzdy = k\int_0^8\left(\frac{1}{6}yz^2 + \frac{1}{4}y^{1/3}z^4\right)\Big|_0^4 dy$$
$$= k\int_0^8\left(\frac{8}{3}y + 64y^{1/3}\right)dy = k\left(\frac{4}{3}y^2 + 48y^{4/3}\right)\Big|_0^8 = \frac{2560}{3}k$$
From Problem 25, $m = 96k$. Thus, $R_g = \sqrt{I_y/m} = \sqrt{\dfrac{2560k/3}{96k}} = \dfrac{4\sqrt{5}}{3}$.

32. We are given $\rho(x,y,z) = k$.
$$I_x = \int_0^1\int_{z^2}^{2-z}\int_0^3 k(y^2+z^2)dydxdz = k\int_0^1\int_{z^2}^{2-z}\left(\frac{1}{3}y^3 + yz^2\right)\Big|_0^3 dxdz = k\int_0^1\int_{z^2}^{2-z}(9+3z^2)dxdz$$
$$= k\int_0^1(9x+3xz^2)\Big|_{z^2}^{2-z} dz = k\int_0^1(18-9z-3z^2-3z^3-3z^4)dz$$
$$= k\left(18z - \frac{9}{2}z^2 - z^3 - \frac{3}{4}z^4 - \frac{3}{5}z^5\right)\Big|_0^1 = \frac{223}{20}k$$
$$m = \int_0^1\int_{z^2}^{2-z}\int_0^3 kdydxdz = k\int_0^1\int_{z^2}^{2-z} 3dxdz = 3k\int_0^1(2-z-z^2)dz$$
$$= 3k\left(2z - \frac{1}{2}z^2 - \frac{1}{3}z^3\right)\Big|_0^1 = \frac{7}{2}k$$
$$R_g = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{223k/20}{7k/2}} = \sqrt{\frac{223}{70}}$$

33. $$I_z = k\int_0^1\int_0^{1-x}\int_0^{1-x-y}(x^2+y^2)dzdydx$$
$$= k\int_0^1\int_0^{1-x}(x^2+y^2)(1-x-y)dydx$$
$$= k\int_0^1\int_0^{1-x}(x^2 - X^3 - x^2y + y^2 - xy^2 - y^3)dydx$$
$$= k\int_0^1\left[(x^2-x^3)y - \frac{1}{2}x^2y^2 + \frac{1}{3}(1-x)y^3 - \frac{1}{4}y^4\right]\Big|_0^{1-x} dx$$
$$= k\int_0^1[\frac{1}{2}x^2 - x^3 + \frac{1}{2}x^4 + \frac{1}{12}(1-x)^4]dx = k\left[\frac{1}{6}x^6 - \frac{1}{4}x^4 + \frac{1}{10}x^5 - \frac{1}{60}(1-x)^5\right]\Big|_0^1 = \frac{k}{30}$$



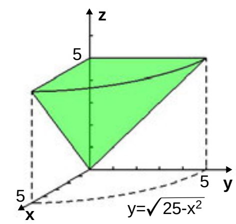
34. We are given $\rho(x,y,z) = kx$.



$$I_y = \int_0^1 \int_0^2 \int_z^{4-z} kx(x^2+z^2)dydxdz = k\int_0^1\int_0^2 (x^3+xz^2)y\Big|_z^{4-z} dxdz$$
$$= k\int_0^1\int_0^2 (x^3+xz^2)(4-2z)dxdz$$
$$= k\int_0^1 (\frac{1}{4}x^4 + \frac{1}{2}x^2z^2)(4-2z)\Big|_0^2 dz$$
$$= k\int_0^1 (4+2z^2)(4-2z)dz = 4k\int_0^1 (4-2z+2z^2-z^3)dz$$
$$= 4k\left(4z - z^2 + \frac{2}{3}z^3 - \frac{1}{4}z^4\right)\Big|_0^1 = \frac{41}{3}k$$

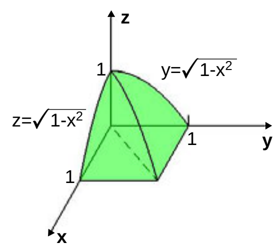
35. We are given $\rho(x,y,z) = k\sqrt{x^2+y^2+z^2}$. Both the region and the integrand are symmetric with respect to the yz- and xz-planes.

$$I_z = 4\int_0^5 \int_0^{\sqrt{25-x^2}} \int_{\sqrt{x^2+y^2}}^5 k(x^2+y^2)\sqrt{x^2+y^2+z^2}dzdydx$$



36. We are given $\rho(x,y,z) = kz$. Both the region and the integrand are symmetric with respect to the xz- and xy-planes.

$$I_y = 4\int_{-1}^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-z^2}} kx(x^2+z^2)dydzdx$$



14.8 Triple Integrals in Other Coordinate Systems

1. $x = 10\cos 3\pi/4 = -5\sqrt{2}$; $y = 10\sin 3\pi/4 = 5\sqrt{2}$; $(-5\sqrt{2}, 5\sqrt{2}, 5)$

2. $x = 2\cos 5\pi/6 = -\sqrt{3}$; $y = 2\sin 5\pi/6 = 1$; $(-\sqrt{3}, 1, -3)$

3. $x = \sqrt{3}\cos\pi/3 = \sqrt{3}/2$; $y = \sqrt{3}\sin\pi/3 = 3/2$; $(\sqrt{3}/2, 3/2, -4)$

4. $x = 4\cos 7\pi/4 = 2\sqrt{2}$; $y = 4\sin 7\pi/4 = -2\sqrt{2}$; $(2\sqrt{2}, -2\sqrt{2}, 0)$

5. $x = 5\cos\pi/2 = 0$; $y = 5\sin\pi/2 = 5$; $(0, 5, 1)$

6. $x = 10\cos 5\pi/3 = 5$; $y = 10\sin 5\pi/3 = -5\sqrt{3}$; $(5, -5\sqrt{3}, 2)$

7. With $x = 1$ and $y = -1$ we have $r^2 = 2$ and $\tan\theta = -1$. The point is $(\sqrt{2}, -\pi/4, -9)$

8. With $x = 2\sqrt{3}$ and $y = 2$ we have $r^2 = 16$ and $\tan\theta = 1/\sqrt{3}$. The point is $(4, \pi/6, 17)$.

9. With $x = -\sqrt{2}$ and $y = \sqrt{6}$ we have $r^2 = 8$ and $\tan\theta = -\sqrt{3}$. The point is $(2\sqrt{2}, 2\pi/3, 2)$.

10. With $x = 1$ and $y = 2$ we have $r^2 = 5$ and $\tan\theta = 2$. The point is $(\sqrt{5}, \tan^{-1} 2, 7)$.

11. With $x = 0$ and $y = -4$ we have $r^2 = 16$ and $\tan\theta$ undefined. The point is $(4, -\pi/2, 0)$.

12. With $x = \sqrt{7}$ and $y = -\sqrt{7}$ we have $r^2 = 14$ and $\tan\theta = -1$. The point is $(\sqrt{14}, -\pi/4, 3)$.

13. $r^2 + z^2 = 25$

14. $r\cos\theta + r\sin\theta - z = 1$

15. $r^2 - z^2 = 1$

16. $r^2\cos^2\theta + z^2 = 16$

17. $z = x^2 + y^2$

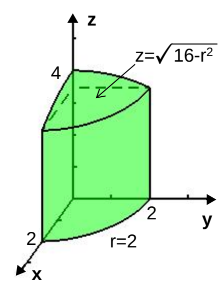
18. $z = 2y$

19. $r\cos\theta = 5, \quad z = 5$

20. $\tan\theta = 1/\sqrt{3}, \quad y/z = 1/\sqrt{3}, \quad z = \sqrt{3}y, \quad x > 0$

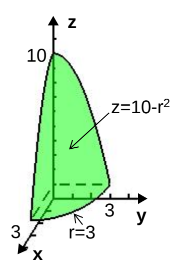
21. The equations are $r^2 = 4, \quad r^+ z^2 = 16$, and $z = 0$.

$$V = \int_0^{2\pi}\int_0^2\int_0^{\sqrt{16-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2 r\sqrt{16-r^2}\,dr\theta$$
$$= \int_0^{2\pi} -\frac{1}{3}(16-r^2)^{3/2}\Big|_0^2 d\theta = \int_0^{2\pi}(64-24\sqrt{3})d\theta = \frac{2\pi}{3}(64-24\sqrt{3})$$



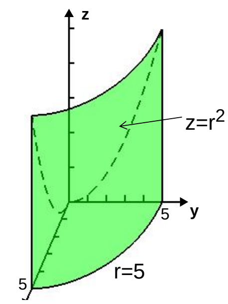
22. The equation is $z = 10 - r^2$.

$$V = \int_0^{2\pi}\int_0^3\int_1^{10-r^2} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^3 r(9-r^2)\,dr\,d\theta$$
$$= \int_0^{2\pi}\left(\frac{9}{2}r^2 - \frac{1}{4}r^4\right)\Big|_0^3 d\theta = \int_0^{2\pi}\frac{81}{4}d\theta = \frac{81\pi}{2}.$$

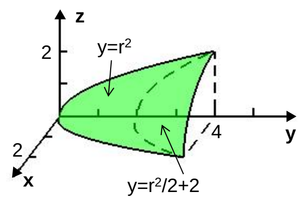


23. The equations are $z = r^2$, $r = 5$, and $z = 0$.

$$V = \int_0^{2\pi}\int_0^5\int_0^{r^2} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^5 r^3\,dr\,d\theta = \int_0^{2\pi} \frac{1}{4}r^4\Big|_0^5 d\theta$$
$$= \int_0^{2\pi} \frac{625}{4}d\theta = \frac{625\pi}{2}$$

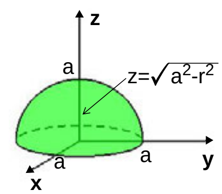


24. Substituting the first equation into the second, we see that the surfaces intersect in the plane $y = 4$. Using polar coordinates in the xz-plane, the equations of the surfaces become $y = r^2$ and $y = \frac{1}{2}r^2 + 2$.



$$V = \int_0^{2\pi}\int_0^2\int_{r^2}^{r^2/2+2} r\,dy\,dr\,d\theta = \int_0^{2\pi}\int_0^2 r\left(\frac{r^2}{2} + 2 - r^2\right)dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^2 \left(2r - \frac{1}{2}r^3\right)dr\,d\theta = \int_0^{2\pi}\left(r^2 - \frac{1}{8}r^4\right)\Big|_0^2 d\theta = \int_0^{2\pi} 2\,d\theta = 4\pi$$

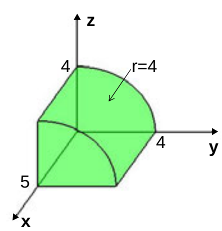
25. The equation is $z = \sqrt{a^2 - r^2}$. By symmetry, $\overline{x} = \overline{y} = 0$.



$$m = \int_0^{2\pi}\int_0^a\int_0^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^a r\sqrt{a^2 - r^2}\,dr\,d\theta$$
$$= \int_0^{2\pi} -\frac{1}{3}(a^2 - r^2)^{3/2}\Big|_0^a d\theta = \int_0^{2\pi}\frac{1}{3}a^3\,d\theta = \frac{2}{3}\pi a^3$$
$$M_{xy} = \int_0^{2\pi}\int_0^a\int_0^{\sqrt{a^2-r^2}} zr\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^a \frac{1}{2}rz^2\Big|_0^{\sqrt{a^2-r^2}} dr\,d\theta$$
$$= \frac{1}{2}\int_0^{2\pi}\int_0^a r(a^2 - r^2)\,dr\,d\theta$$
$$= \frac{1}{2}\int_0^{2\pi}\left(\frac{1}{2}a^2r^2 - \frac{1}{4}r^4\right)\Big|_0^a d\theta = \frac{1}{2}\int_0^{2\pi}\frac{1}{4}a^4\,d\theta = \frac{1}{4}\pi a^4$$

$\overline{z} = M_{xy}/m = \dfrac{\pi a^4/4}{2\pi a^3/3} = 3a/8$. The centroid is $(0, 0, 3a/8)$.

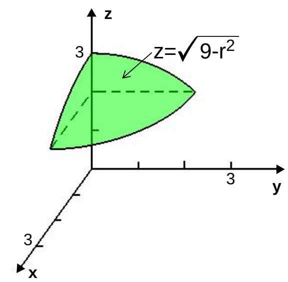
26. We use polar coordinates in the yz-plane. The density is $\rho(x, y, z) = kz$. By symmetry, $\overline{y} = \overline{z} = 0$.



$$m = \int_0^{2\pi}\int_0^4\int_0^5 kxr\,dx\,dr\,d\theta = k\int_0^{2\pi}\int_0^4 \frac{1}{2}rz^2\Big|_0^5 dr\,d\theta$$
$$= \frac{k}{2}\int_0^{2\pi}\int_0^4 25r\,dr\,d\theta = \frac{25k}{2}\int_0^{2\pi}\frac{1}{2}r^2\Big|_0^4 d\theta$$
$$= \frac{25k}{2}\int_0^{2\pi} 8\,d\theta = 200k\pi$$

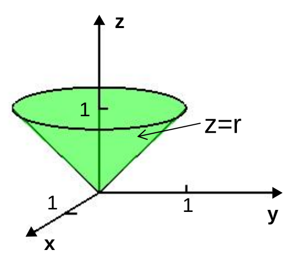
$$M_{yz} = \int_0^{2\pi}\int_0^4\int_0^5 kx^2 r\,dx\,dr\,d\theta = k\int_0^{2\pi}\int_0^4 \frac{1}{3}rx^3\Big|_0^5 dr\,d\theta = \frac{1}{3}k\int_0^{2\pi}\int_0^4 125r\,dr\,d\theta$$
$$= \frac{1}{3}k\int_0^{2\pi}\frac{125}{2}r^2\Big|_0^4 d\theta = \frac{1}{3}k\int_0^{2\pi}1000\,d\theta = \frac{2000}{3}k\pi$$
$\overline{x} = M_{yz}/m = \dfrac{2000k\pi/3}{200k\pi} = 10/3$, The center of mass of the given solid is $(10/3, 0, 0)$.

27. The equation is $z = \sqrt{9-r^2}$ and the density is $\rho = k/r^2$. When $x = 2$, $r = \sqrt{5}$.



$$I_z = \int_0^{2\pi}\int_0^{\sqrt{5}}\int_2^{\sqrt{9-r^2}} r^2(k/r^2)r\,dz\,dr\,d\theta(k/r^2)$$
$$= k\int_0^{2\pi}\int_0^{\sqrt{5}} rz\Big|_2^{\sqrt{9-r^2}} dr\,d\theta$$
$$= k\int_0^{2\pi}\int_0^{\sqrt{5}}\left(r\sqrt{9-r^2}-2r\right)dr\,d\theta$$
$$= k\int_0^{2\pi}\left[-\frac{1}{3}(9-r^2)^{3/2}-r^2\right]\Big|_0^{\sqrt{5}} d\theta$$
$$= k\int_0^{2\pi}\frac{4}{3}d\theta = \frac{8}{3}\pi k$$

28. The equation is $z = r$ and the density is $\rho = kr$.



$$I_x = \int_0^{2\pi}\int_0^1\int_r^1 (y^2+z^2)(kr)r\,dz\,dr\,d\theta$$
$$= k\int_0^{2\pi}\int_0^1\int_r^1 (r^4\sin^2\theta + r^2z^2)\,dz\,dr\,d\theta$$
$$= k\int_0^{2\pi}\int_0^1\left[(r^4\sin^2\theta)z+\frac{1}{3}r^2z^3\right]\Big|_r^1 dr\,d\theta$$
$$= k\int_0^{2\pi}\int_0^1\left(r^4\sin^2\theta+\frac{1}{3}r^2-r^5\sin^2\theta-\frac{1}{3}r^4\right)dr\,d\theta$$
$$= k\int_0^{2\pi}\left(\frac{1}{5}r^5\sin^2\theta+\frac{1}{9}r^3-\frac{1}{6}r^6\sin^2\theta-\frac{1}{18}r^6\right)\Big|_0^1 d\theta$$
$$= k\int_0^{2\pi}\left(\frac{1}{30}\sin^2\theta+\frac{1}{18}\right)d\theta$$
$$= k\left(\frac{1}{60}\theta-\frac{1}{120}\sin 2\theta+\frac{1}{18}\theta\right)\Big|_0^{2\pi} = \frac{13}{90}\pi k$$

29. (a) $x = (2/3)\sin(\pi/2)\sin(\pi/2)\cos(\pi/6) = \sqrt{3}/3$; $y = (2/3)\sin(\pi/2)\sin(\pi/6) = 1/3$;

(b) With $x = \sqrt{3}/3$ and $y = 1/3$ we have $r^2 = 4/9$ and $\tan\theta = \sqrt{3}/3$. The point is $(2/3, \pi/6, 0)$.

30. (a) $x = 5\sin(5\pi/4)\cos(2\pi/3) = 5\sqrt{2}/4$; $y = 5\sin(5\imath/4)\sin(2\pi/3) = -5\sqrt{6}/4$; $z = 5\cos(5\pi/4) = -5\sqrt{2}/2$; $(5\sqrt{2}/4, \ -5\sqrt{6}/4, -5\sqrt{2}/2)$

    (b) With $x = 5\sqrt{2}/4$ and $y = -5\sqrt{6}/4$ we have $r^2 = 25/2$ and $\tan\theta = -\sqrt{3}$. The point is $(5/\sqrt{2}, 2\pi/3, -5\sqrt{2}/2)$.

31. (a) $x = 8\sin(\pi/4)\cos(3\pi/4) = -4$; $y = 8\sin(\pi/4)\sin(3\pi/4) = 4$; $z = 8\cos(\pi/4) = 4\sqrt{2}$; $(-4, 4, 4\sqrt{2})$

    (b) With $x = -4$ and $y = 4$ we have $r^2 = 32$ and $\tan\theta = -1$. The point is $(4\sqrt{2}, 3\pi/4, 4\sqrt{2})$.

32. (a) $x = (1/3)\sin(5\pi/3)\cos(\pi/6) = -1/4$; $y = (1/3)\sin(5\pi/3)\sin(\pi/6) = -\sqrt{3}/12$; $z = (1/3)\cos(5\pi/3) = 1/6$; $(-1/4, -\sqrt{3}/12, 1/6)$

    (b) With $x = -1/4$ and $y = -\sqrt{3}/12$ we have $r^2 = 1/12$ and $\tan\theta = \sqrt{3}/3$. The point is $(1/2\sqrt{3}, \pi/6, 1/6)$.

33. (a) $x = 4\sin(3\pi/4)\cos 0 = 2\sqrt{2}$; $y = 4\sin(3\pi/4)\sin 0 = 0$; $z = 4\cos(3\pi/4) = -2\sqrt{2}$; $(2\sqrt{2}, 0, -2\sqrt{2})$

    (b) With $x = 2\sqrt{2}$ and $y = 0$ we have $r^2 = 8$ and $\tan\theta = 0$. The point is $(2\sqrt{2}, 0, -2\sqrt{2})$.

34. (a) $x = 1\sin(11\pi/6)\cos\pi = 1/2$; $y = 1\sin(11\pi/6)\sin\pi = 0$; $z = 1\cos(11\pi/6) = \sqrt{3}/2$; $(1/2, 0, (\sqrt{3}/2)$

    (b) With $x = 1/2$ and $y = 0$ we have $r^2 = 1/4$ and $\tan\theta = 0$. The point is $(1/2, 0\sqrt{3}/2)$.

35. With $x = -5$, $y = -5$, and $z = 0$, we have $\rho^2 = 50$, $\tan\theta = 1$, and $\cos\phi = 0$. The point is $(5\sqrt{2}, \pi/2, 5\pi/4)$.

36. With $z = 1$, $y = -\sqrt{3}$, and $z = 1$, we have $\rho^2 = 5$, $\tan\theta = -\sqrt{3}$, and $\cos\phi = 1/\sqrt{5}$. The point is $(\sqrt{5}, \cos^{-1} 1/\sqrt{5}, -\pi/3)$.

37. With $x = \sqrt{3}/2$, $y = 1/2$, and $z = 1$, we have $\rho^2 = 2$, $\tan\theta = 1/\sqrt{3}$, and $\cos\phi = 1/\sqrt{2}$. The point is $(\sqrt{2}, \pi/4, \pi/6)$.

38. With $x = -\sqrt{3}/2$, $y = 0$, and $z = -1/2$, we have $\rho^2 = 1$, $\tan\theta = 0$, and $\cos\phi = -1/2$. The point is $(1, 2\pi/3, 0)$.

39. With $x = 3$, $y = -3$, and $z = 3\sqrt{2}$, we have $\rho^2 = 36$, $\tan\theta = -1$, and $\cos\phi = -\sqrt{2}/2$. The point is $(6, \pi/4, -\pi/4)$

40. With $x = 1$, $y = 1$, and $z = -\sqrt{6}$, we have $\rho^2 = 8$, $\tan\theta = 1$, and $\cos\phi = -\sqrt{3}/2$ The point is $(2\sqrt{2}, 5\pi/6, \pi/4)$.

41. $\rho = 8$

42. $\rho^2 = 4\rho\cos\phi$; $\rho = 4\cos\phi$

43. $4z^2 = 3x^2 + 3y^2 + 3z^2$; $4\rho^2\cos^2\phi = 3\rho^2$; $\cos\phi = \pm\sqrt{3}/2$; $\phi = \pi/6$, $5\pi/6$

44. $-x^2 - y^- z^2 = 1 - 2z^2$; $-\rho^2 = 1 - 2\rho^2\cos^2\phi$; $\rho^2(2\cos^2\phi - 1) = 1$

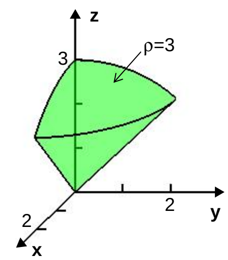
45. $x^2 + y^2 + z^2 = 100$

46. $\cos\phi = 1/2$; $\rho^2\cos^2\phi = \rho^2/4$; $4z^2 = x^2+y^2+z^2$; $z^2+y^2 = 3z^2$

47. $\rho\cos\phi = 2$; $z = 2$

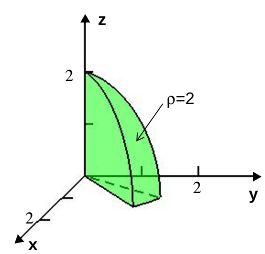
48. $\rho(1-\cos^2\phi) = \cos\phi$; $\rho^2 - \rho^2\cos^2\phi = \rho\cos\phi$; $x^2+y^2+z^2-z^2 = z$; $z = x^2+y^2$

49. The equations are $\phi = \pi/4$ and $\rho = 3$.



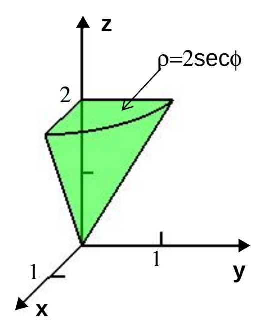
$$V = \int_0^{2\pi}\int_0^{\pi/4}\int_0^3 \rho^2\sin\phi drhod\phi d\theta = \int_0^{2\pi}\int_0^{\pi/4}\frac{1}{3}\rho^3\sin\phi\Big|_0^3 d\phi d\theta$$
$$= \int_0^{2\pi}\int_0^{\pi/4} 9\sin\phi d\phi d\theta = \int_0^{2\pi} -9\cos\phi\Big|_0^{\pi/4} d\theta$$
$$= -9\int_0^{2\pi}\left(\frac{\sqrt{2}}{2}-1\right)d\theta = 9\pi(2-\sqrt{2})$$

50. The equations are $\rho = 2$, $\theta = \pi/4$, and $\theta = \pi/3$.



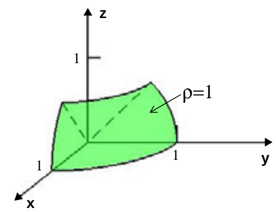
$$\int_{\pi/4}^{\pi/3}\int_0^{\pi/2}\int_0^2 \rho^2\sin\phi d\rho d\phi d\theta = \int_{\pi/4}^{\pi/3}\int_0^{\pi/2}\frac{1}{3}\rho^3\sin\phi\Big|_0^2 d\phi d\theta$$
$$= \int_{\pi/4}^{\pi/3}\int_0^{\pi/2}\frac{8}{3}\sin\phi d\phi d\theta$$
$$= \frac{8}{3}\int_{\pi/4}^{\pi/3} -\cos\phi\Big|_0^{\pi/2} d\theta$$
$$= \frac{8}{3}\int_{\pi/4}^{\pi/3}(0+1)d\theta = \frac{2\pi}{9}$$

51. Using Problem 43, the equations are $\phi = \pi/6$, $\theta = \pi/2$, and $\rho\cos\phi = 2$.



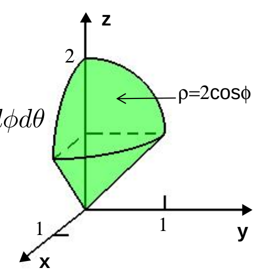
$$V = \int_0^{\pi/2}\int_0^{\pi/6}\int_0^{2\sec\phi}\rho^2\sin\phi d\rho d\phi d\theta$$
$$= \int_0^{\pi/2}\int_0^{\pi/6}\frac{1}{3}\rho^2\sin\phi\Big|_0^{2\sec\phi} d\phi d\theta$$
$$= \frac{8}{3}\int_0^{\pi/2}\int_0^{\pi/6}\sec^3\phi\sin\phi d\phi d\theta = \frac{8}{3}\int_0^{\pi/2}\int_0^{\pi/6}\sec^2\phi\tan\phi d\phi d\theta$$
$$= \frac{8}{3}\int_0^{\pi/2}\frac{1}{2}\tan^2\phi\Big|_0^{\pi/6} d\theta = \frac{4}{3}\int_0^{\pi/2}\frac{1}{3}d\theta = \frac{2}{9}\pi$$

52. The equations are $\rho = 1$ and $\phi = \pi/4$. We find the volume above the xy-plane and double.



$$V = 2\int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^1 \rho^2 \sin\phi d\rho d\phi d\theta = 2\int_0^{2\pi}\int_{\pi/4}^{\pi/2} \frac{1}{3}\rho^3 \sin\phi\bigg|_0^1 d\phi d\theta$$
$$= \frac{2}{3}\int_0^{2\pi}\int_{\pi/4}^{\pi/2} \sin\phi d\phi d\theta = \frac{2}{3}\int_0^{2\pi} -\cos\phi\bigg|_{\pi/4}^{\pi/2} = \frac{2}{3}\int_0^{2\pi}\frac{\sqrt{2}}{2}d\theta$$
$$= \frac{2\pi\sqrt{2}}{3}$$

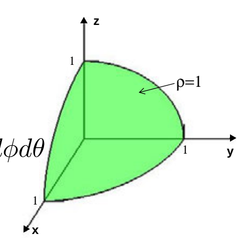
53. By symmetry, $\overline{x} = \overline{y} = 0$. The equations are $\phi = \pi/4$ and $\rho = 2\cos\phi$.



$$m = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{2\cos\phi} \rho^2 \sin\phi d\rho d\phi d\theta = \int_0^{2\pi}\int_0^{\pi/4}\frac{1}{3}\rho^3\sin\phi\bigg|_0^{2\cos\phi} d\phi d\theta$$
$$= \frac{8}{3}\int_0^{2\pi}\int_0^{\pi/4}\sin\phi\cos^3\phi d\phi d\theta = \frac{8}{3}\int_0^{2\pi} -\frac{1}{4}\cos^4\phi\bigg|_0^{\pi/4} d\theta$$
$$= -\frac{2}{3}\int_0^{2\pi}\left(\frac{1}{4}-1\right)d\theta = \pi$$
$$M_{xy} = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{2\cos\phi} z\rho^2\sin\phi d\rho d\phi d\theta \int_0^{2\pi}\int_0^{\pi/4}\int_0^{2\cos\phi}\rho^3\sin\phi\cos\phi d\rho d\phi d\theta$$
$$= \int_0^{2\pi}\int_0^{\pi/4}\frac{1}{4}\rho^4\sin\phi\cos\phi\bigg|_0^{2\cos\phi} d\phi d\theta = 4\int_0^{2\pi}\int_0^{\pi/4}\cos^5\phi\sin\phi d\phi d\theta$$
$$= 4\int_0^{2\pi} -\frac{1}{6}\cos^6\phi\bigg|_0^{\pi/4} = -\frac{2}{3}\int_0^{2\pi}\left(\frac{1}{8}-1\right)d\theta = \frac{7}{6}\pi$$

$\overline{z} = M_{xy}/m = \dfrac{7\pi/6}{\pi} = 7/6$. The centroid is $(0, 0, 7/6)$.

54. We are given density$= kz$. By symmetry, $\overline{x} = \overline{y} = 0$. The equation is $\rho = 1$.

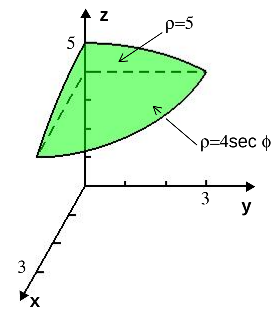


$$m = \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 kz\rho^2\sin\phi d\rho d\phi d\theta = k\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho^3\sin\phi\cos\phi d\rho d\phi d\theta$$
$$= k\int_0^{2\pi}\int_0^{\pi/2}\frac{1}{4}\rho^4\sin\phi\cos\phi\bigg|_0^1 d\phi d\theta$$
$$= \frac{1}{4}k\int_0^{2\pi}\int_0^{\pi/2}\sin\phi\cos\phi d\phi d\theta$$
$$= \frac{1}{4}k\int_0^{2\pi}\frac{1}{2}\sin^2\phi\bigg|_0^{\pi/2} d\theta = \frac{1}{8}k\int_0^{2\pi}d\theta = \frac{k\pi}{4}$$

$$M_{xy} = \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 kz^2\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = k\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho^3\cos^2\phi\sin\phi\, d\rho\, d\phi\, d\theta$$
$$= k\int_0^{2\pi}\int_0^{\pi/2} \frac{1}{5}\rho^5\cos^2\phi\sin\phi\Big|_0^1 d\phi\, d\theta = \frac{1}{5}k\int_0^{2\pi}\int_0^{\pi/2}\cos^2\phi\sin\phi\, d\phi\, d\theta$$
$$= \frac{1}{5}k\int_0^{2\pi} -\frac{1}{3}\cos^3\phi\Big|_0^{\pi/2} = -\frac{1}{15}k\int_0^{2\pi}(0-1)d\theta = \frac{2}{15}k\pi$$

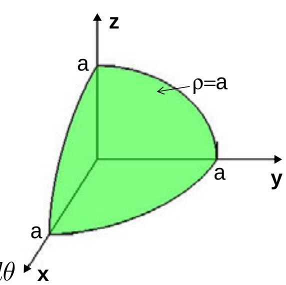
$\overline{z} = M_{xy}/m = \dfrac{2k\pi/15}{k\pi/4} = 8/15$. The center of mass is $(0, 0, 8/15)$.

55. We are given density$= k/\rho$.



$$m = \int_0^{2\pi}\int_0^{\cos^{-1}4/5}\int_{4\sec\phi}^5 \frac{k}{\rho}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$
$$= k\int_0^{2\pi}\int_0^{\cos^{-1}4/5} \frac{1}{2}\rho^2\sin\phi\Big|_{4\sec\phi}^5 d\phi\, d\theta$$
$$= \frac{1}{2}k\int_0^{2\pi}\int_0^{\cos^{-1}4/5}(25\sin\phi - 16\tan\phi\sec\phi)d\phi\, d\theta$$
$$\frac{1}{2}k\int_0^{2\pi}(-25\cos\phi - 16\sec\phi)\Big|_0^{\cos^{-1}4/5} d\theta$$
$$= \frac{1}{2}k\int_0^{2\pi}[-25(4/5) - 16(5/4) - (-25-16)]d\theta$$
$$= \frac{1}{2}k\int_0^{2\pi} d\theta = k\pi$$

56. $$I_z = \int_0^{2\pi}\int_0^{\pi}\int_0^a (x^2+y^2)(k\rho)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$



$$= k\int_0^{2\pi}\int_0^{\pi}\int_0^a \sin^2\pi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta)\rho^3\sin\phi\, d\rho\, d\phi\, d\theta$$
$$= k\int_0^{2\pi}\int_0^{\pi}\int_0^a \rho^5\sin^3\phi\, d\rho\, d\phi\, d\theta = k\int_0^{2\pi}\int_0^{\pi}\frac{1}{6}\rho^6\sin^3\phi\Big|_0^a d\phi\, d\theta$$
$$= \frac{1}{6}ka^6\int_0^{2\pi}\int_0^{\pi}\sin^3\phi\, d\phi\, d\theta = \frac{1}{6}ka^3\int_0^{2\pi}\int_0^{\pi}(1-\cos^2\phi)\sin\phi\, d\phi\, d\theta$$
$$= \frac{1}{6}ka^3\int_0^{2\pi}\left(-\cos\phi + \frac{1}{3}\cos^3\phi\right)\Big|_0^{\pi} d\theta = \frac{1}{6}ka^3\int_0^{2\pi}\frac{4}{3}d\theta = \frac{4\pi}{9}ka^6$$

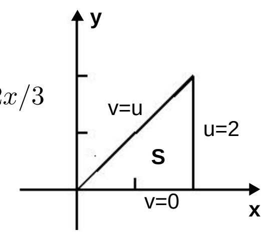
14.9 Change of Variables in Multiple Integrals

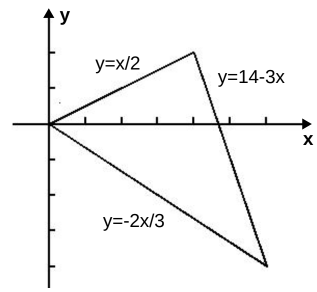
1. $T:\ (0,0) \longrightarrow (0,0);\ \ (0,2) \longrightarrow (-2,8);\ \ (4,0) \longrightarrow (16,20);\ \ (4,2) \longrightarrow (14,28)$

2. Writing $x^2 = v - u$ and $y = v + u$ and solving for u and v, we obtain $u = (y - x^2)/2$ and $v = (x^2+y)/2$. Then the images under T^{-1} are $(1,1) \longrightarrow (0,1)$; $(1,3) \longrightarrow (1,2)$; $(\sqrt{2}, 2) \longrightarrow$

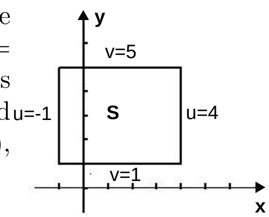
$(0, 2)$.

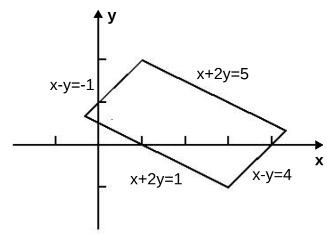
3. The uv-corner points $(0,0)$, $(2,0)$, $(2,2)$ correspond to xy-points $(0,0)$, $(4,2)$, $(6,-4)$.
$v = 0: \quad x = 2u, \quad y = u \Longrightarrow y = x/2$
$u = 2: \quad x = 4 + v, \quad y = 2 - 3v \Longrightarrow$
$y = 2 - 3(x - 4) = -3 + 14$
$v = u: \quad x = 3u, \quad y = -2u \Longrightarrow y = -2x/3$



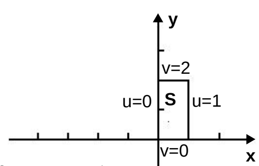


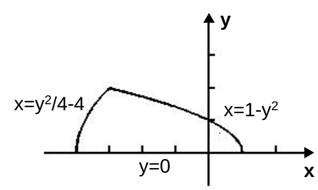
4. Solving for x and y we see that the transformation is $x = 2u/3 + v/3$, $y = -u/3 + v/3$ > The uv-corner points $(-1,1)$, $(4,1)$, $(4,5)$, $(-1,5)$ correspond to the xy-points $(-1/3, 2/3)$, $(3,-1)$, $(13/3, 1/3)$, $(1,2)$. $v = 1: \quad x + 2y = 1$;
$v = 5: \quad x + 2y = 5$;
$u = -1: \quad x - y = -1$;
$u = 4: x - y = 4$



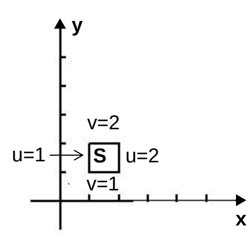


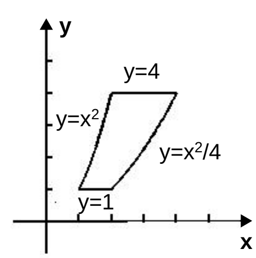
5. The uv-corner points (0.0), $(1,0)$, $(1,2)$, $(0,2)$ correspond to the xy-points $(0,0)$, $(1,0)$, $(-3,2)$, $(-4,0)$.
$v = 0: \quad x = u^2, \quad y = 0 \Longrightarrow y = 0$ and $0 \le x \le 1$
$u = 1: \quad x = 1 - v^2, \quad y = v \Longrightarrow x = 1 - y^2$
$v = 2: \quad x = u^2 - 4, \quad y = 2u \Longrightarrow x = y^2/4 - 4$
$u = 0: \quad x = -v^2, \quad y = 0 \Longrightarrow y = 0$ and $-4 \le x \le 0$





6. The uv-corner points $(1,1)$, $(2,1)$, $(2,2)$, $(1,2)$ correspond to the xy-points $(1,1)$, $(2,1)$, $(4,4)$, $(2,4)$.
$v = 1: \quad x = u, \quad y = 1 \Longrightarrow y = 1, \quad 1 \le x \le 2$
$u = 2: \quad x = 2v, \quad y = v^2 \Longrightarrow y = x^2/4$
$v = 2: \quad x = 2u, \quad y = 4 \Longrightarrow y = 4, \quad 2 \le x \le 4$
$u = 1: \quad x = v, \quad y = v^2 \Longrightarrow y = x^2$





7. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -ve^{-u} & e^{-u} \\ ve^{u} & e^{u} \end{vmatrix} = -2v$

8. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 3e^{3u}\sin v & e^{3u}\cos v \\ 3e^{3u}\cos v & -e^{3u}\sin v \end{vmatrix} = -3e^{6u}$

9. $\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -2y/x^3 & 1/x^2 \\ -y^2/x^2 & 2y/x \end{vmatrix} = -\dfrac{3y^2}{x^4} = -3(\dfrac{y}{x^2})^2 = -3u^2$; $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{-3u^2} = -\dfrac{1}{3u^2}$

10. $\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{2(y^2-x^2)}{x^2+y^2)^2} & \dfrac{-4xy}{(x^2+y^2)^2} \\ \dfrac{4xy}{(x^2+y^2)^2} & \dfrac{2(y^2-x^2)}{(x^2+y^2)^2} \end{vmatrix} = \dfrac{4}{(x^2+y^2)^2}$

From $u = 2x/(x^2+y^2)$ and $v = -2y(x^2+y^2)$ we obtain $u^2 = v^2 = 4/(x^2+y^2)$. Then $x^2+y^2 = 4/(u^2+v^2)$ and $\partial(x,y)/\partial(u,v) = (x^2+y^2)^2/4 = 4/(u^2+v^2)^2$.

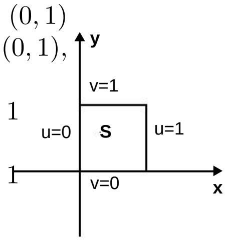
11. (a) The uv-corner points $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$ correspond to the xy-points $(0,0)$, $(1,0)$, $(0,1)$, $(0,0)$.

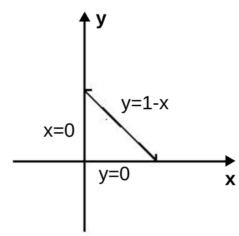
$v = 0: \quad x = u, \quad y = 0 \Longrightarrow y = 0, \quad 0 \le x \le 1$

$u = 1: \quad x = 1 - v, \quad y = v \Longrightarrow y = 1 - x$

$v = 1: \quad x = 0, \quad y = u \Longrightarrow x = 0, \quad 0 \le y \le 1$

$u = 0: \quad x = 0, \quad y = 0$





(b) Since the segment $u = 0$, $0 \le v \le 1$ in the uv-plane maps to the origin in the xy-plane, the transformation is not one-to-one.

12. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1-v & v \\ -u & u \end{vmatrix} = u$. The transformation is 0 when u is 0, for $0 \le v \le 1$.

13. $R1: \quad x + y = -1 \Longrightarrow v = -1$

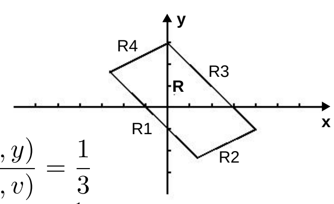
$R2: \quad x - 2y = 6 \Longrightarrow u = 6$

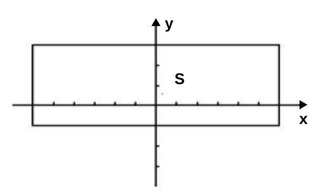
$R3: \quad x + y = 3 \Longrightarrow v = 3$

$R4: \quad x - 2y = -6 \Longrightarrow u = -6$

$\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} = 3 \Longrightarrow \dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{3}$

$\int\int_R (x+y)dA = \int\int_S v(\dfrac{1}{3})dA' = \dfrac{1}{3}\int_{-1}^{3}\int_{-6}^{6} v\,du\,dv = \dfrac{1}{3}(12)\int_{-1}^{3} v\,dv = 4(\dfrac{1}{2})v^2\Big|_{-1}^{3} = 16$

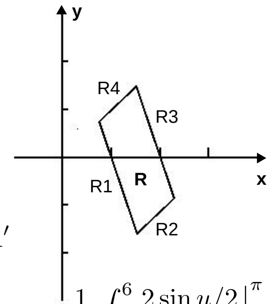


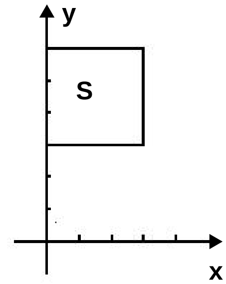


14. $R1: \quad y = -3x + 3 \Longrightarrow v = 3$
$R2: \quad y = x - \pi \Longrightarrow u = \pi$
$R3: \quad y = -3x + 6 \Longrightarrow v = 6$
$R4: \quad y = x \Longrightarrow u = 0$

$$\frac{\partial(u,v)}{\partial(x,y)} \begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix} = 4 \Longrightarrow \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{4}$$

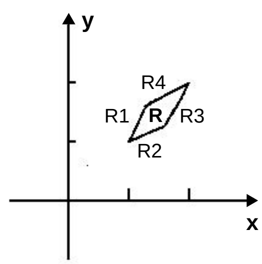
$$\int\int_R \frac{\cos\frac{1}{2}(x-y)}{3x+y} dA = \int\int_S \frac{\cos u/2}{v}\left(\frac{1}{4}\right)dA'$$
$$= \frac{1}{4}\int_3^6\int_0^\pi \frac{\cos u/2}{v} dudv = \frac{1}{4}\int_3^6 \frac{2\sin u/2}{v}\bigg|_0^\pi dv$$
$$= \frac{1}{2}\int_3^6 \frac{dv}{v} = \frac{1}{2}\ln v\bigg|_3^6 = \frac{1}{2}\ln 2$$

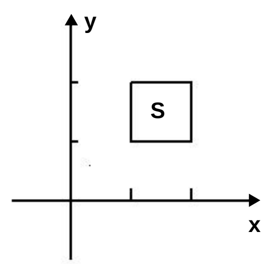




15. $R1: \quad y = x^2 \Longrightarrow u = 1$
$R2: \quad x = y^2 \Longrightarrow v = 1$
$R3: \quad y = \frac{1}{2}x^2 \Longrightarrow u = 2$
$R4: \quad x = \frac{1}{2}y^2 \Longrightarrow v = 2$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2x/y & -x^2/y^2 \\ -y^2/x^2 & 2y/x \end{vmatrix} = 3$$
$$\Longrightarrow \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{3}$$
$$\int\int_R \frac{y^2}{x} dA = \int\int_S v\left(\frac{1}{3}\right)dA' = \frac{1}{3}\int_1^2\int_1^2 vdudv = \frac{1}{3}\int_1^2 vdv = \frac{1}{6}v^2\bigg|_1^2 = \frac{1}{2}$$



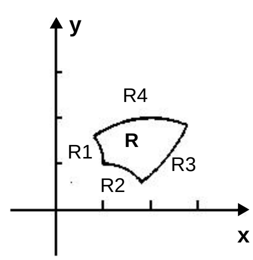


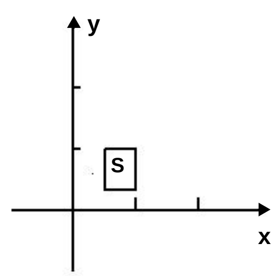
16. $R1: \quad x^2 + y^2 = 2y \Longrightarrow v = 1$
$R2: \quad x^2 + y^2 = 2x \Longrightarrow u = 1$
$R3: \quad x^2 + y^2 = 6y \Longrightarrow v = 1/3$
$R4: \quad x^2 + y^2 = 4x \Longrightarrow u = 1/2$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{2(y^2-x^2)}{(x^2+y^2)^2} & \dfrac{-4xy}{(x^2+y^2)^2} \\ \dfrac{-4xy}{(x^2+y^2)^2} & \dfrac{2(x^2-y^2)}{(x^2+y^2)^2} \end{vmatrix}$$
$$= \frac{-4}{(x^2+y^2)^2}$$

Using $u^2 + v^2 = 4/(x^2+y^2)$ we see that $\partial(x,y)/\partial(u,v) = -4/(u^2+v^2)^2$.

$$\int\int_R (x^2+y^2)^{-3} dA = \int\int_S \left(\frac{4}{u^2+v^2}\right)^{-3}\left|\frac{-4}{(u^2+v^2)^2}\right| dA' = \frac{1}{16}\int_{1/3}^1\int_{1/2}^1 (u^2+v^2)dudv = \frac{115}{5184}$$



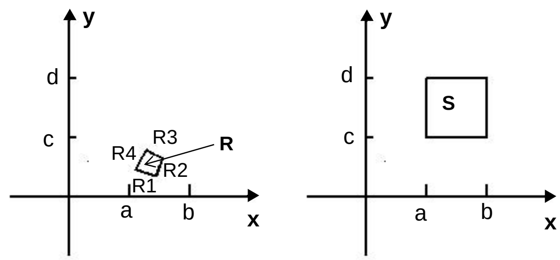


17. $R1: \quad 2xy = c \Longrightarrow v = c$
$R2: \quad x^2 - y^2 = b \Longrightarrow u = b$
$R3: \quad 2xy = d \Longrightarrow v = d$
$R4: \quad x^2 - y^2 = a \Longrightarrow u = a$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4(x^2+y^2)$$

$$\Longrightarrow \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{4(x^2+y^2)}$$

$$\textstyle\int\int_R (x^2+y^2)dA = \int\int_S (x^2+y^2)\frac{1}{4(x^2+y^2)}dA' = \frac{1}{4}\int_c^d\int_a^b dudv = \frac{1}{4}(b-a)(d-c)$$



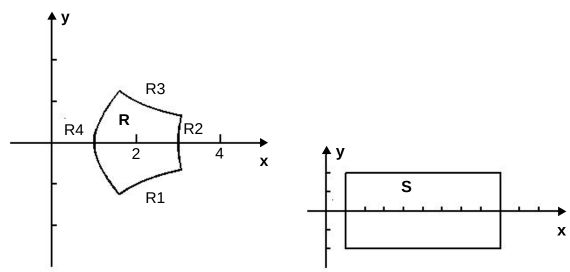
18. $R1: \quad xy = -2 \Longrightarrow v = -2$
$R2: \quad x^2 - y^2 = 9 \Longrightarrow u = 9$
$R3: \quad xy = 2 \Longrightarrow v = 2$
$R4: \quad x^2 - y^2 = 1 \Longrightarrow u = 1$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2x & -2y \\ y & x \end{vmatrix} = 2(x^2+y^2)$$

$$\Longrightarrow \frac{\partial x,y)}{\partial(u,v)} = \frac{1}{2(x^2+y^2)}$$

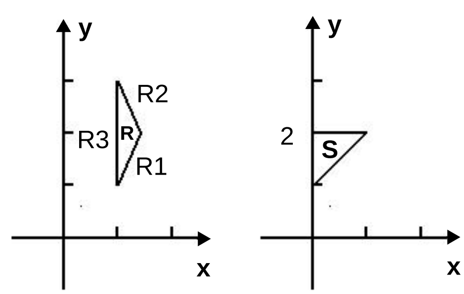
$$\int\int_R (x^2+y^2)\sin xy dA = \int\int_S (x^2+y^2)\sin v(\frac{1}{2(x^2+y^2)})dA' = \frac{1}{2}\int_{-2}^{2}\int_1^9 \sin v du dv$$

$$= \frac{1}{2}\int_{-2}^{2} 8\sin v dv = 0$$



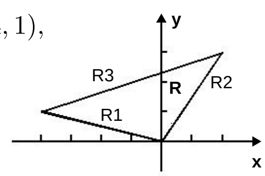
19. $R1: \quad y = x^2 \Longrightarrow v + u = v - u \Longrightarrow u = 0$
$R2: \quad y = 4 - x^2 \Longrightarrow v + u = 4 - (v-u)$
$\Longrightarrow v + u = 4 - v + u \Longrightarrow v = 2$
$R3: \quad x = 1 \Longrightarrow v - u = 1 \Longrightarrow v = 1 + u$

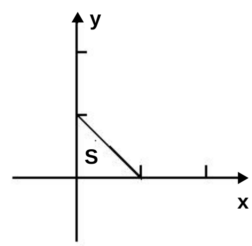
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -\dfrac{1}{2\sqrt{v-u}} & \dfrac{1}{2\sqrt{v-u}} \\ 1 & 1 \end{vmatrix} = -\frac{1}{\sqrt{v-u}}$$



$$\int\int_R \frac{x}{y+x^2}dA = \int\int_S \frac{\sqrt{v-u}}{2v}\left|-\frac{1}{\sqrt{v-u}}\right|dA'$$
$$= \frac{1}{2}\int_0^1\int_{1+u}^2 \frac{1}{v}dvdu = \frac{1}{2}\int_0^1[\ln 2 - \ln(1+u)]du$$
$$= \frac{1}{2}\ln 2 - \frac{1}{2}[(1+u)\ln(1+u) - (1+u)]\Big|_0^1 = \frac{1}{2}\ln 2 - \frac{1}{2}[2\ln 2 - 2 - (0-1)]$$
$$= \frac{1}{2} - \frac{1}{2}\ln 2$$

20. Solving $x = 2u - 4v,\ \ y = 3u + v$ for u and v we obtain $u = \frac{1}{14}x + \frac{2}{7}y$, $v = -\frac{3}{14}x + \frac{1}{7}y$. The xy-corner points $(-4,1)$, $(0,0)$, $(2,3)$ correspond to the uv-points $(0,1)$, $(0,0)$, $(1,0)$.

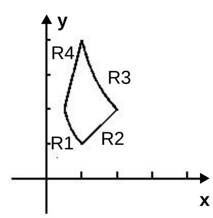


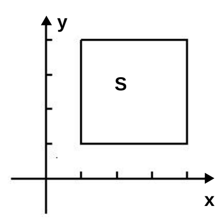


$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2 & -4 \\ 3 & 1 \end{vmatrix} = 14$$

$$\int\int_R ydA = \int\int_S (3u+v)(14)dA' = 14\int_0^1\int_0^{1-u}(3u+v)dvdu = 14\int_0^1 (3uv + \frac{1}{2}v^2)\Big|_0^{1-u} du$$
$$= 14\int_0^1 (\frac{1}{2} + 2u - \frac{5}{2}u^2)du = (7u + 14u^2 - \frac{35}{3}u^3)\Big|_0^1 = \frac{28}{3}$$

21. $R1:\ \ y = 1/x \Longrightarrow u = 1$
$R2:\ \ y = x \Longrightarrow v = 1$
$R3:\ \ y = 4/x \Longrightarrow u = 4$
$R4:\ \ y = 4x \Longrightarrow v = 4$





$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} y & x \\ -y/x^2 & 1/x \end{vmatrix} = \frac{2y}{x} \Longrightarrow \frac{\partial(x,y)}{\partial(u,v)} = \frac{x}{2y}$$

$$\int\int_R y^4 dA = \int\int_S u^2v^2(\frac{1}{2v})dudv = \frac{1}{2}\int_1^4 u^2 vdudv = \frac{1}{2}\int_1^4 \frac{1}{3}u^3v\Big|_1^4 dv = \frac{1}{6}\int_1^4 63vdv$$
$$= \frac{21}{4}v^2\Big|_1^4 = \frac{315}{4}$$

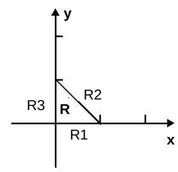
22. Under the transformation $u = y + z,\ \ v = -y + z,\ \ w = x - y$ the parallelepiped D is mapped

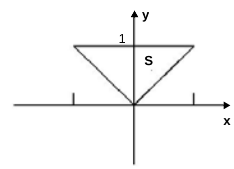
to the parallelepiped $E: \ 1 \le u \le 3, \ \ -1 \le v \le 1, \ \ 0 \le w \le 3.$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & -1 & 0 \end{vmatrix} = 2 \Longrightarrow \frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{2}$$

$$\begin{aligned} \int\int_D (4z+2x-2y)dV &= \int\int_E (2u+2v+2w)\frac{1}{2}dV' = \frac{1}{2}\int_0^3\int_{-1}^1\int_1^3 (2u+2v+2w)dudvdw \\ &= \frac{1}{2}\int_0^3\int_{-1}^1 (u^2+2uv+2uw)\Big|_1^3 dvdw = \frac{1}{2}\int_0^3 int_{-1}^1(8+4v+4w)dvdw \\ &= \int_0^3 (4v+v^2+2vw)\Big|_{-1}^1 dw = \int_0^3 (8+4w)dw = (8w+2w^2)\big|_0^3 = 42 \end{aligned}$$

23. We let $u = y - x$ and V=y+x.





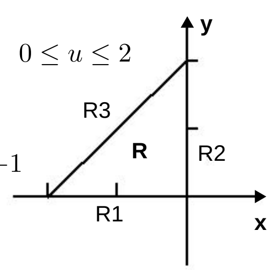
$R1: \ \ y = 0 \Longrightarrow u = -x, \ \ v = x \Longrightarrow v = -u$
$R2: \ \ x + y = 1 \Longrightarrow v = 1$
$R3: \ \ x = 0 \Longrightarrow u = y, \ \ v = y, \ \ \Longrightarrow v = u$

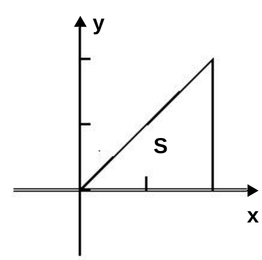
$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -2$$

$$\Longrightarrow \frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{2}$$

$$\begin{aligned} \int\int_R e^{(y-x)/(y+x)}dA &= \int\int_S e^{u/v}\left|-\frac{1}{2}\right| dA' \\ &= \frac{1}{2}\int_0^1\int_{-v}^v e^{u/v}dudv = \frac{1}{2}\int_0^1 ve^{u/v}\Big|_{-v}^v dv \\ &= \frac{1}{2}\int_0^1 v(e-e^{-1})dv = \frac{1}{2}(e-e^{-1})\frac{1}{2}v^2\Big|_0^1 = \frac{1}{4}(e-e^{-1}) \end{aligned}$$

24. We let $u = y - x$ and $v = y$.





$R1: \ \ y = 0 \Longrightarrow v = 0, \ \ u = -x \Longrightarrow v = 0, \ \ 0 \le u \le 2$
$R2: \ \ x = 0 \Longrightarrow v = u$
$R3: \ \ y = x + 2 \Longrightarrow u = 2$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} = -1 \Longrightarrow \frac{\partial(x,y)}{\partial(u,v)} = -1$$

$$\begin{aligned} \int\int_R e^{y^2-2xy+x^2}dA &= \int\int_S e^{u^2}\,|-1|\,dA' \\ &= \int_0^2\int_0^u e^{u^2}dvdu = \int_0^2 ue^{u^2}du = \frac{1}{2}e^{u^2}\Big|_0^2 = \frac{1}{2}(e^4-1) \end{aligned}$$

25. Noting that $R2$, $R3$, and $R4$ have equations $y + 2x = 8$, $y - 2x = 0$, and $y + 2x = 2$, we let $u = y/x$ and $v = y + 2x$.

$R1: \ y = 0 \Longrightarrow u = 0, \ \ v = 2x \Longrightarrow u = 0, 2 \le v \le 8$

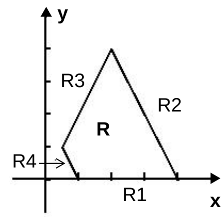
$R2: \ y + 2x = 8 \Longrightarrow v = 8$

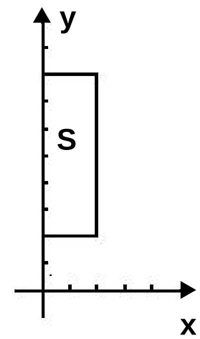
$R3: \ y - 2x = 0 \Longrightarrow u = 2$

$R4: \ y + 2x = 2 \Longrightarrow v = 2$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -y/x^2 & 1/x \\ 2 & 1 \end{vmatrix} = \frac{y+2x}{x^2}$$

$$\Longrightarrow \frac{\partial(x,y)}{\partial(u,v)} = \frac{x^2}{y+2x}$$





$\int\int_R (6x + 3y)dA = 3\int\int_S (y + 2x)\left|-\dfrac{x^2}{y+2x}\right| dA' = 3\int\int_S x^2 dA'$

From $y = ux$ we see that $v = ux + 2x$ and $x = v/(u+2)$. Then

$3\int\int_S x^2 dA' = 3\int_0^2\int_2^8 v^2(u+2)^2 dv du = \int_0^2 \left.\dfrac{v^3}{(u+2)^2}\right|_2^8 du =$

$504\int_0^2 \dfrac{du}{(u+2)^2} = -\left.\dfrac{504}{u+2}\right|_0^2 = 126.$

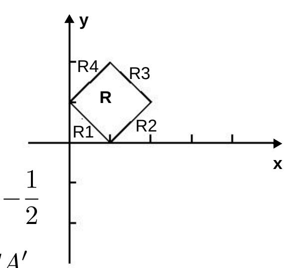
26. We let $u = x + y$ and $v = x - y$.

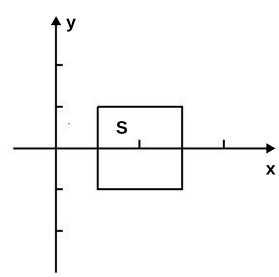
$R1: \ x + y = 1 \Longrightarrow u = 1$

$R2: \ x - y = 1 \Longrightarrow v = 1$

$R3: \ x + y = 3 \Longrightarrow u = 3$

$R4: \ x - y = -1 \Longrightarrow v = -1$





$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \Longrightarrow \frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{2}$$

$$\int\int_R (x+y)^4 e^{x-y} dA = \int\int_S u^4 e^v \left|-\frac{1}{2}\right| dA'$$

$$= \frac{1}{2}\int_1^3\int_{-1}^1 u^4 e^v dv du = \frac{1}{2}\int_1^3 u^4 e^v\Big|_{-1}^1 du$$

$$= \frac{e - e^{-1}}{2}\int_1^3 u^4 du = \frac{e-e^{-1}}{10}u^5\Big|_1^3 = \frac{242(e-e^{-1})}{10} = \frac{121}{5}(e - e^{-1})$$

27. The image of the ellipse is the unit circle $x^2 + y^2 = 1$. From $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 5 & 0 \\ 0 & 3 \end{vmatrix} = 15$ we

obtain

$$\int\int_R (\frac{x^2}{25}+\frac{y^2}{9})dA = \int\int_s (u^2+v^2)15dA' = 15\int_0^{2\pi}\int_0^1 r^2 r dr d\theta = \frac{15}{4}\int_0^{2\pi} r^4\Big|_0^1 d\theta$$
$$= \frac{15}{4}\int_0^{2\pi} d\theta = \frac{15\pi}{2}.$$

28. $$\frac{\partial(x,y,z)}{\partial(\rho,\omega,\theta)} = \begin{vmatrix} \sin\omega\cos\theta & \rho\cos\omega\cos\theta & -\rho\sin\omega\sin\theta \\ \sin\omega\sin\theta & \rho\cos\omega\sin\theta & \rho\sin\omega\cos\theta \\ \cos\omega & -\rho\sin\omega & 0 \end{vmatrix}$$
$$= \cos\omega(\rho^2\sin\omega\cos\omega\cos^2\theta + \rho^2\sin\omega\cos\omega\sin^2\theta) + \rho\sin\omega(\rho\sin^2\omega\cos^2\theta + \rho\sin^2\omega\sin^2\theta)$$
$$= \rho^2\sin\omega\cos^2\omega(\cos^2\theta+\sin^2\theta) + \rho^2\sin^3\omega(\cos^2\theta+\sin^2\theta) = \rho^2\sin\omega(\cos^2\omega+\sin^2\omega)$$
$$= \rho^2\sin\omega$$

29. The image of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ under the transformation $u = x/a,\ \ v = y,\ \ w = z/c$, is the unit sphere $u^2+v^2+w^2 = 1$. The volume of this sphere is $\frac{4}{3}\pi$. Now

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$$

and

$$\int\int\int_D dV = \int\int\int EabcdV' = abc\int\int\int_E dV' = abc\left(\frac{4}{3}\pi\right) = \frac{4}{3}\pi abc.$$

30. Let $u = xy$ and $v = xy^{1.4}$. Then $xy^{1.4} = c \Longrightarrow v = c;\ \ xy = b \Longrightarrow u = b;\ \ xy^{1.4} = d \Longrightarrow v = d;\ \ xy = a \Longrightarrow u = a.$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} y & x \\ y^{1.4} & 1.4xy^{0.4} \end{vmatrix} = 0.4xy^{1.4} = 0.4v \Longrightarrow \frac{\partial(x,y)}{\partial(u,v)} = \frac{5}{2v}$$
$$\int\int_R dA = \int\int_S \frac{5}{2v}dA' = \int_c^d\int_a^b \frac{5}{2v}dudv = \frac{5}{2}(b-a)\int_c^d \frac{dv}{v} = \frac{5}{2}(b-a)(\ln d - \ln c)$$

Chapter 14 in Review

A. True/False

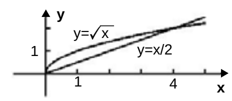
1. True; use $e^{x^2-y} = e^{x^2}e^{-y}$ and Problem 53 in Section 14.2
2. True
3. True
4. False; consider $f(x,y) = x$.
5. False; both the density function and the lamina must be symmetric about an axis.
6. True; the equation of the plane is $\theta = \frac{\pi}{4},\ \ \theta = \frac{5\pi}{4}$

B. Fill in the Blanks

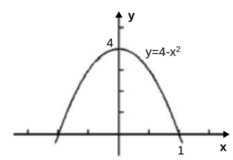
1. $\displaystyle\int_{y^2+1}^{5}\left(8y^3-\frac{5y}{x}\right)dx=(8xy^3-5y\ln x)\Big|_{y^2+1}^{5}=40y^3-5y\ln 5-[8(y^2+1)y^3-5y\ln(y^2+1)]$

$$=-8y^5+32y^3+5y\ln\frac{y^2+1}{y}$$

2. 16
3. square
4. II
5. $f(x,4)-f(x,2)$
6. $\displaystyle\int_{-a}^{a}\int_{-b\sqrt{1-x^2/a^2}}^{b\sqrt{1-x^2/a^2}}\int_{-c\sqrt{1-x^2/a^2-y^2/b^2}}^{c\sqrt{1-x^2/a^2-y^2/b^2}}\rho(x,y,z)dzdydx$
7. $\int_0^4\int_{x/2}^{\sqrt{x}}f(x,y)dydx$



8. $x=6\sin(5\pi/3)\cos(5\pi/6)=9/2$; $y=6\sin(5\pi/3)\sin(5\pi/6)=-3\sqrt{3}/2$; $z=6\cos(5\pi/3)=3$
   The point is $(9/2,\ -3\sqrt{3}/2,3)$.
9. $r=2\sin(\pi/4)=\sqrt{2}$; $\theta=2\pi/3$; $z=2\cos(\pi/4)=\sqrt{2}$; $(\sqrt{2},2\pi/3,\sqrt{2})$
10. $\displaystyle\int_0^4\int_{-\sqrt{4-y}}^{\sqrt{4-y}}f(x,y)dxdy$



11. $z=r^2$; $\rho=\cot\phi\csc\phi$
12. circle

C. Exercises

1. Holding x fixed, $\displaystyle\int(12x^2e^{-4xy}-5x+1)dy=\frac{12x^2e^{-4xy}}{-4x}=-5xy+y+c_1(x)$

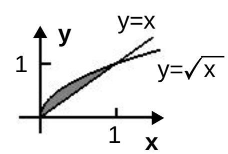
$$=-3xe^{-4xy}-5xy+y+c_1(x)$$

2. Holding y fixed,
   $\displaystyle\int\frac{1}{4+3xy}dx=\frac{\ln|3xy+4|}{3y}+c_2(y)$
3. $\displaystyle\int_{y^3}^{y}y^2\sin xydx=-y\cos xy|_{y^3}^{y}=y(\cos y^4-\cos y^2)$
4. $\displaystyle\int_{1/x}^{e^x}\frac{x}{y^2}dy=-\frac{x}{y}\Big|_{1/x}^{e^x}=x^2-x/e^x$

5. $$\int_0^2\int_0^{2x} ye^{y-x}dydx = \int_0^2 (ye^{y-x}-e^{y-x})\Big|_0^{2x} dx \quad \boxed{\text{Integration by parts}}$$
$$= \int_0^2 (2xe^x - e^x + e^{-x})dx \quad \boxed{\text{Integration by parts}}$$
$$= (2xe^x - 2e^x - e^x - e^{-x})\Big|_0^2 = e^2 - e^{-2} + 4$$

6. $$\int_0^4\int_x^4 \frac{1}{16+x^2}dydx = \int_0^4 \frac{y}{16+x^2}\Big|_x^4 = \int_0^4 \left(\frac{4}{16+x^2} - \frac{x}{16+x^2}\right)dx = \left[\tan^{-1}\frac{x}{4} - \frac{1}{2}\ln(16+x^2)\right]\Big|_0^4$$
$$= \left(\frac{\pi}{4} - \frac{1}{2}\ln 32\right) - \left(0 - \frac{1}{2}\ln 16\right) = \frac{\pi}{4} + \frac{1}{2}(\ln 16 - \ln 32) = \frac{\pi}{4} + \frac{1}{2}\ln\frac{1}{2}$$

7. $$\int_0^1\int_x^{\sqrt{x}} \frac{\sin y}{y}dydx = \int_0^1\int_{y^2}^y \frac{\sin y}{y}dxdy = \int_0^1 \frac{\sin y}{y}x\Big|_{y^2}^y dy$$



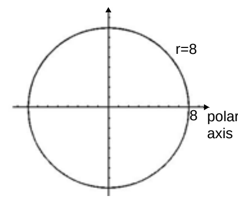
$$= \int_0^1 (\sin y - y\sin y)dy \quad \boxed{\text{Integration by parts}}$$
$$= (-\cos y - s\in y + y\cos y)\Big|_0^1 = (-\cos 1 - \sin 1 + \cos 1) - (-1) = 1 - \sin 1$$

8. $$\int_e^{e^2}\int_0^{1/x} \ln x dydx = \int_e^{e^2} y\ln x\Big|_0^{1/x} dx = \int_e^{e^2} \frac{1}{x}\ln x dx = \frac{1}{2}(\ln x)^2\Big|_e^{e^2} = \frac{1}{2}(2x^2 - 1^2) = \frac{2}{3}$$

9. $$\int_0^5\int_0^{\pi/2}\int_0^{\cos\theta} 3r^2drd\theta dz = \int_0^5\int_0^{\pi/2} r^3\Big|_0^{\cos\theta} d\theta dz = \int_0^5\int_0^{\pi/2} \cos^3\theta d\theta dz$$
$$= \int_0^5\int_0^{\pi/2} (1-\sin^2\theta)\cos\theta d\theta dz = \int_0^5 \left(\sin\theta - \frac{1}{3}\sin^3\theta\right)\Big|_0^{\pi/2} dz$$
$$= \int_0^5 2/3dz = 10/3$$

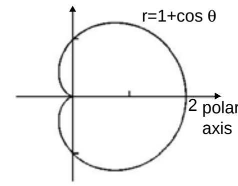
10. $$\int_{\pi/4}^{\pi/2}\int_0^{\sin x}\int_0^{\ln x} e^y dydxdz = \int_{\pi/4}^{\pi/2}\int_0^{\sin x} e^y\Big|_0^{\ln x} dxdz = \int_{\pi/4}^{\pi/2}\int_0^{\sin x} (x-1)dxdz$$
$$= \int_{\pi/4}^{\pi/2} \left(\frac{1}{2}x^2 - x\right)\Big|_0^{\sin z} = \int_{\pi/4}^{\pi/2} \left(\frac{1}{2}\sin^2 z - \sin z\right)dz$$
$$= \left(\frac{1}{4}z - \frac{1}{8}\sin 2z + \cos z\right)\Big|_{\pi/4}^{\pi/2} = \frac{\pi}{8} - \left(\frac{\pi}{16} - \frac{1}{8} + \frac{\sqrt{2}}{2}\right)$$
$$= \frac{\pi + 2 - 8\sqrt{2}}{16}$$

11. Using polar coorindates,



$$\int\int_R 5dA = \int_0^{2\pi}\int_0^8 5rdrd\theta = 5\int_0^{2\pi} \frac{1}{2}r^2\Big|_0^8 d\theta = 5\int_0^{2\pi} 32d\theta = 320\pi.$$

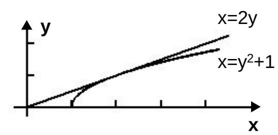
12. Using symmetry, $\displaystyle\int\int_R dA = 2\int_0^{\pi}\int_0^{1+\cos\theta} rdrd\theta = 2\int_0^{\pi} \frac{1}{2}r^2\Big|_0^{1+\cos\theta} d\theta$



$$= \int_0^{\pi}(1 + 2\cos\theta + \cos^2\theta)d\theta$$

$$= \left(\theta + 2\sin\theta + \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right)\Big|_0^{\pi} = 3\pi/2.$$

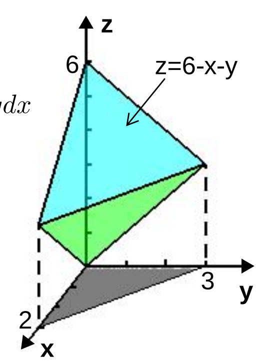
13. $\displaystyle\int\int_R (2x+y)dA = \int_0^1\int_{2y}^{y^2+1}(2x+y)dxdy = \int_0^1 (x^2+xy)\Big|_{2y}^{y^2+1} dy$



$$= \int_0^1 [(y^2+1)^2 + (y^2+1)y - (4y^2+2y^2)]dy$$

$$= (y^4 + y^3 - 4y^2 + y + 1)dy = \left(\frac{1}{5}y^5 + \frac{1}{4}y^4 - \frac{4}{3}y^3 + \frac{1}{2}y^2 + y\right)\Big|_0^1 = \frac{37}{60}$$

14. Substracting $z = 6 - x - y$ from $z = x + y$, we obtain $x + y = 3$.



$$\int\int\int_R xdV = \int_0^3\int_0^{3-x}\int_{x+y}^{6-x-y} xdzdydx = \int_0^3\int_0^{3-x} xz\Big|_{x+y}^{6-x-y} dydx$$

$$= \int_0^3\int_0^{3-x}(6x - 2x^2 - 2xy)dydx$$

$$= \int_0^3 (6xy - 2x^2y - xy^2)\Big|_0^{3-x} dx$$

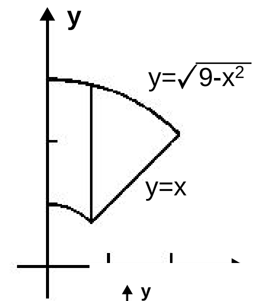
$$= \int_0^3 [6x(3-x) - 2x^2(3-x) - x(3-x)^2]dx$$

$$= \int_0^3 (9x - 6x^2 + x^3)dx$$

$$= \left(\frac{9}{2}x^2 - 2x^3 + \frac{1}{4}x^4\right)\Big|_0^3 = \frac{27}{4}$$

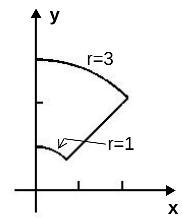
15. The circle $x^2 + y^2 = 1$ intersects $y = x$ at $x = 1/\sqrt{2}$. The circle $x^2 + y^2 = 9$ intersects $y = x$ at $x = 3/\sqrt{2}$.

$$\int\int_R \frac{1}{x^2+y^2}dA = \int_0^{1/\sqrt{2}} \int_{\sqrt{1-x^2}}^{\sqrt{9-x^2}} \frac{1}{x^2+y^2}dydx + \int_{1/\sqrt{2}}^{3/\sqrt{2}} \int_x^{\sqrt{9-x^2}} \frac{1}{x^2+y^2}dydx$$

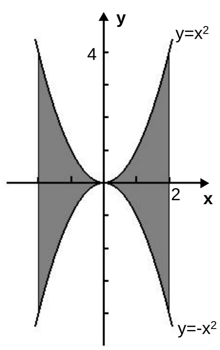


16. The circles are $r = 1$ and $r = 3$; the line is $\theta = \pi/4$.

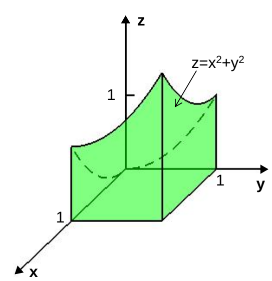
$$\int\int_R \frac{1}{x^2+y^2}dA = \int_{\pi/4}^{\pi/2} \int_1^3 \frac{1}{r^2} r dr d\theta = \int_{\pi/4}^{\pi/2} \ln r \Big|_1^3 d\theta = \int_{\pi/4}^{\pi/2} \ln 3 d\theta = \frac{\pi}{4}\ln 3$$



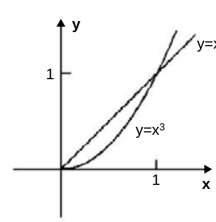
17.



18. The region is symmetric with respect to the xz- and yz-planes and is shown in the first octant.



19. $$\int_0^1 \int_y^{\sqrt[3]{y}} \cos x^2 dxdy = \int_0^1 \int_{x^3}^x \cos x^2 dydx = \int_0^1 y\cos x^2\Big|_{x^3}^x dx = \int_0^1 (x\cos x^2 - x^3\cos x^2)dx$$



$$= \frac{1}{2}\sin x^2\Big|_0^1 = \int_0^1 x^2(x\cos x^2)dx$$

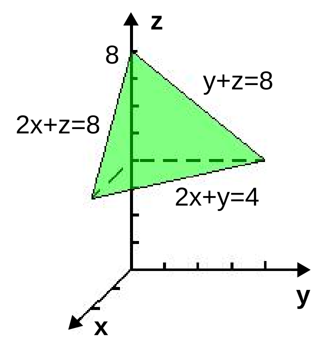
$\boxed{\text{Integration by parts}}$

$$= \frac{1}{2}\sin 1 - \left(\frac{1}{2}x^2\sin x^2 + \frac{1}{2}\cos x^2\right)\Big|_0^1$$

$$= \frac{1}{2}\sin 1 - \left(\frac{1}{2}\sin 1 + \frac{1}{2}\cos 1 - \frac{1}{2}\right)$$

$$= \frac{1-\cos 1}{2}$$

20. The six forms of the integral are:



$$\int_0^2\int_0^{4-2x}\int_4^{8-2x-y} F(x,y,z)dzdydx;$$
$$\int_0^4\int_0^{2-y/2}\int_4^{8-2x-y} F(x,y,z)dzdxdy;$$
$$\int_4^8\int_0^{8-z}\int_0^{4-y/2-z/2} F(x,y,z)dxdydz;$$
$$\int_0^4\int_4^{8-y}\int_0^{4-y/2-z/2} F(x,y,z)dxdzdy;$$
$$\int_4^8\int_0^{4-z/2}\int_0^{8-2x-z} F(x,y,z)dydxdz;$$
$$\int_0^0\int_4^{8-2x}\int_0^{8-2x-z} F(x,y,z)dydzdx.$$

21. $$\int_0^2\int_{1/2}^1\int_0^{\sqrt{x-x^2}}(4z+1)dydxdz = \int_0^2\int_{1/2}^1(4z-1)\sqrt{x-^2}dxdz$$

$$= \int_0^2\int_{1/2}^1(4z+1)\sqrt{\frac{1}{4}-\left(x-\frac{1}{2}\right)^2}dxdz \quad \boxed{\text{Trig substitution}}$$

$$= \int_0^2\left[(4z+1)\left(\frac{x-1/2}{2}\sqrt{x-x^2}+\frac{1}{8}\sin^{-1}\frac{x-1/2}{1/2}\right)\right]\Bigg|_{1/2}^1 dz$$

$$= \int_0^2(4z+1)\left(\frac{\pi}{16}-0\right)dz = \frac{\pi}{16}(2z^2+z)\Big|_0^2 = \frac{5\pi}{8}$$

22. The region is the portion of the sphere of radius 1 centered at the origin in the first octant and the octant below that. Using spherical coordinates, we have

$$\int_0^1\int_0^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}}(x^2+y^2+z^2)^4 dzdydx = \int_0^{\pi/2}\int_0^{\pi}\int_0^1 \rho^8\rho^2\sin\phi d\rho d\phi d\theta$$
$$= \int_0^{\pi/2}\int_0^{\pi}\left(\frac{1}{11}\rho^1 1\sin\phi\right)\Bigg|_0^1 d\phi d\theta = \int_0^{\pi/2}\int_0^{\pi}\frac{1}{11}\sin\phi d\phi d\theta$$
$$= \frac{1}{11}\int_0^{\pi/2} -\cos\phi\Bigg|_0^{\pi} d\theta = \frac{1}{11}\int_0^{\pi/2} 2d\theta = \frac{\pi}{11}$$

23. $f_x = y;\;\; f_y = z,\;\; 1+f_x^2+x_y^2 = 11+x^2+y^2$. Using cylindrical coordinates,

$$A = \int_0^{2\pi}\int_0^1 \sqrt{1+r^2}rdrd\theta = \int_0^{2\pi}\frac{1}{3}(1+r^2)^{3/2}\Bigg|_0^1 d\theta = \frac{1}{3}\int_0^{2\pi}(2^{3/2}-1)d\theta = \frac{2\pi}{3}(2\sqrt{2}-1).$$

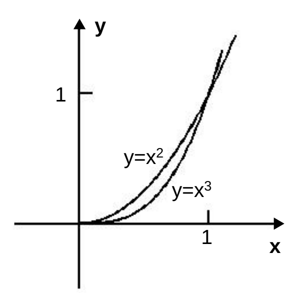
24. $V = \int_0^{\sqrt{3}}\int_{x^2}^3\left(6-\frac{2}{3}y^2\right)dydx = \int_0^{\sqrt{3}}\left(6y-\frac{2}{9}y^3\right)\Bigg|_{x^2}^3 dx = \int_0^{\sqrt{3}}\left[(18-6)-\left(6x^2-\frac{2}{9}x^6\right)\right]dx$

$$= \int_0^{\sqrt{3}}\left(12x-6x^2+\frac{2}{9}x^6\right)dx = \left(12x-2x^3+\frac{2}{63}x^7\right)\Bigg|_0^{\sqrt{3}} = 12\sqrt{3}-6\sqrt{3}+\frac{6}{7}\sqrt{3} = \frac{48}{7}\sqrt{3}$$

25. (a) $V = \int_0^1\int_x^{2x}\sqrt{1-x^2}dydx = \int_0^1 y\sqrt{1-x^2}\Bigg|_x^{2x} dx = \int_0^1 x\sqrt{1-x^2}dx$

$$= -\frac{1}{3}(1-x^2)^{3/2}\Bigg|_0^1 = \frac{1}{3}$$

(b) $V = \int_0^1\int_{y/2}^{y}\sqrt{1-x^2}dxdy + \int_1^2\int_{y/2}^{1}\sqrt{1-x^2}dxdy$



26. We are given $\rho = k(x^2+y^2)$.

$$m = \int_0^1\int_{x^3}^{x} k(x^2+y^2)dydx = k\int_0^1\left(x^2y+\frac{1}{3}y^3\right)\Bigg|_{x^3}^{x^2} dx$$
$$= k\int_0^1\left(x^4+\frac{1}{3}x^{-}x^5-\frac{1}{3}x^9\right)dx$$
$$= k\left(\frac{1}{5}x^5+\frac{1}{21}x^7-\frac{1}{6}x^6-\frac{1}{30}x^{10}\right)\Bigg|_0^1 = \frac{k}{21}$$

$$M_y = \int_0^1 \int_{x^3}^{x} k(x^3 + xy^2)dydx = k\int_0^1 \left(x^3y + \frac{1}{3}xy^3\right)\Bigg|_{x^3}^{x^2} dx = k\int_0^1 \left(x^5 + \frac{1}{3}x^7 - x^6 - \frac{1}{3}x^{10}\right)dx$$
$$= k\left(\frac{1}{6}x^6 + \frac{1}{24}x^8 - \frac{1}{7}x^7 - \frac{1}{33}x^{11}\right)\Bigg|_0^1 = \frac{65k}{1848}$$
$$M_x = \int_0^1 \int_{x^3}^{x} k(x^2y + y^3)dydx = k\int_0^1 \left(\frac{1}{2}x^2y^2 + \frac{1}{4}y^4\right)\Bigg|_{x^3}^{x^2} dx$$
$$= k\int_0^1 \left(\frac{1}{2}x^6 + \frac{1}{4}x^8 - \frac{1}{2}x^8 - \frac{1}{4}x^{12}\right)dx$$
$$= k\left(\frac{1}{14}x^7 - \frac{1}{36}x^9 - \frac{1}{52}x^{13}\right)\Bigg|_0^1 = \frac{20k}{819}$$
$$\overline{x} = M_y/m = \frac{65k/1848}{k/21} = 65/88; \overline{y} = M_x/m = \frac{20k/819}{k/21} = 20/39$$

The center of mass is $(65/88, 20/39)$.

27. $$I_y = \int_0^1 \int_{x^3}^{x^2} k(x^4 + x^2y^2)dydx = k\int_0^1 \left(x^4y + \frac{1}{3}x^2y^3\right)\Bigg|_{x^3}^{x^2} dx$$
$$= k\int_0^1 \left(x^6 + \frac{1}{3}x^8 - x^7 - \frac{1}{3}x^{11}\right)dx = k\left(\frac{1}{7}x^7 + \frac{1}{27}x^9 - \frac{1}{8}x^8 - \frac{1}{36}x^{12}\right)\Bigg|_0^1 = \frac{41}{1512}k$$

28. (a) Using symmetry,
$$V = 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dzdydx = 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}dydx$$
$\boxed{\text{Trig substitution}}$
$$= 8\int_0^a \left(\frac{y}{2}\sqrt{a^2-x^2-y^2} + \frac{a^2-x^2}{2}\sin^{-1}\frac{y}{\sqrt{a^2-x^2}}\right)\Bigg|_0^{\sqrt{a^2-x^2}} dx = 8\int_0^a \frac{\pi}{2}\frac{a^2-x^2}{2}dx$$
$$= 2\pi\left(a^2x - \frac{1}{3}x^3\right)\Bigg|_0^a = \frac{4}{3}\pi a^3$$

(b) Using symmetry,
$$V = 2\int_0^{2\pi}\int_0^a \int_0^{\sqrt{a^2-r^2}} rdzdrd\theta = 2\int_0^{2\pi}\int_0^a r\sqrt{a^2-r^2}drd\theta$$
$$= 2\int_0^{2\pi} -\frac{1}{3}(a^2-r^2)^{3/2}\Bigg|_0^a d\theta = \frac{2}{3}\int_0^{2\pi} a^3 d\theta = \frac{4}{3}\pi a^3$$

(c) $$V = \int_0^{2\pi}\int_0^{\pi}\int_0^a \rho^2\sin\phi d\rho d\phi d\theta = \int_0^{2\pi}\int_0^{\pi} \frac{1}{3}\rho^3\sin\phi\Bigg|_0^a d\phi d\theta$$
$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi} a^3\sin\phi d\phi d\theta = \frac{1}{3}\int_0^{2\pi} -a^3\cos\phi\Bigg|_0^{\pi} d\theta = \frac{1}{3}\int_0^{2\pi} 2a^3 d\theta = \frac{4}{3}\pi a^3$$

29. We use spherical coordinates.

$$
\begin{aligned}
V &= \int_0^{2\pi}\int_{\tan^{-1}1/2}^{\pi/4}\int_0^{3\sec\phi}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta\\
&= \int_0^{2\pi}\int_{\tan^{-1}1/2}^{\pi/4}\frac{1}{3}\rho^3\sin\phi\Big|_0^{3\sec\phi} d\phi\, d\theta\\
&= \frac{1}{3}\int_0^{2\pi}\int_{\tan^{-1}1/2}^{\pi/4} 27\sec^3\phi\sin\phi\, d\phi\, d\theta = 9\int_0^{2\pi}\int_{\tan^{-1}1/2}^{\pi/4}\tan\phi\sec^2\phi\, d\phi\, d\theta\\
&= 9\int_0^{2\pi}\frac{1}{2}\tan^2\phi\Big|_{\tan^{-1}1/2}^{\pi/4} d\theta = \frac{9}{2}\int_0^{2\pi}\left(1-\frac{1}{9}\right)d\theta = 8\pi
\end{aligned}
$$

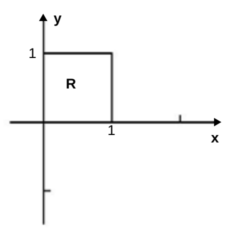
30.

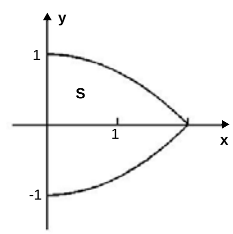
$$
\begin{aligned}
V &= \int_0^{2\pi}\int_0^{\pi/6}\int_1^2\rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi/6}\frac{1}{3}\rho^2\sin\phi\Big|_1^2 d\phi\, d\theta\\
&= \int_0^{2\pi}\int_0^{\pi/6}\left(\frac{8}{3}\sin\phi - \frac{1}{3}\sin\phi\right)d\phi\, d\theta = \frac{7}{3}\int_0^{2\pi}\int_0^{\pi/6}\sin\phi\, d\phi\, d\theta = \frac{7}{3}\int_0^{2\pi} -\cos\phi\Big|_0^{\pi/6} d\theta\\
&= \frac{7}{3}\int_0^{2\pi}\left[-\frac{\sqrt{3}}{2}-(-1)\right]d\theta = \frac{7}{3}\left(1-\frac{\sqrt{3}}{2}\right)2\pi = \frac{7\pi}{3}(2-\sqrt{3})
\end{aligned}
$$

31. $x = 0 \Longrightarrow u = 0,\ \ v = -y^2 \Longrightarrow u = 0,\ \ -1 \le v \le 0$
$x = 1 \Longrightarrow u = 2y,\ \ v = 1 - y^2 = 1 - u^2/4$
$x = 1 \Longrightarrow u = 2y,\ \ v = 1 - y^2 = 1 - u^2/4$
$y = 0 \Longrightarrow u = 0,\ \ v = x^2 \Longrightarrow u = 0,\ \ 0 \le v \le 1$
$y = 1 \Longrightarrow u = 2x,\ \ v = x^2 - 1 = u^2/4 - 1$

$$
\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2y & 2x \\ 2x & -2y \end{vmatrix} = -4(x^2+y^2)
$$

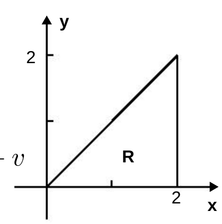
$$
\Longrightarrow \frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{4(x^2+y^2)}
$$

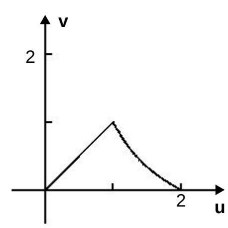




$$
\begin{aligned}
\int\int_R (x^2+y^2)\sqrt[3]{x^2+y^2}\,dA &= \int\int_S (x^2+y^2)\sqrt[3]{v}\left|-\frac{1}{4(x^2+y^2)}\right| dA' = \frac{1}{4}\int_0^2\int_{u^2/4-1}^{1-u^2/4} v^{1/3}\,dv\,du\\
&= \frac{1}{4}\int_0^2 \frac{3}{4}v^{4/3}\Big|_{u^2/4-1}^{1-u^2/4} du = \frac{3}{16}\int_0^2\left[(1-u^2/4)^{4/3}-(u^2/4-1)^{4/3}\right]du\\
&= \frac{3}{16}\int_0^2\left[(1-u^2/4)^{4/3}-(1-u^2/4)^{4/3}\right]du = 0
\end{aligned}
$$

32. $y = x \Longrightarrow u + uv = v + uv2 \Longrightarrow v = u$
$x = 2 \Longrightarrow u + uv = 2 \Longrightarrow v = (2-u)/u$
$y = 0 \Longrightarrow v = 0$ or $u = -1$
we take $v = 0\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1-w & u \\ v & 1+u \end{vmatrix} = 1 + u + v$





Using $x = u + uv$ and $y = v + uv$ we find

$$(x-y)^2 = (u + uv - v - uv)^2 = (u-v)^2 = u^2 - 2uv + v^2$$
$$x + y = u + uv + v + uv = u + v + 2uv$$
$$(x+y)^2 + 2(x+y) + 1 = u^2 + 2uv + v^2 + 2(u+v) + 1 = (u+v)^2 + 2(u+v) + 1 = (u+v+1)^2.$$

Then

$$\int\int_R \frac{1}{\sqrt{(x-y)^2 + 2(x+y) + 1}} dA = \int\int_S \frac{1}{u+v+1}(u+v+1)dA' = \int_0^1 \int_v^{2/(1+v)} du dv$$
$$= \int_0^1 \left(\frac{2}{1+v} - v\right) dv = \left[2\ln(1+v) - \frac{1}{2}v^2\right]\Bigg|_0^1 = 2\ln 2 - \frac{1}{2}.$$

Chapter 15

Vector Integral Calculus

15.1 Line Integrals

1. $$\int_C 2xy \ \mathrm{d}x = \int_0^{\pi/4} 2(5\cos t)(5\sin t)(-5\sin t) \ \mathrm{d}t = -250\int_0^{\pi/4} \sin^2 t\cos t \ \mathrm{d}t$$
$$= -250\left(\frac{1}{3}\sin^3 t\right)\Bigg|_0^{\pi/4} = -\frac{125\sqrt{2}}{6}$$
$$\int_C 2xy \ \mathrm{d}y = \int_0^{\pi/4} 2(5\cos t)(5\sin t)(5\cos t) \ \mathrm{d}t = 250\int_0^{\pi/4} \cos^2 t\sin t \ \mathrm{d}t = 250\left(-\frac{1}{3}\cos^3 t\right)\Bigg|_0^{\pi/4}$$
$$= \frac{250}{3}\left(1-\frac{\sqrt{2}}{4}\right) = \frac{125}{6}(4-\sqrt{2})$$
$$\int_C 2xy \ \mathrm{d}s = \int_0^{\pi/4} 2(5\cos t)(5\sin t)\sqrt{25\sin^2 t+25\cos^2 t} \ \mathrm{d}t = 250\int_0^{\pi/4} \sin t\cos t \ \mathrm{d}t$$
$$= 250\left(\frac{1}{2}\sin^2 t\right)\Bigg|_0^{\pi/4} = \frac{125}{2}$$

2. $$\int_C (x^3+2xy^2+2x) \ \mathrm{d}x = \int_0^1 [8t^3+2(2t)(t^4)+2(2t)]2 \ \mathrm{d}t = 2\int_0^1 (8t^3+4t^5+4t) \ \mathrm{d}t$$
$$= 2\int_0^1 (8t^3+4t^5+4t) \ \mathrm{d}t$$
$$= 2\left(2t^4+\frac{2}{3}t^6+2t^2\right)\Bigg|_0^1 = \frac{28}{3}$$
$$\int_C (x^3+2xy^2+2x) \ \mathrm{d}y = \int_0^1 [8t^3+2(2t)(t^4)+2(2t)]2t \ \mathrm{d}t = 2\int_0^1 (8t^4+4t^6+4t^2) \ \mathrm{d}t$$
$$= 2\left(\frac{8}{5}t^5+\frac{4}{7}t^7+\frac{4}{3}t^3\right)\Bigg|_0^1 = \frac{736}{105}$$

$$\int_C (x^3 + 2xy^2 + 2x)\ \mathrm{d}s = \int_0^1 [8t^3 + 2(2t)(t^4) + 2(2t)]\sqrt{4+4t^2}\ \mathrm{d}t = 8\int_0^1 t(1+t^2)^{5/2}\ \mathrm{d}t$$
$$= \left(\frac{1}{7}(1+t^2)^{7/2}\right)\Big|_0^1 = \frac{8}{7}(2^{7/2} - 1)$$

3. $$\int_C (3x^2 + 6y^2)\ \mathrm{d}x = \int_{-1}^0 [3x^2 + 6(2x+1)^2]\ \mathrm{d}x = \int_{-1}^0 (27x^2 + 24x + 6)\ \mathrm{d}x = (9x^3 + 12x^2 + 6x)\big|_{-1}^0$$
$$= -(-9 + 12 - 6) = 3$$
$$\int_C (3x^2 + 6y^2)\ \mathrm{d}y = \int_{-1}^0 [3x^2 + 6(2x+1)^2]2\ \mathrm{d}x = 6$$
$$\int_C (3x^2 + 6y^2)\ \mathrm{d}s = \int_{-1}^0 [3x^2 + 6(2x+1)^2]\sqrt{1+4}\ \mathrm{d}x = 3\sqrt{5}$$

4. $$\int_C \frac{x^2}{y^3}\ \mathrm{d}x = \int_1^8 \frac{x^2}{27x^2/8}\ \mathrm{d}x = \frac{8}{27}\int_1^8\ \mathrm{d}x = \frac{56}{27}$$
$$\int_C \frac{x^2}{y^3}\ \mathrm{d}y = \int_1^8 \frac{x^2}{27x^2/8}x^{-1/3}\ \mathrm{d}x = \frac{8}{27}\int_1^8 x^{-1/3}\ \mathrm{d}x = \frac{4}{9}x^{2/3}\Big|_1^8 = \frac{4}{3}$$
$$\int_C \frac{x^2}{y^3}\ \mathrm{d}s = \int_1^8 \frac{x^2}{27x^2/8}\sqrt{1+x^{-2/3}}\ \mathrm{d}x = \int_1^8 x^{-1/3}\sqrt{1+x^{2/3}}\ \mathrm{d}x = \frac{8}{27}(1+x^{2/3})^{3/2}\Big|_1^8$$
$$= \frac{8}{27}(5^{3/2} - 2^{3/2})$$

5. $$\int_C (x^2+y^2)ds = \int_0^{2\pi} (25\cos^t - 25\sin^2 t)\sqrt{25\sin^2 t + 25\cos^2 t}dt = 125\int_0^{2\pi} (\cos^2 t - \sin^2 t)dt$$
$$= 125\int_0^{2\pi} \cos 2t dt = \frac{125}{2}\sin 2t\Big|_0^{2\pi} = 0$$

6. $$\int_C (2x+3y)\ \mathrm{d} = \int_0^{\pi} (6\sin 2t + 6\cos 2t)(-4\sin 2t)\ \mathrm{d}t$$
$$= \int_0^{\pi} -24\sin^2 2t - 24\sin 2t\cos 2t\ \mathrm{d}t$$
$$= \int_0^{\pi} -24\left[\frac{1}{2}(1-\cos 2t)\right] - 24\sin 2t\cos 2t\ \mathrm{d}t$$
$$= -12t + 6\sin^2 2t - 12\sin^2 2t\big|_0^{\pi}$$
$$= -12\pi$$

7. $$\int_C z\ \mathrm{d}x = \int_0^{\pi/2} t(-\sin t)\ \mathrm{d}t \quad \boxed{\text{Integration by parts}}$$
$$= (t\cos t - \sin t)\big|_0^{\pi/2} = -1$$
$$\int_C z\ \mathrm{d}y = \int_0^{\pi/2} t\cos t\ \mathrm{d}t \quad \boxed{\text{Integration by parts}}$$
$$= (t\sin t + \cos t)\big|_0^{\pi/2} = \frac{\pi}{2} - 1$$

$$\int_C z \ \mathrm{d}z = \int_0^{\pi/2} t \ \mathrm{d}t = \frac{1}{2}t^2\bigg|_0^{\pi/2} = \frac{\pi^2}{8}$$
$$\int_C z \ \mathrm{d}x = \int_0^{\pi/2} t\sqrt{\sin^2 t + \cos^2 t + 1} \ \mathrm{d}t = \sqrt{2}\int_0^{\pi/2} t \ \mathrm{d}t = \frac{\pi^2\sqrt{2}}{8}$$

8. $$\int_C 4xyz \ \mathrm{d}x = \int_0^1 4\left(\frac{1}{3}t^3\right)(t^2)(2t)t^2 \ \mathrm{d}t = \frac{8}{3}\int_0^1 t^8 \ \mathrm{d}t = \frac{8}{27}t^9\bigg|_0^1 = \frac{8}{27}$$
$$\int_C 4xyz \ \mathrm{d}y = \int_0^1 4\left(\frac{1}{3}t^3\right)(t^2)(2t)2t \ \mathrm{d}t = \frac{16}{3}\int_0^1 t^7 \ \mathrm{d}t = \frac{2}{3}t^8\bigg|_0^1 = \frac{2}{3}$$
$$\int_C 4xyz \ \mathrm{d}z = \int_0^1 4\left(\frac{1}{3}t^3\right)(t^2)(2t)2 \ \mathrm{d}t = \frac{16}{21}\int_0^1 t^6 \ \mathrm{d}t = \frac{16}{21}\bigg|_0^1 = \frac{16}{21}$$
$$\int_C 4xyz \ \mathrm{d}s = \int_0^1 4\left(\frac{1}{3}t^3\right)(2t)\sqrt{t^4+4t^2+4} \ \mathrm{d}t = \frac{8}{3}\int_0^1 t^6(t^2+2) \ \mathrm{d}t = \frac{8}{3}\left(\frac{1}{9}t^9 + \frac{2}{7}t^7\right)\bigg|_0^1 = \frac{200}{189}$$

9. Using x as the parameter, $\mathrm{d}y = \mathrm{d}x$ and

$$\int_C = (2x+y) \ \mathrm{d}x + xy \ \mathrm{d}y = \int_{-1}^2 (2x+x+3+x^2+3x) \ \mathrm{d}x = \int_{-1}^2 (x^2+6x+3) \ \mathrm{d}x$$
$$= \left(\frac{1}{3}x^3 + 3x^2 + 3x\right)\bigg|_{-1}^2 = 21.$$

10. Using x as the parameter, $\mathrm{d}y = 2x \ \mathrm{d}x$ and
$$\int_C (2x+y) \ \mathrm{d}x + xy \ \mathrm{d}y = \int_{-1}^2 (2x+x^2+1) \ \mathrm{d}x + \int_{-1}^2 x(x^2+1)2x \ \mathrm{d}x = \int_{-1}^2 (2x^4+3x^2+2x+1) \ \mathrm{d}x$$
$$= \left(\frac{2}{5}x^5 + x^3 + x^2 + x\right)\bigg|_{-1}^2 = \frac{141}{5}.$$

11. From $(-1,2)$ to $(2,2)$ we use x as a parameter with $y=2$ and $dy=0$. From $(2,2)$ to $(2,5)$ we use y as a parameter with $x=2$ and $dx=0$.

$$\int_C (2x+y) \ \mathrm{d} + xy \ \mathrm{d}y = \int_{-1}^2 (2x+2) \ \mathrm{d}x + \int_2^5 2y \ \mathrm{d}y = \left.(x^2+2x)\right|_{-1}^2 + \left.y^2\right|_{-1}^2 = 9+21 = 30$$

12. From $(-1,2)$ to $(-1,0)$ we use y as a parameter with $x=-1$ and $dx=0$. From $(-1,0)$ to $(2,0)$ we use x as a parameter with $y=dy=0$. From $(2,0)$ to $(2,5)$ we use y as a parameter with $x=2$ and $dx=0$.

$$\int_C (2x+y) \ \mathrm{d} + xy \ \mathrm{d}y = \int_2^0 (-1)y \ \mathrm{d}y + \int_{-1}^2 2x \ \mathrm{d}x + \int_0^5 2y \ \mathrm{d}y = -\frac{1}{2}y^2\bigg|_2^0 + \left.x^2\right|_{-1}^2 + \left.y^2\right|_0^2$$
$$= 2+3+25 = 30$$

13. Using x as a the parameter, $dy = 2xdx$.

$$\int_C y \ \mathrm{d}x + x \ \mathrm{d}y = \int_0^1 x^2 \ \mathrm{d}x + \int_0^1 x(2x) \ \mathrm{d}x = \int_0^1 3x^2 \ \mathrm{d}x = x^3\Big|_0^1 = 1$$

14. Using x as a the parameter, $dydx$.

$$\int_C y \ \mathrm{d}x + x \ \mathrm{d}y = \int_0^1 x \ \mathrm{d}x + \int_0^1 x \ \mathrm{d}x = \int_0^1 2x \ \mathrm{d}x = x^2\Big|_0^1 = 1$$

15. From $(0,0)$ to $(0,1)$ we use y as a parameter with $x = dx = 0$. From $(0,1)$ to $(1,1)$ we use x as a parameter with $y = 1$ and $dy = 0$.

$$\int_C y \ \mathrm{d}x + x \ \mathrm{d}y = 0 + \int_0^1 1 \ \mathrm{d}x = 1$$

16. From $(0,0)$ to $(1,0)$ we use x as a parameter with $y = dy = 0$. From $(1,0)$ to $(1,1)$ we use y as a parameter with $x = 1$ and $dx = 0$.

$$\int_C y \ \mathrm{d}x + x \ \mathrm{d}y = 0 + \int_0^1 1 \ \mathrm{d}y = 1$$

17. $$\begin{aligned}\int_C (6x^2 + 2y + 2) \ \mathrm{d}x + 4xy \ \mathrm{d}y &= \int_4^9 (6t + 2t^2)\frac{1}{2}t^{-1/2} \ \mathrm{d}t + \int_4^9 4\sqrt{t}t \ \mathrm{d}t + \int_4^9 (3t^{1/2} + 5t^{3/2}) \ \mathrm{d}t \\ &= (2t^{3/2} + 2t^{3/2})\Big|_4^9 = 460\end{aligned}$$

18. $\int_C (-y^2) \ \mathrm{d}x + xy \ \mathrm{d}y = \int_0^2 (-t^6)2 \ \mathrm{d}t + \int_0^2 (2t)(t^3)3t^2 \ \mathrm{d}t = \int_0^2 4t^6 \ \mathrm{d}t = \left.\frac{4}{7}t^7\right|_0^2 = \frac{512}{7}$

19. $$\begin{aligned}\int_C 2x^3y \ \mathrm{d} + (3x + y) \ \mathrm{d}y &= \int_{-1}^1 2(y^6)y2y \ \mathrm{d}y + \int_{-1}^1 (3y^2 + y) \ \mathrm{d}y = \int_{-1}^1 (4y^8 + 3y^2 + y) \ \mathrm{d}y \\ &= \left.\left(\frac{4}{9}y^9 + y^3 + \frac{1}{2}y^2\right)\right|_{-1}^1 = \frac{26}{9}\end{aligned}$$

20. $$\begin{aligned}\int_C 4x \ \mathrm{d}x + 2y \ \mathrm{d}y &= \int_{-1}^2 4(y^3 + 1)3y^2 \ \mathrm{d}y + \int_{-1}^2 2y \ \mathrm{d}y = \int_{-1}^2 (12y^5 + 12y^2 + 2y) \ \mathrm{d}y \\ &= 2y^6 + 4y^3 + y^2\Big|_{-1}^2 = 165\end{aligned}$$

21. From $(-2,0)$ to $(2,0)$ we use x as a parameter with $y = dy = 0$. From $(2,0)$ to $(-2,0)$ we parameterize the semicircle as $x = 2\cos\theta$ and $y = 2\sin\theta$ for $0 \le \theta \le \pi$.

$$\begin{aligned}\int_C (x^2+y^2)\ \mathrm{d}x - 2xy\ \mathrm{d}y &= \int_{-2}^{2} x^2\ \mathrm{d}x + \int_0^{\pi} 4(-2\sin\theta\ \mathrm{d}\theta) - \int_0^{\pi} 8\cos\theta\sin\theta(2\cos\theta\ \mathrm{d}\theta)\\ &= \frac{1}{3}x^3\Big|_{-2}^{2} - 8\int_0^{\pi}(\sin\theta + 2\cos^2\theta\sin\theta)\ \mathrm{d}\theta\\ &= \frac{16}{3} - 8\left(-\cos\theta - \frac{2}{3}\cos^3\theta\right)\Big|_0^{\pi} = \frac{16}{3} - \frac{80}{3} = -\frac{64}{3}\end{aligned}$$

22. We start at $(0,0)$ and use x as a parameter.

$$\begin{aligned}\int_C (x^2+y^2)\ \mathrm{d}x - 2xy\ \mathrm{d}y &= \int_0^1 (x^2+x^4)\ \mathrm{d}x - 2\int_0^1 xx^2(2x\ \mathrm{d}x) + \int_1^0 (x^2+x)\ \mathrm{d}x\\ &\quad - 2\int_1^0 x\sqrt{x}\left(\frac{1}{2}x^{-1/2}\right)\ \mathrm{d}x\\ &= \int_0^1 (x^2-3x^4)\ \mathrm{d}x + \int_1^0 x^2\ \mathrm{d}x = \int_0^1(-3x^4)\ \mathrm{d}x = -\frac{3}{5}x^5\Big|_0^1 = -\frac{3}{5}\end{aligned}$$

23. From $(1,1)$ to $(-1,1)$ and $(-1,-1)$ to $(1.-1)$ we use x as a parameter with $y=1$ and $y=-1$, respectively, and $dy=0$. From $(-1,1)$ to $(-1,-1)$ and $(1,-1)$ to $(1,1)$ we use y as a parameter with $x=-1$ and $z=1$, respectively, and $dx=0$.

$$\begin{aligned}\int_C x^2y^3\ \mathrm{d}x - xy^2\ \mathrm{d}y &= \int_1^{-1} x^2(1)\ \mathrm{d}x + \int_1^{-1} -(-1)y^2\ \mathrm{d}y + \int_{-1}^{1} x^2(-1)^3\ \mathrm{d}x + \int_{-1}^{1} -(1)y^2\ \mathrm{d}y\\ &= \frac{1}{3}x^3\Big|_1^{-1} + \frac{1}{3}y^3\Big|_{-1}^{1} - \frac{1}{3}x^3\Big|_{-1}^{1} - \frac{1}{3}y^3\Big|_{-1}^{1} = -\frac{8}{3}\end{aligned}$$

24. From $(2,4)$ to $(0,4)$ we use x as a parameter with $y=4$ and $dy=0$. From $(0,4)$ to $(0,0)$ we use y as a parameter with $x=dx=0$. From $(0,0)$ to $(2,4)$ we use $y=2x$ and $dy=2dx$.

$$\begin{aligned}\int_C x^2y^3\ \mathrm{d}x - xy^2\ \mathrm{d}y &= \int_2^0 x^2(64)\ \mathrm{d}x - \int_4^0 0\ \mathrm{d}y + \int_0^2 x^2(8x^3)\ \mathrm{d}x - \int_0^2 x(4x^2)2\ \mathrm{d}x\\ &= \frac{64}{3}x^3\Big|_2^0 + \frac{4}{3}x^6\Big|_0^2 - 2x^4\big|_0^2 = -\frac{512}{3} + \frac{256}{3} - 32 = -\frac{352}{3}\end{aligned}$$

25. $$\begin{aligned}\int_C y\ \mathrm{d}x - x\ \mathrm{d}y &= \int_0^{\pi} 3\sin t(-2\sin t)\ \mathrm{d}t - \int_0^{\pi} 2\cos t(3\cos t)\ \mathrm{d}t = -6\int_0^{\pi}(\sin^2 t + \cos^2)\ \mathrm{d}t\\ &= -6\int_0^{\pi}\ \mathrm{d}t = -6\pi\end{aligned}$$

Thus, $\displaystyle\int_{-C} y\ \mathrm{d}x - x\ \mathrm{d}y = 6\pi$.

26. $$\int_C x^2y^3 + x^3y^2\ \mathrm{d}y = \int_{-1}^{1} x^2(x^{12})\ \mathrm{d}x + \int_{-1}^{1} x^3(x^8)(4x^3)\ \mathrm{d}x$$
$$= \int_{-1}^{1} x^{14}\ \mathrm{d}x + \int_{-1}^{1} 4x^{14}\ \mathrm{d}x$$
$$= \int_{-1}^{1} 5x^{14}\ \mathrm{d}x = \frac{5}{15}x^{15}\Big|_{-1}^{1}$$
$$= \frac{5}{15} + \frac{5}{15} = \frac{2}{3}$$

27. We parameterize the line segment from $(0,0,0)$ to $(2,3,4)$ by $x = 2t,\ \ y = 3y,\ \ z = 4t$ for $0 \le t \le 1$. We parameterize the line segment from $(2,3,4)$ to $(6,8,5)$ by $x = 2+2t,\ \ y = 3+5t,\ \ z = 4+t$, $0 \le t \le 1$.

$$\int_C y\ \mathrm{d}x + z\ \mathrm{d}y + x\ \mathrm{d}z = \int_0^1 3t(2\ \mathrm{d}t) + \int_0^1 4t(3\ \mathrm{d}t) + \int_0^1 2t(4\ \mathrm{d}t) + \int_0^1 (3+5t)(4\ \mathrm{d}t)$$
$$\int_0^1 (4+t)(5\ \mathrm{d}t) + \int_0^1 (2+4t)\ \mathrm{d}t$$
$$= \int_0^1 (55t+34)\ \mathrm{d}t = \left(\frac{55}{2}t^2 + 34t\right)\Big|_0^1 = \frac{123}{2}$$

28. $$\int_C y\ \mathrm{d}x + z\ \mathrm{d}y + x\ \mathrm{d}z = \int_0^2 t^3(3\ \mathrm{d}t) + \int_0^2 \left(\frac{5}{4}t^2\right)(3t^2\ \mathrm{d}t) + \int_0^2 (3t)\left(\frac{5}{2}t\ \mathrm{d}t\right)$$
$$= \int_0^2 \left(3t^3 + \frac{15}{4}t^4 + \frac{15}{2}t^2\right)\ \mathrm{d}t = \left(\frac{3}{4}t^4 + \frac{3}{4}t^5 + \frac{5}{2}t^3\right)\Big|_0^2 = 56$$

29. From $(0,0,0)$ to $(6,0,0)$ we use x as a parameter with $y = dy = 0$ and $z = dz = 0$. From $(6,0,0)$ to $(6,0,5)$ we use z as a parameter with $x = 6$ and $dx = 0$ and $y = dy = 0$. From $(6,0,5)$ to $(6,8,5)$ we use y as a parameter with $x = 6$ and $dz = 0$ and $z = 5$ and $dz = 0$.

$$\int_C y\ \ \mathrm{d}x + z\ \ \mathrm{d}y + z\ \ \mathrm{d}z = \int_0^6 0\ \ \mathrm{d}x + \int_0^5 6\ \ \mathrm{d}z + \int_0^8 5\ \ \mathrm{d}y = 70$$

30. We parameterize the line segment from $(0,0,0)$ to $(6,8,0)$ by $x = 6t,\ \ y = 8t,\ \ z = 0$ for $0 \le t \le 1$. From $(6,8,0)$ to $(6,8,5)$ we use z as a parameter with $x = 6,\ \ dx = 0$, and $y = 8,\ \ dy = 0$.

$$\int_C y\ \mathrm{d}x + z\ \mathrm{d}y + z\ \mathrm{d}z = \int_0^1 8t(6\ \mathrm{d}t) + \int_0^5 6\ \mathrm{d}z = 24t^2\big|_0^1 + 30 = 54$$

31. $$\int_C 10x\ \mathrm{d}x - 2xy^2\ \mathrm{d}y + 6xz\ \mathrm{d}z = \int_0^1 10(t)\ \mathrm{d}t - \int_0^1 2(t)(t^2)^2(2t)\ \mathrm{d}t + \int_0^1 6(t)(t^3)(3t^2)\ \mathrm{d}t$$
$$= 5t^2\big|_0^1 - 4t^6\big|_0^1 + 18t^6\big|_0^1$$
$$= 5 - 4 + 18 = 19$$

32. Parametrize the line segments as follows:
$C_1: \ \mathbf{r}_1(t) = t\mathbf{i} + t\mathbf{j}, \ \ 0 \le t \le 1$
$C_2: \ \mathbf{r}_2(t) = \mathbf{i} + \mathbf{j} + t\mathbf{k}, \ \ 0 \le t \le 1$
$C_3: \ \mathbf{r}_3(t) = (1-t)\mathbf{i} + (1-t)\mathbf{j} + (1-t)\mathbf{k}, \ \ 0 \le t \le 1$
We then have

$$\begin{aligned}\int_{C_1} 3x \ \mathrm{d}x - y^2 \ \mathrm{d}y + z^2 \ \mathrm{d}z &= \int_0^1 3t \ \mathrm{d}t - \int_0^1 t^2 \ \mathrm{d}t \\ &= \frac{3}{2} - \frac{1}{3} = \frac{7}{6} \\ \int_{C_2} 3x \ \mathrm{d}x - y^2 \ \mathrm{d}y + z^2 \ \mathrm{d}z &= \int_0^1 3(1)(0) \ \mathrm{d}t - \int_0^1 (1)^2(0) \ \mathrm{d}t + \int_0^1 t^2 \ \mathrm{d}t \\ &= \frac{1}{3} \\ \int_{C_3} 3x \ \mathrm{d}x - y^2 \ \mathrm{d}y + z^2 \ \mathrm{d}z &= \int_0^1 3(1-t)(-1) \ \mathrm{d}t - \int_0^1 (1-t)^2(-1) \ \mathrm{d}t + \int_0^1 (1-t)^2(-1) \ \mathrm{d}t \\ &= \left(-\frac{3}{2}\right) - \left(-\frac{1}{3}\right) + \left(-\frac{1}{3}\right) = -\frac{3}{2}\end{aligned}$$

33. $$\begin{aligned}\int_{C_1} y^2dx + xydy &= \int_0^1 (4t+2)^2 2dt + \int_0^1 (2t+1)(4t+2)4dt = \int_0^1 (64t^2+64t+16)dt \\ &= \left(\frac{64}{3}t^3 + 32t^2 + 16t\right)\Big|_0^1 = \frac{64}{3} + 32 + 16 = \frac{208}{3} \\ \int_{C_2} y^2dx + xydy &= \int_1^{\sqrt{3}} 4y^4(2t)dt + \int_1^{\sqrt{3}} 2t^4(4t)dt = \int_1^{\sqrt{3}} 16t^5 dt = \frac{8}{3}t^6\Big|_1^{\sqrt{3}} = 72 - \frac{8}{3} = \frac{208}{3} \\ \int_{C_3} y^2dx + xydy &= \int_e^{e^3} 4(\ln t)^2\frac{1}{t}dt + \int_e^{e^3} 2(\ln t)^2\frac{2}{t}dt = \int_e^{e^3} \frac{8}{t}(\ln t)^2 dt = \frac{8}{3}(\ln t)^3\Big|_e^{e^3} \\ &= \frac{8}{3}(27-1) = \frac{208}{3}\end{aligned}$$

34. $\int_{C_1} xyds = \int_0^2 t(2t)\sqrt{1+4}dt = 2\sqrt{5}\int_0^2 t^2dt = 2\sqrt{5}\left(\frac{1}{3}t^3\right)\Big|_0^2 = \frac{16\sqrt{5}}{3}$

$$\begin{aligned}\int_{C_2} xyds &= \int_0^2 t(t^2)\sqrt{1+4t^2}dt = \int_0^2 t^3\sqrt{1+4t^2}dt \quad \boxed{u = 1+4t^2, \ \ du = 8tdt; \ \ t^2 = \frac{1}{4}(u-1)} \\ &= \int_1^{17} \frac{1}{4}(u-1)u^{1/2}\frac{1}{8}du = \frac{1}{32}\int_1^{17}(u^{3/2} - u^{1/2})du \\ &= \frac{1}{32}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right)\Big|_1^{17} \\ &= \frac{391\sqrt{17}+1}{120}\end{aligned}$$

$$\int_{C_3} xyds = \int_2^3 (2t-4)(4t-8)\sqrt{4+16}dt = 16\sqrt{5}\int_2^3 (t-2)^2dt \ 16\sqrt{5}\left[\frac{1}{3}(t-2)^3\right]\Big|_2^3 = \frac{16\sqrt{5}}{3}$$

C_1 and C_3 are different parameterization of the same curve, while C_1 and C_2 are different

curves.

35. We are given $\rho = kx$. Then

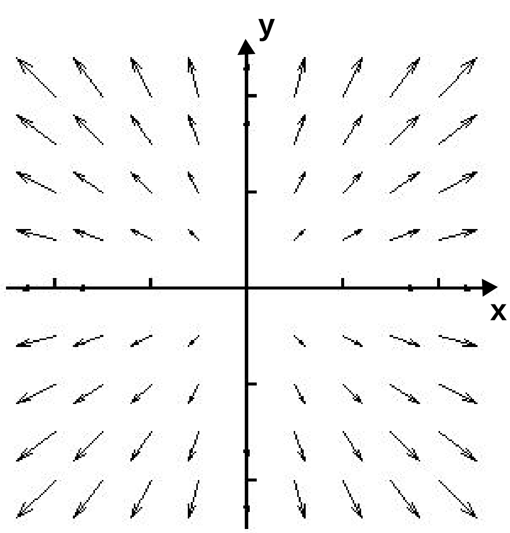
$$m = \int_C \rho \, \mathrm{d}x = \int_0^\pi kx \, \mathrm{d}s = k \int_0^\pi (1+\cos t)\sqrt{\sin^2 t + \cos^2 t} \, \mathrm{d}t = k \int_0^\pi (1+\cos t) \, \mathrm{d}t$$
$$= k \left.(t+\sin t)\right|_0^\pi = k\pi.$$

36. From Problem 35, $m = k\pi$ and $ds = dt$.

$$M_x = \int_C y\rho ds = \int_C kxyds = k\int_0^\pi (1+\cos t)\sin t dt k \left.\left(-\cos t + \frac{1}{2}\sin^2 t\right)\right|_0^\pi = 2k$$
$$M_y = int_C x\rho ds = int_C kx^2 ds = k\int_0^\pi (1+\cos t)^2 dt = k\int_0^\pi (1+2\cos t+\cos^2 t)dt$$
$$= k\left.\left(t + 2\sin t + \frac{1}{2}t + \frac{1}{4}\sin 2t\right)\right|_0^\pi = \frac{3}{2}k\pi$$
$$\overline{x} = M_y/m = \frac{3k\pi/2}{k\pi} = \frac{3}{2}; \overline{y} = M_x/m = \frac{2k}{k\pi} = \frac{2}{\pi}.$$ The center of mass is $(3/2, 2/\pi)$.

15.2 Line Integrals of Vector Fields

1.



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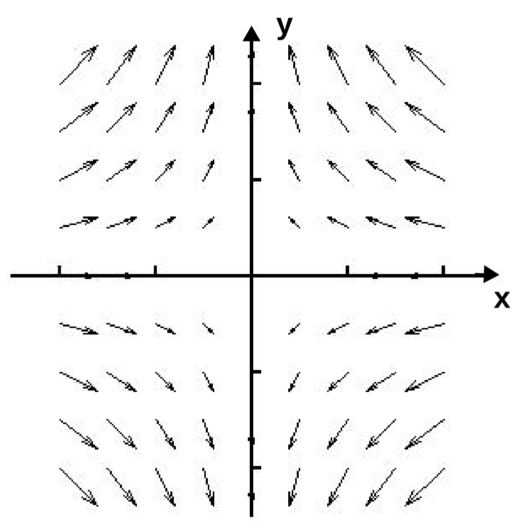

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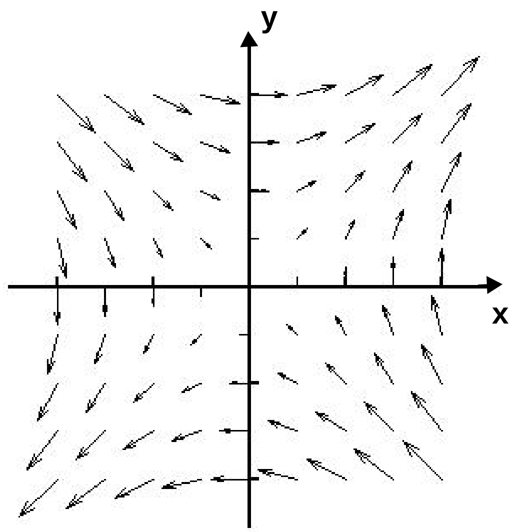
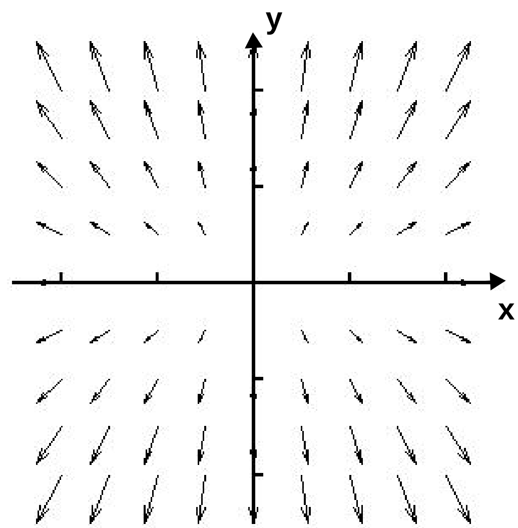
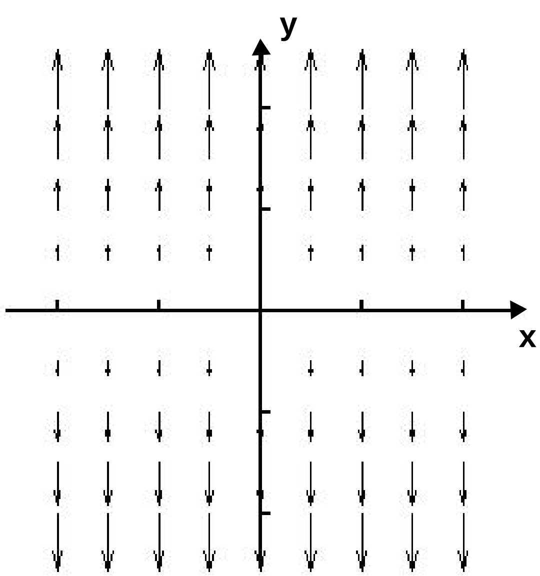


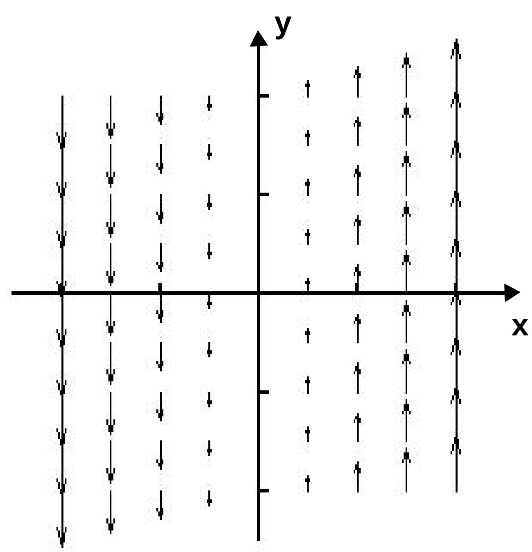
4.



5.



6.



7. Since each vector points in a northeasterly direction, the vector field must have positive **i** and **j** components. Therefore, the answer is **(b)**.

8. Since each vector points in a northwesterly direction, the vector field must have negative **i** and positive **j** components. Therefore, the answer is **(a)**.

9. Since each vector points in a southwesterly direction, the vector field must have negative **i** and **j** components. Therefore, the answer is **(d)**.

10. Since each vector points in a southeasterly direction, the vector field must have positive **i** and negative **j** components. Therefore, the answer is **(c)**.

11. Note that the **k** component of each vector is always positive. Therefore, the answer is **(d)**.

12. Note that the **i** component of each vector is always positive. Therefore, the answer is **(c)**.

13. Note that each vector points directly away from the origin. Therefore, the answer is **(a)**.

14. Note that the **i** and **j** components of each vector are zero. Therefore, the answer is **(b)**.

15. $\mathbf{F} = e^{3t}\mathbf{i} - (e^{-4t})e^t\mathbf{j} = e^{3t}\mathbf{i} - e^{-3t}\mathbf{j};\quad d\mathbf{r} = (-2e^{-2t}\mathbf{i} + e^t\mathbf{j})dt;\quad \mathbf{F}\cdot d\mathbf{r} = (-2e^t - e^{-2t})dt;$

$$\int_c \mathbf{F}\cdot d\mathbf{r} = \int_0^{\ln 2} (-2e^t - e^{-2t})dt = (-2e^t + \frac{1}{2}e^{-2t})\Big|_0^{\ln 2} = -\frac{31}{8} - (-\frac{3}{2}) = -\frac{19}{8}$$

16. $\mathbf{F} = 2(t)(t^2)\mathbf{i} + t^2\mathbf{j} = 2t^3\mathbf{i} + t^2\mathbf{j};\quad d\mathbf{r} = (\mathbf{i} + 2t\mathbf{j})dt;$
$\mathbf{F}\cdot d\mathbf{r} = 4t^3dt;$
$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^2 4t^2\ \mathrm{d}t = t^4\big|_0^2 = 16$

17. $\mathbf{F} = 2(2t-1)\mathbf{i} - 2(6t+1)\mathbf{j} = (4t-2)\mathbf{i} + (12t+2)\mathbf{j};\quad d\mathbf{r} = (2\mathbf{i} + 6\mathbf{j})dt;$
$\mathbf{F}\cdot d\mathbf{r} = -64t - 16;$
$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{-1}^1 (-64t - 16)\ \mathrm{d}t = -32t^2 - 16t\big|_{-1}^1 = -32$

18. $\mathbf{F} = \cos^2 t\mathbf{i} + \sin t\mathbf{j};\quad d\mathbf{r} = (-\sin t\mathbf{i} + \cos t\mathbf{j});$
$\mathbf{F}\cdot d\mathbf{r} = (-\cos^2 t\sin t + \sin t\cos t)dt;$

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{\pi/6} (-\cos^2 t\sin t + \sin t\cos t)\ \mathrm{d}t\\ &= \frac{\cos^3 t}{3} + \frac{\sin^2 t}{2}\Big|_0^{\pi/6}\\ &= \frac{1}{3}\left(\frac{\sqrt{3}}{2}\right)^3 + \frac{1}{2}\left(\frac{1}{2}\right)^2 - \left[\frac{1}{3} + 0\right]\\ &= \frac{\sqrt{3}}{8} + \frac{1}{8} - \frac{1}{3} = \frac{\sqrt{3}}{8} - \frac{5}{24}\end{aligned}$$

19. $\mathbf{F} = -3\sin t\mathbf{i} + 2\cos t\mathbf{j} + 6t\mathbf{k};\quad d\mathbf{r} = (-2\sin t\mathbf{i} + 3\cos t\mathbf{j} + 3\mathbf{k})dt;$
$\mathbf{F}\cdot d\mathbf{r} = (-6\sin^2 t + 6\cos^2 t + 18t)dt;$

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{\pi} (-6\sin^2 t + 6\cos^2 t + 18t)\ \mathrm{d}t\\ &= \int_0^{\pi} -6\left[\frac{1}{2}(1-\cos 2t)\right] + 6\left[\frac{1}{2}(1+\cos 2t)\right] + 18t\ \mathrm{d}t \quad = 3\sin 2t + 9t^2\big|_0^{\pi}\\ &= 9\pi^2\end{aligned}$$

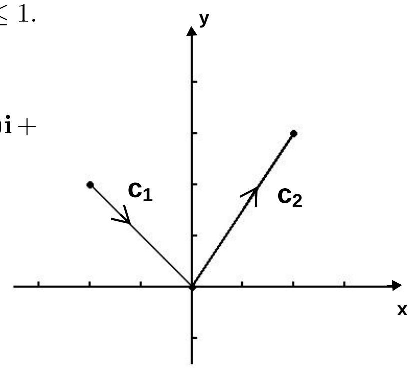
20. $\mathbf{F} = e^t\mathbf{i} + te^{t^3}\mathbf{j} + t^3e^{t^6}\mathbf{k}$; $d\mathbf{r} = (\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k})dt$;

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 (e^t + 2t^2e^{t^3} + 3t^5e^{t^6})dt = \left(e^t + \frac{2}{3}e^{t^3} + \frac{1}{2}e^{t^6}\right)\Big|_0^1 = \frac{13}{6}(e-1)$$

21. Using x as a parameter, $\mathbf{r}(x) = x\mathbf{i} + \ln x\mathbf{j}$. Then $\mathbf{F} = \ln x\mathbf{i} + x\mathbf{j}$, $d\mathbf{r} = (\mathbf{i} + \frac{1}{x}\mathbf{j})dx$, and

$$W = \int_c \mathbf{F}\cdot d\mathbf{r} = \int_1^e (\ln x + 1)dx = (x\ln x)|_1^e = e.$$

22. Let $\mathbf{r}_1 = (-2+2t)\mathbf{i}+(2-2t)\mathbf{j}$ and $\mathbf{r}_2 = 2t\mathbf{i}+3t\mathbf{j}$ for $0 \le t \le 1$. Then



$$d\mathbf{r}_1 = 2\mathbf{i} - 2\mathbf{j}, \quad d\mathbf{r}_2 = 2\mathbf{i} + 3\mathbf{j},$$

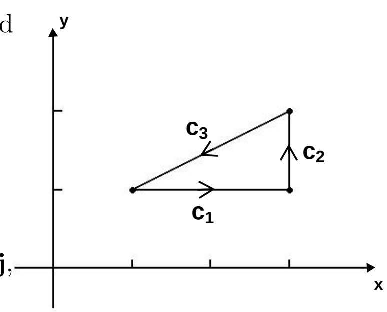
$\mathbf{F}_1 = 2(-2+2t)(2-2t)\mathbf{i} + 4(2-2t)^2\mathbf{j} = (-8t^2+16t-8)\mathbf{i} + (16t^2 - 32t + 16)\mathbf{j}$,

$$\mathbf{F}_2 = 2(2t)(3t)\mathbf{i} + 4(3t)^2\mathbf{j} = 12t^2\mathbf{i} + 36t^2\mathbf{j},$$

and

$$\begin{aligned} W &= \int_{C_1} \mathbf{F}_1\cdot d\mathbf{r}_1 + \int_{cC_2} \mathbf{F}_2\cdot d\mathbf{r}_2 \\ &= \int_0^1 (-16t^2 + 32t - 16 - 32t^2 + 64t - 32)dt + \int_0^1 (24t^2 + 108t^2)dt \\ &= \int_0^1 (84t^2 + 96t - 48)dt = (28t^3 + 48t^2 - 48t)\Big|_0^1 = 28. \end{aligned}$$

23. Let $\mathbf{r}_1 = (1+2t)\mathbf{i} + \mathbf{j}$, $\mathbf{r}_2 = 3\mathbf{i} + (1+t)\mathbf{j}$, and $\mathbf{r}_3 = (3-2t)\mathbf{i} + (2-t)\mathbf{j}$ for $0 \le t \le 1$. Then



$$d\mathbf{r}_1 = 2\mathbf{i}, \quad d\mathbf{r}_2 = \mathbf{j}, \quad d\mathbf{r}_3 = -2\mathbf{i} - \mathbf{j},$$

$$\mathbf{F}_1 = (1+2t+2)\mathbf{i} + (6-2-4t)\mathbf{j} = (3+2t)\mathbf{i} + (4-4t)\mathbf{j},$$

$$\mathbf{F}_2 = (3+2+2t)\mathbf{i} + (6+6t-6)\mathbf{j} = (5+2t)\mathbf{i} + 6t\mathbf{j},$$

$$\mathbf{F}_3 = (3-2t+4-2t)\mathbf{i}+(12-6t-6+4t)\mathbf{j} = (7-4t)\mathbf{i}+(6-2t)\mathbf{j},$$

and

$$\begin{aligned} W &= \int_{C_1} \mathbf{F}_1\cdot d\mathbf{r}_1 + \int_{c_2} \mathbf{F}_2\cdot d\mathbf{r}_2 + \int_{c_3} \mathbf{F}_3\cdot d\mathbf{r}_3 \\ &= \int_0^1 (6+4t)dt + \int_0^1 6t dt + \int_0^1 (-14 + 8t - 6 + 2t)dt \\ &= \int_0^1 (-14 + 20t)dt = (-14t + 10t^2)\Big|_0^1 = -4. \end{aligned}$$

24. $\mathbf{F} = t^3\mathbf{i} + t^4\mathbf{j} + t^5\mathbf{k}$; $d\mathbf{r} = 3t^2\mathbf{i} + 2t\mathbf{j} + \mathbf{k}$;
$W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_1^3 (3t^5 + 2t^5 = t^5)dt = \int_1^3 6t^5 dt = t^6\big|_1^3 = 728$

25. $\mathbf{r} = 3\cos t\mathbf{i} + 3\sin t\mathbf{j}$, $0 \le t \le 2\pi$; $d\mathbf{r} = -3\sin t\mathbf{i} + 3\cos t\mathbf{j}$; $\mathbf{F} = a\mathbf{i} + b\mathbf{j}$;
$W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}(-3a\sin t + 3b\cos t)dt = (3a\cos t + 3b\sin t)\big|_0^{2\pi} = 0$

26. Let $\mathbf{r} = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}$ for $1 \le t \le 3$. Then $d\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, and

$$\mathbf{F} = \frac{c}{|r|^3}(t\mathbf{i} + t\mathbf{j} + t\mathbf{k}) = \frac{ct}{(\sqrt{3}t^2)^3}(\mathbf{i} + \mathbf{j} + \mathbf{k}) = \frac{c}{3\sqrt{3}t^2}(\mathbf{i} + \mathbf{j} + \mathbf{k}),$$

$$\begin{aligned} W &= \int_C \mathbf{F}\cdot d\mathbf{r} = \int_1^3 \frac{c}{3\sqrt{3}t^2}(1+1+1)dt \quad = \frac{c}{\sqrt{3}}\int_1^3 \frac{1}{t^2}dt = \frac{c}{\sqrt{3}}\left(-\frac{1}{t}\right)\Bigg|_1^3 \\ &= \frac{c}{\sqrt{3}}(-\frac{1}{3} + 1) = \frac{2c}{3\sqrt{3}}. \end{aligned}$$

27. $\mathbf{F} = 10\cos t\mathbf{i} - 10\sin t\mathbf{j}$; $d\mathbf{r} = (-5\sin t\mathbf{i} + 5\cos t\mathbf{j})dt$;
$\mathbf{F}\cdot d\mathbf{r} = (-100\cos t\sin t)dt$;
$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}(-100\cos t\sin t)\ \mathrm{d}t = 50\cos^2 t\big|_0^{2\pi} = 0$

28. $\mathbf{F} = 10\cos t\mathbf{i} + 2\sin t\mathbf{j}$; $d\mathbf{r} = (2\cos t\mathbf{i} - 10\sin t\mathbf{j})dt$;
$\mathbf{F}\cdot d\mathbf{r} = (10\cos^2 t - 20\sin^2 t)dt$;

$$\begin{aligned} \int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi}(10\cos^2 t - 20\sin^2 t)\ \mathrm{d}t \\ &= \int_0^{2\pi} 10\left[\frac{1}{2}(1+\cos 2t)\right] - 20\left[\frac{1}{2}(1-\cos 2t)\right]\ \mathrm{d}t \quad = -5t + \frac{15}{2}\sin 2t\Bigg|_0^{\pi} \\ &= -10\pi \end{aligned}$$

29. On $C_1, \mathbf{T} = \mathbf{i}$ and $\mathbf{F}\cdot\mathbf{T}\mathrm{comp}_{\mathbf{T}}\mathbf{F} \approx 1$. On $C_2, \mathbf{T} = -\mathbf{j}$ and $\mathbf{F}\cdot\mathbf{T} = \mathrm{comp}_{\mathbf{T}}\mathbf{F} \approx 2$. On$C_3, \mathbf{T} = -\mathbf{i}$ and $\mathbf{F}\cdot\mathbf{T} = \mathrm{comp}_{\mathbf{T}}\mathbf{F} \approx 1.5$. Using the fact that the lengths of C_1, C_2, and C_3 are 4, 5, and 5, respectively, we have
$W = \int_C \mathbf{F}\cdot\mathbf{T}ds = \int_{C_1}\mathbf{F}\cdot\mathbf{T}ds + \int_{C_2}\mathbf{F}\cdot\mathbf{T}ds + \int_{C_3}\mathbf{F}\cdot\mathbf{T}ds \approx$ 1(4)+2(5)+1.5(5)=21.5 ft-lb.

30. $$\begin{aligned} W &= \int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b (m\mathbf{a}\cdot\mathbf{r}'(t))\ \mathrm{d}t \\ &= \int_a^b m(\mathbf{a}\cdot\mathbf{v})\ \mathrm{d}t = \int_a^b m\left(\frac{d\mathbf{v}}{dt}\cdot\mathbf{v}\right)\ \mathrm{d}t \\ &= \int_a^b \frac{m}{2}\left(\frac{d\mathbf{v}}{dt}\cdot\mathbf{v} + \mathbf{v}\cdot\frac{d\mathbf{v}}{dt}\right)\ \mathrm{d}t = \int_a^b \frac{m}{2}\frac{d}{dt}(\mathbf{v}\cdot\mathbf{v})\ \mathrm{d}t \\ &= \int_a^b \frac{m}{2}\frac{d}{dt}(v^2)\ \mathrm{d}t = \frac{m}{2}v^2\Big|_a^b \\ &= \frac{1}{2}m[v(b)]^2 - \frac{1}{2}m[v(a)]^2 \\ &= K(B) - K(A) \end{aligned}$$

31. $$\begin{aligned} \nabla f(x,y) &= \frac{1}{3}(3x-6y)3\mathbf{i} + \frac{1}{3}(3x-6y)(-6)\mathbf{j} \\ &= (3x-6y)\mathbf{i} + (-6x+12y)\mathbf{j} \end{aligned}$$

32. $\nabla f(x, y) = (1 + 2\cos 5xy - 10xy\sin 5xy)\mathbf{i} + (-1 - 10x^2 \sin 5xy)\mathbf{j}$

33. $\nabla f(x, y, z) = \tan^{-1} zy\mathbf{i} + \dfrac{xz}{y^2z^2+1}\mathbf{j} + \dfrac{xy}{y^2z^2+1}\mathbf{k}$

34. $\nabla f(x, y, z) = (1 - 2xyz^4)\mathbf{i} - x^2z^4\mathbf{j} - 4x^2yz^3\mathbf{k}$

35. $\nabla f(x, y, z) = e^{-y^2}\mathbf{i} + \left(1 + 2xye^{-y^2}\right)\mathbf{j} + \mathbf{k}$

36. $\nabla f(x, y, z) = \dfrac{2x}{x^2+2y^4+3z^6}\mathbf{i} + \dfrac{8y^3}{x^2+2y^4+3z^6}\mathbf{j} + \dfrac{18z^5}{x^2+2y^4+3z^6}\mathbf{k}$

37. $\nabla\left(x^2 + \frac{1}{2}y^2\right) = 2x\mathbf{i} + y\mathbf{j} = \mathbf{F}(x, y)$. Therefore, the answer is **(b).**

38. $\nabla\left(\frac{1}{2}x^2 + y^2 - 4\right) = x\mathbf{i} + 2y\mathbf{j} = \mathbf{F}(x, y)$. Therefore, the answer is **(c).**

39. $\nabla\left(2x + \frac{1}{2}y^2 + 1\right) = 2\mathbf{i} + y\mathbf{j} = \mathbf{F}(x, y)$. Therefore, the answer is **(d).**

40. $\nabla\left(\frac{1}{2}x^2 + \frac{1}{3}y^3 - 5\right) = x\mathbf{i} + y^2\mathbf{j} = \mathbf{F}(x, y)$. Therefore, the answer is **(a).**

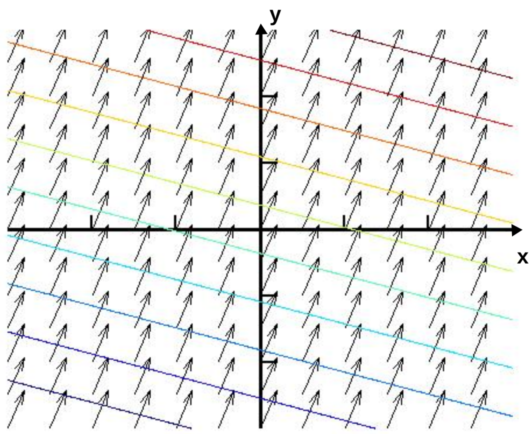
41. $\phi(x, y) = \sin x + y + \cos y$

42. $\phi(x, y) = xe^{-y}$

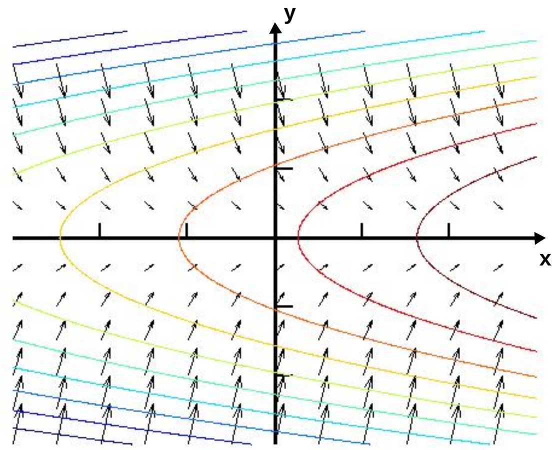
43. $\phi(x, y) = x + y^2 - 4z^3$

44. $\phi(x, y) = xy^2z^3$

45.



46.



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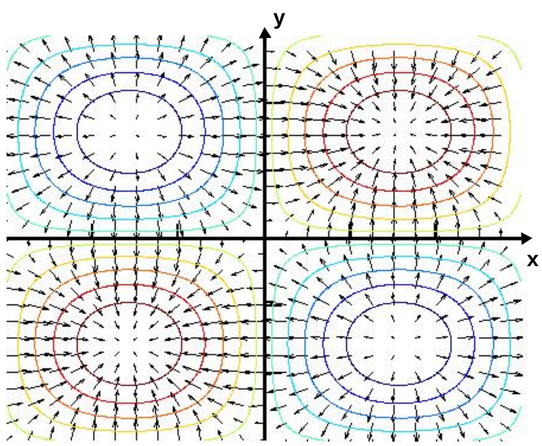
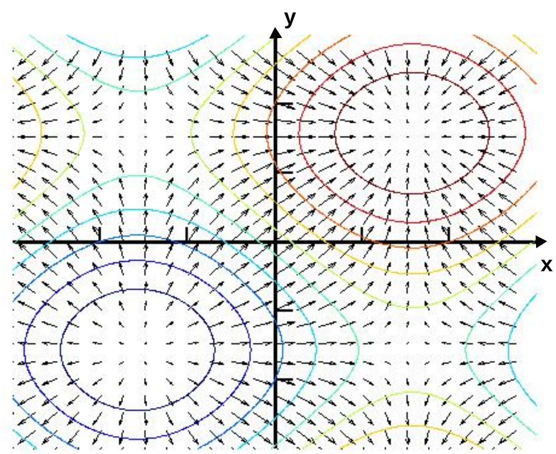


48.



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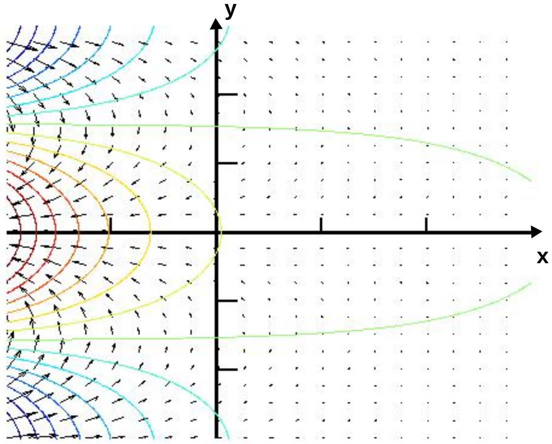
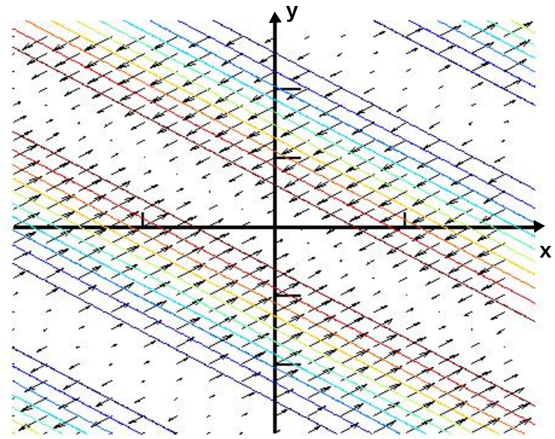


50.



51. Let $\phi(x,y,z) = -c(x^2+y^2+z^2)^{-1/2}$. Then

$$\begin{aligned}\nabla\phi(x,y,z) &= \frac{cx}{(x^2+y^2+z^2)^{3/2}}\mathbf{i} + \frac{cy}{(x^2+y^2+z^2)^{3/2}}\mathbf{j} + \frac{cz}{(x^2+y^2+z^2)^{3/2}}\mathbf{k} \\ &= \frac{c(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})}{(x^2+y^2+z^2)^{3/2}} \\ &= \frac{c\mathbf{r}}{|\mathbf{r}|^3} = \mathbf{F}\end{aligned}$$

52. Yes; if f and g differ by a constant, they will have the same gradient field.

15.3 Independence of the Path

1. (a) $P_y = 0 = Q_x$ and the integral is independent of path. $\phi_x = x^2$, $\phi = \frac{1}{3}x^3 + g(y)$, $\phi_y = g'(y) = y^2$, $g(y) = \frac{1}{3}y^3$, $\phi = \frac{1}{3}x^3 + \frac{1}{3}y^3$, $\int_{(0,0)}^{(2,2)} x^2dx + y^2dy = \frac{1}{3}(x^3+y^3)\Big|_{(0,0)}^{(2,2)} = \frac{16}{3}$

   (b) Use $y = x$ for $0 \le x \le 2$. $\int_{(0,0)}^{(2,2)} x^2dx + y^2dy = \int_0^2 (x^2+x^2)dx = \frac{2}{3}x^3\Big|_0^2 = \frac{16}{3}$

2. (a) $P_y = 2x = Q_x$ and the integral is independent of path. $\phi_x = 2xy$, $\phi = x^2y + g(y)$, $\phi_y = x^2 + g'(y) = x^2$, $g(y) = 0$, $\phi = x^2y$, $\int_{(1,1)}^{(2,4)} 2xydx + x^2dy = x^2y\big|_{(1,1)}^{(2,4)} = 16 - 1 = 15$

   (b) Use $y = 3x-2$ for $1 \le x \le 2$.
   $\int_{(1,1)}^{(2,4)} 2xydx + x^2dy = \int_1^2 [2x(3x-2) + x^2(3)]dx = \int_1^2 (9x^2-4x)dx = (3x^3-2x^2)\big|_1^2 = 15$

3. (a) $P_y = 2 = Q_x$ and the integral is independent of path. $\phi_x = x + 2y$, $\phi = \frac{1}{2}x^2 + 2xy + g(y)$, $\phi_y = 2x + g'(y) = 2x - y$, $g(y) = -\frac{1}{2}y^2$, $\phi = \frac{1}{2}x^2 + 2xy - \frac{1}{2}y^2$, $\int_{(1,0)}^{(3,2)} (x+2y)dx + (2x-y)dy = \left(\frac{1}{2}x^2 + 2xy - \frac{1}{2}y^2\right)\Big|_{(1,0)}^{(3,2)} = 14$

   (b) Use $y = x - 1$ for $1 \le x \le 3$.

$$\int_{(1,0)}^{(3,2)} (x+2y)dx + (2x-y)dy = \int_1^3 [x + 2(x-1) + 2x - (x-1)]dx$$
$$= \int_1^3 (4x-1)dx = (2x^2 - x)\Big|_1^3 = 14$$

4. (a) $P_y = -\cos x \sin y = Q_x$ and the integral is independent of path. $\phi_x = \cos x \cos y$, $\phi = \sin x \cos y + g(y)$, $\phi_y = -\sin x \sin y + g'(y) = 1 - \sin x \sin y$, $g(y) = y$, $\phi = \sin x \cos y + y$, $\int_{(0,0)}^{(\pi/2,0)} \cos x \cos y dx + (1 - \sin x \sin y)dy = (\sin x \cos y + y)\big|_{(0,0)}^{(\pi/2,0)} = 1$

   (b) Use $y = 0$ for $0 \le x \le \pi/2$.

$$\int_{(0,0)}^{(\pi/2,0)} \cos x \cos y dx + (1 - \sin x \sin y)dy = \int_0^{\pi/2} \cos x dx = \sin x\big|_0^{\pi/2} = 1$$

5. (a) $P_y = 1/y^2 = Q_x$ and the integral is independent of path. $\phi_x = -\frac{1}{y}$, $\phi = -\frac{x}{y} + g(y)$, $\phi_y = \frac{x}{y^2} + g'(x) = \frac{x}{y^2}$, $g(y) = 0$, $\phi = -\frac{x}{y}, \int_{(4,1)}^{(4,4)} -\frac{1}{y}dx + \frac{x}{y^2}dy = (-\frac{x}{y})\Big|_{(4,1)}^{(4,4)} = 3$

   (b) Use $x = 4$ for $1 \le y \le 4$.

$$\int_{(4,1)}^{(4,4)} -\frac{1}{y}dx + \frac{x}{y^2}\ dy = \int_1^4 \frac{4}{y^2}dy = -\frac{4}{y}\Big|_1^4 = 3$$

6. (a) $P_y = -xy(x^2+y^2)^{-3/2} = Q_x$ and the integral is independent of path. $\phi_x = \frac{x}{\sqrt{x^2+y^2}}$, $\phi = \sqrt{x^2+y^2} + g(y)$, $\phi_y = \frac{y}{\sqrt{x^2+y^2}} + g'(y) = \frac{y}{\sqrt{x^2+y^2}}$, $g(y) = 0$, $\phi = \sqrt{x^2+y^2}$,
$$\int_{(1,0)}^{(3,4)} \frac{xdx + ydy}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2}\Big|_{(1,0)}^{(3,4)} = 4$$

   (b) Use $y = 2x - 2$ for $1 \le x \le 3$.
$$\int_{(1,0)}^{(3,4)} \frac{xdx + ydy}{\sqrt{x^2+y^2}} = \int_1^3 \frac{x + (2x-2)2}{\sqrt{x^2 + (2x-2)^2}}\ dx = \int_1^3 \frac{5x-4}{\sqrt{5x^2 - 8x + 4}}$$
$$= \sqrt{5x^2 - 8x + 4}\Big|_1^3 = 4$$

7. (a) $P_y = 4xy = Q_x$ and the integral is independent of path. $\phi_x = 2y^2x - 3$, $\phi = x^2y^2 - 3x + g(y)$, $\phi_y = 2x^2y + g'(y) = 2x^2y + 4$, $g(y) = 4y$, $\phi = x^2y^2 - 3x + 4y$, $\int_{(1,2)}^{(3,6)}(2y^2x-3)dx + (2yx^2+4)dy = \left(x^2y^2 - 3x + 4y\right)\Big|_{(1,2)}^{(3,6)} = 330$

   (b) Use $y = 2x$ for $1 \le x \le 3$.

$$\int_{(1,2)}^{(3,6)}(2y^2x-3)dx + (2yx^2+4)dy = \int_1^3([2(2x)^2x - 3] + [2(2x)x^2 + 4]2)dx$$
$$= \int_1^3(16x^3 + 5)dx = (4x^4 + 5x)\Big|_1^3 = 330$$

8. (a) $P_y = 4 = Q_x$ and the integral is independent of path. $\phi_x = 5x + 4y$, $\phi = \dfrac{5}{2}x^2 + 4xy + g(y)$, $\phi_y = 4x + g'(y) = 4x - 8y^3$, $g(y) = -2y^4$, $\phi = \dfrac{5}{2}x^2 + 4xy - 2y^4$, $\int_{(-1,1)}^{(0,0)}(5x+4y)dx + (4x-8y^3)dy = \left(\dfrac{5}{2}x^2 + 4xy - 2y^4\right)\Bigg|_{(-1,1)}^{(0,0)} = \dfrac{7}{2}$

   (b) Use $y = -x$ for $-1 \le x \le 0$.

$$\int(-1,1)^{(0,0)}(5x+4y)dx + (4x-8y^3)dy = \int_{-1}^0[(5x-4x) + (4x+8x^3)(-1)]dx$$
$$= \int_{-1}^0(-3x - 8x^3)dx = (-\frac{3}{2}x^2 - 2x^4)\Bigg|_{-1}^0 = \frac{7}{2}$$

9. (a) $P_y = 3y^2 + 3x^2 = Q_x$ and the integral is independent of path. $\phi_x = y^3 + 3x^2y$, $;\phi = xy^3 + x^3y + g(y)$, $\phi_y = 3xy^2 + x^3 + g'(y) = x^3 + 3y^2x + 1$, $g(y) = y$, $\phi = xy^3 + x^3y + y$, $\int_{(0,0)}^{(2,8)}(y^3 + 3x^2y)dx + (x^3 + 3y^2x + 1)dy = .(xy^3 + x^3y + y)\Big|_{(0,0)}^{(2,8)} = 1096$

   (b) Use $y = 4x$ for $0 \le x \le 2$.

$$\int_{(0,0)}^{(2,8)}(y^3 + 3x^2y)dx + (x^3 + 3y^2x + 1)dy = \int_0^2[(64x^3 + 12x^3) + (x^3 + 48x^3 + 1)(4)]dx$$
$$= \int_0^2(272x^3 + 4)dx = (68x^4 + 4x)\Big|_0^2 = 1096$$

10.

11. $P_y = 12x^3y^2 = Q_x$ throughout the plane and the vector field is a conservative field. $\phi_x = 4x^3y^3 + 3$, $\phi = x^4y^3 + 3x + g(y)$, $\phi_y = 3x^4y^2 + g'(y) = 3x^4y^2 + 1$, $g(y) = y$, $\phi = x^4y^3 + 3x + y$

12. $P_y = 6xy^2 = Q_x$ throughout the plane and the vector field is a conservative field. $\phi_x = 2xy^3$, $\phi = x^2y^3 + g(y)$, $\phi_y = 3x^2y^2 + g'(y) = 3x^2y^2 + 3y^2$, $g(y) = y^3$, $\phi = x^2y^3 + y^3$

13. $P_y = -2xy^3\sin xy^2 + 2y\cos xy^2$, $Q_x = -2xy^3\cos xy^2 - 2y\sin xy^2$ throughout the plane and the vector is not a conservative field.

14. $P_y = -4xy(x^2+y^2+1)^{-3} = Q_x$ throughout the plane and the vector field is a conservative field. $\phi_x = x(x^2+y^2+1)^{-2}$, $\phi = -\frac{1}{2}(x^2+y^2+1)^{-1} + g(y)$, $\phi_y = y(x^2+y^2+1)^{-2} + g'(y) = y(x^2+y^2+1)^{-2}$, $g(y) = 0$, $\phi = -\frac{1}{2}(x^2+y^2+1)^{-1}$

15. $P_y = 1 = Q_x$ throughout the plane and the vector field is a conservative field. $\phi_x = x^3 + y$, $\phi = \frac{1}{4}x^4 + xy + g(y)$, $\phi_y = x + g'(y) = x + y^3$, $g(y) = -\frac{1}{4}y^4$, $\phi = \frac{1}{4}x^4 + xy + \frac{1}{4}y^4$

16. $P_y = 4e^{2y}$, $Q_x = e^{2y}$ throughout the plane and the vector field is not a conservative field.

17. $P_y = 0 = Q_x$, $P_x = 0 = R_x$, $Q_z = -1 = R_y$ throughout 3-space and the vector field is a conservative field.
$\phi_x = 2x$, $\phi = x^2 + g(y,z)\phi_y = \dfrac{\partial g}{\partial y} = 3y^2 - x$,
$g(y,z) = y^3 - yz + h(z), \phi = x^2 + y^3 - yz + h(z)$,
$\phi_z = -y + h'(z) = -y$, $h(z) = 0, \phi = x^2 + y^3 - yz$

18. $P_y = 2x = Q_x$, $P_z = 0 = R_x$, $Q_z = -e^{-y} = R_y$ throughout 3-space and the vector field is a conservative field.
$\phi_x = 2xy$, $\phi = x^2y + g(y,z)$, $\phi_y = x^2 + \dfrac{\partial g}{\partial y} = x^2 - ze^{-y}$,
$g = ze^{-y} + h(z)$, $\phi = x^2y + ze^{-y} + h(z)$,
$\phi_z = e^{-y} + h'(z) = e^y - 1$, $h(z) = -z$, $\phi = x^2y + ze^{-y} - z$

19. Since $P_y = -e^{-y} = Q_x$, $\mathbf{F}$ is conservative and $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of the path. Thus, instead of the given curve we may use the simpler curve $C_1:\ y = x,\ 0 \le x \le 1$. Then

$$\begin{aligned}
W &= \int_{C_1} (2x + e^{-y})dx + (4y - xe^{-y})dy \\
&= \int_0^1 (2x + e^{-x})dx + \int_0^1 (4x - e^{-x})dx \quad \boxed{\text{Integration by parts}} \\
&= (x^2 - e^{-x})\Big|_0^1 + (2x^2 + xe^{-x})\Big|_0^1 \\
&= [(1 - e^{-1}) - (-1)] + [(2 + e^{-1} + e^{-1}) - (1)] = 3 + e^{-1}.
\end{aligned}$$

20. Since $P_y = -e^{-y} = Q_x, \mathbf{F}$ is conservative and $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of the path. Thus, instead of the given curve we may use the simpler curve $C_1:\ y = 0\ \ -2 \le -x \le 2$. Then $dy = 0$ and
$W = \int_{C_1}(2x+e^{-y})dx + (4y - xe^{-y})dy = \int_2^{-2}(2x+1)dx = (x^2+x)\Big|_2^{-2} = (4-2)-(4+2) = -4.$

21. $P_y = z = Q_x$, $Q_z = x = R_y$, $R_x = y = P_z$, and the integral is independent of path. Parameterize the line segment between the points by $x = 1+t$, $y = 1+3t$, $z = 1+7t$, for $0 \le t \le 1$. Then $dx = dt$, $dy = 3dt$, $dz = 7dt$ and

$$\begin{aligned}
\int_{(1,1,1)}^{(2,4,8)} yzdx + xzdy + xydz &= \int_0^1 [(1+3t)(1+7t) + (1+t)(1+7t)(3) + (1+t)(1+3t)(7)]dt \\
&= \int_0^1 (11 + 62t + 63t^2)dt = (11t + 31t^2 + 21t^3)\Big|_0^1 = 63.
\end{aligned}$$

22. $P_y = 0 = Q_x, \quad Q_z = 0 = R_y, \quad R_x = 0 = P_z$ and the integral is independent of path. Parameterize the line segment between the points by $x = t, \quad y = t, \quad z = t$, for $0 \le t \le 1$. Then $dx = dy = dz = dt$ and
$\int_{(0,0,0)}^{(1,1,1)} 2xdx + 3y^2dy + 4z^3dz = \int_0^1 (2t + 3t^2 + 4t^3)dt = (t^2 + t^3 + t^4)\big|_0^1 = 3.$

23. $P_y = 2x\cos y = Q_x, \quad Q_z = 0 = R_y, \quad R_x = 3e^{3z} = P_z$, and the integral is independent of path. Integrating $\phi_x = 2x\sin y + e^{3z}$ we find $\phi = x^2\sin y + xe^{3z} + g(y,z)$. Then $\phi_y = x^2\cos y + g_y = Q = x^2\cos y$, so $g_y = 0, \quad g(y,z) = h(z)$, and $\phi = x^2\sin y + xe^{3z} + h(z)$. Now $\phi_z = 3xe^{3z} + h'(z) = R = 3xe^{3z} + 5$, so $h'(z) = 5$ and $h(z) = 5z$. Thus $\phi = x^2\sin y + xe^{3z} + 5z$ and

$$\int_{(1,0,0)}^{(2,\pi/2,1)} (2x\sin y + e^{3z})dx + x^2\cos y dy_(3xe^{3z} + 5)dz$$
$$= (x^2\sin y + xe^{3z} + 5z)\big|_{(1,0,0)}^{(2,\pi/2,1)} = [4(1) + 2e^3 + 5] - [0 + 1 + 0] = 8 + 2e^3.$$

24. $P_y = 0 = Q_x, \quad Q_z = 0 = R_y, \quad R_x = 0 = P_z$, and the integral is independent of path. Parameterize the line segment between the points by $x = 1 + 2t, \quad y = 2 + 2t, \quad z = 1$, for $0 \le t \le 1$. Then $dx = 2dt, \quad dz = 0$ and

$$\int_{(1,2,1)}^{(3,4,1)} (2x+1)dx + 3y^2dy + \frac{1}{z}dz = \int_0^1 [(2 + 4t + 1)2 + 3(2 + 2t)^2 2]dt$$
$$= \int_0^1 (24t^2 + 56t + 30)dt = (8t^3 + 28t^2 + 30t)\big|_0^1 = 66.$$

25. $P_y = 0 = Q_x; \quad Q_z = 0 = R_y, \quad R_x = 2e^{2z} = P_z$ and the integral is independent of path. Parameterize the line segment between the points by $x = 1 + t, \quad y = 1 + t, \quad z = \ln 3$, for $0 \le t \le 1$. Then $dx = dy = dt, \quad dz = 0$ and

$$\int_{(1,1,\ln 3)}^{(2,2,\ln 3)} e^{2z}dx + 3y^2dy + 2xe^{2z}dz = \int_0^1 [e^{2\ln 3} + 3(1+t)^2]dt = [9t + (1+t)^3]\big|_0^1 = 16$$

26. $P_y = 0 = Q_x, \quad Q_z = 2y = R_y, \quad R_x = 2x = P_z$ and the integral is independent of path. Parameterize the line segment between the points by $x = -2(1-t), \quad y = 3(1-t), \quad z = 1-t$, for $0 \le t \le 1$. Then $dx = 2dt, \quad dy = -3dt, \quad dz = -dt$, and

$$\int_{(-2,3,1)}^{(0,0,0)} 2xzdx + 2yzdy + (x^2 + y^2)dz = \int_0^1 [-4(1-t)^2(2) + 6(1-t)^2(-3)$$
$$+ 4(1-t)^2(-1) + 9(1-t)^2(-1)]dt$$
$$= \int_0^1 -39(1-t)^2dt = 13(1-t)^3\Big|_0^1 = -13.$$

27. $P_y = 1 - z\sin x = Q_x, Q_z = \cos x = R_y, \quad R_x = -y\sin x = P_z$ and the integral is independent of path. Integrating $\theta_x = y - yz\sin x$ we find $\theta = xy + yz\cos x + g(y,z)$. Then $\theta_y = x + z\cos x + g_y(y,z) = Q = x + z\cos x$, so $g_y = 0, \quad g(y,z) = h(z)$, and

$\theta = xy + yz\cos x + h(z)$. Now $\theta_z = y\cos x + h(z) = R = y\cos x$, so $h(z) = 0$ and $\theta = xy + yz\cos x$. Since $\mathbf{r}(0) = 4\mathbf{j}$ and $\mathbf{r}(\pi/2) = \pi\mathbf{i} + \mathbf{j} + 4\mathbf{k}$,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = (xy + yz\cos x)\Big|_{(0,4,0)}^{\pi,1,4)} = (\pi - 4) - (0 + 0) = \pi - 4.$$

28. $P - y = 0 = Q_x$, $Q_z = 0 = R_y$, $R_z = -e^z = P_z$ and the integral is independent of path. Integrating $\phi_x = 2 - e^z$ we find $\phi = 2x - xe^z + g(y,z)$. Then $\phi - Y = g_y = 2y - 1$, so $g(y,z) = y^2 - y + h(z)$ and $\phi = 2x - xe^z + y^2 - y + h(z)$. Now $\phi_z = -xe^z + h'(z) = R = 2 - xe^z$, so $h'(z) = 2$, $h(z) = 2z$, and $\phi = 2x - xe^z + y^2 - y + 2z$. Thus

$$\int_C \mathbf{F}\cdot d\mathbf{r} = (2x - xe^z + y^2 - y + 2z)\Big|_{(-1,1,-1)}^{(2,4,8)}$$
$$= (4 - 2e^8 + 16 - 4 + 16) - (-2 + e^{-1} + 1 - 1 - 2) = 36 - 2e^8 - e^{-1}$$

29. Since $P_y = Gm_1m_2(2xy/|r|^5) = Q_x$, $Q_z = Gm_1m_2(2yz/|r|^5) = R_y$, and $R_x = Gm_1m_2(2xz/|r|^5) = P_z$, the force field is conservative.
$\theta_x = -Gm_1m_2\dfrac{x}{(x^2+y^2+z^2)^{3/2}}$, $\theta = Gm_1m_2(x^2+y^2+z^2)^{-1/2} + g(y,z)$,
$\theta_y = -Gm_1m_2\dfrac{y}{(x^2+y^2+z^2)^{3/2}} + g_y(y,z) = -Gm_1m_2\dfrac{y}{(x^2+y^2+z^2)^{3/2}}$, $g(y,z) = h(z)$,
$\theta = Gm_1m_2(x^2+y^2+z^2)^{-1/2} + h(z)$,
$\theta_z = -Gm_1m_2\dfrac{z}{(x^2+y^2+z^2)^{3/2}} + h'(z) = -Gm_1m_2\dfrac{z}{(x^2+y^2+z^2)^{3/2}}$,
$h(z) = 0$, $\theta = \dfrac{Gm_1m_2}{\sqrt{x^2+y^2+z^2}} = \dfrac{Gm_1m_2}{|r|}$

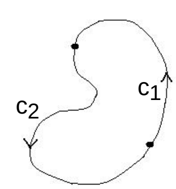
30. Since $P_y = 24xy^2z = Q_x$, $Q_z = 12x^2y^2 = R_y$, and $R_x = 8xy^3 = P_z$, $\mathbf{F}$ is conservative. Thus, the work done between two points is independent of the path. From $\theta_x = 8xy^3z$ we obtain $\theta = 4x^2y^3z$ which is a potential function for $\mathbf{F}$. Then

$$W = \int_{(2,0,0)}^{(1,\sqrt{3},\pi/3)} \mathbf{F}\cdot d\mathbf{r} = 4x^2y^3z\Big|_{(2,0,0)}^{(1,\sqrt{3},\pi/3)} = 4\sqrt{3}\pi$$

and $W = \int_{(2,0,0)}^{(0,2,\pi/2)} \mathbf{F}\cdot d\mathbf{r} = 0$.

31. Since $\mathbf{F}$ is conservative, $\int_{C_1}\mathbf{F}\cdot d\mathbf{r} = \int_{-C_2}\mathbf{F}\cdot d\mathbf{r}$. Then, since the simply closed curve C is composed of C_1 and C_2,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{C_1}\mathbf{F}\cdot d\mathbf{r} + \int_{C_2}\mathbf{F}\cdot d\mathbf{r} = \int_{C_1}\mathbf{F}\cdot d\mathbf{r} - \int_{-C_2}\mathbf{F}\cdot d\mathbf{r} = 0.$$



32. From $\mathbf{F} = (x^2+y^2)^{n/2}(x\mathbf{i} + y\mathbf{j})$ we obtain $P_y = nxy(x^2+y^2)^{n/2-1} = Q_x$, so that $\mathbf{F}$ is conservative. From $\theta_x = x(x^2+y^2)^{n/2}$ we obtain the potential function $\theta + (x^2+y^2)^{(n+2)/2}/(n+2)$. Then

$$W = \int_{(x_1,y_1)}^{(x_2,y_2)} \mathbf{F}\cdot d\mathbf{r} = \left(\frac{(x^2+y^2)^{(n+2)/2}}{n+2}\right)\Bigg|_{(x_1,y_1)}^{(x_2,y_2)} = \frac{1}{n+2}\left[(x_2^2+y_2^2)^{(n+2)/2} - (x_1^2+y_1^2)^{(n+2)/2}\right].$$

33. $P - y = -2x \sin y = Q_x$ throughout the plane and the vector field **F** is a conservative field. The path starts at point $(1, 0)$ and ends at point $(2, 1)$. Since **F** is conservative, the integral is path independent so we can use any path C starting at $(1, 0)$ and ending at $(2, 1)$. Use the path $y = x - 1,\ \ 1 \le x \le 2$. Then

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_1^2 (2x \cos(x-1) - x^2 \sin(x-1))dx \\ &= x^2 \cos(x-1)\Big|_1^2 = 4\cos 1 - 1\end{aligned}$$

34. $P_y = \cos y + Q_x$, $P - z = 0 = R_x$, $Q_z = 0 = R_y$ throughout the plane and the vector field **F** is a conservative field. The path starts at the point $(0, 0, 0)$ and ends at the point $(1, 1, 1)$. Since **F** is conservative, the integral is path independent so we can use any path C starting at $(0, 0, 0)$ and ending at $(1, 1, 1)$. Use the path $y = z = x,\ \ 0 \le x \le 1$. Then

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 (\sin x + x\cos x + x^2)dx \\ &= x \sin x + \frac{x^3}{3}\Bigg|_0^1 = \sin 1 + \frac{1}{3}\end{aligned}$$

35. **F** cannot be a conservative field in the region.

36. (a) $P_y = \dfrac{y^2 - x^2}{(y^2 + x^2)^2} = Q_x$. Using the hint, we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (\sin^2 t + \cos^2 t)dt = \int_0^{2\pi} dt = 2\pi.$$

Since C is a closed path, the integral would be zero if **F** were conservative.

(b) Any simply connected region containing the path C would have to contain the origin. But **F** and the partials P_y and Q_x are not defined at the origin. Therefore, the theorem does not apply.

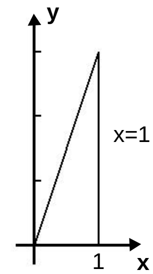
37. From Problem 45 in Exercises 15.2, $\dfrac{dv}{dt} \cdot \dfrac{dr}{dt} = \dfrac{dv}{dt} \cdot v = \dfrac{1}{2}\dfrac{d}{dt}v^2$. Then, using $\dfrac{dp}{dt} = \dfrac{\partial p}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial p}{\partial y}\dfrac{dy}{dt} = \nabla p \cdot \dfrac{dr}{dt}$, we have

$$\begin{aligned}\int m\frac{dv}{dt} \cdot dr dt dt + \int \nabla p \cdot \frac{dr}{dt} &= \int 0 dt \\ \frac{1}{2}m \int \frac{d}{dt}v^2 dt + \int \frac{dp}{dt}dt &= \text{ constant} \\ \frac{1}{2}mv^2 + p &= \text{constant.}\end{aligned}$$

38. By Problem 37, the sum of kinetic and potential energies in a conservative force field is constant. That is, it is independent of points A and B, so $p(B) + K(B) = p(A) + K(A)$.

15.4 Green's Theorem

1. The sides of the triangle are $C_1: \ y = 0, \ 0 \le x \le 1$; $C_2: \ x = 1, \ 0 \le y \le 3$; $C_3: \ y = 3x, \ 0 \le -x \le 1$.

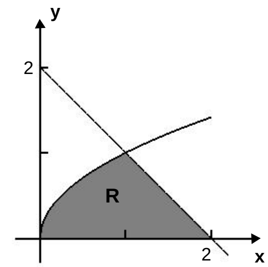


$$\int_C (x-y)dx + xydy = \int_0^1 xdx + \int_0^3 ydy + \int_1^0 (x-3x)dx + \int_1^0 x(3x)dx$$
$$= \left(\frac{1}{2}x^2\right)\Big|_0^1 + \left(\frac{1}{2}y^2\right)\Big|_0^3 + (-x^2)\big|_0^1 + (3x^2)\big|_1^0$$
$$= \frac{1}{2} + \frac{9}{2} + 1 - 3 = 3$$
$$\int\int_R (y+1)dA = \int_0^1 \int_0^{3x} (y+1)dydx = \int_0^1 \left(\frac{1}{2}y^2 + y\right)\Big|_0^{3x} dx = \int_0^1 \left(\frac{9}{2}x^2 + 3x\right) dx$$
$$= \left(\frac{3}{2}x^3 + \frac{3}{2}x^2\right)\Big|_0^1 = 3$$

2. The sides of the rectangle are $C_1: \ y = 0, \ -1 \le x \le 1$; $C_2: \ x = 1, \ 0 \le y \le 1$; $C_3: \ y = 1, \ 1 \ge x \ge -1$; $C_4: \ x = -1, \ 1 \ge y \ge 0$.
$$\int_C 3x^2ydx + (x^2 - 5y)dy = \int_{-1}^1 0dx + \int_0^1 (1-5y)dy = \int_{-1}^1 0dx + \int_0^1 (1-5y)dy$$
$$= \int_1^{-1} 3x^2dx + \int_1^0 (1-5y)dy = \left(y - \frac{5}{2}y^2\right)\Big|_0^1 + x^3\big|_1^{-1} + \left(y - \frac{5}{2}y^2\right)\Big|_0^1 = -2$$
$\int\int_R (2x - 3x^2)dA = \int_0^1 \int_{-1}^1 (2x - 3x^2)dxdy = \int_0^1 (x^2 - x^3)\Big|_{-1}^1 dy = \int_0^1 (-2)dy = -2$

3. $$\int_C -y^2dx + x^2dy = \int_0^{2\pi} (-9\sin^2 t)(-3\sin t)dt + \int_0^{2\pi} 9\cos^2 t(3\cos t)dt$$
$$= 27\int_0^{2\pi} [(1-\cos^2 t)\sin t + (1-\sin^2 t)\cos t]dt$$
$$= 27\left(-\cos t + \frac{1}{3}\cos^3 t + \sin t - \frac{1}{3}\sin^3 t\right)\Big|_0^{2\pi} = 27(0) = 0$$
$$\int\int_R (2x+2y)dA = 2\int_0^{2\pi}\int_0^3 (r\cos\theta + r\sin\theta)rdrd\theta = 2\int_0^{2\pi}\int_0^3 r^2(\cos\theta + \sin\theta)drd\theta$$
$$= 2\int_0^{2\pi} \left[\frac{1}{3}r^3(\cos\theta + \sin\theta)\right]\Big|_0^3 d\theta = 18\int_0^{2\pi} (\cos\theta + \sin\theta)d\theta$$
$$= 18(\sin\theta - \cos\theta)\big|_0^{2\pi} = 18(0) = 0$$

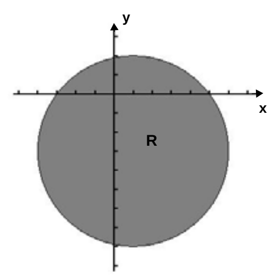
4. The sides of the region are $C_1: \quad y=0, \quad 0 \le x \le 2$; $C_2:$ $y=-x+2, \ 2 \ge x \ge 1$; $C_3: \quad y=\sqrt{x}, \ 1 \ge x \ge 0$.



$$\int_C -2y^2dx + 4xydy = \int_0^2 0dx + \int_2^1 -2(-x+2)^2dx + \int_2^1 4x(-x+2)(-dx) + \int_1^0 -2xdx + \int_1^0 4x\sqrt{x}\left(\frac{1}{2\sqrt{x}}\right)dx = 0 + \frac{2}{3} + \frac{8}{3} + 1 - 1 = \frac{10}{3}$$

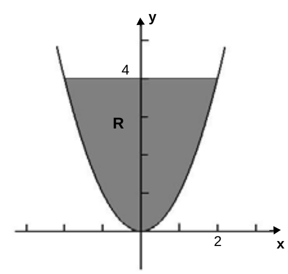
$$\int\int_R 8ydA = \int_0^1\int_{y^2}^{2-y} 8ydxdy = \int_0^1 8y(2-y-y^2)dy = \left(8y^2 - \frac{8}{3}y^3 - 2y^4\right)\Big|_0^1 = \frac{10}{3}$$

5. $P=2y, \ \ P_y=2, \ \ Q=5x, \ \ Q_x=5$



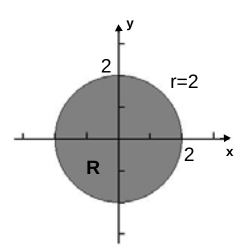
$$\int_C 2ydx + 5xdy = \int\int_R (5-2)dA = 3\int\int_R dA = 3(25\pi) = 75\pi$$

6. $P=x+y^2, \ \ P_y=2y, \ \ Q=2x^2-y, \ \ Q_x=4x$



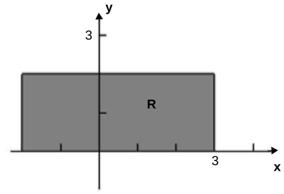
$$\int_C (x+y^2)dx + (2x^2-y)dy = \int\int_R (4x-2y)dA = \int_{-2}^2\int_{x^2}^4 (4x-2y)dydx = \int_{-2}^2 (4xy-y^2)\Big|_{x^2}^4 dx = \int_{-2}^2 (16x-16-4x^3+x^4)dx = \left(8x^2-16x-x^4+\frac{1}{5}x^5\right)\Big|_{-2}^2 = -\frac{96}{5}$$

7. $P=x^4-2y^3, \ \ P_y=-6y^2, \ \ Q=2x^3-y^4, \ \ Q_x=6x^2$.
Using polar coordinates,



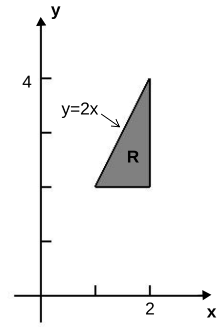
$$\int_C (x^4-2y^3)dx + (2x^3-y^4)dy = \int\int_R (6x^2+6y^2)dA = \int_0^{2\pi}\int_0^2 6r^2rdrd\theta = \int_0^{2\pi}\left(\frac{3}{2}r^4\right)\Big|_0^2 d\theta = \int_0^{2\pi} 24d\theta = 48\pi.$$

8. $P = x - 3y,\ \ P_y = -3,\ \ Q = 4x + y,\ \ Q_x = 4$
$\int_C (x - 3y)dx + 4(x + y)dy = \int\int_R (4 + 3)dA = 7(10) = 70$



9. $P = 2xy,\ \ P_y = 2x,\ \ Q = 3xy^2,\ \ Q_x = 3y^2$

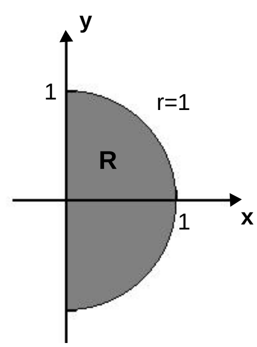
$$\int_C 2xydx + 3xy^2dy = \int\int_R (3y^2 - 2x)dA = \int_1^2 \int_2^{2x} (3y^2 - 2x)dydx$$
$$= \int_1^2 (y^3 - 2xy)\Big|_2^{2x} dx = \int_1^2 (8x^3 - 4x^2 - 8 + 4x)dx$$
$$= \left(2x^4 - \frac{4}{3}x^3 - 8x + 2x^2\right)\Big|_1^2 = \frac{40}{3} - \left(-\frac{16}{3}\right) = \frac{56}{3}$$



10. $P = e^{2x}\sin 2y,\ \ P_y = 2e^{2x}\cos 2y,\ \ Q = e^{2x}\cos 2y,\ \ Q_x = 2e^{2x}\cos 2y$
$\int_C = e^{2x}\sin 2ydx + e^{2x}\cos 2ydy = \int\int_R 0dA = 0$

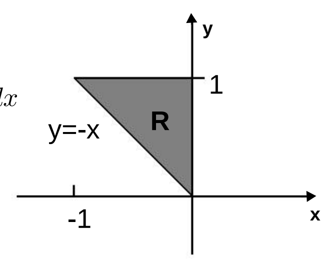
11. $P = xy,\ \ P_y = x,\ \ Q = x^2,\ \ Q_x = 2x$. Using polar coordinates,

$$\int_C xydx + x^2dy = \int\int_R (2x - x)dA = \int_{-\pi/2}^{\pi/2} \int_0^1 r\cos\theta rdrd\theta$$
$$= \int_{-\pi/2}^{\pi/2} \left(\frac{1}{3}r^3\cos\theta\right)\Big|_0^1 d\theta = \int_{-\pi/2}^{\pi/2} \frac{1}{3}\cos\theta d\theta$$
$$= \frac{1}{3}\sin\theta\Big|_{-\pi/2}^{\pi/2} = \frac{2}{3}$$

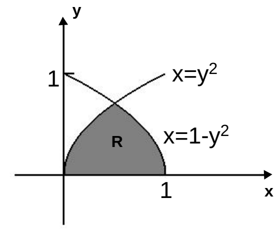


12. $P = e^{x^2},\ \ P_y = 0,\ \ Q = 2\tan^{-1}x,\ \ Q_x = \dfrac{2}{1 + x^2}$

$$\int_C e^{x^2}dx + 2\tan^{-1}xdy = \int\int_R \frac{2}{1 + x^2}dA = \int_{-1}^0 \int_{-x}^1 \frac{2}{1 + x^2}dydx$$
$$= \int_{-1}^0 \left(\frac{2y}{1 + x^2}\right)\Big|_{-x}^1 dx$$
$$= \int_{-1}^0 \left(\frac{2}{1 + x^2} + \frac{2x}{1 + x^2}\right)dx$$
$$= [2\tan^{-1}x + \ln(1 + x^2)]\Big|_{-1}^0 = 0 - \left(-\frac{\pi}{2} + \ln 2\right) = \frac{\pi}{2} - \ln 2$$

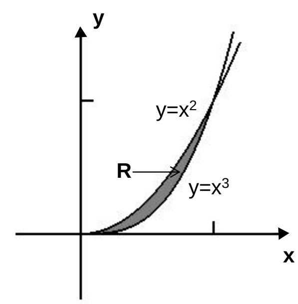


13. $P = \frac{1}{3}y^3,\ \ P_y = y^2,\ \ Q = xy + xy^2,\ \ Q_x = y + y^2$



$$\int_C \frac{1}{3}y^3 dx + (xy + xy^2)dy = \int\int_R y dA = \int_0^{1/\sqrt{2}} \int_{y^2}^{1-y^2} y dx dy$$
$$= \int_0^{1/\sqrt{2}} (xy)\Big|_{y^2}^{1-y^2} dy$$
$$= \int_0^{1/\sqrt{2}} (y - y^2 - y^3) dy$$
$$= \left(\frac{1}{2}y^2 - \frac{1}{2}y^4\right)\Big|_0^{1/\sqrt{2}} = \frac{1}{4} - \frac{1}{8} = \frac{1}{8}$$

14. $P = xy^2,\ \ P_y = 2xy,\ \ Q = 3\cos y,\ \ Q_x = 0$



$$\int_C xy^2 dx + 3\cos y dy = \int\int_R (-2xy) dA = -\int_0^1 \int_{x^3}^{x^2} 2xy dy dx$$
$$= -\int_0^1 (xy)\Big|_{x^3}^{x^2} dx = -\int_0^1 (x^3 - x^4) dx$$
$$= \left(\frac{1}{4}x^4 - \frac{1}{5}x^5\right)\Big|_0^1 = -\frac{1}{20}$$

15. $P = ay,\ \ P_y = a,\ \ Q = bx,\ \ Q_x = b.$
$\int_C aydx + bxdy = \int\int_R (b - a)dA = (b - a) \times$ (area bounded by C)

16. $P = P(x),\ \ P_y = 0,\ \ Q = Q(y),\ \ Q_x = 0.\ \ \int_C P(x)dx + Q(y)dy = \int\int_R 0dA = 0$

17. For the first integral: $P = 0,\ \ P_y = 0,\ \ Q = x,\ \ Q_x = 1;\ \ \int_C xdy = -\int\int_R 1dA =$area of R. For the second integral: $P = y,\ \ P_y = 1,\ \ Q = 0,\ \ Q = 0;\ -\int_C ydx = -\int\int_R -dA =$area of R. Thus, $\int_C xdy = -\int_C ydx$.

18. $P = -y,\ \ P_y = -1,\ \ Q + x,\ \ Q_x = 1.\ \ \frac{1}{2}\int_C -ydx + xdy = \frac{1}{2}\int\int_R 2dA = \int\int_R dA =$area of R

19. $$A = \int\int_R dA = \int_C xdy = \int_0^{2\pi} a\cos^3 t(3a\sin^2 t\cos t dt) = 3a^2\int_0^{2\pi} \sin^2 t\cos^4 t dt$$
$$= 3a^2\left(\frac{1}{16}t - \frac{1}{64}\sin 4t + \frac{1}{48}\sin^3 2t\right)\Big|_0^{2\pi} = \frac{3}{8}\pi a^2$$

20. $$A = \int\int_R dA = \int_C xdy = \int_0^{2\pi} a\cos t(b\cos t dt) = ab\int_0^{2\pi} \cos^2 t dt$$
$$= ab\left(\frac{1}{2}t + \frac{1}{4}\sin 2t\right)\Big|_0^{2\pi} = \pi ab$$

21. (a) Parameterize C by $x = x_1 + (x_2 - x_1)t$ and $y = y_1 + (y_2 - y_1)t$ for $0 \le t \le 1$. Then

$$\begin{aligned}
\int_C -ydx + xdy &= \int_0^1 -[y_1 + (y_2 - y_1)t](x_2 - x_1)dt + \int_0^1 [x_1 + (x_2 - x_1)t](y_2 - y_1)dt \\
&= -(x_2 - x_1)[y_1 t + \frac{1}{2}(y_2 - y_1)t^2]\Big|_0^1 + (y_2 - y_1)[x_1 t + \frac{1}{2}(x_2 - x_1)t^2]\Big|_0^1 \\
&= -(x_2 - x_1)\left[y_1 + \frac{1}{2}(y_2 - y_1)\right] + (y_2 - y_1)\left[x_1 + \frac{1}{2}(x_2 - x_1)\right] \\
&= x_1 y_2 - x_2 y_1.
\end{aligned}$$

(b) Let $C_{\mathbf{i}}$ be the line segment from $(x_{\mathbf{i}}, y_{\mathbf{i}})$ to $(x_{\mathbf{i}+1}, y_{\mathbf{i}+1})$ for $\mathbf{i} = 1, \ 2, \ \cdots, n-1$, and C_2 the line segment from $(x_n, y_n)$to (x_1, y_1). Then

$$\begin{aligned}
A &= \frac{1}{2}\int_C -ydx + xdy \quad \boxed{\text{Problem 18}} \\
&= \frac{1}{2}\left[\int_{C_1} -ydx + xdy + \int_{C_2} -ydx + xdy + \cdots + \int_{C_{n-1}} -ydx + xdy + \int_{C_n} -ydx + xdy\right] \\
&= \frac{1}{2}(x_1 y_2 - x_2 y_1) + \frac{1}{2}(x_2 y_3 - x_3 y_2) + \frac{1}{2}(x_{n-1} y_n - x_n y_{n-1}) + \frac{1}{2}(x_n y_1 - x_1 y_n).
\end{aligned}$$

22. From part (**b**) of Problem 21

$$\begin{aligned}
A &= \frac{1}{2}\left[(-1)(1) - (1)(3)] + \frac{1}{2}[(1)(2) - (4)(1)\right] + \frac{1}{2}\left[(4)(5) - (3)(2)] + \frac{1}{2}[(3)(3) - (-1)(5)\right] \\
&= \frac{1}{2}(-4 - 2 + 14 + 14) = 11.
\end{aligned}$$

23. $P = 4x^2 - y^3, \ \ P_y = -3y^2; \ \ Q = x^3 + y^2, \ \ Q_x = 3x^2.$

$$\begin{aligned}
\int_C (4x^2 - y^3)dx + (x^3 + y^2)dy &= \int\int_R (3x^2 + 3y^2)dA = \int_0^{2\pi}\int_1^2 3r^2(rdrd\theta) = \int_0^{2\pi}\left(\frac{3}{4}r^4\right)\Big|_1^2 d\theta \\
&= \int_0^{2\pi}\frac{45}{4}d\theta = \frac{45\pi}{2}
\end{aligned}$$

24. $P = \cos^2 x - y, \ \ P_y = -1; \ \ Q = \sqrt{y^2 + 1}, \ \ Q_x = 0$

$$\begin{aligned}
\oint_C (\cos^2 x - y)dx + \sqrt{y^2 + 1}dy &= \int\int_R (0 + 1)dA = \int\int_R dA \\
&= (6\sqrt{2})^2 - \pi(2)(4) = 72 - 8\pi
\end{aligned}$$

25. We first observe that $P_y + (y^4 - 3x^2y^2)/(x^2 + y^2)^3 = Q_x$. Letting C' be the circle $x^2 + y^2 = \dfrac{1}{4}$ we have

$$\int_C \frac{-y^3dx+xy^2dy}{(x^2+y^2)^2} = \int_{C'} \frac{-y^3dx+xy^2dy}{(x_2+y^2)^2}$$

$$\boxed{x=\frac{1}{4}\cos t,\quad dx=-\frac{1}{4}\sin tdt,\quad y=\frac{1}{4}\sin t,\quad dy=\frac{1}{4}\cos tdt}$$

$$= \int_0^{2\pi} \frac{-\frac{1}{64}\sin^3 t\left(-\frac{1}{4}\sin tdt\right)+\frac{1}{4}\cos t\left(\frac{1}{16}\sin^2 t\right)\left(\frac{1}{4}\cos tdt\right)}{1/256}$$

$$= \int_0^{2\pi}(\sin^4 t+\sin^2 t\cos^2 t)dt = \int_0^{2\pi}(\sin^4 t+(\sin^2 t-\sin^4 t)dt$$

$$= \int_0^{2\pi}\sin^2 tdt = \left(\frac{1}{2}t-\frac{1}{4}\sin 2t\right)\Big|_0^{2\pi} = \pi$$

26. We first observe that $P_y=[4y^2-(x+1)^2]/[(x+1)^2+4y^2]^2=Q_x$. Letting C' be the ellipse $(x+1)^2+4y^2=4$ we have

$$\int_C \frac{-y}{(x+1)^2+4y^2}dx+\frac{x+1}{(x+1)^2+4y^2}dy = \int_{C'} \frac{-y}{(x+1)^2+4y^2}dx+\frac{x+1}{(x+1)^2+4y^2}dy$$

$$\boxed{x+1=2\cos t,\quad dx=-2\sin tdt,\quad y=\sin t,\quad dy=\cos tdt}$$

$$= \int_0^{2\pi}\left[\frac{-\sin t}{4}(-2\sin t)+\frac{2\cos t}{4}\cos t\right]dt = \frac{1}{2}\int_0^{2\pi}(\sin^2 t+\cos^2 t)dt=\pi.$$

27. Writing $\int\int_R x^2dA = \int\int_R(Q_x-P_y)dA$ we identify $Q=0$ and $P=-x^2y$. Then, with $C:\ \ x=3\cos t,\ \ y=2\sin t,\ \ 0\le t\le 2\pi$, we have

$$\int\int_R x^2dA = \int_C Pdx+Qdy = \int_C -x^2ydx = -\int_0^{2\pi}9\cos^2 t(2\sin t)(-3\sin t)dt$$

$$= \frac{54}{4}\int_0^{2\pi}4\sin^2 t\cos^2 tdt = \frac{27}{2}\int_0^{2\pi}\sin^2 2tdt = \frac{27}{4}\int_0^{2\pi}(1-\cos 4t)dt$$

$$= \frac{27}{4}\left(t-\frac{1}{4}\sin 4t\right)\Big|_0^{2\pi} = \frac{27\pi}{2}.$$

28. Writing $\int\int_R[1-2(y-1)]dA=\int\int_R(Q_x-P_y)dA$ we identify $Q=x$ and $P=(y-1)^2$. Then, with $C_1:\ \ x=\cos t,\ \ y-1=\sin t,\ \ -\pi/2\le t\le\ \pi/2$, and $C_2:\ \ x=0,\ \ 2\ge y\ge 0$,

$$\int\int_R[1-2(y-1)]dA = \int_{C_1}Pdx+Qdy+\int_{C_2}Pdx+Qdy = \int_{C_1}(y-1)^2dx+xdy+\int_{C_2}0dy$$

$$= \int_{-\pi/2}^{\pi/2}[\sin^2 t(-\sin t)+\cos t\cos t]dt = \int_{-\pi/2}^{\pi/2}[\cos^2 t-(1-cos^2 t)\sin t]dt$$

$$= \int_{-\pi/2}^{\pi/2}\left[\frac{1}{2}(1+\cos 2t)-\sin t+\cos^2 t\sin t\right]dt$$

$$= \left(\frac{1}{2}t+\frac{1}{4}\sin 2t+\cos t-\frac{1}{3}\cos^3 t\right)\Big|_{-\pi/2}^{\pi/2} = \frac{\pi}{4}-\left(-\frac{\pi}{4}\right)=\frac{\pi}{2}.$$

29. $P = x - y,\ \ P_y = -1,\ \ Q = x + y,\ \ Q_x = 1;$
$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int\int_R 2dA = 2 \times \text{area} = 2(\frac{3\pi}{4}) = \frac{3}{2}\pi$

30. $P = -xy^2,\ \ P_y = -2xy,\ \ Q = x^2y,\ \ Q_x = 2xy$. Using polar coordinates,
$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int\int_R 4xydA = \int_0^{\pi/2}\int_1^2 4(r\cos\theta)(r\sin\theta)rdrd\theta = \int_0^{\pi/2}(r^4\cos\theta\sin\theta)\Big|_1^2 d\theta$$
$$= 15\int_0^{\pi/2}\sin\theta\cos\theta d\theta = \frac{15}{2}\sin^2\theta\Big|_0^{\pi/2} = \frac{15}{2}.$$

31. Let $P = 0$ and $W = x^2$. Then $Q_x - P_y = 2x$ and
$$\frac{1}{2A}\oint_C x^2dy = \frac{1}{2A}\int\int_R 2xdA = \frac{\int\int_R xdA}{A} = \overline{x}.$$
Let $P = y^2$ and $Q = 0$. Then $Q_x - P_y = -2y$ and
$$-\frac{1}{2A}\oint_C y^2dx = -\frac{1}{2A}\oint_C\int\int_R -2ydA = \frac{\int\int_R ydA}{A} = \overline{y}.$$

32. Using Green's Theorem,
$$\begin{aligned} W &= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C -ydx + xdy = \int\int_R 2dA \ = 2\int_0^{2\pi}\int_0^{1+\cos\theta} rdrd\theta \\ &= 2\int_0^{2\pi}\left(\frac{1}{2}r^2\right)\Big|_0^{1+\cos\theta} d\theta = \int_0^{2\pi}(1 + 2\cos\theta + \cos^2\theta)d\theta \\ &= \left(\theta + 2\sin\theta + \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right)\Big|_0^{2\pi} = 3\pi. \end{aligned}$$

33. Since $\int_A^B Pdx + Qdy$ is independent of path, $P_y = Q_x$ by Theorem 17.3. Then, by Green's Theorem
$$\int_C Pdx + Qdy = \int\int_R (Q_x - P_y)dA = \int\int_R 0dA = 0.$$

34.

15.5 Parametric Surfaces and Area

1. $x = u,\ \ y = v,\ \ z = 4u + 3v - 2,\ \ -\infty < u < \infty,\ \ -\infty < v < \infty$

2. $x = u,\ \ y = 1 - 2u,\ \ z = v,\ \ -\infty < u < \infty,\ \ -\infty < v < \infty$

3. $x = u,\ \ y = -\sqrt{1 + u^2 + v^2},\ \ z = v,\ \ -\infty < u < \infty,\ \ -\infty < v < \infty$

4. $x = u,\ \ y = v,\ \ z = 5 - u^2 - v^2,\ \ -\infty < u < \infty,\ \ -\infty < v < \infty$

5. $\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + (1 - v^2)\mathbf{k},\ \ -2 \le u \le 2,\ \ -3 \le v \le 3$

6. $\mathbf{r}(u,v) = 2\cos u\mathbf{i} + 3\sin u\mathbf{j} + v\mathbf{k}, \quad 0 \le u \le 2\pi, \quad -\infty < v < \infty$

7. $x^2 + y^2 = \cos^2 u + \sin^2 u = 1$, circular cylinder

8. $z = x^2 + y^2$, paraboloid

9. $x = \sin u, \quad y = \sin u \cos v, \quad z = \sin u \sin v$
   $y^2 + z^2 = \sin^2 u \cos^2 v + \sin^2 u \sin^2 v = \sin^2(\cos^2 v + \sin^2 v) = \sin^2 u = z^2$,
   so $x^2 = y^2 + z^2$, portion of a circular cone

10. $x = 2\sin\phi\cos\theta, \quad y = 3\sin\phi\sin\theta, \quad z = 4\cos\phi,$
$$\begin{aligned}\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} &= \sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta + \cos^2\phi \\ &= \sin^2(\cos^2\theta + \sin^2\theta) + \cos^2\phi \\ &= \sin^2\phi + \cos^2\phi = 1,\end{aligned}$$
    ellipsoid

11. Surface is parameterzied by $x = u, \quad y = \sin v, \quad z = \cos v$ so R is defined by $0 \le u \le 4, \quad 0 \le v \le \frac{\pi}{2}$

12. Surface is parameterzied by $x = u, \quad y = \sin v, \quad z = \cos v$ so R is defined by $-2 \le u \le 2, \quad -\frac{\pi}{2} \le v \le \frac{\pi}{2}$

13. Surface is parameterzied by $x = \sin\phi\cos\theta, \quad y = \sin\phi\sin\theta, \quad z = \cos\phi$ so R is defined by $0 \le \theta \le 2\pi, \quad \frac{\pi}{2} \le \phi \le \pi$

14. Surface is parameterzied by $x = \sin\phi\cos\theta, \quad y = \sin\phi\sin\theta, \quad z = \cos\phi$ so R is defined by $0 \le \theta \le \pi, \quad 0 \le \phi \le \frac{\pi}{2}$

15. At $u = \pi/6, \quad v = 2$, we have $x = 5, \quad y = 5\sqrt{3}, \quad z = 2$.
    $\dfrac{\partial x}{\partial u}\left(\frac{\pi}{6}, 2\right) = 5\sqrt{3}, \quad \dfrac{\partial y}{\partial u}\left(\frac{\pi}{6}, 2\right) = -5, \quad \dfrac{\partial z}{\partial u}\left(\frac{\pi}{6}, 2\right) = 0$
    $\dfrac{\partial x}{\partial v}\left(\frac{\pi}{6}, 2\right) = 0, \quad \dfrac{\partial y}{\partial v}\left(\frac{\pi}{6}, 2\right) = 0, \quad \dfrac{\partial z}{\partial v}\left(\frac{\pi}{6}, 2\right) = 1.$
    A normal vector is given by $\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5\sqrt{3} & -5 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -5\mathbf{i} - 5\sqrt{3}\mathbf{j}.$
    The tangent plane is $-5(x-5) - 5\sqrt{3}(y - 5\sqrt{3}) = 0$ or $x + \sqrt{3}y = 20$.

16. At $u = 1, \quad v = 0$, we have $x = 1, \quad y = 0, \quad z = 1$.
    $\dfrac{\partial x}{\partial u}(1,0) = 1, \quad \dfrac{\partial y}{\partial u}(1,0) = 0, \quad \dfrac{\partial z}{\partial u}(1,0) = 2$
    $\dfrac{\partial x}{\partial v}(1,0) = 0, \quad \dfrac{\partial y}{\partial v}(1,0) = 1, \quad \dfrac{\partial z}{\partial v}(1,0) = 0.$
    A normal vector is given by $\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{vmatrix} = -2\mathbf{i} + \mathbf{k}.$
    The tangent plane is $-2(x-1) + (z-1) = 0$ or $-2x + z = -1$.

17. At $u = 1,\ \ v = 2$, we have $x = 3,\ \ y = 3,\ \ z = -3$.
$\dfrac{\partial x}{\partial u}(1,2) = 2,\ \ \dfrac{\partial y}{\partial u}(1,2) = 1,\ \ \dfrac{\partial z}{\partial u}(1,2) = 2$
$\dfrac{\partial x}{\partial v}(1,2) = 1,\ \ \dfrac{\partial y}{\partial v}(1,2) = 1,\ \ \dfrac{\partial z}{\partial v}(1,2) = -4$.
A normal vector is given by $\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 2 \\ 1 & 1 & -4 \end{vmatrix} = -6\mathbf{i} + 10\mathbf{j} + \mathbf{k}$.
The tangent plane is $-6(x-3) + 10(y-3) + (z+3) = 0$ or $-6x + 10y + z = 9$.

18. At $u = -1,\ \ v = \frac{\pi}{3}$, we have $x = -4,\ \ y = \frac{3}{2},\ \ z = \frac{3\sqrt{3}}{2}$.
$\dfrac{\partial x}{\partial u}\left(-1, \frac{\pi}{3}\right) = 4,\ \ \dfrac{\partial y}{\partial u}\left(-1, \frac{\pi}{3}\right) = -3,\ \ \dfrac{\partial z}{\partial u}\left(-1, \frac{\pi}{3}\right) = -3\sqrt{3}$
$\dfrac{\partial x}{\partial v}\left(-1, \frac{\pi}{3}\right) = 0,\ \ \dfrac{\partial y}{\partial v}\left(-1, \frac{\pi}{3}\right) = \dfrac{-3\sqrt{3}}{2},\ \ \dfrac{\partial z}{\partial v}\left(-1, \frac{\pi}{3}\right) = \dfrac{3}{2}$.
A normal vector is given by $\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -3 & -3\sqrt{3} \\ 0 & -\dfrac{3\sqrt{3}}{2} & \dfrac{3}{2} \end{vmatrix} = -18\mathbf{i} - 6\mathbf{j} - 6\sqrt{3}\mathbf{k}$.
The tangent plane is $-18(x+4) - 6(y - \frac{3}{2}) - 6\sqrt{3}\left(z - \frac{3\sqrt{3}}{2}\right) = 0$ or $3x + y + \sqrt{3}z = -6$.

19. At $u = 3,\ \ v = 3$, we have $x = 3,\ \ y = 3,\ \ z = 9$.
$\dfrac{\partial x}{\partial u}(3,3) = 1,\ \ \dfrac{\partial y}{\partial u}(3,3) = 0,\ \ \dfrac{\partial z}{\partial u}(3,3) = 3$
$\dfrac{\partial x}{\partial v}(3,3) = 0,\ \ \dfrac{\partial y}{\partial v}(3,3) = 1,\ \ \dfrac{\partial z}{\partial v}(3,3) = 3$.
A normal vector is given by $\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 3 \\ 0 & 1 & 3 \end{vmatrix} = -3\mathbf{i} - 3\mathbf{j} + \mathbf{k}$.
The tangent plane is $-3(x-3) - 3(y-3) + (z-9) = 0$ or $3x + 3y - z = 9$.

20. At $u = 1,\ \ v = \pi/4$, we have $x = \frac{\sqrt{2}}{2},\ \ y = \frac{\sqrt{2}}{2},\ \ z = 1$.
$\dfrac{\partial x}{\partial u}\left(1, \frac{\pi}{4}\right) = \dfrac{\sqrt{2}}{2},\ \ \dfrac{\partial y}{\partial u}\left(1, \frac{\pi}{4}\right) = \dfrac{\sqrt{2}}{2},\ \ \dfrac{\partial z}{\partial u}\left(1, \frac{\pi}{4}\right) = 1$
$\dfrac{\partial x}{\partial v}\left(1, \frac{\pi}{4}\right) = \dfrac{\sqrt{2}}{2},\ \ \dfrac{\partial y}{\partial v}\left(1, \frac{\pi}{4}\right) = -\dfrac{\sqrt{2}}{2},\ \ \dfrac{\partial z}{\partial v}\left(1, \frac{\pi}{4}\right) = 0$.
A normal vector is given by $\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 1 \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \end{vmatrix} = \dfrac{\sqrt{2}}{2}\mathbf{i} + \dfrac{\sqrt{2}}{2}\mathbf{j} - \mathbf{k}$.
The tangent plane is $\dfrac{\sqrt{2}}{2}\left(x - \dfrac{\sqrt{2}}{2}\right) + \dfrac{\sqrt{2}}{2}\left(y - \dfrac{\sqrt{2}}{2}\right) - (z-1) = 0$ or $x + y - \sqrt{2}z = 0$.

21. At $u = -2,\ \ v = 1$, we have $x = -1,\ \ y = 3,\ \ z = -2$.
$\dfrac{\partial x}{\partial u}(-2,1) = 1,\ \ \dfrac{\partial y}{\partial u}(-2,1) = -1,\ \ \dfrac{\partial z}{\partial u}(-2,1) = 1$
$\dfrac{\partial x}{\partial v}(-2,1) = 1,\ \ \dfrac{\partial y}{\partial v}(-2,1) = 1,\ \ \dfrac{\partial z}{\partial v}(-2,1) = -2$.

A normal vector is given by $\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 1 & 1 & -2 \end{vmatrix} = \mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$.
The tangent plane is $(x+1) + 3(y-3) + 2(z+2) = 0$ or $x + 3y + 2z = 4$.

22. At $u = 0,\ \ v = \ln 3$, we have $x = 0,\ \ y = \ln 3 + 1,\ \ z = 3$.
$\dfrac{\partial x}{\partial u}(0, \ln 3) = \ln 3,\ \ \dfrac{\partial y}{\partial u}(0, \ln 3) = 1,\ \ \dfrac{\partial z}{\partial u}(0, \ln 3) = 1$
$\dfrac{\partial x}{\partial v}(0, \ln 3) = 0,\ \ \dfrac{\partial y}{\partial v}(0, \ln 3) = 1,\ \ \dfrac{\partial z}{\partial v}(0, \ln 3) = 3$.
A normal vector is given by $\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \ln 3 & 1 & 1 \\ 0 & 1 & 3 \end{vmatrix} = 2\mathbf{i} - 3\ln 3\mathbf{j} + \ln 3\mathbf{k}$.
The tangent plane is $2(x-0) - 3\ln 3(y - \ln 3 - 1) + \ln 3(z-3) = 0$ or $2x - 3(\ln 3)y + (\ln 3)z = -3(\ln 3)^2$.

23. At $(1, 7, 5)$, we have $u = 2,\ \ v = 1$.
$\dfrac{\partial x}{\partial u}(2,1) = 1,\ \ \dfrac{\partial y}{\partial u}(2,1) = 2,\ \ \dfrac{\partial z}{\partial u}(2,1) = 4$
$\dfrac{\partial x}{\partial v}(2,1) = -1,\ \ \dfrac{\partial y}{\partial v}(2,1) = 3,\ \ \dfrac{\partial z}{\partial v}(2,1) = 2$.
A normal vector is given by $\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 4 \\ -1 & 3 & 2 \end{vmatrix} = -8\mathbf{i} - 6\ln 3\mathbf{j} + 5\mathbf{k}$.
The tangent plane is $-8(x-1) - 6(y-7) + 5(z-35) = 0$ or $8x + 6y - 5z = 25$.

24. At $(1, 3, 16)$, we have $u = 4,\ \ v = 1$.
$\dfrac{\partial x}{\partial u}(4,1) = 0,\ \ \dfrac{\partial y}{\partial u}(4,1) = 1,\ \ \dfrac{\partial z}{\partial u}(4,1) = 8$
$\dfrac{\partial x}{\partial v}(4,1) = 2,\ \ \dfrac{\partial y}{\partial v}(4,1) = -1,\ \ \dfrac{\partial z}{\partial v}(4,1) = 0$.
A normal vector is given by $\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 8 \\ 2 & -1 & 0 \end{vmatrix} = 8\mathbf{i} - 16\mathbf{j} - 2\mathbf{k}$.
The tangent plane is $8(x-1) - 16(y-3) - 2(z-16) = 0$ or $4x - 8y - z = -36$.

25. $\dfrac{\partial \mathbf{r}}{\partial u} = \langle 2, 1, \rangle,\ \ \dfrac{\partial \mathbf{r}}{\partial v} = \langle -1, 1, 0 \rangle$
$\dfrac{\partial \mathbf{r}}{\partial u} \times \dfrac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ -1 & 1 & 0 \end{vmatrix} = \langle -1, -1, 3 \rangle$
$\left| \dfrac{\partial \mathbf{r}}{\partial u} \times \dfrac{\partial \mathbf{r}}{\partial v} \right| = \sqrt{1+1+9} = \sqrt{11}$
$A = \int_0^2 \int_{-1}^1 \sqrt{11}\ dv\ du = 4\sqrt{11}$

26. Let $x = u,\ \ y = v,\ \ z = 1 - u - v$.
Then $A = \int_{-1}^1 \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} \sqrt{1 + (-1)^2 + (-1)^2}\ dv\ du = \sqrt{3}\pi$

27. $\dfrac{\partial \mathbf{r}}{\partial u} = \langle 1, 0, 2u \rangle,\ \ \dfrac{\partial \mathbf{r}}{\partial v} = \langle 0, 1, 2v \rangle$

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = \langle -2u, -2v, 1 \rangle$$

$$\left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| = \sqrt{4u^2 + 4v^2 + 1}$$

Since $0 \le z \le 4$, we have $0 \le u^2 + v^2 \le 4$. So

$$A = \int_{-2}^{2} \int_{-\sqrt{4-u^2}}^{\sqrt{4-u^2}} \sqrt{4u^2 + 4v^2 + 1}\ \mathrm{d}v\ \mathrm{d}u$$

$$= \int_{-2}^{2} \frac{v}{2}\sqrt{4u^2+4v^2+1} + \frac{(4u^2+1)}{4} \ln\left|2v + \sqrt{4u^2+4v^2+1}\right| \Bigg|_{-\sqrt{4-u^2}}^{\sqrt{4-u^2}}$$

$$= \frac{1}{4}\left[2(4u^2+1)\ln\left|2\sqrt{4-u^2} + \sqrt{17}\right| - (4u^2+1)\ln(4u^2+1) + 4\sqrt{-17(u^2-4)}\right]$$

$$= \int_0^{2\pi} \int_0^2 \sqrt{4r^2+1}r\ \ dr d\theta \quad \boxed{\text{polar transformation}}$$

$$= \int_0^{2\pi} \frac{(4r^2+1)^{3/2}}{12}\Bigg|_0^2\ d\theta = \int_0^{2\pi} \frac{17\sqrt{17}-1}{12}\ d\theta$$

$$= \frac{17\sqrt{17}-1}{12}\theta\Bigg|_0^{2\pi} = \frac{(17\sqrt{17}-1)\pi}{6}$$

28. $\dfrac{\partial \mathbf{r}}{\partial r} = \langle \cos\theta, \sin\theta, 1 \rangle, \quad \dfrac{\partial \mathbf{r}}{\partial \theta} = \langle -r\sin\theta, r\cos\theta, 0 \rangle$

$$\frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 1 \\ r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \langle -r\cos\theta, -r\sin\theta, r \rangle$$

$$\left| \frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta} \right| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2} = \sqrt{r^2+r^2} = r\sqrt{2}.$$

$A = \int_0^2 \int_0^{2\pi} r\sqrt{2}\ \mathrm{d}\theta\ \mathrm{d}r = 4\pi\sqrt{2}$

29. $\mathbf{r} = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + \theta\mathbf{k}$

$\dfrac{\partial \mathbf{r}}{\partial r} = \langle \cos\theta, \sin\theta, 0 \rangle, \quad \dfrac{\partial \mathbf{r}}{\partial \theta} = \langle -r\sin\theta, r\cos\theta, f \rangle$

$$\frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 1 \end{vmatrix} = \langle \sin\theta, -\cos\theta, r \rangle$$

$$\left| \frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta} \right| = \sqrt{\sin^2\theta + \cos^2\theta + r^2} = \sqrt{1+r^2}.$$

$A = \int_0^{2\pi} \int_0^2 \sqrt{1+r^2}\ \mathrm{d}r\ \mathrm{d}\theta = 2\sqrt{5}\pi + \pi\ln(2+\sqrt{5})$

30. $\mathbf{r} = (a\sin\phi\cos\theta)\mathbf{i} + (a\sin\phi\sin\theta)\mathbf{j} + (a\cos\phi)\mathbf{k}$

$\dfrac{\partial \mathbf{r}}{\partial \theta} = \langle -a\sin\phi\sin\theta,\ \ a\sin\phi\cos\theta,\ \ 0 \rangle, \qquad \dfrac{\partial \mathbf{r}}{\partial \phi} = \langle a\cos\phi\cos\theta,\ \ a\cos\phi\sin\theta,\ \ -a\sin\phi \rangle$

$$\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \phi} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \end{vmatrix}$$

$$= \langle -a^2\sin^2\phi\cos\theta,\ \ -a\sin^2\phi\sin\theta,\ \ -a^2\sin\phi\cos\phi\rangle$$

$$\left|\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \phi}\right| = \sqrt{a^4\sin^4\phi\cos^2\theta + a^4\sin^4\phi\sin^2\theta + a^4\sin^2\phi\cos^2\theta}$$

$$= \sqrt{a^4\sin^4\phi + a^4\sin^2\phi\cos^2\phi} = \sqrt{a^4\sin^2\phi} = a^2\sin\phi$$

$A = \int_0^\pi \int_0^{2\pi} a^2 \sin\phi d\theta d\phi = 4a^2\pi$

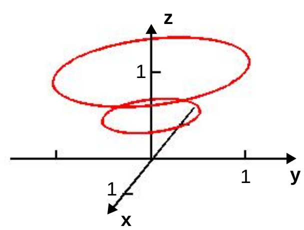
31. We have $a = 2$, so $x = 2\sin\phi\cos\theta$, $y = 2\sin\phi\sin\theta$, $z = 2\cos\phi$, $\frac{\pi}{3} \le \phi \le \pi$, $0 \le \theta \le 2\pi$,
$A = \int_{\pi/3}^{\pi}\int_0^{2\pi} 4\sin\phi \ \mathrm{d}\theta \ \mathrm{d}\phi = 12\pi$

32. $x = 2\sin\phi\cos\theta$, $y = 2\sin\phi\sin\theta$, $z = 2\cos\phi$, $\frac{\pi}{3} \le \phi \le \frac{\pi}{2}$, $0 \le \theta \le 2\pi$,
$A = \int_{\pi/3}^{\pi}\int_0^{2\pi} 4\sin\phi \ \mathrm{d}\theta \ \mathrm{d}\phi = 4\pi$

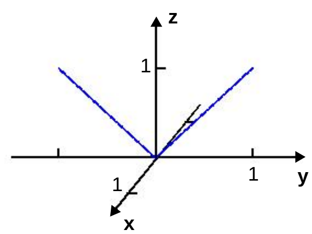
33. $x = 2\sin\phi\cos\theta$, $y = 2\sin\phi\sin\theta$, $z = 2\cos\phi$, $0 \le \phi \le \frac{\pi}{4}$, $0 \le \theta \le 2\pi$,
$A = \int_0^{\pi/4}\int_0^{2\pi} 4\sin\phi \ \mathrm{d}\theta \ \mathrm{d}\phi = 4\pi(2-\sqrt{2})$

34. $x = 2\sin\phi\cos\theta$, $y = 2\sin\phi\sin\theta$, $z = 2\cos\phi$; the spehre intersects the cylinder when $z^2 = 2$, so the region outside the cylinder is described by $\frac{\pi}{4} \le \phi\frac{3\pi}{4}$, $0 \le \theta \le 2\pi$
$A = \int_{\pi/4}^{3\pi/4}\int_0^{2\pi} 4\sin\phi d\theta d\phi = 8\pi\sqrt{2}$

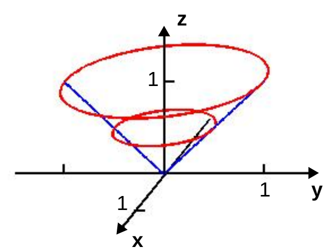
35. **(a)**



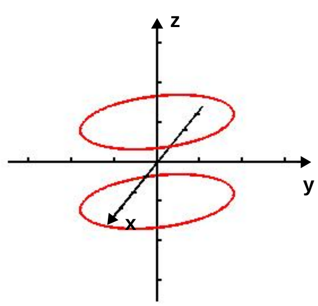
(b)



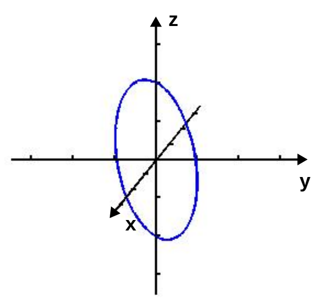
(c)



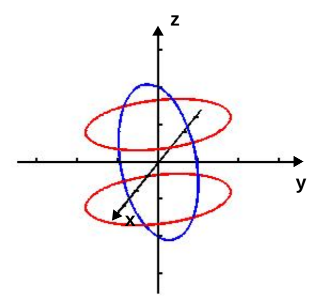
36. **(a)**



(b)



(c)



37. **(f)**

38. **(e)**

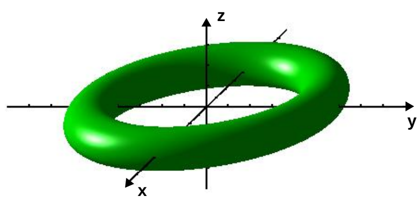
39. **(d)**

40. **(a)**

41. **(c)**

42. **(b)**

43.



44. $\dfrac{\partial \mathbf{r}}{\partial \phi} = \langle -\cos\phi\cos\theta,\ \ -\cos\phi\sin\theta,\ \ -\sin\phi\rangle,$
$\dfrac{\partial \mathbf{r}}{\partial \theta} = \langle (\sin\phi - R)\sin\theta,\ \ (R - s\in\phi)\cos\theta,\ \ 0\rangle$

$$\frac{\partial \mathbf{r}}{\partial \phi} \times \frac{\partial \mathbf{r}}{\partial \theta} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\cos\phi\cos\theta & -\cos\phi\sin\theta & -\sin\phi \\ (\sin\phi - R)\sin\theta & (R-\sin\phi)\cos\theta & 0 \end{vmatrix}$$
$$= \langle (r-\sin\phi)\sin\phi\cos\theta,\ \ (R-\sin\phi)\sin\phi\sin\theta,\ \ -(R-\sin\phi)\cos\phi\rangle$$
$$\left|\frac{\partial \mathbf{r}}{\partial \phi} \times \frac{\partial \mathbf{r}}{\partial \theta}\right| = \sqrt{(R-\sin\phi)^2\sin^2\phi\cos^2\theta + (R-\sin\phi)^2\sin^2\phi\sin^2\theta + (R-\sin\phi)^2\cos^2\phi}$$
$$= \sqrt{(R-\sin\phi)^2\sin^2\phi + (R-\sin\phi)^2\cos^2\phi}$$
$$= \sqrt{(R-\sin\phi)^2} = R - \sin\phi$$
$A = \int_0^{2\pi}\int_0^{2\pi} R - \sin\phi d\theta d\phi = 4\pi^2 R$

45. $x = 2u,\ \ y = 2v,\ \ z = 8u + 6v - 2,\ \ -\infty < u < \infty,\ \ -\infty < v < \infty$

46. The surface area of a circular cylinder with height h and radius r is $A = 2\pi rh$. The surface pictures is one quarter of a circular cylinder with height 4 and radius 1. Therefore, $A = 2\pi$.

47. We have $\mathbf{r} = u\mathbf{i} + f(u)\cos v\mathbf{j} + f(u)\sin v\mathbf{k}$.
$\dfrac{\partial r}{\partial u} = \langle 1, f'(u)\cos v, f'(u)\sin v\rangle,\ \ \dfrac{\partial r}{\partial v} = \langle 0, -f(u)\sin v, f(u)\cos v\rangle,$

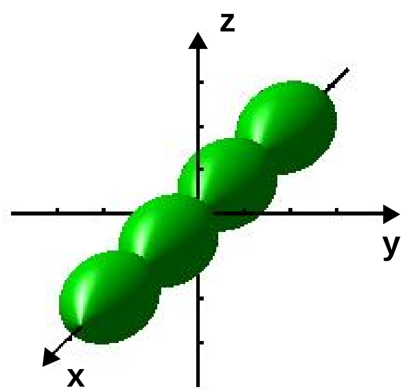
$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & f'(u)\cos v & f'(u)\sin v \\ 0 & -f(u)\sin v & f(u)\cos v \end{vmatrix}$$
$$= \langle f(u)f'(u), -f(u)\cos v \qquad -f(u)\sin v\rangle$$

$$\left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right| = \sqrt{[f(u)f'(u)]^2 + [f(u)]^2\cos^2 v + [f(u)]^2\sin^2 v}$$
$$= \sqrt{[f(u)]^2[f'(u)]^2 + [f(u)]^2}$$
$$= f(u)\sqrt{1 + [f'(u)]^2}$$

By (11), $A = \int_a^b \int_0^{2\pi} f(u)\sqrt{1+[f'(u)]^2}dvdu = 2\pi \int_a^b f(u)\sqrt{1+[f'(u)]^2}du$

48. (a) $x = u, \;\; y = \sin u \cos v, \; z = \sin u \sin v, \; -2\pi \leq u \leq 2\pi \;\; 0 \leq v \leq 2\pi$

(b)



(c) Let S_1 be the surface corresponding to the parameter domain $0 \leq u \leq pi$, $0 \leq v \leq 2\pi$. Then

$$A(S_1) = 2\pi \int_0^{\pi} \sin(x)\sqrt{1+\cos^2 x}dx$$
$$= \left(\ln(\sqrt{2}+1) - \ln(\sqrt{2}-1) = 2\sqrt{2}\right)\pi$$

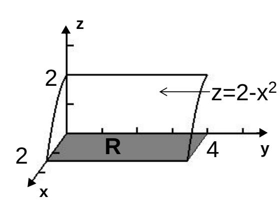
Finding the area of the entire surface S, we have $A(S) = 4A(S_1) = \left(\ln(\sqrt{2}+1) - \ln(\sqrt{2}-1) = 2\sqrt{2}\right)4\pi$

49. The surface is a plane passing through the point (x_0, y_0, z_0) with a normal vector $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2$.

50. $x = 5\sin\phi\cos\theta + 2, \;\; y = 5\sin\phi\sin\theta + 3, \;\; z = 5\cos\phi + 4$

15.6 Surface Integrals

1. $z_x = -2x, \;\; z_y = 0; \;\; dS = \sqrt{1+4x^2}dA$



$$\int\int_S xdS = \int_0^4 \int_0^{\sqrt{2}} x\sqrt{1+4x^2}dxdy = \int_0^4 \frac{1}{12}(1+4x^2)^{3/2}\bigg|_0^{\sqrt{2}} dy$$
$$= \int_0^4 \frac{13}{6}dy = \frac{26}{3}$$

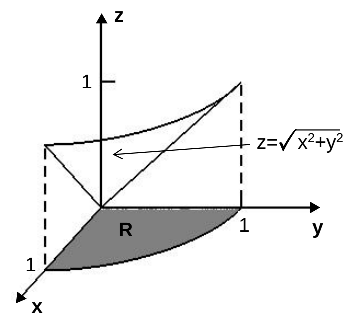
2. See Problem 1.

$$\int\int_S xy(9-4z)dS = \int\int_S xy(1+4x^2)dS = \int_0^4 \int_0^{\sqrt{2}} xy(1+4x^2)^{3/2}dxdy$$

$$= \int_0^4 \frac{y}{20}(1+4x^2)^{5/2}\Big|_0^{\sqrt{2}} dy = \int_0^4 \frac{242}{20} ydy = \frac{121}{10}\int_0^4 ydy = \frac{121}{10}\left(\frac{1}{2}y^2\right)\Big|_0^4 = \frac{484}{5}$$

3. $z_x = \dfrac{x}{\sqrt{x^2+y^2}}$, $z_y = \dfrac{y}{\sqrt{x^2+y^2}}$; $dS = \sqrt{2}dA$.

Using polar coordinates,



$$\int\int_S xz^3 dS = \int\int_R x(x^2+y^2)^{3/2}\sqrt{2}dA$$

$$= \sqrt{2}\int_0^{2\pi}\int_0^1 (r\cos\theta)r^{3/2}rdrd\theta$$

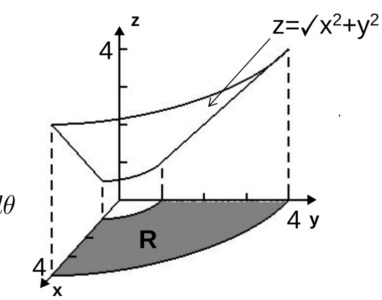
$$= \sqrt{2}\int_0^{2\pi}\int_0^1 r^{7/2}\cos\theta drd\theta$$

$$= \sqrt{2}\int_0^{2\pi} \frac{2}{9}r^{9/2}\cos\theta\Big|_0^1 d\theta$$

$$= \sqrt{2}\int_0^{2\pi} \frac{2}{9}\cos\theta d\theta = \frac{2\sqrt{2}}{9}\sin\theta\Big|_0^{2\pi} = 0.$$

4. $z_x = \dfrac{x}{\sqrt{x^2+y^2}}$, $z_y = \dfrac{y}{\sqrt{x^2+y^2}}$; $dS = \sqrt{2}dA$.

Using polar coordinates,



$$\int\int_S (x+y+z)dS = \int\int R(x+y+\sqrt{x^2+y^2})\sqrt{2}dA$$

$$= \sqrt{2}\int_0^{2\pi}\int_1^4 (r\cos\theta + r\sin\theta + r)rdrd\theta$$

$$= \sqrt{2}\int_0^{2\pi}\int_1^4 r^2(1+\cos\theta+\sin\theta)drd\theta$$

$$= \sqrt{2}\int_0^{2\pi} \frac{1}{3}r^3(1+\cos\theta+\sin\theta)\Big|_1^4 d\theta$$

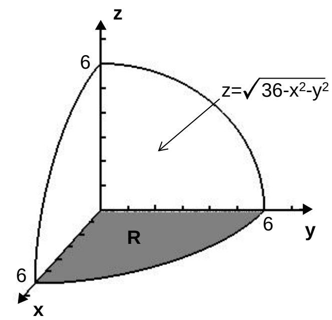
$$= \frac{63\sqrt{2}}{3}\int_0^{2\pi}(1+\cos\theta+\sin\theta)d\theta = 21\sqrt{2}(\theta+\sin\theta-\cos\theta)\Big|_0^{2\pi} = 42\sqrt{2}\pi.$$

5. $z = \sqrt{36 - x^2 - y^2}$, $z_x = -\dfrac{x}{\sqrt{36 - x^2 - y^2}}$,

$z_y = -\dfrac{y}{36 - x^2 - y^2}$;

$$dS = \sqrt{1 + \frac{x^2}{36 - x^2 - y^2} + \frac{y^2}{36 - x^2 - y^2}}dA$$
$$= \frac{6}{\sqrt{36 - x^2 - y^2}}dA.$$

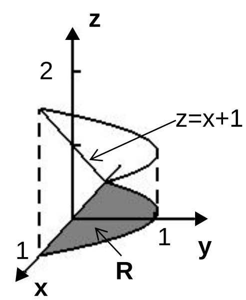
Using polar coordinates,



$$\int\int_S (x^2 + y^2)zdS = \int\int_R (x^2 + y^2)\sqrt{36 - x^2 - y^2}\frac{6}{\sqrt{36 - x^2 = y^2}}dA = 6\int_0^{2\pi}\int_0^6 r^2 rdrd\theta$$
$$= 6\int_0^{2\pi} \frac{1}{4}r^4\Big|_0^6 d\theta = 6\int_0^{2\pi} 324d\theta = 972\pi.$$

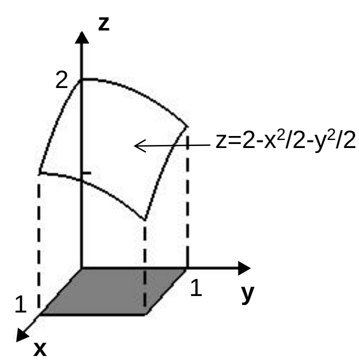
6. $z_x = 1$, $z_y = 0$; $dS = \sqrt{2}dA$

$$\int\int_S z^2 dS = \int_{-1}^1\int_0^{1-x^2} (x + 1)^2\sqrt{2}dydx = \sqrt{2}\int_{-1}^1 y(x + 1)^2\Big|_0^{1-x^2} dx$$
$$= \sqrt{2}\int_{-1}^1 (1 - x^2)(x + 1)^2 dx$$
$$= \sqrt{2}\int_{-1}^1 (1 + 2x - 2x^3 - x^4)dx$$
$$= \sqrt{2}\left(x + x^2 - \frac{1}{2}x^4 - \frac{1}{5}x^5\right)\Big|_{-1}^1 = \frac{8\sqrt{2}}{5}$$



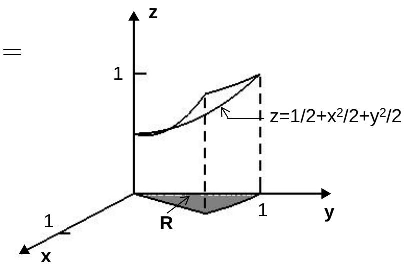
7. $z_x = -x$, $z_y = -y$; $dS = \sqrt{1 + x^2 + y^2}dA$

$$\int\int_S xydS = \int_0^1\int_0^1 xy\sqrt{1 + x^2 + y^2}dxdy$$
$$= \int_0^1 \frac{1}{3}y(1 + x^2 + y^2)^{3/2}\Big|_0^1 dy$$
$$= \int_0^1 \left[\frac{1}{3}y(2 + y^2)^{3/2} - \frac{1}{3}y(1 + y^2)^{3/2}\right] dy$$
$$= \left[\frac{1}{15}(2 + y^2)^{5/2} - \frac{1}{15}(1 + y^2)^{5/2}\right]\Big|_0^1$$
$$= \frac{1}{15}(3^{5/2} - 2^{7/2} + 1)$$



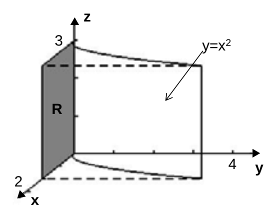
8. $z = \frac{1}{2} + \frac{1}{2}x^2 + \frac{1}{2}y^2, \quad z_x = x, \quad z_y = y; \quad dS = \sqrt{1+x^2+y^2}dA$.
Using polar coordinates,

$$\begin{aligned}\int\int_S 2z dS &= \int\int_R (1+x^2+y^2)\sqrt{1+x^2+y^2}dA \\ &= \int_{\pi/3}^{\pi/2}\int_0^1 (1+r^2)\sqrt{1+r^2}rdrd\theta \\ &= \int_{\pi/3}^{\pi/2}\int_0^1 (1+r^2)^{3/2}rdrd\theta \\ &= \int_{\pi/3}^{\pi/2} \frac{1}{5}(1+r^2)^{5/2}\Big|_0^1 d\theta = \frac{1}{5}\int_{\pi/3}^{\pi/2}(2^{5/2}-1)d\theta \\ &= \frac{4\sqrt{2}-1}{5}\left(\frac{\pi}{2}-\frac{\pi}{3}\right) = \frac{(4\sqrt{2}-1)\pi}{30}.\end{aligned}$$



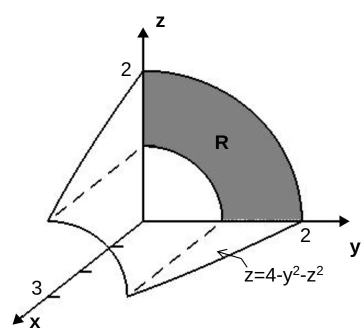
9. $y_x = 2x, \quad y_z = 0; \quad dS = \sqrt{1+4x^2}dA$

$$\begin{aligned}\int\int_S 24\sqrt{y}z dS &= \int_0^3\int_0^2 24xz\sqrt{1+4x^2}dxdz \\ &= \int_0^3 2z(1+4x^2)^{3/2}\Big|_0^2 dz \\ &= 2(17^{3/2}-1)\int_0^3 zdz = 2(17^{3/2}-1)\left(\frac{1}{2}z^2\right)\Big|_0^3 \\ &= 9(17^{3/2}-1)\end{aligned}$$



10. $x_y = -2y, \quad x_z = -2z; \quad dS = \sqrt{1+4y^2+4z^2}dA$
Using polar coordinates,

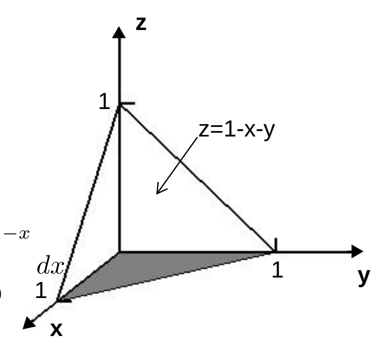
$$\begin{aligned}\int\int_S (1+4y^2+4z^2)^{1/2}dS &= \int_0^{\pi/2}\int_1^2 (1+4r^2)rdrd\theta \\ &= \int_0^{\pi/2} \frac{1}{16}(1+4r^2)^2\Big|_1^2 d\theta \\ &= \frac{1}{16}\int_0^{\pi/2} 12d\theta = \frac{3\pi}{8}.\end{aligned}$$



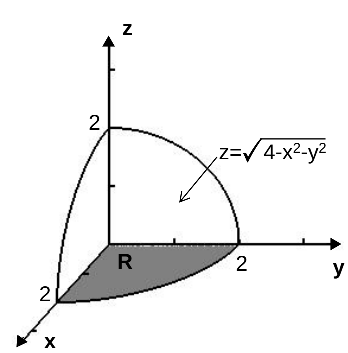
11. Write the equation of the surface as $y = \frac{1}{2}(6-x-3z)$.
$y_z = -\frac{1}{2}, \quad y_z = -\frac{3}{2}; \quad dS = \sqrt{1+1/4+9/4} = \frac{\sqrt{14}}{2}$.

$$\int\int_S (3z^2+4yz)dS = \int_0^2 \int_0^{6-3z} \left[3z^2 + 4z\frac{1}{2}(6-x-3z)\right] \frac{\sqrt{14}}{2} dxdz$$
$$= \frac{\sqrt{14}}{2} \int_0^2 \left[3z^2x - z(6-x-3z)^2\right]\Big|_0^{6-3z} dz$$
$$= \frac{\sqrt{14}}{2} \int_0^2 [3z^2(6-3z) - 0] - [0 - z(6-3z)^2] dz$$
$$= \frac{\sqrt{14}}{2} \int_0^2 (36z - 18z^2) dz = \frac{\sqrt{14}}{2}(18z^2 - 6z^3)\Big|_0^2 = \frac{\sqrt{14}}{2}(72-48) = 12\sqrt{14}$$

12. Write the equation of the surface as $x = 6 - 2y - 3z$. Then $x_y = -2,\quad x_z = -3;\quad dS = \sqrt{1+4+9} = \sqrt{14}$.
$$\int\int_S (3z^2+4yz)dS = \int_0^2 \int_0^{3-3z/2} (3z^2+4yz)\sqrt{14}dydz = \sqrt{14}\int_0^2 (3yz+2y^2z)\Big|_0^{3-3z/2} dz$$
$$= \sqrt{14}\int_0^2 \left[9z\left(1-\frac{z}{2}\right) + 18z\left(1-\frac{z}{2}\right)^2\right] dz = \sqrt{14}\int_0^2 \left(27z - \frac{45}{2}z^2 + \frac{9}{2}z^3\right) dz$$
$$= \sqrt{14}\left(\frac{27}{2}z^2 - \frac{15}{2}z^3 + \frac{9}{8}z^4\right)\Big|_0^2 = \sqrt{14}(54-60+18) = 2\sqrt{14}$$



13. The density is $\rho = kx^2$. The surface is $z = 1 - x - y$. Then $z_x = -1,\ \ z_y = -1;\ \ dS = \sqrt{3}dA$.
$$m = \int\int_S kx^2 dS = k\int_0^1 \int_0^{1-x} x^2\sqrt{3}dydx = \sqrt{3}k\int_0^1 \frac{1}{3}x^3\Big|_0^{1-x} dx$$
$$= \frac{\sqrt{3}}{3}k\int_0^1 (1-x)^3 dx = \frac{\sqrt{3}}{3}k\left[-\frac{1}{4}(1-x)^4\right]\Big|_0^1 = \frac{\sqrt{3}}{12}k$$



14. $z_x = -\dfrac{x}{\sqrt{4-x^2-y^2}}$, $\quad z_y = -\dfrac{y}{\sqrt{4-x^2-y^2}}$; $\quad dS = \sqrt{1+\dfrac{x^2}{4-x^2-y^2}+\dfrac{y^2}{4-x^2-y^2}}\,dA = \dfrac{2}{\sqrt{4-x^2-y^2}}dA.$
Using symmetry and polar coordinates,

$$m = 4\int\int_S |xy|dS = 4\int_0^{\pi/2}\int_0^2 (r^2\cos\theta\sin\theta)\frac{2}{\sqrt{4-r^2}}rdrd\theta$$

$$= 4\int_0^{\pi/2}\int_0^2 r^2(4-r^2)^{-1/2}\sin 2\theta(rdr)d\theta$$

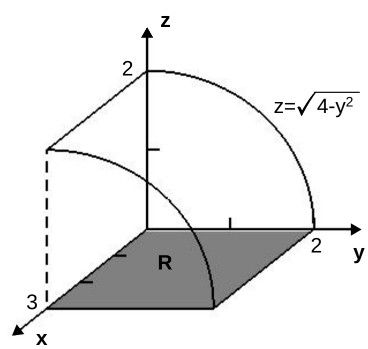
$$\boxed{u = 4-r^2,\ \ du = -2rdr,\ \ r^2 = 4-u}$$

$$= 4\int_0^{\pi/2}\int_4^0 (4-u)u^{-1/2}\sin 2\theta\left(-\frac{1}{2}du\right)d\theta = -2\int_0^{\pi/2}\int_4^0 (4u^{-1/2}-u^{1/2})\sin 2\theta dud\theta$$

$$= -2\int_0^{\pi/2}\left(8u^{1/2}-\frac{2}{3}u^{3/2}\right)\Big|_4^0 \sin 2\theta d\theta = -2\int_0^{\pi/2}\left(-\frac{32}{3}\sin 2\theta\right)d\theta$$

$$= \frac{64}{3}\left(-\frac{1}{2}\cos 2\theta\right)\Big|_0^{\pi/2} = \frac{64}{3}.$$

15. The surface is $g(x,y,z) = y^2+z^2-4 = 0$. $\nabla g = 2y\mathbf{j}+2z\mathbf{k}$, $|\nabla g| = 2\sqrt{y^2+z^2}$; $\nabla\dfrac{y\mathbf{i}+z\mathbf{k}}{\sqrt{y^2+z^2}}$;
$\mathbf{F}\cdot\nabla = \dfrac{2yz}{\sqrt{y^2+z^2}} + \dfrac{yz}{\sqrt{y^2+z^2}} = \dfrac{3yz}{\sqrt{y^2+z^2}}$;; $\quad z = \sqrt{4-y^2}$, $z_x = 0$,
$z_y = -\dfrac{y}{\sqrt{4-y^2}}$; $dS = \sqrt{1+\dfrac{y^2}{4-y^2}}dA = \dfrac{2}{\sqrt{4-y^2}}dA$

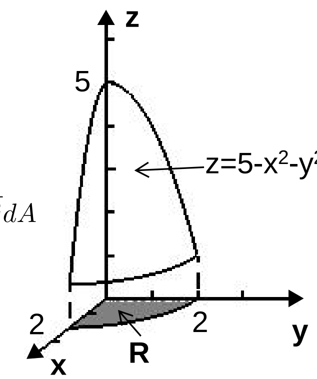


$$\text{Flux} = \int\int_S \mathbf{F}\cdot\mathbf{n}dS = \int\int_R \frac{3yz}{\sqrt{y^2+z^2}}\frac{2}{\sqrt{4-y^2}}dA$$

$$= \int\int_R \frac{3y\sqrt{4-y^2}}{\sqrt{y^2+4-y^2}}\frac{2}{\sqrt{4-y^2}}dA$$

$$= \int_0^3\int_0^2 3ydydx = \int_0^3 \frac{3}{2}y^2\Big|_0^2 dx = \int_0^3 6dx = 18$$

16. The surface is $g(x,y,z) = x^2 + y^2 + z - 5 = 0$. $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$, $|\nabla g| = \sqrt{1+4x^2+4y^2}$; $\mathbf{n} = \dfrac{2x\mathbf{i}+2y\mathbf{j}+\mathbf{k}}{\sqrt{1+4x^2+4y^2}}$; $\mathbf{F}\cdot\mathbf{n} = \dfrac{z}{\sqrt{1+4x^2+4y^2}}$; $z_x = -2x$, $z_y = -2y$, $dS = \sqrt{1+4x^2+4y^2}\,dA$. Using polar coordinates,

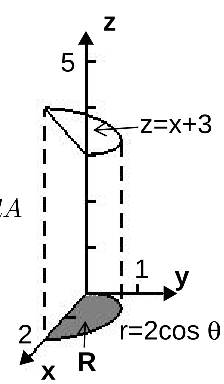


$$\text{Flux} = \int\int_S \mathbf{F}\cdot\mathbf{n}dS = \int\int_R \frac{z}{\sqrt{1+4x^2+4y^2}}\sqrt{1+4x^2+4y^2}dA$$
$$= \int\int_R (5-x^2-y^2)dA$$
$$= \int_0^{2\pi}\int_0^2 (5-r^2)rdrd\theta$$
$$= \int_0^{2\pi}\left(\frac{5}{2}r^2 - \frac{1}{4}r^4\right)\Bigg|_0^2 d\theta = \int_0^{2\pi} 6d\theta = 12\pi.$$

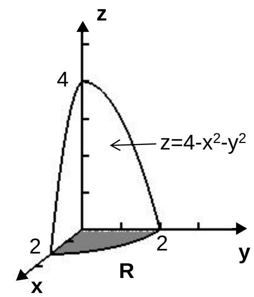
17. From Problem 16, $\mathbf{n} = \dfrac{2x\mathbf{i}+2y\mathbf{j}+\mathbf{k}}{\sqrt{1+4x^2+4y^2}}$. Then $\mathbf{F}\cdot\mathbf{n} = \dfrac{2x^2+2y^2+z}{\sqrt{1+4x^2+4y^2}}$. Also, from Problem 16, $dS = \sqrt{1+4x^2+4y^2}dA$. Using polar coordinates,

$$\text{Flux} = \int\int_S \mathbf{F}\cdot\mathbf{n}dS = \int\int_R \frac{2x^2+2y^2+z}{\sqrt{1+4x^2+4y^2}}\sqrt{1+4x^2+4y^2}dA = \int\int_R (2x^2+2y^2+5-x^2-y^2)dA$$
$$= \int_0^{2\pi}\int_0^2 (r^2+5)rdrd\theta = \int_0^{2\pi}\left(\frac{1}{4}r^4 + \frac{5}{2}r^2\right)\Bigg|_0^2 d\theta = \int_0^{2\pi} 14d\theta = 28\pi.$$

18. The surface is $g(x,y,z) = z - x - 3 = 0$. $\nabla g = -\mathbf{i}+\mathbf{k}$, $|\nabla g| = \sqrt{2}$; $\mathbf{n} = \dfrac{-\mathbf{i}+\mathbf{k}}{\sqrt{2}}$;
$\mathbf{F}\cdot\mathbf{n} = \dfrac{1}{\sqrt{2}}x^3y + \dfrac{1}{\sqrt{2}}xy^3$ $z_x = 1$, $z_y = 0$, $dS = \sqrt{2}dA$.
Using polar coordinates,

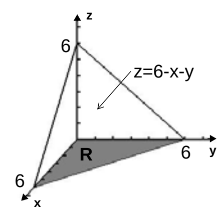


$$\text{Flux} = \int\int_S \mathbf{F}\cdot\mathbf{n}dS = \int\int_R \frac{1}{\sqrt{2}}(x^3y+xy^3)\sqrt{2}dA = \int\int_R xy(x^2+y^2)dA$$
$$= \int_0^{\pi/2}\int_0^{2\cos\theta} (r^2\cos\theta\sin\theta)r^2rdrd\theta$$
$$= \int_0^{\pi/2}\int_0^{2\cos\theta} r^5\cos\theta\sin\theta drd\theta$$
$$= \int_0^{\pi/2}\frac{1}{6}r^6\cos\theta\sin\theta\Bigg|_0^{2\cos\theta}d\theta = \frac{1}{6}\int_0^{\pi/2}64\cos^7\theta\sin\theta d\theta = \frac{32}{3}\left(-\frac{1}{8}\cos^8\theta\right)\Bigg|_0^{\pi/2} = \frac{4}{3}.$$



19. The surface is $g(x,y,z) = x^2+y^2+z-4$. $\nabla g = 2x\mathbf{i}+2y\mathbf{j}+\mathbf{k}$, $|\nabla g| = \sqrt{4x^2+4y^2+1}$;
$\mathbf{n} = \dfrac{2x\mathbf{i}+2y\mathbf{j}+\mathbf{k}}{\sqrt{4x^2+4y^2+1}}$; $\mathbf{F}\cdot\mathbf{n} = \dfrac{x^3+y^3+z}{\sqrt{4x^2+4y^2+1}}$; $z_x = -2x$, $z_y = -2y$,
$dS = \sqrt{1+4x^2+4y^2}dA$. Using polar coordinates,

$$\begin{aligned}
\text{Flux} &= \int\int_S \mathbf{F}\cdot\mathbf{n}dS = \int\int_R (x^3+y^3+z)dA \\
&= \int\int_R (4-x^2-y^2+x^3+y^3)dA \\
&= \int_0^{2\pi}\int_0^2 (4-r^2+r^3\cos^3\theta+r^3\sin^3\theta)rdrd\theta \\
&= \int_0^{2\pi}\left(2r^2-\frac{1}{4}r^4+\frac{1}{5}r^5\cos^3\theta+\frac{1}{5}r^5\sin^3\theta\right)\Big|_0^2 d\theta \\
&= \int_0^{2\pi}\left(4+\frac{32}{5}\cos^3\theta+\frac{32}{5}\sin^3\theta\right)d\theta = 4\theta\big|_0^{2\pi}+0+0 = 8\pi.
\end{aligned}$$



20. The surface is $g(x,y,z) = x+y+z-6$. $\nabla g = \mathbf{i}+\mathbf{j}+\mathbf{k}$, $|\nabla g| = \sqrt{3}$; $\mathbf{n} = (\mathbf{i}+\mathbf{j}+\mathbf{k})/\sqrt{3}$; $\mathbf{F}\cdot\mathbf{n} = (e^y+e^x+18y)/\sqrt{3}$; $z_x = -1$, $z_y = -1$, $dS = \sqrt{1+1+1}dA = \sqrt{3}dA$.

$$\begin{aligned}
\text{Flux} &= \int\int_S \mathbf{F}\cdot\mathbf{n}dS = \int\int_r (e^y+e^x+18y)dA \\
&= \int_0^6\int_0^{6-x}(e^y+e^x+18y)dydx \\
&= \int_0^6 (e^y+ye^x+9y^2)\Big|_0^{6-x} dx \\
&= \int_0^6 [e^{6-x}+(6-x)e^x+9(6-x)^2-1]dx \\
&= [-e^{6-x}+6e^x-xe^x+e^x-3(6-x)^3-x]\big|_0^6 \\
&= (-1+6e^6-6e^6+e^6-6)-(-e^6+6+1-648) = 2e^6+634 \approx 1440.86
\end{aligned}$$

21. For S_1: $g(x,y,z) = x^2+y^2-z$, $\nabla g = 2x\mathbf{i}+2y\mathbf{j}-\mathbf{k}$, $|\nabla g| = \sqrt{4x^2+4y^2+1}$; $\mathbf{n}_1 = \dfrac{2x\mathbf{i}+2y\mathbf{j}-\mathbf{k}}{\sqrt{4x^2+4y^2+1}}$; $\mathbf{F}\cdot\mathbf{n}_1 = \dfrac{2xy^2+2x^2y-5z}{\sqrt{4x^2+4y^2+1}}$; $z_x = 2x$, $z_y = 2y, dS_1 = \sqrt{1+4x^2+4y^2}dA$.
For S_2: $g(x,y,z) = z-1$, $\nabla g = \mathbf{k}$; $|\nabla g| = 1$; $\mathbf{n}_2 = \mathbf{k}; \mathbf{F}\cdot\mathbf{n}_2 = 5z$; $z_x = 0, z_y = 0, dS_2 = dA$. Using polar coordinates and $R:\ x^2+y^2 \le 1$ we have

$$\begin{aligned}
\text{Flux} &= \int\int_{S_1} \mathbf{F}\cdot\mathbf{n}_1 dS_1 + \int\int_{S_2} \mathbf{F}\cdot\mathbf{n}_2 dS_2 = \int\int_R (2xy^2+2x^2y-5z)dA + \int\int_R 5z dA \\
&= \int\int_R [2xy^2+2x^2y-5(x^2+y^2)+5(1)]dA \\
&= \int_0^{2\pi}\int_0^1 (2r^3\cos\theta\sin^2\theta + 2r^3\cos^2\theta\sin\theta - 5r^2+5)r dr d\theta \\
&= \int_0^{2\pi}\left(\frac{2}{5}r^5\cos\theta\sin^2\theta + \frac{2}{5}r^5\cos^2\theta\sin\theta - \frac{5}{4}r^4 + \frac{5}{2}r^2\right)\Big|_0^1 d\theta \\
&= \int_0^{2\pi}\left[\frac{2}{5}(\cos\theta\sin^2\theta+\cos^2\theta\sin\theta)+\frac{5}{4}\right]d\theta = \frac{2}{5}\left(\frac{1}{3}\sin^3\theta - \frac{1}{3}\cos^3\theta\right)\Big|_0^{2\pi} + \frac{5}{4}\theta\Big|_0^{2\pi} \\
&= \frac{2}{5}\left[-\frac{1}{3}-\left(-\frac{1}{3}\right)\right]+\frac{5}{2}\pi = \frac{5}{2}\pi.
\end{aligned}$$

22. For S_1 : $g(x,y,z)=x^2+y^2+z-4$, $\nabla g = 2x\mathbf{i}+2y\mathbf{j}+\mathbf{k}$, $|\nabla g| = \sqrt{4x^2+4y^2+1}$; $\mathbf{n}_1 = \dfrac{2x\mathbf{i}+2y\mathbf{j}+\mathbf{k}}{\sqrt{4x^2+4y^2+1}}$; $\mathbf{F}\cdot\mathbf{n}_1 = 6z^2/\sqrt{4x^2+4y^2+1}$; $z_x=-2x$, $z_y=-2y$, $dS_1=\sqrt{1+4x^2+4y^2}dA$.
For S_2 : $g(x,y,z)=x^2+y^2-z$, $\nabla g = 2x\mathbf{i}+2y\mathbf{j}-\mathbf{k}$, $|\nabla g| = \sqrt{4x^2+4y^2+1}$; $\mathbf{n}_2 = \dfrac{2x\mathbf{i}+2y\mathbf{j}-\mathbf{k}}{\sqrt{4x^2+y^2+1}}$; $\mathbf{F}\cdot\mathbf{n}_2 = -6z^2/\sqrt{4x^2+4y^2+1}$; $z_x=2x$, $z_y=2y$, $dS_2=\sqrt{1+4x^2+4y^2}dA$.
Using polar coordinates and $R:\ x^2+y^2\le 2$ we have

$$\begin{aligned}
\text{Flux} &= \int\int_{S_1}\mathbf{F}\cdot\mathbf{n}_1 dS_1 + \int\int_{S_1}\mathbf{F}\cdot\mathbf{n}_2 dS_2 = \int\int_R 6z^2 dA + \int\int -6z^2 dA \\
&= \int\int_R [6(4-x^2-y^2)^2 - 6(x^2+y^2)^2]dA = 6\int_0^{2\pi}\int_0^{\sqrt{2}}[(4-r^2)^2-r^4]r dr d\theta \\
&= 6\int_0^{2\pi}\left[-\frac{1}{6}(4-r^2)^3 - \frac{1}{6}r^6\right]\Big|_0^{\sqrt{2}} d\theta = -\int_0^{2\pi}[(2^3-4^3)+(\sqrt{2})^6]d\theta = \int_0^{2\pi}48 d\theta = 96\pi.
\end{aligned}$$

23. The surface is $g(x,y,z)=x^2+y^2+z^2-a^2=0$. $\nabla g = 2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}$, $|\nabla g| = 2\sqrt{x^2+y^2+z^2}$; $\mathbf{n} = \dfrac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{\sqrt{x^2+y^2+z^2}}$; $\mathbf{F}\cdot\mathbf{n} = -(2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k})\cdot\dfrac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{\sqrt{x^2+y^2+z^2}} = -\dfrac{2x^2+2y^2+2z^2}{\sqrt{x^2+y^2+z^2}} = -2\sqrt{x^2+y^2+z^2} = -2a$.
Flux $= \int\int_S -2a dS = -2a\times\text{area} = -2a(4\pi a^2) = -8\pi a^3$

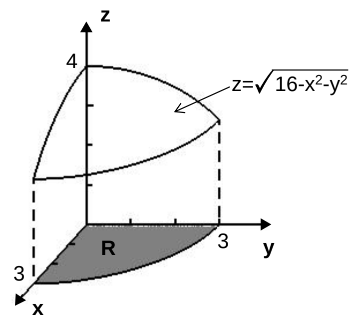
24. $\mathbf{n}_1=\mathbf{k}$, $\mathbf{n}_2=-\mathbf{i}$, $\mathbf{n}_3=\mathbf{j}$, $\mathbf{n}_4=-\mathbf{k}, \mathbf{n}_5=\mathbf{i}$, $\mathbf{n}_6=-\mathbf{j}$; $\mathbf{F}\cdot\mathbf{n}_1=z=1$, $\mathbf{F}\cdot\mathbf{n}_2=-x=0$, $\mathbf{F}\cdot\mathbf{n}_3=y=1$, $\mathbf{F}\cdot\mathbf{n}_4=-z=0$, $\mathbf{F}\cdot\mathbf{n}_5=x=1$, $\mathbf{F}\cdot\mathbf{n}_6=-y=0$;
Flux $= \int\int_{S_1} 1dS + \int\int_{S_3} 1dS + \int\int_{S_5} 1dS = 3$

25. Refering to the solution to Problem 23, we find $\mathbf{n} = \dfrac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{\sqrt{x^2+y^2+z^2}}$ and $dS = \dfrac{a}{\sqrt{a^2-x^2-y^2}}dA$.

$$\text{Now}\quad \mathbf{F}\cdot\mathbf{n} = kq\frac{r}{|r|^3}\cdot\frac{r}{|r|} = \frac{kq}{|r|^4}|r|^2 = \frac{kq}{|r|^2} = \frac{kq}{x^2+y^2+z^2} = \frac{kq}{a^2}$$

$$\text{and} \quad \text{Flux} = \int\int_S \mathbf{F}\cdot\mathbf{n}dS = \int\int_S \frac{kq}{a^2}dS = \frac{kq}{a^2} \times \text{area} = \frac{kq}{a^2}(4\pi a^2) = 4\pi kq.$$

26. We are given $\sigma = kz$. Now $z_x - \dfrac{x}{\sqrt{16-x^2-y^2}}$,



$z_y = -\dfrac{y}{\sqrt{16-x^2-y^2}}$;

$$dS = \sqrt{1+\frac{x^2}{16-x^2-y^2}+\frac{y^2}{16-x^2-y^2}}dA$$

$$= \frac{4}{\sqrt{16-x^2-y^2}}dA$$

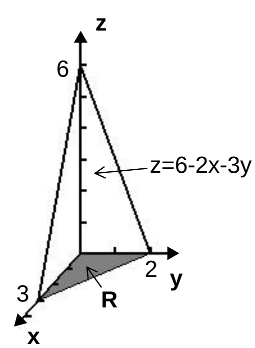
Using polar coordinates,

$$Q = \int\int_S kzdS = k\int\int_R \sqrt{16-x^2-y^2}\frac{4}{\sqrt{16-x^2-y^2}}dA$$

$$= 4k\int_0^{2\pi}\int_0^3 rdrd\theta$$

$$= 4k\int_0^{2\pi}\frac{1}{2}r^2\Big|_0^3 d\theta = 4k\int_0^{2\pi}\frac{9}{2}d\theta = 36\pi k.$$

27. The surface is $z = 6 - 2x - 3y$. Then $z_x = -2$, $z_y = -3$, $dS = \sqrt{1+4+9} = \sqrt{14}dA$. The area of the surface is



$$A(s) = \int\int_S dS = \int_0^3\int_0^{2-2x/3}\sqrt{14}dydx = \sqrt{14}\int_0^3\left(2-\frac{2}{3}x\right)dx$$

$$= \sqrt{14}(2x - \frac{1}{3}x^2)\Big|_0^3 = 3\sqrt{14}.$$

$$\overline{x} = \frac{1}{3\sqrt{14}}\int\int_s xdS = \frac{1}{3\sqrt{14}}\int_0^3\int_0^{2-2x/3}\sqrt{14}xdydx$$

$$= \frac{1}{3}\int_0^3 xy\Big|_0^{2-2x/3}dx = \frac{1}{3}\int_0^3\left(2x-\frac{2}{3}x^2\right)dx$$

$$= \frac{1}{3}\left(x^2-\frac{2}{9}x^3\right)\Big|_0^3 = 1$$

$$\overline{y} = \frac{1}{3\sqrt{14}}\int\int_S ydS = \frac{1}{3\sqrt{14}}\int_0^3\int_0^{2-2x/3}\sqrt{14}ydydx = \frac{1}{3}\int_0^3\frac{1}{2}y^2\Big|_0^{2-2x/3}dx$$

$$= \frac{1}{6}\int_0^3\left(2-\frac{2}{3}x\right)^2dx = \frac{1}{6}\left[-\frac{1}{2}(2-\frac{2}{3}x)^3\right]\Big|_0^3 = \frac{2}{3}$$

$$\overline{z} = \frac{1}{3\sqrt{14}}\int\int_S zdS = \frac{1}{3\sqrt{14}}\int_0^3\int_0^{2-2x/3}(6-2x-3y)\sqrt{14}dydx$$

$$= \frac{1}{3}\int_0^3\left(6y-2xy-\frac{3}{2}y^2\right)\Big|_0^{2-2x/3}dx = \frac{1}{3}\int_0^3\left(6-4x+\frac{2}{3}x^2\right)dx = \frac{1}{3}\left(6x-2x^2+\frac{2}{9}x^3\right)\Big|_0^3 = 2$$

The centroid is $(1, 2/3, 2)$.

28. The area of the hemisphere is $A(s) = 2\pi a^2$. By symmetry, $\overline{x} = \overline{y} = 0$.

$z_x = -\dfrac{x}{\sqrt{a^2-x^2-y^2}}$, $z_y = -\dfrac{y}{\sqrt{a^2-x^2-y^2}}$; $dS = \sqrt{1+\dfrac{x^2}{a^2-x^2-z^2}+\dfrac{y^2}{a^2-x^2-y^2}}dA = \dfrac{a}{\sqrt{a^2-x^2-y^2}}dA$

Using polar coordinates,

$$z = \int\int_S \frac{zdS}{2\pi a^2} = \frac{1}{2\pi a^2}\int\int_R \sqrt{a^2-x^2-y^2}\frac{a}{\sqrt{a^2-x^2-y^2}}dA = \frac{1}{2\pi a}\int_0^{2\pi}\int_0^a rdrd\theta$$
$$= \frac{1}{2\pi a}\int_0^{2\pi}\frac{1}{2}r^2\Big|_0^a d\theta = \frac{1}{2\pi a}\int_0^{2\pi}\frac{1}{2}s^2 d\theta = \frac{a}{2}.$$

The centroid is $(0, 0, a/2)$.

29. (a) The region in the xy-plane is $x^2+y^2 \le 16$. From $z_x = -x/\sqrt{x^2+y^2}$ and $z_y = -y/\sqrt{x^2+y^2}$ we see that

$$dS = \sqrt{1+x^2/(x^2+y^2)+y^2/(x^2+y^2)}dA = \sqrt{2}da$$

and $$A(S) = \int\int_S dS = \int\int_R \sqrt{2}dA = \sqrt{2}\pi 4^2 = 16\sqrt{2}\pi.$$

Then

$$\overline{x} = \frac{1}{16\sqrt{2}\pi}\int\int_S dS = \frac{1}{16\sqrt{2}\pi}\int\int_R \sqrt{2}xdA = \frac{1}{16\pi}\int_0^{2\pi}\int_0^4 r\cos\theta rdrd\theta$$
$$= \frac{1}{16\pi}\int_0^{2\pi}\frac{1}{3}r^3\cos\theta\Big|_0^4 d\theta = \frac{4}{3\pi}\int_0^{2\pi}\cos\theta d\theta = 0$$
$$\overline{y} = \frac{1}{16\sqrt{2}\pi}\int\int_S ydS = \frac{1}{16\sqrt{2}\pi}\int\int_R \sqrt{2}ydA = \frac{1}{16\pi}\int_0^{2\pi}\int_0^4 r\sin\theta rdrd\theta$$
$$= \frac{1}{16\pi}\int_0^{2\pi}\frac{1}{3}r^3\cos\theta\Big|_0^4 d\theta = \frac{4}{3\pi}\int_0^{2\pi}\sin\theta d\theta = 0$$
$$\overline{z} = \frac{1}{16\sqrt{2}\pi}\int\int_S zdS = \frac{1}{16\sqrt{2}\pi}\int\int_R \sqrt{2}(4-\sqrt{x^2+y^2})dA = \frac{1}{16\pi}\int_0^{2\pi}\int_0^4 (4-r)rdrd\theta$$
$$= \frac{1}{16\pi}\int_0^{2\pi}\left(2r^2-\frac{1}{3}r^3\right)\Big|_0^4 d\theta = \frac{2}{3\pi}\int_0^{2\pi}d\theta = \frac{4}{3}.$$

The centroid is (0,0,4/3).

(b) $$I_z = \int\int_S (x^2+y^2)kdS = k\sqrt{2}\int\int_R (x^2+y^2)dA = k\sqrt{2}\int_0^{2\pi}\int_0^4 r^2 rdrd\theta$$
$$= \frac{k\sqrt{2}}{4}\int_0^{2\pi} r^4\Big|_0^4 d\theta = 64k\sqrt{2}\int_0^{2\pi}d\theta = 128k\pi\sqrt{2}$$

30. The surface is $g(x,y,z) = z - f(x,y) = 0$. $\nabla g = -f_x\mathbf{i} - f_y\mathbf{j} + \mathbf{k}$, $|\nabla g| = \sqrt{f_x^2+f_y^2+1}$; $\mathbf{n} =$

$\dfrac{-f_x\mathbf{i} - f_y\mathbf{j} + \mathbf{k}}{\sqrt{1 + f_x^2 + f_y^2}}$; $\mathbf{F}\cdot\mathbf{n} = \dfrac{-Pf_x - Qf_y + R}{\sqrt{1 + f_x^2 + f_y^2}}$;

$dS = \sqrt{1 + f_x^2 + f_y^2}dA$

$\int\int_s \mathbf{F}\cdot\mathbf{n}dS = \int\int_R \dfrac{-Pf_x - Qf_y + R}{\sqrt{1 + f_x^2 + f_y^2}}\sqrt{1 + f_x^2 + f_y^2}dA = \int\int_R(-Pf_x - Qf_y + R)dA$

15.7 Curl and Divergence

1. $\text{curl}\mathbf{F} = (x-y)\mathbf{i} + (x-y)\mathbf{j}$; $\text{div}\mathbf{F} = 2z$
2. $\text{curl}\mathbf{F} = -2x^2\mathbf{i} + (10y - 18x^2)\mathbf{j} + (4xz - 10z)\mathbf{k}$; $\text{div}\mathbf{F} = 0$
3. $\text{curl}\mathbf{F} = 0$; $\text{div}\mathbf{F} = 4y + 8z$
4. $\text{curl}\mathbf{F} = (xe^{2y} + ye^{-yz} + 2xye^{2y})\mathbf{i} - ye^{2y}\mathbf{j} + 3(x-y)^2\mathbf{k}; \text{div}\mathbf{F} = 3(x-y)^2 - ze^{-yz}$
5. $\text{curl}\mathbf{F} = (4y^3 - 6xz^2)\mathbf{i} + (2x^3 - 3x^2)\mathbf{k}; \text{div}\mathbf{F} = 6xy$
6. $\text{curl}\mathbf{F} = -x^3z\mathbf{i} + (3x^2yz - z)\mathbf{j} + (\frac{3}{2}x^2y^2 - y - 15y^2)\mathbf{k}$; $\text{div}\mathbf{F} = (x^3y - x) - (x^3y - x) = 0$
7. $\text{curl}\mathbf{F} = (3e^{-z} - 8yz)\mathbf{i} - xe^{-z}\mathbf{j}$; $\text{div}\mathbf{F} = e^{-z} + 4z^2 - 3ye^{-z}$
8. $\text{curl}\mathbf{F} = (2xyz^3 + 3y)\mathbf{i} + (y\ln x - y^2z^3)\mathbf{j} + (2 - z\ln x)\mathbf{k}$; $\text{div}\mathbf{F} = \dfrac{yz}{x} - 3z + 3xy^2z^2$
9. $\text{curl}\mathbf{F} = (xy^2e^y + 2xye^y + x^3ye^z + x^3yze^z)\mathbf{i} - y^2e^y\mathbf{j} + (-3x^2yze^z - xe^x)\mathbf{k}$;
   $\text{div}\mathbf{F} = xye^x + ye^x - x^3ze^z$
10. $\text{curl}\mathbf{F} = (5xye^{5xy} + e^{5xy} + 3xz^3\sin xz^3 - \cos xz^3)\mathbf{i} + (x^2y\cos yz - 5y^2e^{5xy})\mathbf{j}$
    $+ (-z^4\sin xz^3 - x^2z\cos yz)\mathbf{k}$; $\text{div}\mathbf{F} = 2x\sin yz$
11. $\text{div}\ \mathbf{r} = 1 + 1 + 1 = 3$
12. $\text{curl}\ \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix} = 0\mathbf{i} - 0\mathbf{j} + 0\mathbf{k} = 0$
13. $a\times\nabla = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \end{vmatrix} = \left(a_2\dfrac{\partial}{\partial z} - a_3\dfrac{\partial}{\partial y}\right)\mathbf{i} + \left(a_3\dfrac{\partial}{\partial x} - a_1\dfrac{\partial}{\partial z}\right)\mathbf{j} + \left(a_1\dfrac{\partial}{\partial y} - a_2\dfrac{\partial}{\partial x}\right)\mathbf{k}$

    $(a\times\nabla)\times\mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_2\dfrac{\partial}{\partial z} - a_3\dfrac{\partial}{\partial y} & a_3\dfrac{\partial}{\partial x} - a_1\dfrac{\partial}{\partial z} & a_1\dfrac{\partial}{\partial y} - a_2\dfrac{\partial}{\partial x} \\ x & y & z \end{vmatrix}$

    $= (-a_1 - a_1)\mathbf{i} - (a_2 + a_2)\mathbf{j} + (-a_3 - a_3)\mathbf{k} = -2\mathbf{a}$
14. $\nabla\times(\mathbf{a}\times\mathbf{r}) = (\nabla\cdot\mathbf{r})\mathbf{a} - (\nabla\cdot\mathbf{a})\mathbf{r} = (1+1+1)\mathbf{a} - \left(a_1\dfrac{\partial}{\partial x} + a_2\dfrac{\partial}{\partial y} + a_3\dfrac{\partial}{\partial z}\right)\mathbf{r}$

    $= 3\mathbf{i} - (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) = 2\mathbf{a}$

15. $\nabla \cdot (\mathbf{a} \times \mathbf{r}) = \begin{vmatrix} \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = \dfrac{\partial}{\partial x}(a_2 z - a_3 y) - \dfrac{\partial}{\partial y}(a_1 z - a_3 x) + \dfrac{\partial}{\partial z}(a_1 y - a_2 x) = 0$

16. $\nabla \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix} = \mathbf{0}$; $\mathbf{a} \times (\nabla \times \mathbf{r}) = \mathbf{a} \times \mathbf{0} = \mathbf{0}$

17. $\mathbf{r} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ a_1 & a_2 & a_3 \end{vmatrix} = (a_3 y - a_2 z)\mathbf{i} - (a_3 x - a_1 z)\mathbf{j} + (a_2 x - a_1 y)\mathbf{k}$; $\mathbf{r} \cdot \mathbf{r} = x^2 + y^2 + z^2$

$$\nabla \times [(\mathbf{r} \cdot \mathbf{r})\mathbf{a}] = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (\mathbf{r} \cdot \mathbf{r})a_1 & (\mathbf{r} \cdot \mathbf{r})a_2 & (\mathbf{r} \cdot \mathbf{r})a_3 \end{vmatrix}$$
$$= (2ya_3 - 2za_2)\mathbf{i} - (2xa_3 - 2za_1)\mathbf{j} + (2xa_3 - 2ya_1)\mathbf{k} = 2(\mathbf{r} \times \mathbf{a})$$

18. $\mathbf{r} \cdot \mathbf{a} = a_1 x + a_2 y + a_3 z$; $\mathbf{r} \cdot \mathbf{r} = x^2 + y^2 + z^2$; $\nabla \cdot [(\mathbf{r} \times \mathbf{r})\mathbf{a}] = 2xa_1 + 2ya_2 + 2za_3 = 2(\mathbf{r} \cdot \mathbf{a})$

19. Let $\mathbf{F} = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$ and $\mathbf{G} = S(x, y, z)\mathbf{i} + T(x, y, z)\mathbf{j} + U(x, y, z)\mathbf{k}$.
$$\nabla \cdot (\mathbf{F} + \mathbf{G}) = \nabla \cdot [(P + S)\mathbf{i} + (Q + T)\mathbf{j} + (R + U)\mathbf{k}] = P_x + S_x + Q_y + T_y + R_x + U_z$$
$$= (P_x + Q_y + R_z) + (S_x + T_y + U_z) = \nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G}$$

20. Let $\mathbf{F} = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$ and $\mathbf{G} = S(x, y, z)\mathbf{i} + T(x, y, z)\mathbf{j} + U(x, y, z)\mathbf{k}$.
$$\nabla \times (\mathbf{F} + \mathbf{G}) = \begin{vmatrix} \mathbf{j} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P + S & Q + T & R + U \end{vmatrix}$$
$$= (R_y + U_y - Q_z - T_z)\mathbf{i} - (R_x + U_x - P_z - S_z)\mathbf{j} + (Q_x + T_x - P_y - S_y)\mathbf{k}$$
$$= (R_y - Q_z)\mathbf{i} - (R_x - R_z)\mathbf{j} + (Q_x - P_y)\mathbf{k} + (U_y - T_z)\mathbf{i} - (U_x - S_z)\mathbf{j} + (T_x - S_y)\mathbf{k}$$
$$= \nabla \times \mathbf{F} + \nabla \times \mathbf{G}$$

21. $\nabla \cdot (f\mathbf{F}) = \nabla \cdot (fP\mathbf{i} + fQ\mathbf{j} + fR\mathbf{k}) = fP_x + Pf_x + fQ_y + Qf_y + fR_z + Rf_z$
$$= f(P_x + Q_y + R_z) + (Pf_x + Qf_y + Rf_z) = f(\nabla \cdot \mathbf{F}) + \mathbf{F} \cdot (\nabla f)$$

22. $\nabla \times (f\mathbf{F}) = \begin{vmatrix} \mathbf{j} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ fP & fQ & fR \end{vmatrix}$
$$= (fR_y + Rf_y - fQ_z - Qf_z)\mathbf{i} - (fR_x + Rf_x - fP_z - Pf_z)\mathbf{j}$$
$$(fQ_x + Qf_x - fP_y - Pf_y)\mathbf{k}$$
$$= (fR_y - fQ_z)\mathbf{i} - (fR_x - fP_z)\mathbf{j} + (fQ_x - fP_y)\mathbf{k} + (Rf_y - Qf_z)\mathbf{i}$$
$$- (Rf_x - Pf_z)\mathbf{j} + (Qf_x - Pf_y)\mathbf{k}$$
$$f[(R_y - Q_z)\mathbf{i} - (R_x - P_z)\mathbf{j} + (Q_x - P_y)\mathbf{k}] + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ f_x & f_y & f_z \\ P & Q & R \end{vmatrix} = f(\nabla \times \mathbf{F}) + (\nabla f) \times \mathbf{F}$$

23. Assuming continuous second partial derivatives,

$$\begin{aligned}\operatorname{curl}(\operatorname{grad} f) = \nabla \times (f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}) &= \begin{vmatrix} \mathbf{j} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_x & f_y & f_z \end{vmatrix} \\ &= (f_{zy} - f_{yz})\mathbf{i} - (f_{zx} - f_{xz})\mathbf{j} + (f_{yx} - f_{xy})\mathbf{k} = \mathbf{0}.\end{aligned}$$

24. Assuming continuous second partial derivatives,

$$\begin{aligned}\operatorname{div}(\operatorname{curl}\mathbf{F}) &= \nabla \cdot [(R_y - Q_z)\mathbf{i} - (R_x - P_z)\mathbf{j} + (Q_x - P_y)\mathbf{k}] \\ &= (R_{yx} - Q_{zx} - (R_{xy} - P_{zy}) + (Q_{xz} - P_{yz}) = \mathbf{0}.\end{aligned}$$

25. Let $\mathbf{F} = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$ and $\mathbf{G} = S(x,y,z)\mathbf{i} + Y(x,y,z)\mathbf{j} + U(x,y,z)\mathbf{k}$.

$$\mathbf{F} \times \mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ P & Q & R \\ S & T & U \end{vmatrix} = (QU - RT)\mathbf{i} - (PU - RS)\mathbf{j} + (PT - QS)\mathbf{k}$$

$$\begin{aligned}\operatorname{div}(\mathbf{F} \times \mathbf{G}) &= (QU_x + Q_xU - RT_x - R_xT) - (PU_y + P_yU - RS_y - R_yS) \\ &\quad + (PT_z + P_zT - QS_z - Q_zS) \\ &= S(R_y - Q_z) + T(P_z - R_x) + U(Q_x - P_y) - P(U - y - T_z) - Q(S_z - U_x) \\ &\quad - R(R_x - S_y) \\ &= \mathbf{G} \cdot (\operatorname{curl}\mathbf{F}) - \mathbf{F} \cdot (\operatorname{curl}\mathbf{G})\end{aligned}$$

26. Using Problems 20 and 23,

$$\begin{aligned}\operatorname{curl}(\operatorname{curl}\mathbf{F} + \operatorname{grad} f) &= \nabla \times (\operatorname{curl}\mathbf{F} + \operatorname{grad} f) = \nabla \times (\operatorname{curl}\mathbf{F}) + \nabla \times (\operatorname{grad} f) \\ &= \operatorname{curl}(\operatorname{curl}\mathbf{F}) + \operatorname{curl}(\operatorname{grad} f) = \operatorname{curl}(\operatorname{curl}\mathbf{F}) + \mathbf{0} = \operatorname{curl}(\operatorname{curl}\mathbf{F}).\end{aligned}$$

27. curl $\mathbf{F} = -8yz\mathbf{i} - 2z\mathbf{j} - x\mathbf{k}$; curl (curl $\mathbf{F}$) $= 2\mathbf{i} - (8y - 1)\mathbf{j} + 8z\mathbf{k}$

28. For $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$,

$$\begin{aligned}\operatorname{curl}(\operatorname{curl}\mathbf{F}) &= (Q_{xy} - P_{yy} - P_{zz} + R_{xz})\mathbf{i} + (R_{yz} - Q_{zz} - Q_{xx} + P_{yx})\mathbf{j} \\ &\quad + (P_{zx} - R_{xx} - R_{yy} + Q_{zy})\mathbf{k}\end{aligned}$$

and

$$\begin{aligned}-\nabla^2\mathbf{F} + \operatorname{grad}(\operatorname{div}\mathbf{F}) &= -(P_{xx} + P_{yy} + P_{zz})\mathbf{i} - (Q_{xx} + Q_{yy} + Q_{zz})\mathbf{j} - (R_{xx} + R_{yy} + R_{zz})\mathbf{k} \\ &\quad + \operatorname{grad}(P_x + Q_y + R_z) \\ &= -P_{xx}\mathbf{i} - Q_{yy}\mathbf{j} - R_{zz}\mathbf{k} + (-P_{yy} - P_{zz})\mathbf{i} + (-Q_{xx} - Q_{zz})\mathbf{j} \\ &\quad + (-R_{xx} - R_{yy})\mathbf{k} + (P_{xx} + Q_{yx} + R_{zx})\mathbf{i} + (P_{xy} + Q_{yy} + R_{zy})\mathbf{i} \\ &\quad + (P_{xz} + Q_{yz} + R_{zz})\mathbf{k} \\ &= (-P - P + Q + R)\mathbf{i} + (-Q_{xx} - Q_{zz} + P_{xy} + R_{zy})\mathbf{j} \\ &\quad + (-R_{xx} - R_{yy} + P_{xz} + Q_{yz})\mathbf{k}.\end{aligned}$$

Thus, $\operatorname{curl}(\operatorname{curl}\mathbf{F}) = -\nabla^2\mathbf{F} + \operatorname{grad}(\operatorname{div}\mathbf{F})$.

29. $f_z = 6x + 4y - 9z$; $f_{xx} = 6$; $f_y = 10y + 4z$; $f_{yy} = 10$; $f_z = -9x - 16z$; $f_{zz} = -16$;
$\nabla^2 f = f_{xx} + f_{yy} + f_{zz} = 6 + 10 - 16 = 0$

30. Using Problem 21, $\nabla \cdot (f\nabla f) = f(\nabla \cdot \nabla f) + \nabla f \cdot \nabla f = f(\nabla^2 f) + |\nabla f|^2$.

31. $f_x = 6x + 4y - 9z$; $f_{xx} = 6$; $f_y = 10y + 4x$; $f_{yy} = 10$; $f_z = -9x - 16z$; $f_{zz} = -16$;
$\nabla^2 f + f_x x + f_y y + z_z z = 6 + 10 - 16 = 0$

32. $$\frac{\partial f}{\partial x} = \frac{(a-x)A}{\left[(x-a)^2 + (y-b)^2 + (z-c)^2\right]^{3/2}}$$
$$\frac{\partial^2 f}{\partial x^2} = \frac{A\left[2(x-a)^2 - (y-b)^2 - (z-c)^2\right]}{\left[(x-a)^2 + (y-b)^2 + (z-c)^2\right]^{5/2}}$$
$$\frac{\partial f}{\partial y} = \frac{(b-y)A}{\left[(x-a)^2 + (y-b)^2 + (z-c)^2\right]^{3/2}}$$
$$\frac{\partial^2 f}{\partial y^2} = \frac{A\left[2(y-b)^2 - (x-a)^2 - (z-c)^2\right]}{\left[(x-a)^2 + (y-b)^2 + (z-c)^2\right]^{5/2}}$$
$$\frac{\partial f}{\partial z} = \frac{(c-z)A}{\left[(x-a)^2 + (y-b)^2 + (z-c)^2\right]^{3/2}}$$
$$\frac{\partial^2 f}{\partial z^2} = \frac{A\left[2(z-c)^2 - (x-a)^2 - (y-b)^2\right]}{\left[(x-a)^2 + (y-b)^2 + (z-c)^2\right]^{5/2}}$$
Now $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2} = 0$. Hence f is harmonic.

33. $$f_x = \frac{1}{1 + \dfrac{4y^2}{(x^2+y^2-1)^2}}\left(-\frac{4xy}{(x^2+y^2-1)^2}\right) = -\frac{4xy}{(x^2+y^2-1)^2 + 4y^2}$$
$$f_{xx} = -\frac{[(x^2+y^2-1)^2 + 4y^2]4y - 4xy[4x(x^2+y^2-1)]}{[(x^2+y^2-1) + 4y^2]^2} = \frac{12x^4y - 4y^5 + 8x^2y^3 - 8x^2y - 8y^3 - 4y}{[(x^2+y^2-1)^2 + 4y^2]^2}$$
$$f_y = \frac{1}{1 + \dfrac{4y^2}{(x^2+y^2-1)^2}}\left[\frac{2(x^2+y^2-1) - 4y^2}{(x^2+y^2-1)^2}\right] = \frac{2(x^2+y^2-1)^2}{(x^2+y^2-1)^2 + 4y^2}$$
$$f_{yy} = \frac{[(x^2+y^2-1)^2 + 4y^2](-4y) - 2(x^2+y^2-1)^2[4y(x^2+y^2-1)^2 + 8y]}{[(x^2+y^2-1)^2 + 4y^2]^2}$$
$$= \frac{-12x^4y + 4y^5 - 8x^2y^3 + 8x^2y + 8y^3 + 4y}{[(x^2+y^2-1)^2 + 4y^2]^2}$$
$\nabla^2 f = f_{xx} + f_{yy} = 0$

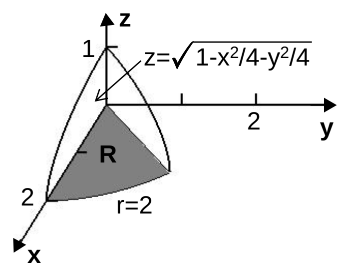
34. $\dfrac{\partial f}{\partial x} = 4x^3 - 12xy^2$, $\dfrac{\partial^2 f}{\partial x^2} = 12x^2 - 12y^2$,
$\dfrac{\partial f}{\partial y} = -12x^2y + 4y^3$, $\dfrac{\partial^2 f}{\partial y^2} = -12x^2 + 12y^2$,
Now $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$. Hence f is harmonic.

35. Using Problems 25 and 23,
$$\nabla \cdot \mathbf{F} = \text{div }(\nabla f \times \nabla g) = \nabla g \dot{(}\text{curl } \nabla f) - \nabla f \cdot (\text{curl } g) = \nabla g \cdot \mathbf{0} - \nabla f \cdot \mathbf{0} = 0.$$

36. Recall that $\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b})=0$. Then, using Problmems 25, 23, and 22,

$$\begin{aligned}\nabla\cdot\mathbf{F} &= (\nabla f\times f\nabla g) = f\nabla g\cdot(\text{curl }\nabla f) - \nabla f\cdot(\text{curl }f\nabla g) = f\nabla g\cdot\mathbf{0} - \nabla f\cdot(\nabla\times f\nabla g)\\ &= -\nabla f\cdot[f(\nabla\times\nabla g)+(\nabla f\times\nabla g)] = -\nabla f\cdot[f\text{curl }\nabla g+(\nabla f\times\nabla g)]\\ &= -\nabla f\cdot[f\mathbf{0}+(\nabla f\times\nabla g)] = -\nabla f\cdot(\nabla f\times\nabla g) = \mathbf{0}.\end{aligned}$$

37. The surface is $g(x,y)=x^2+y^2+4z^2-4=0$.



$\nabla g = 2x\mathbf{i}+2y\mathbf{j}+8z\mathbf{k},\quad |\nabla g| = 2\sqrt{x^2+y^2+16z^2}$;

$\mathbf{n} = \dfrac{2x\mathbf{i}+2y\mathbf{j}+8z\mathbf{k}}{2\sqrt{x^2+y^2+16z^2}} = \dfrac{x\mathbf{i}+y\mathbf{j}+4z\mathbf{k}}{\sqrt{x^2+y^2+16z^2}}$;

$\nabla\times\mathbf{F} = (3x^2-3y^2)\mathbf{k},\quad (\nabla\times\mathbf{F})\cdot\mathbf{n} = \dfrac{12z(x^2+y^2)}{\sqrt{x^2+y^2+16z^2}}$

Writing the equation of the surface as $z = \sqrt{1-x^2/4-y^2/4}$, we have $z_x = -\dfrac{x}{4\sqrt{1-x^2/4-y^2/4}}$, $z_y = -\dfrac{y}{4\sqrt{1-x^2/4-y^2/4}}$, and $dS = \dfrac{\sqrt{16-3x^2-3y^2}}{2\sqrt{4-x^2-y^2}}dA$.

Then, using polar coordinates,

$$\begin{aligned}\text{Flux} &= \int\int_S(\nabla\times\mathbf{F})\cdot\nabla dS = \int\int_R\frac{12z(x^2-y^2)}{\sqrt{x^2+y^2+16z^2}}\frac{\sqrt{16-3x^2-3y^2}}{2\sqrt{4-x^2-y^2}}dA\\ &= \int\int_R\frac{6\sqrt{1-x^2/4-y^2/4}(x^2-y^2)\sqrt{16-3x^2-3y^2}dA}{\sqrt{x^2+y^2+16-4x^2-4y^2}\sqrt{4-x^2-y^2}}\\ &= \int_0^{\pi/4}\int_0^2\sqrt{1-r^2/4}(r^2\cos^2\theta-r^2\sin^2\theta)rdrd\theta\sqrt{4-r^2} = \int_0^{\pi/4}\int_0^2 3r^2\cos 2\theta drd\theta\\ &= \int_0^{\pi/4}\frac{3}{4}r^4\cos 2\theta\Big|_0^2 d\theta = \int_0^{\pi/4}12\cos 2\theta d\theta = 6\sin 2\theta|_0^{\pi/4} = 6.\end{aligned}$$

38. $$\frac{1}{2}\text{curl }\mathbf{v} = \frac{1}{2}\text{curl }(\omega\times\mathbf{r}) = \frac{1}{2}\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ \omega_2 z-\omega_3 y & \omega_3 x-\omega_1 z & \omega_1 y-\omega_2 x\end{vmatrix}$$
$$= \frac{1}{2}[(\omega_1+\omega_1)\mathbf{i}-(\omega_2-\omega_2)\mathbf{j}+(\omega_3+\omega_3)\mathbf{k}] = \omega_1\mathbf{i}+\omega_2\mathbf{j}+\omega_3\mathbf{k} = \omega$$

39. $$\text{curl }\mathbf{F} = -Gm_1m_2\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ x/|\mathbf{r}|^3 & y/|\mathbf{r}|^3 & z/|\mathbf{r}|^3\end{vmatrix}$$
$$= -Gm_1m_2[(-3yz/|\mathbf{r}|^5+3yz/|\mathbf{r}|^5)\mathbf{i}-(-3xz/|\mathbf{r}|^5+3xz/|\mathbf{r}|^5)\mathbf{j}+(-3xy/|\mathbf{r}|^5+3xy/|\mathbf{r}|^5)\mathbf{k}]$$
$$= \mathbf{0}$$
$$\text{div }\mathbf{F} = -Gm_1m_2\left[\frac{-2x^2+y^2+z^2}{|\mathbf{r}|^{5/2}}+\frac{x^2-2y^2+z^2}{|\mathbf{r}|^{5/2}}+\frac{x^2+y^2-2z^2}{|\mathbf{r}|^{5/2}}\right] = \mathbf{0}$$

40. (a) Expressing the vertical component of $\mathbf{V}$ in polar coordinates, we have

$$\frac{2xy}{(x^2+y^2)^2} = \frac{2r^2\sin\theta\cos\theta}{r^4} = \frac{\sin 2\theta}{r^2}$$

Similarly,

$$\frac{x^2-y^2}{(x^2+y^2)^2} = \frac{r^2(\cos^2\theta-\sin^2\theta)}{r^4} = \frac{\cos 2\theta}{r^2}.$$

Since $\lim\limits_{r\to\infty}(\sin 2\theta)/r^2 = \lim\limits_{r\to\infty}(\cos 2\theta)/r^2$, $\mathbf{V} \approx A\mathbf{i}$ for r large or (x,y) far from the origin.

(b) Identify $P(x,y) = A\left[1 - \dfrac{x^2-y^2}{(x^2-y^2)^2}\right]$, $Q(x,y) = -\dfrac{2Axy}{(x^2-y^2)^2}$, and $R(x,y) = 0$, we have

$$P_y = \frac{2Ay(3x^2-y^2)}{(x^2+y^2)^3},\quad Q_x = \frac{2Ay(3x^2-y^2)}{(x^2+y^2)^3},\quad \text{and } P_z = Q_z = R_x = R_y = 0.$$

Thus, $\text{curl}\mathbf{V} = (R_y - Q_z)\mathbf{i} + (P_z - R_x)\mathbf{j} + (Q_x - P_y)\mathbf{k} = 0$ and $\mathbf{V}$ is irrotational.

(c) Since $P_x = \dfrac{2Ax(x^2-3y^2)}{(x^2+y^2)^3}$, $Q_y = \dfrac{2Ax(3y^2-x^2)}{(x^2+y^2)^3}$, and $R_z = 0$, $\nabla\cdot\mathbf{F} = P_x + Q_y + R_z = 0$ and $\mathbf{V}$ is incompressible.

41. We first note that curl $(\partial\mathbf{H}/\partial t) = \partial(\text{curl }\mathbf{H})/\partial t$ and curl $(\partial\mathbf{E}/\partial t) = \partial(\text{curl }\mathbf{E})/\partial t$. Then, from Problem 30,

$$\begin{aligned}-\nabla^2\mathbf{E} &= -\nabla^2\mathbf{E} + \mathbf{0} = -\nabla^2\mathbf{E} + \text{grad } 0 = -\nabla^2\mathbf{E} + \text{grad (div }\mathbf{E}) = \text{curl (curl }\mathbf{E})\\ &= \text{curl }\left(-\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}\right) = -\frac{1}{c}\frac{\partial}{\partial t}\text{curl }\mathbf{H} = -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}\right) = -\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t}\end{aligned}$$

and $\nabla^2\mathbf{E} = \frac{1}{c^2}\partial^2\mathbf{E}/\partial t^2$. Similarly,

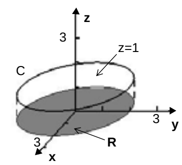
$$\begin{aligned}-\nabla^2\mathbf{H} &= -\nabla^2\mathbf{H} + \text{grad (div }\mathbf{H}) = \text{curl (curl }\mathbf{H}) = \text{curl }\left(\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}\right) = \frac{1}{c}\frac{\partial}{\partial t}\text{curl }\mathbf{E}\\ &= \frac{1}{c}\frac{\partial}{\partial t}\left(-\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}\right) = -\frac{1}{c^2}\frac{\partial^2\mathbf{H}}{\partial t^2}\end{aligned}$$

and $\nabla^2\mathbf{H} = \dfrac{1}{c^2}\partial^2\mathbf{H}/\partial t^2$.

42. We note that div $\mathbf{F} = 2xyz - 2xyz + 1 = 1 \neq 0$. If $\mathbf{F} = \text{curl }\mathbf{G}$, then div (curl $\mathbf{G}$) = div $\mathbf{F} = 1$. But, by problem 24, for any vector field $\mathbf{G}$, div (curl $\mathbf{G}$) = 0. Thus, $\mathbf{F}$ cannot be the curl of $\mathbf{G}$.

15.8 Stokes' Theorem

1. **Surface Integral:** $\text{curl}\mathbf{F} = -10\mathbf{k}$. Letting $g(x,y,z) = -1$, we have $\nabla g = \mathbf{k}$ and $\mathbf{n} = \mathbf{k}$. Then

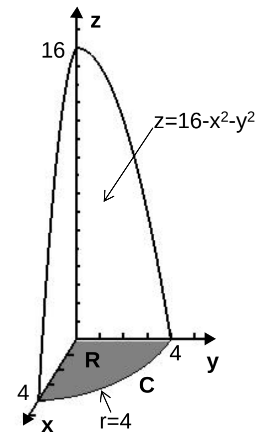


$$\int\int_S (\text{curl}\mathbf{F})\cdot\mathbf{n}dS = \int\int_S (-10)dS = -10\times(\text{area of S}) = -10(4\pi) = -40\pi.$$

Line Integral: Parameterize the curve C by $x = 2\cos t$, $y = 2\sin t$, $z = 1$, for $0 \le t \le 2\pi$. Then

$$\oint \mathbf{F}\cdot d\mathbf{r} = \oint 5ydx - 5xdy + 3dz = \int_0^{2\pi} [10\sin t(-2\sin t) - 10\cos t(2\cos t)]dt$$
$$= \int_0^{2\pi} (-20\sin^2 t - 20\cos^2 t)dt = \int_0^{2\pi} -20dt = -40\pi.$$

2. **Surface Integral:** $\text{curl}\mathbf{F} = 4\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$. Letting $g(x,y,z) = x^2 + y^2 + z - 16$, $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$, and $\mathbf{n} = (2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k})/\sqrt{4x^2 + 4y^2 + 1}$. Thus,



$$\int\int_S (\text{curl}\mathbf{F})\cdot\mathbf{n}dS = \int\int_S \frac{8x - 4y - 3}{\sqrt{4x^2 + 4y^2 + 1}}dS.$$

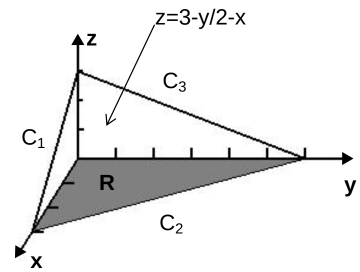
Letting the surface be $z = 16 - x^2 - y^2$, we have $z_x = -2x$, $z_y = -2y$, and $dS = \sqrt{1 + 4x^2 + 4y^2}dA$. Then, using polar coordinates,

$$\int\int_S (\text{curl}\mathbf{F})\cdot\mathbf{n}dS = \int\int_R (8x - 4y - 3)dA$$
$$= \int_0^{2\pi}\int_0^4 (8r\cos\theta - 4r\sin\theta - 3)rdrd\theta$$
$$= \int_0^{2\pi} \left(\frac{8}{3}r^3\cos\theta - \frac{4}{3}r^3\sin\theta - \frac{3}{2}r^2\right)\Big|_0^4 d\theta =$$
$$\int_0^{2\pi} \left(\frac{512}{3}\cos\theta - \frac{256}{3}\sin\theta - 24\right)d\theta$$
$$= \left(\frac{512}{3}\sin\theta + \frac{256}{3}\cos\theta - 24\theta\right)\Big|_0^{2\pi} = -48\pi.$$

Line Integral: Parameterize the curve C by $x = 4\cos t$, $y = 4\sin t$, $z = 0$, for $0 \le t \le 2\pi$. Then,

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_c 2zdx - 3xdy + 4ydz = \int_0^{2\pi} [-12\cos t(4\cos t)]dt$$
$$= \int_0^{2\pi} -48\cos^2 t dt = (-24t - 12\sin 2t)|_0^{2\pi} = -48\pi.$$

3. **Surface Integral:** curl$\mathbf{F} = \mathbf{i} + \mathbf{j} + \mathbf{k}$. Letting $g(x, y, z) = 2x + y + 2z - 6$, we have $\nabla g = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{n} = (2\mathbf{i} + \mathbf{j} + 2\mathbf{k})/3$. Then $\int\int_S (\text{curl}\mathbf{F}) \cdot \mathbf{n} dS = \int\int_S \frac{5}{3} dS$. Letting the surface be $z = 3 - \frac{1}{2}y - x$ we have $z_x = -1$, $z_y = -\frac{1}{2}$, and $dS = \sqrt{1 + (-1)^2 + (-\frac{1}{2})^2} dA = \frac{3}{2} dA$. Then



$$\int\int_S (\text{curl}\mathbf{F}) \cdot \mathbf{n} dS = \int\int_R \frac{5}{3}\left(\frac{3}{2}\right) dA = \frac{5}{2} \times (\text{area of } R) = \frac{5}{2}(9) = \frac{45}{2}.$$

Line Integral: $C_1 : z = 3 - x, 0 \leq x \leq 3,\ \ y = 0$; $C_2 :\ \ y = 6 - 2x,\ \ 3 \geq x \geq 0, z = 0$; $C_3 :\ \ z = 3 - y/2,\ \ 6 \geq y \geq 0,\ \ x = 0$.

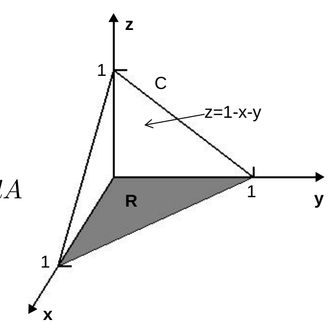
$$\begin{aligned} \oint_C z dx + x dy + y dz &= \int\int_{C_1} z dx + \int_{C_2} x dy + \int_{C_3} y dz \\ &= \int_0^3 (3 - x) dx + \int_3^0 x(-2dx) + \int_6^0 y(-dy/2) \\ &= \left(3x - \frac{1}{2}x^2\right)\Big|_0^3 - x^2\Big|_3^0 - \frac{1}{4}y^2\Big|_6^0 = \frac{9}{2} - (0 - 9) - \frac{1}{4}(0 - 36) = \frac{45}{2} \end{aligned}$$

4. **Surface Integral:** curl$\mathbf{F} = 0$ and $\int\int_S (\text{curl}\mathbf{F}) \cdot \mathbf{n} dS = 0$.
Line Integral: The curve is $x = \cos t,\ \ y = \sin t,\ \ z = 0,\ \ 0 \leq t \leq 2\pi$.

$$\oint_C x dx + y dy + z dz = \int_0^{2\pi} [\cos t(-\sin t) + \sin t(\cos t)] dt = 0.$$

5. curl$\mathbf{F} = 2\mathbf{i} + \mathbf{j}$. A unit vector normal to the plane is $\mathbf{n} = (\mathbf{i} + \mathbf{j} + \mathbf{k})/\sqrt{3}$. Taking the equation of the plane to be $z = 1 - x - y$, we have $z_x = z_y = -1$. Thus, $dS = \sqrt{1 + 1 + 1} dA = \sqrt{3} dA$ and

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \int\int S(\text{curl}\mathbf{F}) \cdot \mathbf{n} dS = \int\int_S \sqrt{3} dS = \sqrt{3}\int\int_R \sqrt{3} dA \\ &= 3 \times (\text{area of } R) = 3(1/2) = 3/2. \end{aligned}$$



6. curl$\mathbf{F} = -2xz\mathbf{i} + x^2\mathbf{k}$. A unit vector normal to the plane is $\mathbf{n} = (\mathbf{j} + \mathbf{k})/\sqrt{2}$. From $z = 1 - y$, we have $z_x = 0$ and $z_y = -1$. Thus, $dS = \sqrt{1 + 1} dA = \sqrt{2} dA$ and

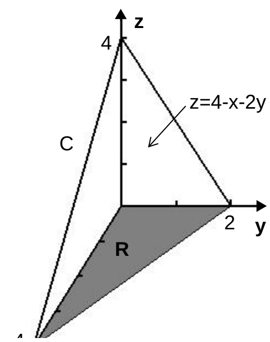
$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \int\int_S (\text{curl}\mathbf{F}) \cdot \mathbf{n} dS = \int\int_R \frac{1}{\sqrt{2}} z^2 \sqrt{2} dA = \int\int_R (1 - y)^2 dA \\ &= \int_0^2 \int_0^1 (1 - y)^2 dy dx = \int_0^2 -\frac{1}{3}(1 - y)^3\Big|_0^1 dx = \int_0^2 \frac{1}{3} dx = \frac{2}{3}. \end{aligned}$$

7. curl$\mathbf{F} = -2y\mathbf{i} - z\mathbf{j} - x\mathbf{k}$. A unit vector normal to the plane is $\mathbf{n} = (\mathbf{j} + \mathbf{k})/\sqrt{2}$. From $z = 1 - y$ we have $z_x = 0$ and $z_y = -1$. Then $dS = \sqrt{1+1}dA = \sqrt{2}dA$ and

$$\begin{aligned}\oint_c \mathbf{F}\cdot d\mathbf{r} &= \int\int_S (\text{curl}\mathbf{F})\cdot\mathbf{n}dS = \int\int_R \left[-\frac{1}{\sqrt{2}}(z+x)\right]\sqrt{2}dA = \int\int_R (y-x-1)dA \\ &= \int_0^2\int_0^1 (y-x-1)dydx = \int_0^2 \left(\frac{1}{2}y^2 - xy - y\right)\bigg|_0^1 dx = \int_0^2 \left(-x-\frac{1}{2}\right)dx \\ &= \left(-\frac{1}{2}x^2 - \frac{1}{2}x\right)\bigg|_0^2 = -3.\end{aligned}$$

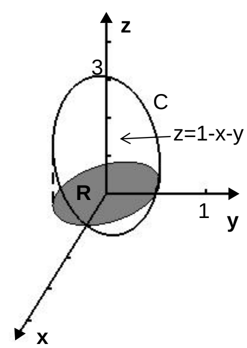
8. curl$\mathbf{F} = 2\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$. Letting $g(x,y,z) = x + 2y + z - 4$, we have $\nabla g = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{n} = (\mathbf{i} + 2\mathbf{j} + \mathbf{k})/\sqrt{6}$. From $z = 4 - x - 2y$ we have $z_x = -1$ and $z_y = -2$. Then $dS = \sqrt{6}dA$ and

$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int\int_S (\text{curl}\mathbf{F})\cdot\mathbf{n}dS = \int\int_R \frac{1}{\sqrt{6}}(9)\sqrt{6}dA = \int\int_R 9dA = 9$x (area of R)$= 9(4) = 36$.



9. curl$\mathbf{F} = (-3x^2 - 3y^2)\mathbf{k}$. A unit vector normal to the plane is $\mathbf{n} = (\mathbf{i}+\mathbf{j}+\mathbf{k})/\sqrt{3}$. From $z = 1-x-y$, we have $z_x = z_y = -1$ and $dS = \sqrt{3}dA$. Then, using polar coordinates,

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r} &= \int\int_S (\text{curl}\mathbf{F})\cdot\mathbf{n}dS = \int\int_R (-\sqrt{3}x^2 - \sqrt{3}y^2)\sqrt{3}dA \\ &= 3\int\int_R (-x^2-y^2)dA = 3\int_0^{2\pi}\int_0^1 (-r^2)rdrd\theta \\ &= 3\int_0^{2\pi} -\frac{1}{4}r^4\bigg|_0^1 d\theta = 3\int_0^{2\pi} -\frac{1}{4}d\theta = \frac{3\pi}{2}.\end{aligned}$$



10. curl$\mathbf{F} = 2xyz\mathbf{i} - y^2z\mathbf{j} + (1-x^2)\mathbf{k}$. A unit vector normal to the surface is $\mathbf{n} = \dfrac{2y\mathbf{j}+\mathbf{k}}{\sqrt{4y^2+1}}$. From $z = 9 - y^2$ we have $z_x = 0$, $z_y = -2y$ and $dS = \sqrt{1+4y^2}dA$. Then

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r} &= \int\int_S (\text{curl}\mathbf{F})\cdot\mathbf{n}dS = \int\int_R (-2y^3z + 1 - x^2)dA = \int_0^3\int_0^{y/2} [-2y^3(9-y^2) + 1 - x^2]dxdy \\ &= \int_0^3 \left(-18y^3x + 2y^5x + x - \frac{1}{3}x^3\right)\bigg|_0^{y/2} dy = \int_0^3 \left(-9y^4 + y^6 + \frac{1}{2}y - \frac{1}{24}y^3\right)dy \\ &= \left(-\frac{9}{5}y^5 + \frac{1}{7}y^7 + \frac{1}{4}y^2 - \frac{1}{96}y^4\right)\bigg|_0^3 \approx 123.57.\end{aligned}$$

11. curl$\mathbf{F} = 3x^2y^2\mathbf{k}$. A unit vector normal to the surface is

$$\mathbf{n} = \frac{8x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{\sqrt{64x^2 + 4y^2 + 4z^2}} = \frac{4x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{16x^2 + y^2 + z^2}}.$$

From $z_x = -\dfrac{4x}{\sqrt{4 - 4x^2 - y^2}}$, $z_y = -\dfrac{y}{\sqrt{4 - 4x^2 - y^2}}$ we obtain $dS = 2\sqrt{\dfrac{1 + 3x^2}{4 - 4x^2 - y^2}}dA$. Then

$$\begin{aligned}
\oint_C \mathbf{F} \cdot d\mathbf{r} &= \int\int_S (\text{curl}\mathbf{F}) \cdot \mathbf{n} dS = \int\int_R \frac{3x^2y^2z}{\sqrt{16x^2 + y^2 + z^2}}\left(2\sqrt{\frac{1 + 3x^2}{4 - 4x^2 - y^2}}\right)dA \\
&= \int\int_R 3x^2y^2 dA \quad \boxed{\text{Using symmetry}} \\
&= 12\int_0^1 \int_0^{2\sqrt{1-x^2}} x^2y^2 dydx = 12\int_0^1 \left(\frac{1}{3}x^2y^3\right)\Big|_0^{2\sqrt{1-x^2}} dx \\
&= 32\int_0^1 x^2(1 - x^2)^{3/2}dx \quad \boxed{x = \sin t, \quad dx = \cos t dt} \\
&= 32\int_0^{\pi/2} \sin^2 t \cos^4 t dt = \pi.
\end{aligned}$$

12. curl$\mathbf{F} = \mathbf{i} + \mathbf{j} + \mathbf{k}$. A unit vector normal to the surface is
$\mathbf{n} = \dfrac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \dfrac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}$
$= x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.
From $z = \sqrt{1 - x^2 - y^2}$, we have $z_x = -\dfrac{x}{\sqrt{1 - x^2 - y^2}}$, $z_y = -\dfrac{y}{\sqrt{1 - x^2 - y^2}}$ and $dS = \dfrac{1}{\sqrt{1 - x^2 - y^2}}dA$. Then

$$\begin{aligned}
\oint_C \mathbf{F} \cdot d\mathbf{r} &= \int\int_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} dS = \int\int_R (x + y + z)\left(\frac{1}{\sqrt{1 - x^2 - y^2}}\right)dA \\
&= \int\int_R \frac{x + y + \sqrt{1 - x^2 - y^2}}{\sqrt{1 - x^2 - y^2}}dA = \int\int_R 1dA + \int\int_R \frac{x + y}{\sqrt{1 - x^2 - y^2}}dA \\
&= \int\int_R 1dA + 0 \quad \boxed{\text{Using Symmetry}} \\
&= \frac{\pi}{2}
\end{aligned}$$

since R is the disk $x^2 + y^2 \le \frac{1}{2}$ with radius $\frac{1}{\sqrt{2}}$.

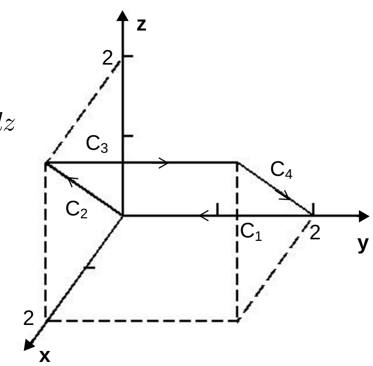
13. Parameterize C by $x = 4\cos t$, $y = 2\sin t, z = 4$, for $0 \le t \le 2\pi$. Then

$$\begin{aligned}\int\int_S (\text{curl}\mathbf{F})\cdot\mathbf{n}dS &= \oint_C \mathbf{F}\cdot d\mathbf{r} = \oint 6yzdx + 5xdy + yze^{x^2}dz \\ &= \int_0^{2\pi} [6(2\sin t)(4)(-4\sin t) + 5(4\cos t)(2\cos t) + 0]dt \\ &= 8\int_0^{2\pi} (-24\sin^2 t + 5\cos^2 t)dt = 8\int_0^{2\pi} (5 - 29\sin^2 t)dt = -152\pi.\end{aligned}$$

14. Parameterize C by $x = 5\cos t$, $y = 5\sin t$, $z = 4$, for $0 \le t \le 2\pi$. Then,

$$\begin{aligned}\int\int_S (\text{curl}\mathbf{F})\cdot\mathbf{n}dS &= \oint_C \mathbf{F}\cdot\mathbf{r} = \oint_C ydx + (y-x)dy + z^2dz \\ &= \int_0^{2\pi} [(5\sin t)(-5\sin t) + (5\sin t - 5\cos t)(5\cos t)]dt \\ &= \int_0^{2\pi} (25\sin t\cos t - 25)dt = \left(\frac{25}{2}\sin^2 t - 25t\right)\Bigg|_0^{2\pi} = -50\pi.\end{aligned}$$

15. Parameterize C by $C_1:\ \ x = 0,\ \ z = 0,\ \ 2 \ge y \ge 0$; $C_2:\ \ z = x,\ \ y = 0,\ \ 0 \le x \le 2$; $C_3:\ \ x = 2,\ \ z = 2,\ \ 0 \le y \le 2$; $C_4:\ \ z = x,\ \ y = 2,\ \ 2 \ge x \ge 0$. Then



$$\begin{aligned}\int\int_S (\text{curl}\mathbf{F})\cdot\mathbf{n}dS &= \oint_C \mathbf{F}\cdot\mathbf{r} = \oint_C 3x^2dx + 8x^3ydy + 3x^2ydz \\ &= \int_{C_1} 0dx + 0dy + 0dz + \int_{C_2} 3x^2dx \\ &\quad + \int_{C_3} 64dy + \int_{C_4} 3x^2dx + 6x^2dx \\ &= \int_0^2 3x^2dx + \int_0^2 64dy + \int_2^0 9x^2dx \\ &= x^3\Big|_0^2 + 64y\Big|_0^2 + 3x^3\Big|_2^0 = 112.\end{aligned}$$

16. Parameterize C by $x = \cos t$, $y = \sin t$, $z = \sin t$, $0 \le t \le 2\pi$. Then

$$\begin{aligned}\int\int_S (\text{curl}\mathbf{F})\cdot\mathbf{n}dS &= \oint_C \mathbf{F}\cdot\mathbf{r} = \oint_C 2xy^2zdx + 2x^2yzdy + (x^2y^2 - 6x)dz \\ &= \int_0^{2\pi} [2\cos t\sin^2 t\sin t(-\sin t) + 2\cos^2 t\sin t\sin t\cos t \\ &\quad + (\cos^2 t\sin^2 t - 6\cos t)\cos t]dt \\ &= \int_0^{2\pi} (-2\cos t\sin^4 t + 3\cos^3 t\sin^2 t - 6\cos^2 t)dt = -6\pi.\end{aligned}$$

17. We take the surface to be $z = 0$. Then $\mathbf{n} = \mathbf{k}$ and $dS = dA$. Since $\text{curl}\mathbf{F} = \dfrac{1}{1+y^2}\mathbf{i} + 2ze^{x^2}\mathbf{j} +$

$y^2\mathbf{k}$,

$$\begin{aligned}\oint_C z^2e^{x^2}dx + xydy + \tan^{-1} ydz &= \int\int_S (\text{curl}\mathbf{F}) \cdot \mathbf{n}dS = \int\int_S y^2 dS = \int\int_R y^2 dA \\ &= \int_0^{2\pi}\int_0^3 r^2 \sin^2\theta r dr d\theta = \int_0^{2\pi} \frac{1}{4}r^4 \sin^2\theta\Big|_0^3 d\theta \\ &= \frac{81}{4}\int_0^{2\pi} \sin^2\theta d\theta = \frac{81\pi}{4}.\end{aligned}$$

18. (a) curl$\mathbf{F} = xz\mathbf{i} - yz\mathbf{j}$. A unit vector normal to the surface is $\mathbf{n} = \dfrac{2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}}{\sqrt{4x^2 + 4y^2 + 1}}$ and $dS = \sqrt{1 + 4x^2 + 4y^2}dA$. Then, using $x = \cos t,\ \ y = \sin t,\ \ 0 \le t \le 2\pi$, we have

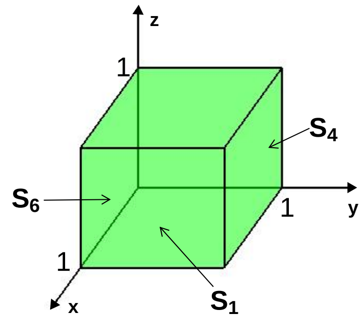
$$\begin{aligned}\int\int_S (\text{curl}\mathbf{F}) \cdot \mathbf{n}dS &= \int\int_R (2x^2z - 2y^2z)dA = \int\int_R (2x^2 - 2y^2)(1 - x^2 - y^2)dA \\ &= \int\int_R (2x^2 - 2y^2 - 2x^4 + 2y^4)dA \\ &= \int_0^{2\pi}\int_0^1 (2r^2\cos^2\theta - 2r^2\sin^2\theta - 2r^4\cos^4\theta + 2r^4\cos^4\theta)r dr d\theta \\ &= 2\int_0^{2\pi}\int_0^1 [r^3\cos 2\theta - r^5(\cos^2\theta - \sin^2\theta)(\cos^2\theta + \sin^2\theta)]dr d\theta \\ &= 2\int_0^{2\pi}\int_0^1 (r^3\cos 2\theta - r^5\cos 2\theta)dr d\theta = 2\int_0^{2\pi} \cos 2\theta\left(\frac{1}{4}r^4 - \frac{1}{6}r^6\right)\Big|_0^1 d\theta \\ &= \frac{1}{6}\int_0^{2\pi}\cos 2\theta d\theta = 0.\end{aligned}$$

(b) We take the surface to be $z = 0$. Then $\mathbf{n} = \mathbf{k}$, curl$\mathbf{F} \cdot \mathbf{n} =$ curl$\mathbf{F} \cdot \mathbf{k} = 0$ and $\int\int_S \text{curl}\mathbf{F} \cdot \mathbf{n}dS = 0$.

(c) By Stoke's Theorem, using $z = 0$, we have

$$\int\int_S \text{curl}\mathbf{F} \cdot \mathbf{n}dS = \oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C xyzdz = \oint xy(0)dz = 0.$$

15.9 Divergence Theorem



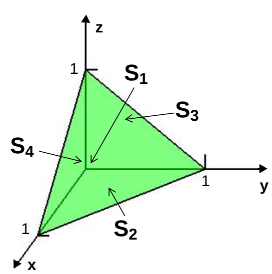
1. div$\mathbf{F} = y + x + z$

The Triple Integral:

$$\int\int\int_D \text{div}\mathbf{F}dV = \int_0^1\int_0^1\int_0^1 (x+y+z)dxdydz$$
$$= \int_0^1\int_0^1 (\frac{1}{2}x^2 + xy + xz)\Big|_0^1 dydz$$
$$= \int_0^1\int_0^1 (\frac{1}{2} + y + z)dydz$$
$$= \int_0^1 (\frac{1}{2}y + \frac{1}{2}y^2 + yz)\Big|_0^1 dz$$
$$= \int_0^1 (1+z)dz = \frac{1}{2}(1+z^2)\Big|_0^1 = 2 - \frac{1}{2} = \frac{3}{2}$$

The Surface Integral: Let the surfaces be S_1 in $z = 0$, S_2 in $z = 1$, S_3 in $y = 0$, S_4 in $y = 1$, S_5 in $x = 0$, and S_6 in $x = 1$. The unit outward normal vectors are $-\mathbf{k}$, $\mathbf{k}$, $-\mathbf{j}$, $\mathbf{j}$, $-\mathbf{i}$ and $\mathbf{i}$, respectively. Then

$$\int\int_S \mathbf{F}\cdot\mathbf{n}dS = \int\int_{S_1}\mathbf{F}\cdot(-\mathbf{k})dS_1 + \int\int_{S_2}\mathbf{F}\cdot\mathbf{k}dS_2 + \int\int_{S_3}\mathbf{F}\cdot(-\mathbf{j})dS_3 + \int\int_{S_4}\mathbf{F}\cdot\mathbf{j}dS_4$$
$$+ \int\int_{S_5}\mathbf{F}\cdot(-\mathbf{i})dS_5 + \int\int_{S_6}\mathbf{F}\cdot\mathbf{i}dS_6$$
$$= \int\int_{S_1}(-xz)dS_1 + \int\int_{S_2} xzdS_2 + \int\int_{S_3}(-yz)dS_3 + \int\int_{S_4} yzdS_4$$
$$+ \int\int_{S_5}(-xy)dS^5 + \int\int_{S_6} xydS_6$$
$$= \int\int_{S_2} xdS_2 + \int\int_{S_4} zdS_4 + \int\int_{S_6} ydS_6$$
$$= \int_0^1\int_0^1 xdxdy + \int_0^1\int_0^1 zdzdx + \int_0^1\int_0^1 ydydz$$
$$= \int_0^1 \frac{1}{2}dy + \int_0^1 \frac{1}{2}dx + \int_0^1 \frac{1}{2}dz = \frac{3}{2}.$$



2. div$\mathbf{F} = 6y + 4z$

The Triple Integral:

$$\int\int\int_D \text{div}\mathbf{F}dV = \int_0^1\int_0^{1-x}\int_0^{1-x-y} (6y+4z)dzdydx$$
$$= \int_0^1\int_0^{1-x} (6yz + 2z^2)\Big|_0^{1-x-y} dydx$$

$$= \int_0^1 \int_0^{1-x} (-4y^2 + 2y - 2xy + 2x^2 - 4x + 2)dydx$$
$$= \int_0^1 \left(-\frac{4}{3}y^3 + y^2 - xy^2 + 2x^2y - 4xy + 2y \right)\Big|_0^{1-x} dx$$
$$= \int_0^1 \left(-\frac{5}{3}x^3 + 5x^2 - 5x + \frac{5}{3} \right) dx = \left(-\frac{5}{12}x^4 + \frac{5}{3}x^3 - \frac{5}{2}x^2 + \frac{5}{3}x \right)\Big|_0^1 = \frac{5}{12}$$

The Surface Integral: Let the surfaces be S_1 in the plane $x+y+z=1$, S_2 in $z=0$, S_3 in $x=0$, and S_4 in $y=0$. The unit outward normal vectors are $\mathbf{n}_1 = (\mathbf{i}+\mathbf{j}+\mathbf{k})/\sqrt{3}$, $\mathbf{n}_2 = -\mathbf{k}$, $\mathbf{n}_3 = -\mathbf{i}$, and $\mathbf{n}_4 = -\mathbf{j}$, respectively. Now on S_1, $dS_1 = \sqrt{3}dA_1$, on S_3, $x=0$, and on S_4, $y=0$, so

$$\int\int_S \mathbf{F}\cdot\mathbf{n}dS = \int\int_{S_1} \mathbf{F}\cdot\mathbf{n}_1 dS_1 + \int\int_{s_2} \mathbf{F}\cdot(-\mathbf{k})dS_2 + \int\int_{S_3} \mathbf{F}\cdot(-\mathbf{j})dS_3 + \int\int_{S_4} \mathbf{F}\cdot(-\mathbf{i})dS_4$$
$$= \int_0^1 \int_0^{1-x} (6xy + 4y(1-x-y) + xe^{-y})dydx + \int_0^1 \int_0^{1-x} (-xe^{-y})dydx$$
$$+ \int\int_{S_3} (-6xy)dS_3 + \int\int_{S_4} (-4yz)dS_4$$
$$= \int_0^1 \left(xy^2 + 2y^2 - \frac{4}{3}y^3 - xe^{-y} \right)\Big|_0^{1-x} dx + \int_0^1 xe^{-y}\Big|_0^{1-x} dx + 0 + 0$$
$$= \int_0^1 \left[x(1-x)^2 + 2(1-x)^2 - \frac{4}{3}(1-x)^3 - xe^{x-1} + x \right] dx + \int_0^1 (xe^{x-1} - x)dx$$
$$= \left[\frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{4}x^4 - \frac{2}{3}(1-x)^3 + \frac{1}{3}(1-x)^4 \right]\Big|_0^1 = \frac{5}{12}.$$

3. div$\mathbf{F} = 3x^2 + 3y^2 + 3z^2$. Using spherical coordinates,

$$\int\int_S \mathbf{F}\cdot\mathbf{n}dS = \int\int\int_D 3(x^2+y^2+z^2)dV = \int_0^{2\pi}\int_0^{\pi}\int_0^a 3\rho^2\rho^2 \sin\phi d\rho d\phi d\theta$$
$$= \int_0^{2\pi}\int_0^{\pi} \frac{3}{5}\rho^5 \sin\phi\Big|_0^a d\phi d\theta = \frac{3a^5}{5}\int_0^{2\pi}\int_0^{\pi} \sin\phi d\phi d\theta$$
$$= \frac{3a^5}{5}\int_0^{2\pi} -\cos\phi\Big|_0^{\pi} d\theta = \frac{6a^5}{5}\int_0^{2\pi} d\theta = \frac{12\pi a^5}{5}.$$

4. div$\mathbf{F} = 4 + 1 + 4 = 9$. Using the formula for the volume of a sphere,

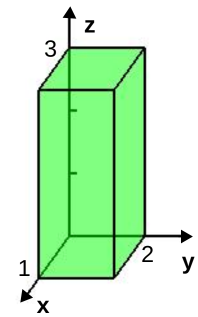
$$\int\int_S \mathbf{F}\cdot\mathbf{n}dS = \int\int\int_D 9dV = 9\left(\frac{4}{3}\pi 2^3\right) = 96\pi.$$

5. div$\mathbf{F} = 2(z-1)$. Using cylindrical coordinates,

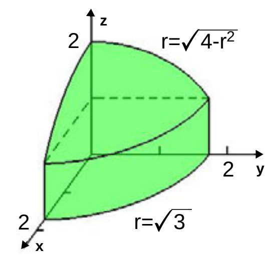
$$\int\int_S \mathbf{F}\cdot\mathbf{n}dS = \int\int\int_D 2(z-1)V = \int_0^{2\pi}\int_0^4\int_1^5 2(z-1)dzrdrd\theta = \int_0^{2\pi}\int_0^4 (z-1)^2\Big|_1^5 rdrd\theta$$
$$= \int_0^{2\pi}\int_0^4 16rdrd\theta = \int_0^{2\pi} 8r^2\Big|_0^4 d\theta = 128\int_0^{2\pi} d\theta = 256\pi.$$

6. $\text{div}\mathbf{F} = 2x + 2z + 12z^2$.

$$\int\int_S \mathbf{F}\cdot\mathbf{n}dS = \int\int\int_D \text{div}\mathbf{F}dV = \int_0^3\int_0^2\int_0^1 (2x+2z+12z^2)dxdydz$$
$$= \int_0^3\int_0^2 (x^2+2xz+12xz^2)\Big|_0^1 dydz$$
$$= \int_0^3\int_0^2 (1+2z+12z^2)dydz$$
$$= \int_0^3 2(1+2z+12z^2)dz = (2z+2z^2+8z^3)\Big|_0^3 = 240$$



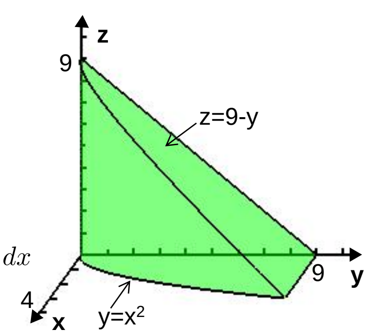
7. $\text{div}\mathbf{F} = 3z^2$. Using cylindrical coordinates,



$$\int\int_S \mathbf{F}\cdot\mathbf{n}dS = \int\int\int_D \text{div}\mathbf{F}dV = \int_0^{2\pi}\int_0^{\sqrt{3}}\int_0^{\sqrt{4-r^2}} 3z^2 rdzdrd\theta$$
$$= \int_0^{2\pi}\int_0^{\sqrt{3}} rz^3\Big|_0^{\sqrt{4-r^2}} drd\theta = \int_0^{2\pi}\int_0^{\sqrt{3}} r(4-r^2)^{3/2}drd\theta$$
$$= \int_0^{2\pi} -\frac{1}{5}(4-r^2)^{5/2}\Big|_0^{\sqrt{3}} d\theta = \int_0^{2\pi} -\frac{1}{5}(1-32)d\theta$$
$$= \int_0^{2\pi}\frac{31}{5}d\theta = \frac{62\pi}{5}.$$

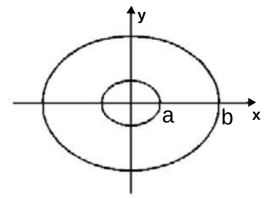
8.

$\text{div}\mathbf{F} = 2x$.



$$\int\int_S \mathbf{F}\cdot\mathbf{n}dS = \int\int\int_D \text{div}\mathbf{F}dV$$
$$= \int_0^3\int_{x^2}^9\int_0^{9-y} 2xdzdydx$$
$$= \int_0^3\int_{x^2}^9 2x(9-y)dydx = \int_0^3 -x(9-y)^2\Big|_{x^2}^9 dx$$
$$= \int_0^3 x(9-x)^2dx$$
$$= \int_0^3 (x^3-18x^2+81x)dx = \left(\frac{1}{4}x^4 - 6x^3 + \frac{81}{2}x^2\right)\Big|_0^3$$
$$= \frac{891}{4}$$

9. $\text{div}\mathbf{F} = \dfrac{1}{x^2+y^2+z^2}$. Using spherical coordinates,



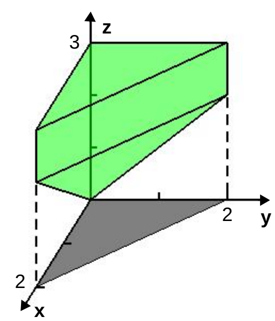
$$\int\int_S \mathbf{F}\cdot\mathbf{n}dS = \int\int\int_D \text{div}\mathbf{F}dV = \int_0^{2\pi}\int_0^{\pi}\int_a^b \frac{1}{\rho^2}\rho^2\sin\phi d\rho d\phi d\theta$$
$$= \int_0^{2\pi}\int_0^{\pi}(b-a)\sin\phi d\phi d\theta = (b-a)\int_0^{2\pi} -\cos\phi\Big|_0^{\pi} d\theta$$
$$= (b-a)\int_0^{2\pi} 2d\theta = 4\pi(b-a).$$

10. Since $\text{div}\mathbf{F}=0$, $\displaystyle\int\int_S \mathbf{F}\cdot\mathbf{n}dS = \int\int\int_D 0dV = 0.$

11. $\text{div}\mathbf{F} = 2z+10y-2z = 10y$.

$$\int\int_S \mathbf{F}\cdot\mathbf{n}dS = \int\int\int_D 10ydV = \int_0^2\int_0^{2-x^2/2}\int_z^{4-z} 10ydydzdx$$
$$= \int_0^2\int_0^{2-x^2/2} 5y^2\Big|_z^{4-z} dzdx = \int_0^2\int_0^{2-x^2/2}(80-40z)dzdx$$
$$= \int_0^2 (80z-20z^2)\Big|_0^{2-x^2/2} dx = \int_0^2 (80-5x^4)dx = (80x-x^5)\big|_0^2 = 128$$

12. $\text{div}\mathbf{F} = 30xy$.



$$\int\int_S \mathbf{F}\cdot\mathbf{n}dS = \int\int\int_D 30xydV = \int_0^2\int_0^{2-x}\int_{x+y}^3 30xydzdydx$$
$$= \int_0^2\int_0^{2-x} 30xyz\Big|_{x+y}^3 dydx$$
$$= \int_0^2\int_0^{2-x}(90xy-30x^2y-30xy^2)dydx$$
$$= \int_0^2 (45xy^2-15x^2y^2-10xy^3)\Big|_0^{2-x} dx$$
$$= \int_0^2(-5x^4+45x^3-120x^2+100x)dx = \left(-x^5+\frac{45}{4}x^4-40x^3+50x^2\right)\Big|_0^2 = 28$$

13. $\text{div}\mathbf{F} = 6xy^2+1-6xy^2 = 1$. Using cylindrical coordinates,

$$\int\int_S \mathbf{F}\cdot\mathbf{n}dS = \int\int\int_D dV = \int_0^{\pi}\int_0^{2\sin\theta}\int_{r^2}^{2r\sin\theta} dzrdrd\theta = \int_0^{\pi}\int_0^{2\sin\theta}(2r\sin\theta-r^2)rdrd\theta$$
$$= \int_0^{\pi}\left(\frac{2}{3}r^3\sin\theta-\frac{1}{4}r^4\right)\Big|_0^{2\sin\theta} d\theta = \int_0^{\pi}\left(\frac{16}{3}\sin^4\theta-4\sin^4\theta\right)d\theta$$
$$= \frac{4}{3}\int_0^{\pi}\sin^4\theta d\theta = \frac{4}{3}\left(\frac{3}{8}\theta-\frac{1}{4}\sin 2\theta+\frac{1}{32}\sin 4\theta\right)\Big|_0^{\pi} = \frac{\pi}{2}$$

14. div$\mathbf{F} = y^2 + x^2$. Using spherical coordinates, we have $x^2 + y^2 = \rho^2 \sin^2 \omega$ and $z = \rho\cos\omega$ or $\rho = z\sec\omega$. Then

$$\int\int_S \mathbf{F}\cdot\mathbf{n}dS = \int\int\int_D (x^2+y^2)dS = \int_0^{2\pi}\int_0^{\pi/4}\int_{2\sec\phi}^{4\sec\phi} \rho^2\sin^2\phi\rho^2\sin\phi d\rho d\phi d\theta$$
$$= \int_0^{2\pi}\int_0^{\pi/4} \frac{1}{5}\rho^5\sin^3\phi\bigg|_{2\sec\phi}^{4\sec\phi} d\phi d\theta = \int_0^{2\pi}\int_0^{\pi/4}\frac{992}{5}\sec^5\phi\sin^3\phi d\phi d\theta$$
$$= \int_0^{2\pi}\int_0^{\pi/4}\frac{992}{5}\tan^3\phi\sec^2\phi d\phi d\theta = \frac{992}{5}\int_0^{2\pi}\frac{1}{4}\tan^4\phi\bigg|_0^{\pi/4} d\theta$$
$$= \frac{992}{5}\int_0^{2\pi}\frac{1}{4}d\theta = \frac{496\pi}{5}.$$

15. Since div $\mathbf{a} = 0$, by the divergence Theorem

$$\int\int_S(\mathbf{a}\cdot\mathbf{n})dS = \int\int\int_D \text{div } \mathbf{a}dV = \int\int\int_D 0dV = 0.$$

16. By the Divergence Theorem and Problem 24 in Section 15.7,

$$\int\int_S(\text{curl}\mathbf{F}\cdot\mathbf{n})dS = \int\int\int_D \text{div}(\text{curl}\mathbf{F})dV = \int\int\int_D 0dV = 0.$$

17. (a) $\text{div}\mathbf{E} = q\left[\frac{-2x^2+y^2+z^2}{(x^2+y^2+z^2)^{5/2}} + \frac{x^2-2y^2+z^2}{(x^2+y^2+z^2)^{5/2}} + \frac{x^2+y^2-2z^2}{(x^2+y^2+z^2)^{5/2}}\right] = 0$
$$\int\int_{SUS_a}(\mathbf{E}\cdot\mathbf{n})dS = \int\int\int_D \text{div}\mathbf{E}dV = \int\int\int_D 0dV = 0$$

(b) From (**a**), $\int\int_S(\mathbf{E}\cdot\mathbf{n})dS + \int\int_{S_a}(\mathbf{E}\cdot\mathbf{n})dS = 0$ and $\int\int_S(\mathbf{E}\cdot\mathbf{n})dS = -\int\int_{S_a}(\mathbf{E}\cdot\mathbf{n})dS$. on S_a, $|\mathbf{r}| = a$, $\mathbf{n} = -(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})/a = -\mathbf{r}/a$ and $\mathbf{E}\cdot\mathbf{n} = (q\mathbf{r}/a^3)\cdot(-\mathbf{r}/a) = -qa^2/a^4 = -qa^2$. Thus
$\int\int_S(\mathbf{E}\cdot\mathbf{n})dS = -\int\int_{S_a}(-\frac{q}{a^2})dS = \frac{q}{a^2}\int\int_{S_a}dS = \frac{q}{a^2}\times(\text{area of } S_a) = \frac{q}{a^2}(4\pi a^2) = 4\pi q.$

18. (a) By Gauss' Law $\int\int(\mathbf{E}\cdot\mathbf{n})dS = \int\int\int_D 4\pi\rho dV$, and by the Divergence Theorem $\int\int_S(\mathbf{E}\cdot\mathbf{n})dS = \int\int\int_D \text{div}\mathbf{E}dV$. Thus $\int\int\int_D 4\pi\rho dV = \int\int\int_D \text{div}\mathbf{E}dV$ and $\int\int\int_D(4\pi\rho - \text{div}\mathbf{E})dV = 0$. Since this holds for all regions D, $4\pi\rho - \text{div}\mathbf{E} = 0$ and $\text{div}\mathbf{E} = 4\pi\rho$.

(b) Since $\mathbf{E}$ is irrotational, $\mathbf{E} = \nabla\phi$ and $\nabla^2\phi = \nabla\cdot\nabla\phi = \nabla\mathbf{E} = \text{div}\mathbf{E} = 4\pi\rho$.

19. By the Divergence Theorem and Problem 21 in Section 15.7,

$$\int\int_S(f\nabla g)\cdot\mathbf{n}dS = \int\int\int_D \text{div}(f\nabla g)dV = \int\int\int_D \nabla\cdot(f\nabla g)dV = \int\int\int_D[f(\nabla\cdot\nabla g) + \nabla g\cdot\nabla f]dV$$
$$= \int\int\int_D(f\nabla^2 g + \nabla g\cdot\nabla f)dV.$$

20. By the Divergence Theorem and Problem 19 and 21 in Section 15.7,

$$\begin{aligned}\int\int_S (f\nabla g - g\nabla f)\cdot \mathbf{n}dS &= \int\int\int_D \mathrm{div}(f\nabla g - g\nabla f)dV = \int\int\int_D \nabla\cdot(f\nabla g - g\nabla f)dV\\ &= \int\int\int_D [f(\nabla\cdot\nabla g) + \nabla g\cdot\nabla f - g(\nabla\cdot\nabla f) - \nabla f\cdot\nabla g]dV\\ &= \int\int\int_D (f\nabla^2 g - g\nabla^2 f)dV.\end{aligned}$$

21. If $G(x, y, z)$ is a vector valued function then we define surface integrals and triple integrals of $\mathbf{G}$ component-wise. In this case, if $\mathbf{a}$ is a constant vector it is easily shown that

$$\int\int_S \mathbf{a}\cdot\mathbf{G}dS = \mathbf{a}\cdot\int\int_S \mathbf{G}dS \text{and} \int\int\int_D \mathbf{a}\cdot\mathbf{G}dV = \mathbf{a}\cdot\int\int\int_D \mathbf{G}dV.$$

Now let $F = f\mathbf{a}$. Then

$$\int\int_S \mathbf{F}\cdot\mathbf{n}dS = \int\int_S (f\mathbf{a})\cdot\mathbf{n}dS = \int\int_S \mathbf{a}\cdot(f\mathbf{n})dS$$

and, using Problem 21 in Section 15.7 and the fact that $\nabla\cdot\mathbf{a} = 0$, we have

$$\int\int\int_D \mathrm{div}\mathbf{F}dV = \int\int\int_D \nabla\cdot(f\mathbf{a})dV = \int\int\int_D [f(\nabla\cdot\mathbf{a}) + \mathbf{a}\cdot\nabla f]dV = \int\int\int_D \mathbf{a}\cdot\nabla f dV.$$

By the Divergence Theorem,

$$\int\int_S \mathbf{a}\cdot(f\mathbf{n})dS = \int\int_S \mathbf{F}\cdot\mathbf{n}dS = \int\int\int_D \mathrm{div}\mathbf{F}dV = \int\int\int_D \mathbf{a}\cdot\nabla f dV$$

and

$$\mathbf{a}\cdot\left(\int\int_S f\mathbf{n}dS\right) = \mathbf{a}\cdot\left(\int\int\int_D \nabla f dV\right) \text{ or } \mathbf{a}\cdot\left(\int\int_S f\mathbf{n}dS - \int\int\int_D \nabla f dV\right) = 0.$$

Since $\mathbf{a}$ is arbitrary,

$$\int\int_S f\mathbf{n}dS - \int\int\int_D \nabla f dV = 0 \text{ and } \int\int_S f\mathbf{n}dS = \int\int\int_D \nabla f dV.$$

22. $$\begin{aligned}\mathbf{B} + \mathbf{W} &= -\int\int_S p\mathbf{n}dS + m\mathbf{g} = m\mathbf{g} - \int\int\int_D \nabla p dV = m\mathbf{g} - \int\int\int_D \rho\mathbf{g}dV\\ &= m\mathbf{g} - \left(\int\int\int_D \rho dV\right)\mathbf{g} = m\mathbf{g} - m\mathbf{g} = 0\end{aligned}$$

Chapter 15 in Review

A. True/False

1. True; the value is 4/3.

2. True; since $2xydx - x^2dy$ is not exact.
3. False; $\int_C xdx + x^2dy = 0$ from $(-1,0)$ to $(1,0)$ along the x-axis and along the semicircle $y = \sqrt{1-x^2}$, but since $xdx + x^2dy$ is not exact, the integral is not independent of path.
4. True
5. True; assuming that the first partial derivatives are continuous.
6. True
7. True
8. True; since curl$\mathbf{F} = \mathbf{0}$ when $\mathbf{F}$ is a conservative vector field.
9. True
10. True
11. True
12. True

B. Fill in the Blanks

1. $\mathbf{F} = \nabla\phi = -x(x^2+y^2)^{-3/2}\mathbf{i} - y(x^2+y^2)^{-3/2}\mathbf{j}$
2. curl$\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f(x) & g(y) & h(z) \end{vmatrix} = \mathbf{0}$
3. $2xy + 2xy + 2xy = 6xy$
4. $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2y & xy^2 & 2xyz \end{vmatrix} = 2xz\mathbf{i} - 2yz\mathbf{j} + (y^2 - x^2)\mathbf{k}$
5. $\dfrac{\partial}{\partial x}(2xz) - \dfrac{\partial}{\partial y}(2yz) + \dfrac{\partial}{\partial z}(y^2 - x^2) = 0$
6. $\nabla(6xy) = 6y\mathbf{i} + 6x\mathbf{j}$
7. 0; since $(y - 7e^{x^3})dx + (x + \ln\sqrt{y})dy$ is exact.
8. Irrotational
9. At $u = 1, \;\; v = 4$, we have $\mathbf{r} = \langle 1, 4, 4\rangle$.
   $\dfrac{\partial\mathbf{r}}{\partial u}(1,4) = \langle 1, 0, 2\rangle, \;\; \dfrac{\partial\mathbf{r}}{\partial v}(1,4) = \langle 0, 1, 1/2\rangle$
   A normal vector is given by $\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 2 \\ 0 & 1 & 1/2 \end{vmatrix} = \langle -2, -1/2, 1\rangle$.
   The tangent plane is $-2(x-1) - \frac{1}{2}(y-4) + (z-4) = 0$ or $4x + y - 2z = 0$.
10. $\mathbf{r}(2, v) = (8+v)\mathbf{i} + (2+2v)\mathbf{j} + (2+v)\mathbf{k}$
    So the parametric equations are $x = 8 + v, \;\; y = 2 + 2v, \;\; z = 2 + v$

C. Exercises

1. $$\int_C \frac{z^2}{x^2+y^2}ds = \int_\pi^{2\pi} \frac{4t^2}{\cos^2 2t + \sin 2t}\sqrt{4\sin^2 2t + 4\cos^2 2t + 4}dt = \int_\pi^{2\pi} 8\sqrt{2}t^2 dt$$
$$= \frac{8\sqrt{2}}{3}t^3\bigg|_\pi^{2\pi} = \frac{56\sqrt{2}\pi^3}{3}$$

2. $$\int_C (xy+4x)ds = \int_0^1 [x(2-2x)+4x]\sqrt{1+4}dx = \sqrt{5}\int_0^1 (6x-2x^2)dx$$
$$= \sqrt{5}\left(3x^2 - \frac{2}{3}x^3\right)\bigg|_1^0 = -\frac{7\sqrt{5}}{3}$$

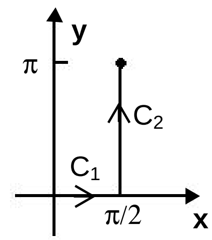
3. Since $P - y = 6x^2y = Q_x$, the integral is independent of path.
$\phi_x = 3x^2y^2,\ \ \phi = x^3y^2 + g(y),\ \ \phi_y = 2x^3y + g'(y) = 2x^3y - 3y^2;$
$g(y) = -y^3;\ \ \phi = x^3y^2 - y^3;$
$\int_{(0,0)}^{(-1,2)} 3x^2y^2dx + (2x^3y - 3y^2)dy = \ (x^3y^2 - y^3)\big|_{(0,0)}^{(-1,2)} = -12$

4. By Green's Theorem,
$$\oint_C (x^2+y^2)dx + (x^2-y^2)dy = \int\int_R (2x-2y)dA = 2\int_0^{2\pi}\int_0^3 (r\cos\theta - r\sin\theta)rdrd\theta$$
$$= 2\int_0^{2\pi}\left(\frac{1}{3}r^3\cos\theta - \frac{1}{3}r^3\sin\theta\right)\bigg|_0^3 d\theta = 2\int_0^{2\pi}\frac{27}{3}(\cos\theta - \sin\theta)d\theta = 0.$$

5. $$\int_C y\sin\pi z dx + x^2e^y dy + 3xyzdz$$
$$= \int_0^1 [t^2\sin\pi t^3 + t^2e^{t^2}(2t) + 3tt^2t^3(3t^2)]dt = \int_0^1 (t^2\sin\pi t^3 + 2t^3e^{t^2} + 9t^8)dt$$
$$= \left(-\frac{1}{3\pi}\cos\pi t^3 + t^9\right)\bigg|_0^1 + 2\int_0^1 t^3e^{t^2}dt \quad \boxed{\text{Integration by parts}}$$
$$= \frac{2}{3\pi} + 1 + (t^2e^{t^2} - e^{t^2})\bigg|_0^1 = \frac{2}{3\pi} + 2$$

6. Parameterize C by $x = \cos t,\ \ y = \sin t;\ \ 0 \le t \le 2\pi$. Then
$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} [4\sin t(-\sin t dt) + 6\cos t(\cos t)dt] = \int_0^{2\pi} (6\cos^2 t - 4\sin^2 t)dt$$
$$= \int_0^{2\pi} (10\cos^2 t - 4)dt = \ (5t + \frac{5}{2}\sin 2t - 4t)\bigg|_0^{2\pi} = 2\pi.$$
Using Green's Theorem, $Q_z - P_y = 6 - 4 = 2$ and $\oint_C \mathbf{F}\cdot d\mathbf{r} = \int\int_R 2dA = 2(\pi\cdot 1^2) = 2\pi$.

7. Let $\mathbf{r}_1 = \frac{\pi}{2}t\mathbf{i}$ and $\mathbf{r}_2 = \frac{\pi}{2}\mathbf{i} + \pi t\mathbf{j}$ for $0 \le t \le 1$. Then $d\mathbf{r}_1 = \frac{\pi}{2}\mathbf{i}$, $d\mathbf{r}_2 = \pi\mathbf{j}$, $\mathbf{F}_1 = 0$,



$$\mathbf{F}_2 = \frac{\pi}{2}\sin\pi t\mathbf{i} + \pi t\sin\frac{\pi}{2}\mathbf{j} = \frac{\pi}{2}\sin\pi t\mathbf{i} + \pi t\mathbf{j},$$

and

$$W = \int_{C_1}\mathbf{F}_1\cdot d\mathbf{r}_1 + \int_{C_2}\mathbf{F}_2\cdot d\mathbf{r}_2 = \int_0^1 \pi^2 t dt = \frac{1}{2}\pi^2 t^2\Big|_0^1 = \frac{\pi^2}{2}.$$

8. Parameterize the line segment from (-1/2,1/2) TO (-1,1) using $y = -x$ as x goes from -1/2 to -1. Parameterize the line segment from (-1,1) to (1,1) using $y = 1$ as x goes from -1 to 1. Parameterize the line segment from (1,1) to $(1, \sqrt{3})$ using $x = 1$ as y goes from 1 to $\sqrt{3}$. Then

$$\begin{aligned} W &= \int_C \mathbf{F}\cdot d\mathbf{r} = \int_{-1/2}^{-1}\mathbf{F}\cdot(dx\mathbf{i} - dx\mathbf{j}) + \int_{-1}^{1}\mathbf{F}\cdot(dx\mathbf{i}) + \int_1^{\sqrt{3}}\mathbf{F}\cdot(dy\mathbf{j}) \\ &= \int_{-1/2}^{-1}\left(\frac{2}{x^2+(-x)^2} - \frac{1}{x^2+(-x)^2}\right)dx + \int_{-1}^{1}\frac{2}{x^2+1}dx + \int_1^{\sqrt{3}}\frac{1}{1+y^2}dy \\ &= \int_{-1/2}^{-1}\frac{1}{2x^2}dx + \int_{-1}^{1}\frac{2}{1+x^2}dx + \int_1^{\sqrt{3}}\frac{1}{1+y^2}dy \\ &= -\frac{1}{2x}\Big|_{-1/2}^{-1} + 2\tan^{-1}x\Big|_{-1}^{1} + \tan^{-1}y\Big|_1^{\sqrt{3}} = -\frac{1}{2} + 2\left(\frac{\pi}{2}\right) + \frac{\pi}{12} = \frac{13\pi-6}{12}. \end{aligned}$$

9. $P_y = 2x = Q_x$, $Q_z = 2y = R_y$, $R_x = 0 = P_z$ and the integral is independent of path. Parameterize the line segment between the points by $x = 1$, $y = 1$, $z = t$, $0 \le t \le \pi$. Then $dx = dy = 0$, $dz = dt$, and

$$\int_{(1,1,0)}^{(1,1,\pi)} 2xydx + (x^2+2yz)dy + (y^2+4)dz = \int_0^{\pi}[2(0) + (1+2t)(0) + (1+4)]dt = 5\pi.$$

10. $P_y = 0 = Q_x$, $Q_z = 0 = R_y$, $R_x = 2e^{2x} = P_z$ and the integral is independent of path. From $\omega = x^2 + y^2 - y + ze^{2x}$ we obtain

$$\int_{(0,0,1)}^{(3,2,0)}(2x+2ze^{2x})dx + (2y-1)dy + e^{2x}dz = (x^2+y^2-y+ze^{2x})\Big|_{(0,0,1)}^{(3,2,0)} = 11 - 1 = 10.$$

11. Using Green's Theorem,

$$\begin{aligned} \oint_C -4ydx + 8xdy &= \int\int_R [8-(-4)]dA = 12\int\int_R dA = 12\times(\text{area of } R) \\ &= 12(16\pi - \pi) = 180\pi. \end{aligned}$$

12. $P_y = [(x-1)^2 - (y-1)^2]/[(x-1)^2 + (y-1)^2]^2 = Q_x$. When (1,1) is outside C, Green's Theorem applies and

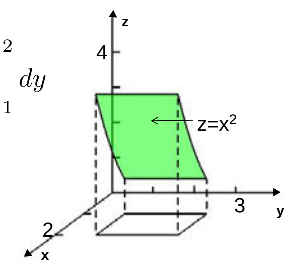
$$\oint_C P dx + Q dy = \int\int_R (Q_x - P_y) dA = \int\int_R 0 dA = 0.$$

For (1,1) inside C, let C_a be a circle of radius a centered at (1,1) and lying entirely inside C. Using $x - 1 = a\cos\theta$ and $y - 1 = a\sin\theta$ for $0 \le \theta \le 2\pi$ we obtain

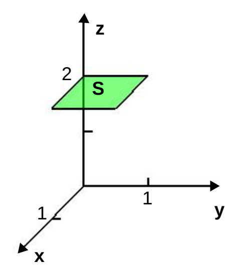
$$\begin{aligned}\oint_C P dx + Q dy = \oint_{C_a} P dx + Q dy &= \frac{1}{a^2}\oint_{C_a} (y-1)dx + (1-x)dy \\ &= \frac{1}{a^2}\int_0^{2\pi} [a\sin\theta(-a\sin\theta) - a\cos\theta(a\cos\theta)]d\theta \\ &= -\int_0^{2\pi} (\sin^2\theta + \cos^2\theta)d\theta = -2\pi.\end{aligned}$$

13. $z_x = 2x, \quad z_y = 0; \quad dS = \sqrt{1+4x^2}dA$

$$\begin{aligned}\int\int_S \frac{z}{xy} dS &= \int_1^3\int_1^2 \frac{x^2}{xy}\sqrt{1+4x^2}dxdy = \int_1^3 \frac{1}{y}\left[\frac{1}{12}(1+4x^2)^{3/2}\right]\Bigg|_1^2 dy \\ &= \frac{1}{12}\int_1^3 \frac{17^{3/2} - 5^{3/2}}{y}dy = \frac{17\sqrt{17} - 5\sqrt{5}}{12}\ln y\Bigg|_1^3 \\ &= \frac{17\sqrt{17} - 5\sqrt{5}}{12}\ln 3\end{aligned}$$



14. $\mathbf{n} = \mathbf{k}, \quad \mathbf{F}\cdot\mathbf{n} = 3;$
flux $= \int\int_S \mathbf{F}\cdot\mathbf{n}dS = 3\int\int_S dS = 3\times(\text{area of } S) = 3(1) = 3$



15. The surface is $g(x,y,z) = y + e^{-x} - 2 = 0$. Then $\nabla g = -e^{-x}\mathbf{i} + \mathbf{j}, \quad \mathbf{n} = (-e^{-x}\mathbf{i} + \mathbf{j})/\sqrt{e^{-2x}+1}$, and $dS = \sqrt{1+e^{-2x}}dA$.

$$\text{flux} = \int\int_S (\mathbf{F}\cdot\mathbf{n})dS = \int\int_R (-4e^{-x} + 2 - y)dA = \int_0^2\int_0^3 (-3e^{-x})dxdz = \int_0^2 3e^{-x}\Big|_0^3 dz$$

$$= \int_0^2 (3e^{-3} - 3)dz = 6e^{-3} - 6.$$

16. Solving $y = 2 - e^{-x}$ for x, we obtain $x = -\ln(2-y)$. The surface is $g(x,y,z) = x + \ln(2-y) = 0$. Then $\nabla g = \mathbf{i} - \dfrac{1}{2-y}\mathbf{j}$, and $|\nabla g| = \sqrt{1 + 1/(2-y)^2}$. Due to the orientation of S we want the $\mathbf{j}$ component of the unit normal vector to be positive. Since $y < 2$ we shall take

$\mathbf{n} = [-\mathbf{i} + (1/2 - y)\mathbf{j}]/\sqrt{1 + 1/(2-y)^2}$. Now $dS = \sqrt{1 + 1/(2-y)^2}dA$ and the region R in the yz -plane is $0 \le z \le 2$ and $1 \le y \le 2 - e^{-3}$. Then

$$\begin{aligned}\text{flux} &= \int\int_S (\mathbf{F}\cdot\mathbf{n})dS = \int\int_R (-4+1)dA = -3 \times (\text{area of } R)\\ &= -3\left(2\left[(2-e^{-3}) - 1\right]\right) = -6 + 6e^{-3}.\end{aligned}$$

17. The surface is $g(x,y,z) = x^2 + y^2 + z^2 - a^2 = 0$. $\nabla g = 2(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = 2\mathbf{r}$ $\mathbf{n} = \mathbf{r}/|\mathbf{r}|$, $\mathbf{F} = c\nabla(1/|\mathbf{r}|) + c\nabla(x^2+y^2+z^2)^{-1/2} = c\dfrac{-x\mathbf{i} - y\mathbf{j} - z\mathbf{k}}{(x^2+y^2+z^2)^{3/2}} = c\mathbf{r}/|\mathbf{r}|^3$
$\mathbf{F}\cdot\mathbf{n} = -\dfrac{\mathbf{r}}{|\mathbf{r}|^3}\cdot\dfrac{\mathbf{r}}{|\mathbf{r}|} = -c\dfrac{\mathbf{r}\cdot\mathbf{r}}{|\mathbf{r}|^4} = -c\dfrac{|\mathbf{r}|^2}{|\mathbf{r}|^4} = -\dfrac{c}{|\mathbf{r}|^2} = -\dfrac{c}{a^2}$
flux $= \int\int_S \mathbf{F}\cdot\mathbf{n}dS = -\dfrac{c}{a^2}\int\int_S dS = -\dfrac{c}{a^2} \times (\text{area of } S) = -\dfrac{c}{a^2}(4\pi a^2) = -4\pi c$

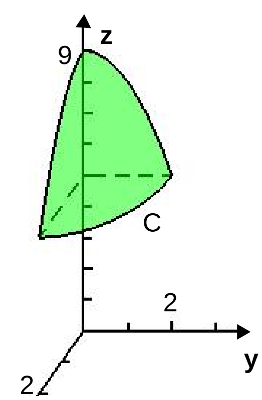
18. In Problem 17, $\mathbf{F}$ is not continuous at $(0,0,0)$ which is in any acceptable region containing the sphere.

19. Since $\mathbf{F} = c\nabla(1/r)$, $\text{div}\mathbf{F} = \nabla\cdot(c\nabla(1/r)) = c\nabla^2(1/r) = c\nabla^2[(x^2+y^2+z^2)^{-1/2}] = 0$ by Problem 31 in Section 17.5. Then, by the Divergence Theorem,

$$\text{flux}\mathbf{F} = \int\int_S \mathbf{F}\cdot\mathbf{n}dS = \int\int\int_D \text{div}\mathbf{F}dV = \int\int\int_D 0dV = 0.$$

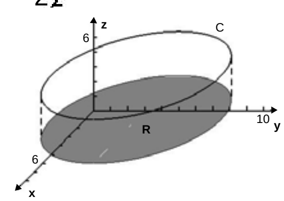
20. Parameterize C by $x = 2\cos t$, $y = 2\sin t$, $z = 5$, for $0 \le t \le 2\pi$. Then

$$\begin{aligned}\int\int_S (\text{curl}\mathbf{F}\cdot\mathbf{n})dS &= \oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C 6xdx + 7zdy + 8ydz\\ &= \int_0^{2\pi}[12\cos t(-2\sin t) + 35(2\cos t)]dt\\ &= \int_0^{2\pi}(70\cos t - 24\sin t\cos t)dt\\ &= (70\sin t - 12\sin^2 t)\Big|_0^{2\pi} = 0.\end{aligned}$$

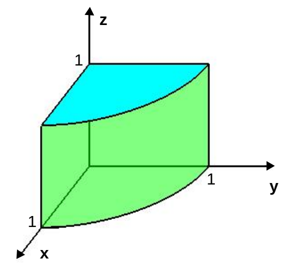


21. Identify $\mathbf{F} = -2y\mathbf{i} + 3x\mathbf{j} + 10z\mathbf{k}$. Then $\text{curl}\mathbf{F} = 5\mathbf{k}$. The curve C lies in the plane $z = 3$, so $\mathbf{n} = \mathbf{k}$ and $dS = dA$. Thus,

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r} &= \int\int_S (\text{curl}\mathbf{F})\cdot\mathbf{n}dS\\ &= \int\int_R 5dA = 5 \times (\text{area of } R) = 5(25\pi) = 125\pi.\end{aligned}$$



22. Since $\text{curl}\mathbf{F} = \mathbf{0}$, $\oint \mathbf{F}\cdot d\mathbf{r} = \int\int_S (\text{curl}\mathbf{F}\cdot\mathbf{n})dS = \int\int_S 0dS = 0$.

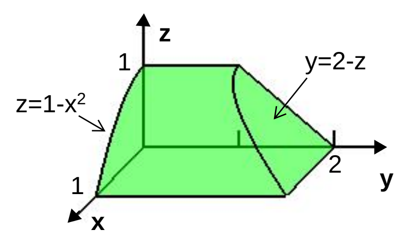


23. $\operatorname{div}\mathbf{F} = 1 + 1 + 1 = 3$;

$$\int\int_S \mathbf{F}\cdot\mathbf{n}dS = \int\int\int_D \operatorname{div}\mathbf{F}dV$$
$$= \int\int\int_D 3dV = 3 \times (\text{volume of } D) = 3\pi$$

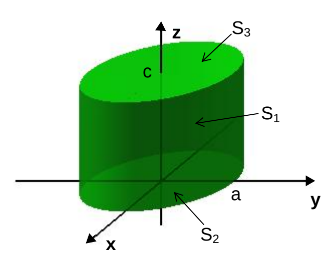
24. $\operatorname{div}\mathbf{F} = x^2 + y^2 + z^2$. Using cylindrical coordinates,

$$\int\int_S \mathbf{F}\cdot\mathbf{n}dS = \int\int\int_D \operatorname{div}\mathbf{F}dV = \int\int\int_D (x^2+y^2+z^2)dV = \int_0^{2\pi}\int_0^1\int_0^1 (r^2+z^2)rdzdrd\theta$$
$$= \int_0^{2\pi}\int_0^1 \left(r^3z + \frac{1}{3}rz^3\right)\Big|_0^1 drd\theta = \int_0^{2\pi}\int_0^1 \left(r^3 + \frac{1}{3}r\right)drd\theta$$
$$= \int_0^{2\pi}\left(\frac{1}{4}r^4 + \frac{1}{6}r^2\right)\Big|_0^1 d\theta = \int_0^{2\pi}\frac{5}{12}d\theta = \frac{5\pi}{6}.$$



25. $\operatorname{div}\mathbf{F} = 2x + 2(x+y) - 2y = 4x$

$$\int\int_S \mathbf{F}\cdot\mathbf{n}dS = \int\int\int_D \operatorname{div}\mathbf{F}dV = \int\int\int_D 4xdV$$
$$= \int_0^1\int_0^{1-x^2}\int_0^{2-z} 4xdydzdx$$
$$= \int_0^1\int_0^{1-x^2} 4x(2-z)dzdx$$
$$= \int_0^1\int_0^{1-x^2}(8x - 4xz)dzdx = \int_0^1 (8xz - 2xz^2)\Big|_0^{1-x^2} dx$$
$$= \int_0^1 [8x(1-x^2) - 2x(1-x^2)^2]dx$$
$$= \left[-2(1-x^2)^2 + \frac{1}{3}(1-x^2)^3\right]\Big|_0^1 = \frac{5}{3}$$



26. For S_1, $\mathbf{n} = (x\mathbf{i} + y\mathbf{j})/\sqrt{x^2+y^2}$; for S_2, $\mathbf{n}_2 = -\mathbf{k}$ and $z = 0$; and for $S_3, \mathbf{n}_3 = \mathbf{k}$ and $z = c$. Then

$$\begin{aligned}\int\int_S \mathbf{F}\cdot\mathbf{n}dS &= \int\int_{S_1} \mathbf{F}\cdot\mathbf{n}_1 dS_1 + \int\int_{S_2} \mathbf{F}\cdot\mathbf{n}_2 dS_2 + \int\int_{S_3} \mathbf{F}\cdot\mathbf{n}_3 dS_3 \\ &= \int\int_{S_1} \frac{x^2+y^2}{\sqrt{x^2+y^2}} dS_1 + \int\int_{s_2} (-z^2-1)dS_2 + \int\int_{S_3} (z^2+1)dS_3 \\ &= \int\int_{S_1} \sqrt{x^2+y^2}dS_1 + \int\int_{S_2} (-1)dS_2 + \int\int_{S_3} (c^2+1)dS_3 \\ &= a\int\int_{S_1} dS_1 - \int\int_{S_2} dS_2 + (c^2+1)\int\int_{S_3} dS_3 \\ &= a(2\pi ac) - \pi a^2 + (c^2+1)\pi a^2 = 2\pi a^2 c + \pi a^2 c^2.\end{aligned}$$

27. $x^2 - y^2 = u^2(\cosh v)^2 - u^2(\sinh v)^2$ hyperbolic paraboloid
$$= u^2\left[(\cosh v)^2 - (\sinh v)^2\right]$$
$$u^2 = z;$$

28. $z = x^2 + y^2$; paraboloid

29. $y = x^2$; parabolic cylinder

30. $x^2 + y^2 - z^2 = (\cos u \cosh v)^2 + (\sin u \cosh v)^2 - (\sinh v)^2$
$$= (\cosh v)^2 - (\sinh v)^2 = 1; \quad z^2 = x^2 + y^2 - 1$$
frustum of a cone

Chapter 16

Higher-Order Differential Equations

16.1 Exact First-Order Equations

1. Since $P_y = 0 = Q_x$, the equation is exact.
   $f_x = 2x + 4,\ \ f = x^2 + 4x + g(y),\ \ f_y = g'(y) = 3y - 1,\ \ g(y) = \dfrac{3}{2}y^2 - y$
   The solution is $x^2 + 4x + \dfrac{3}{2}y^2 - y = C$.

2. Since $P_y = 1$ and $Q_x = -1$, the equation is not exact.

3. Since $P_y = 4 = Q_x$, the equation is exact.
   $f_x = 5x + 4y,\ \ f = \dfrac{5}{2}x^2 + 4xy + g(y),\ \ f_y = 4x + g'(y) = 4x - 8y^3,\ \ g(y) = -2y^4$
   The solution is $\dfrac{5}{2}x^2 + 4xy - 2y^4 = C$.

4. Since $P_y = \cos y - \sin x = Q_x$, the equation is exact.
   $f_x = \sin y - y\sin x,\ \ f = x\sin y + y\cos x + g(y),\ \ f_y = x\cos y + \cos x + g'(y) = \cos x + x\cos y - y,$
   $g(y) = -\dfrac{1}{2}y^2$
   The solution is $x\sin y + y\cos x - \dfrac{1}{2}y^2 = C$.

5. Since $P_y = 4xy = Q_x$, the equation is exact.
   $f_x = 2y^2x - 3,\ \ f = y^2x^2 - 3x + g(y),\ \ f_y = 2yx^2 + g'(y) = 2yx^2 + 4,\ \ g(y) = 4y$
   The solution is $y^2x^2 - 3x + 4y = C$.

6. $\left(\dfrac{y}{x^2} - 4x^3 + 3y\sin 3x\right)dx + \left(2y - \dfrac{1}{x} + \cos 3x\right)dy = 0$. Since $P_y = \dfrac{1}{x^2} + 3\sin 3x$ and $Q_x = \dfrac{1}{x^2} - 3\sin 3x$, the equation is not exact.

7. $(x^2 - y^2)dx + (x^2 - 2xy)dy = 0$. Since $P_y = -2y$ and $Q_x = 2x - 2y$, the equation is not exact.

8. $\left(1+\ln x+\dfrac{y}{x}\right)dx+(\ln x-1)dy=0$. Since $P_y=\dfrac{1}{x}=Q_x$, the equation is exact. $f_y=\ln x-1$, $f=y\ln x-y+g(x)$, $f_x=\dfrac{y}{x}+g'(x)=1+\ln x+\dfrac{y}{x}$, $g'(x)=1+\ln x$, $g(x)=x\ln x$
The solution is $y\ln x-y+x\ln x=C$.

9. $(y^3-y^2\sin x-x)dx+(3xy^2+2y\cos x)dy=0$. Since $P_y=3y^2-2y\sin x=Q_x$, the equation is exact.
$f_x=y^3-y^2\sin x-x$, $f=xy^3+y^2\cos x-\frac{1}{2}x^2+g(y), f_y=3xy^2+2y\cos x+g'(y)=3xy^2+2y\cos$, $g(y)=0$
The solution is $xy^3+y^2\cos x-\frac{1}{2}x^2=C$

10. Since $P_y=3y^2=Q_x$, the equation is exact. $f_x=x^3+y^3$, $f=\dfrac{1}{4}x^4+xy^3+g(y)$, $f_y=3xy^2+g'(y)=3xy^2$, $g(y)=0$
The solution is $\dfrac{1}{4}x^4+xy^3=C$.

11. Since $P_y=1+\ln y+xe^{-xy}$ and $Q_x=\ln y$, the equation is not exact.

12. Since $P_y=3x^2+e^y=Q_x$, the equation is exact. $f_x=3x^2y+e^y$, $f=x^3y+xe^y+g(y)$, $f_y=x^3+xe^y+g'(y)=x^3+xe^y-2y$, $g(y)=-y^2$
The solution is $x^3y+xe^y-y^2=C$.

13. $(2xe^x-y+6x^2)dx-xdy=0$. Since $P_y=-1=Q_x$, the equation is exact. $f_x=2xe^x-y+6x^2$, $f=2xe^x-2e^x-yx+2x^3+g(y)$, $f_y=-x+g'(y)=-x$, $g(y)=0$
The solution is $2xe^x-2e^x-yx+2x^3=C$.

14. $\left(1-\dfrac{3}{x}+y\right)dx+\left(1-\dfrac{3}{y}+x\right)dy=0$. Since $P_y=1=Q_x$, the equation is exact.
$f_x=1-\dfrac{3}{x}+y$, $f=x-3\ln|x|+xy+g(y)$, $f_y=x+g'(y)=1-\dfrac{3}{y}+x$, $g'(y)=1-\dfrac{3}{y}$,
$g(y)=y-3\ln|y|$
The solution is $x-3\ln|xy|+xy+y=C$.

15. Since $P_y=3x^2y^2=Q_x$, the equation is exact.
$f_y=x^3y^2$, $f=\dfrac{1}{3}x^3y^3+g(x)$, $f_x=x^2y^3+g'(x)=x^2y^3-\dfrac{1}{1=9x^2}$, $g'(x)=-\dfrac{1}{1+9x^2}=-\dfrac{1}{9}\dfrac{1}{1/9+x^2}$, $g(x)=-\dfrac{1}{9}\dfrac{1}{1/3}\tan^{-1}\dfrac{x}{1/3}=-\dfrac{1}{3}\tan^{1}3x$
The solution is $\dfrac{1}{3}x^3y^3-\dfrac{1}{3}\tan^{-1}3x=C$ or $x^3y^3=\tan-13x=C_1$.

16. $2ydx-(5y-2x)dy=0$. Since $P_y=2=Q_x$, the equation is exact.
$f_x=2y$, $f=2xy+g(y)$, $f_y=2x+g'(y)=-5y+2x$, $g(y)_=-\dfrac{5}{2}y^2$
The solution is $2xy-\dfrac{5}{2}y^2=C$.

17. Since $P_y=\sin x\cos y=Q_x$, the equation is exact.
$f_y=\cos x\cos y$, $f=\cos x\sin y+g(x)$, $f_x=-\sin x\sin y+g'(x)=\tan x-\sin x\sin y$, $g'(x)=\tan x$, $g(x)=\ln|\sec x|$
The solution is $\cos x\sin y+\ln|\sec x|=C$ or $\cos x\sin y-\ln|\cos x|+C$.

18. $(2y\sin x\cos x - y + 2y^2e^{xy^2})dx + (\sin^2 x + 4xye^{xy^2} - x)dy = 0$. Since $P_y = 2\sin x\cos x - 1 + 4xy^3e^{xy^2} + 4ye^{xy^2} = Q_z$, the equation is exact. $f_x = 2y\sin x\cos x - y + 2y^2e^{xy^2} = y\sin 2x - y + 2y^2e^{xy^2}$, $f = -\frac{1}{2}y\cos 2x - xy + 2e^{xy^2} + g(y)$,
$f_y = -\frac{1}{2}\cos 2x - x + 4xye^{xy^2} + g'(y) = -\frac{1}{2}(1 - 2\sin^2 x) - x + 4xye^{xy^2} + g'(y)$
$= -\frac{1}{2} + \sin^2 x - x + 4xye^{xy^2} + g'(y) = \sin^2 x + 4xye^{xy^2} - x$
$g'(y) = \frac{1}{2}$, $g(y) = \frac{1}{2}y$
The solution is $-\frac{1}{2}y\cos 2x - xy + 2e^{xy^2} + \frac{1}{2}y = C$.

19. Since $P_y = 4t^3 - 1 = Q_t$, the equation is exact.
$f_t = 4t^3y - 15t^2 - y$, $f = t^4y - 5t^3 - yt + g(y)$,
$f_y = t^4 - t + g'(y) = t^4 + 3y^2 - t$, $g'(y) - 3y^2$, $g(y) = y^3$.
The solution is $t^4y - 5t^3 - yt + y^3 = C$.

20. Since $P_y = -\dfrac{(t^2 + y^2) + y(2y)}{(t^2 + y^2)^2} = \dfrac{y^2 - t^2}{(t^2 + y^2)^2}$
and $Q_y = \dfrac{-2t}{(t^2 + y^2)^2}$, the equation is not exact.

21. Since $P_y = 2(x + y) = Q_x$, the equation is exact.
$f_x = (x+y)^2 = x^2+2xy+y^2$, $f = \frac{1}{3}x^3+x^2y+xy^2+g(y)$, $f_y = x^2+2xy+g'(y) = 2xy+x^2-1$
$g'(y) = -1$, $g(y) = -y$ A family of solutions is $\frac{1}{3}x^3 + x^2y + xy^2 - y = C$. Substituting $x = 1$ and $y = 1$ we obtain $\frac{1}{3} + 1 + 1 - 1 = \frac{4}{3} = C$. The solution subject to the given condition is $\frac{1}{3}x^3 + x^2y + xy^2 - y = \frac{4}{3}$.

22. Since $P_y = 1 = Q_x$, the equation is exact.
$f_x = e^x + y$, $f = e^x + xy + g(y)$, $f_y = x + g'(y) = 2 + x + ye^y$, $g'(y) = 2 + ye^y$ Using integration by parts, $g(y) = 2y + ye^y - y$. A family of solutions is $e^x + xy + 2y + ye^y - e^y = C$. Substituting $x = 0$ and $y = 1$ we obtain $1 + 2 + e - e = 3 = C$. The solution subject to the given condition is $e^x + xy + 2y + ye^y - e^y = 3$.

23. Since $P_y = 4 = Q_t$, the equation is exact.
$f_t = 4y+2t-5$, $f = 4ty+t^2-5t+g(y)$, $f_y = 4t+g'(y) = 6y+4t-1, g'(y) = 6y-1$, $g(y) = 3y^2 - y$ A family of solutions is $4ty + t^2 - 5t + 3y^2 - y = C$. Substituting $t = -1$ and $y = 2$ we obtain $-8 + 1 + 5 + 12 - 2 = 8 = C$. The solution subject to the given condition is $4ty + t^2 - 5t + 3y^2 - y = 8$.

24. Since $P_y = 2y\cos x - 3x^2 = Q_x$, the equation is exact.
$f_x = y^2\cos x - 3x^2y - 2x$, $f = y^2\sin x - x^3y - x^2 + g(y)$, $f_y = 2y\sin x - x^3 + g'(y) = 2y\sin x - x^3 + \ln y$,
$g'(y) = \ln y, g(y) = y\ln y - y$ A family of solutions is $y^2\sin x - x^3y - x^2 + y\ln y - y = C$. Substituting $x = 0$ and $y = e$ we obtain $e - e = 0 = C$. The solution subject to the given condition is $y^2\sin x - x^3y - x^2 + y\ln y - y = 0$.

25. We want $P_y = Q_x$ or $3y^2 + 4kxy^3 = 3y^2 + 40xy^3$. Thus, $4k = 40$ and $k = 10$.

26. We want $P_y = Q_x$ or $18xy^2 - \sin y = 4kxy^2 - \sin y$. Thus $4k = 18$ and $k = \frac{9}{2}$.

27. We need $P_y = Q_x$, so we must have $\dfrac{\partial M}{\partial y} = e^{xy} + xye^{xy} + 2y - \frac{1}{x^2}$. This gives $M(x,y) = \frac{1}{x}e^{xy} + \dfrac{(yx-1)e^{xy}}{x} + y^2 - \dfrac{y}{x^2} + g(x)$ for some function g.

28. We need $P_y = Q_x$, so we must have $\dfrac{\partial N}{\partial x} = \frac{1}{2}x^{-1/2}y^{-1/2} - \dfrac{x}{(x^2+y^2)^2}$. This gives $N(x,y) = x^{1/2}y^{-1/2} + \dfrac{1}{2(x^2+y)} + g(y)$ for some function g.

29. Let $\mu(x, y = y^3$. Then $\dfrac{\partial}{\partial y}[\mu(x,y)M(x,y)] = \dfrac{\partial}{\partial y}\left[xy^4\right] = 4xy^3$
$\dfrac{\partial}{\partial z}[\mu(x,y)N(x,y)] = \dfrac{\partial}{\partial x}\left[2x^2y^3 + 3y^5 - 20y^3\right] = 4xy^3$
Therefore, $\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y) = 0$ is exact, and $\mu(x,y)$ is an integrating factor.
Now, if $y^3\left[xydx + (2x^2 + 3y^2 - 20)dy\right] = 0$, then $xydx + (2x^2 + 3y^2 - 20)dy = 0$, provided $y \neq 0$. Therefore, to solve the original DE, we solve $xy^4dx + \left(2x^2y^3 + 3y^5 - 20y^3\right)dy = 0$.
$f_x = xy^4,\ \ f = \frac{1}{2}x^2y^4 + g(y),\ \ f_y = 2x^2y^+g'(y) = 2x^2y^3 + 3y^5 - 20y^3,$
$g'(y) = 3y^5 - 20y^, \ \ g(y) = \frac{1}{2}y^6 - 5y^4,\ \ f = \frac{1}{2}x^2y^4 + \frac{1}{2}y^6 - 5y^4.$
The solution is therefore $\frac{1}{2}x^2y^4 + \frac{1}{2}y^6 - 5y^4 = C$.

30. True; a separable equation can be written as $\dfrac{1}{h(y)}dy - g(x)dx = 0$. Since g is a function of x only and h is a function of y only, we have $P_y = Q_x = 0$.

16.2 Homogeneous Linear Equations

1. $3m^2 - m = 0 \Longrightarrow m(3m-1) = 0 \Longrightarrow m = 0,\ \ 1/3;\ \ y + C_1 + C_2e^{x/3}$

2. $2m^2 + 5m = 0 \Longrightarrow m(2m+5) = 0 \Longrightarrow m = 0,\ \ -5/2;\ \ y = C_1 + C_2e^{-5x/2}$

3. $m^2 - 16 = 0 \Longrightarrow m^2 = 16 \Longrightarrow m = -4,\ \ 4;\ \ y = C_1e^{-4x} + C_2e^{4x}$

4. $m^2 - 8 = 0 \Longrightarrow m^2 = 8 \Longrightarrow m = -2\sqrt{2},\ \ 2\sqrt{2};\ \ y = C_1e^{-2\sqrt{2}x} + C_2e^{2\sqrt{2}x}$

5. $m^2 + 9 = 0 \Longrightarrow m^2 = -9 \Longrightarrow m = -3i,\ \ 3i;\ \ y = C_1\cos 3x + C_2\sin 3x$

6. $4m^2 + 1 = 0 \Longrightarrow m^2 = -1/4 \Longrightarrow m = -i/2,\ \ i/2;\ \ y = C_1\cos\dfrac{1}{2}x + C_2\sin\dfrac{1}{2}x$

7. $m^2 - 3m + 2 = 0 \Longrightarrow (m-1)(m-2) = 0 \Longrightarrow m = 1,\ \ 2;\ \ y = C_1e^x + C_2e^{2x}$

8. $m^2 - m - 6 = 0 \Longrightarrow (m+2)(m-3) = 0 \Longrightarrow m = -2,\ \ 3;\ \ y = C_1e^{-2x} + C_23^{3x}$

9. $m^2 + 8m + 16 = 0 \Longrightarrow (m+4)^2 = 0 \Longrightarrow m = -4,\ \ -4;\ \ y = C_1e^{-4x} + C_2xe^{-4x}$

10. $m^2 - 10m + 25 = 0 \Longrightarrow (m-5)^2 = 0 \Longrightarrow m = 5,\ \ 5;\ \ y = C_1e^{5x} + C_2xe^{5x}$

11. $m^2+3m-5=0 \Longrightarrow m=-3/2\pm\sqrt{29}/2;\ \ y=C_1e^{(-3/2-\sqrt{29}/2)x}+C_2e^{(-3/2+\sqrt{29}/2)x}$

12. $m^2+4m-1=0 \Longrightarrow m=-2\pm\sqrt{5};\ \ y=C_1e^{(-2-\sqrt{5})x}+C_2e^{(-2+\sqrt{5})x}$

13. $12m^2-5m-2=0 \Longrightarrow (3m-2)(4m+1)=0 \Longrightarrow m=-1/4,\ \ 2/3;\ \ y=C_1e^{-x/4}+C_2e^{2x/3}$

14. $8m^2+2m-1=0 \Longrightarrow (4m-1)(2m+1)=0 \Longrightarrow m=-1/2,\ \ 1/4;\ \ y=C_1e^{-x/2}+C_2e^{x/4}$

15. $m^2-4m+5=0 \Longrightarrow m=2\pm i;\ \ y=e^{2x}(C_1\cos x+C_2\sin x)$

16. $2m^2-3m+4=0 \Longrightarrow m=3/4\pm(\sqrt{23}/4)i; y=e^{3x/4}\left(C_1\cos\frac{\sqrt{23}}{4}x+C_2\sin\frac{\sqrt{23}}{4}x\right)$

17. $3m^2+2m+1=0 \Longrightarrow m=-1/3\pm(\sqrt{2}/3)i;\ \ y=e^{-x/3}\left(C_1\cos\frac{\sqrt{2}}{3}x+C_2\sin\frac{\sqrt{2}}{3}x\right)$

18. $2m^2+2m+1=0 \Longrightarrow m=-1/2\pm(1/2)i;\ \ y=e^{-x/2}\left(C_1\cos\frac{1}{2}x+C_2\sin\frac{1}{2}x\right)$

19. $9m^2+6m+1=0 \Longrightarrow (3m+1)^2=0 \Longrightarrow m=-1/3,\ \ -1/3;\ \ y=C_1e^{-x/3}+C_2xe^{-x/3}$

20. $15m^2-16m-7=0 \Longrightarrow (3m+1)(5m-7)=0 \Longrightarrow m=-1/3,\ \ 7/5;\ \ y=C_1e^{-x/3}+C_2e^{7x/5}$

21. $m^2+16=0 \Longrightarrow m^2=-16 \Longrightarrow m=\pm 4i;\ \ y=C_1\cos 4x+C_2\sin 4x;\ \ y'=-4C_1\sin 4x+C_2\cos 4x$
Using $y(0)=2$ we obtain $2=C_1$. Using $y'(0)=-2$ we obtain $-2=4C_2$ or $C_2=-1/2$. The solution is $y=2\cos 4x-\frac{1}{2}\sin 4x$.

22. $m^2-1=0 \Longrightarrow m^2=1 \Longrightarrow m=\pm 1;\ \ y=C_1e^x+C_2e^{-x};\ \ y'=C_1e^x-C_2e^{-x}$. Using $y(0)=Y'(0)=1$ we obtain the system $C_1+C_2=1,\ \ C_1-C_2=1$. Thus, $C_1=1$ and $C_2=0$. The solution is $y=e^x$.

23. $m^2+6m+5=0 \Longrightarrow (m+1)(m+5)=0 \Longrightarrow m=-5,\ \ -1;\ \ y=C_1e^{-5x}+C_2e^{-x}$; $y'=-5C_1e^{-5x}-C_2e^{-x}$. Using $y(0)=0$ and $y'(0)=3$ we obtain the system $C_1+C_2=0$, $-5C_1-C_2=3$. Thus, $C_1=-3/4$ and $c_2=3/4$. The solution is $y=-\frac{3}{4}e^{-5x}+\frac{3}{4}e^{-x}$.

24. $m^2-8m+17=0 \Longrightarrow m=4\pm i;\ \ y=e^{4x}(C_1\cos x+C_2\sin x)$;
$y'=e^{4x}\left[(4C_1+C_2)\cos x+(-C_1+4C_2)\sin x\right]$.
Using $y(0)=4$ and $y'(0)=-1$ we obtain the system $C_1=4,\ \ 4C_1+C_2=-1$. Thus, $C_1=4$ and $C_2=-17$. The solution is $y=e^{4x}(4\cos x-17\sin x)$.

25. $2m^2-2m+1=0 \Longrightarrow m=1/2\pm(1/2)i;\ \ y=e^{x/2}(C_1\cos\frac{1}{2}x+C_2\sin\frac{1}{2}x);\ \ y'=e^{x/2}[\frac{1}{2}(C_1+C_2)\cos\frac{1}{2}x-\frac{1}{2}(C_1-C_2)\sin\frac{1}{2}x]$. Using $y(0)=-1$ and $y'(0)=0$ we obtain the system $C_1=-1$, $\frac{1}{2}C_1+\frac{1}{2}C_2=0$. Thus, $C_1=-1$ and $C_2=1$. The solution is $y=e^{x/2}\left(\sin\frac{1}{2}x-\cos\frac{1}{2}x\right)$.

26. $m^2 - 2m + 1 = 0 \Longrightarrow (m-1)^2 = 0 \Longrightarrow m = 1,\ \ 1;\ \ y = C_1e^x + C_2xe^x;\ \ y' = (C_1 + C_2)e^x + C_2xe^x$.
Using $y(0) = 5$ and $y'(0) = 10$ we obtain the system $C_1 = 5,\quad C_1 + C_2 = 10$. Thus, $C_1 = C_2 = 5$. The solution is $y = 5e^x + 5xe^x$.

27. $m^2 + m + 2 = 0 \Longrightarrow m = -1/2 \pm (\sqrt{7}/2)i;\ \ y = e^{-x/2}(C_1 \cos \frac{\sqrt{7}}{2}x + C_2 \sin \frac{\sqrt{7}}{2}x)$;
$$y' = e^{-x/2}\left[(-\frac{1}{2}C_1 + \frac{\sqrt{7}}{2}C_2)\cos\frac{\sqrt{7}}{2}x + (-\frac{\sqrt{7}}{2}C_1 - \frac{1}{2}C_2)\sin\frac{\sqrt{7}}{2}x\right].$$
Using $y(0) = y'(0) = 0$ we obtain the system $C_1 = 0,\ \ -\frac{1}{2}C_1 + \frac{\sqrt{7}}{2}C_2 = 0$. Thus, $C_1 = C_2 = 0$. The solution is $y = 0$.

28. $4m^2 - 4 - 3 = 0 \Longrightarrow (2m-3)(2m+1) = 0 \Longrightarrow m = -1/2,\ \ 3/2;\ \ y = C_1e^{-x/2} + C_2e^{3x/2}$;
$y' = -\frac{1}{2}C_1e^{-x/2} + \frac{3}{2}C_2e^{3x/2}$. Using $y(0) = 1$ and $y'(0) = 5$ we obtain the system $C_1 + C_2 = 1$, $-\frac{1}{2}C_1 + \frac{3}{2}C_2 = 5$. Thus, $c_1 = -7/4$ and $C_2 = 11/4$. The solution is $y = -\frac{7}{4}e^{-x/2} + \frac{11}{4}e^{3x/2}$.

29. $m^2 - 3m + 2 = 0 \Longrightarrow (m-1)(m-2) = 0 \Longrightarrow m = 1,\ \ 2;\ \ y = C_1e^x + C_2e^{2x};\ \ y' = C_1e^x + 2C_2e^{2x}$.
Using $y(1) = 0$ and $y'(1) = 1$ we obtain the system $eC_1 + e^2C_2 = 0,\ \ eC_1 + 2e^2C_2 = 1$. Thus, $C_1 = -e^{-1}$ and $C_2 = e{-}2$. The solution is $y = -e^{x-1} + e^{2x-2}$.

30. $m^2 + 1 = 0 \Longrightarrow m^2 = -1 \Longrightarrow m = \pm i;\ \ y = C_1 \cos x + C_2 \sin x;\ \ y' = C_1 \sin x + C_2 \cos x$. Using $y(\pi/3) = 0$ and $y'(\pi/3) = 2$ we obtain the system $\frac{1}{2}C_1 + \frac{\sqrt{3}}{2}C_2 = 0,\ -\frac{\sqrt{3}}{2}C_1 + \frac{1}{2}C_2 = 2$. Thus, $C_1 = -\sqrt{3}$ and $C_2 = 1$. The solution is $y = -\sqrt{3}\cos x + \sin x$.

31. The auxiliary equation is $(m-4)(m+5) = m^2 + m - 20 = 0$. The differential equation is $y'' + y' - 20y = 0$.

32. The auxiliary equation is $[(m-3) - i]\,[(m-3) + i] = (m-3)^2 - i^2 = m^2 - 6m + 10 = 0$. The differential equation is $y'' = 6y' + 10y = 0$.

33. The auxiliary equation is $m^2 + 1 = 0$, so $m = \pm i$. The general solution is $y = C_1 \cos x + C_2 \sin x$. The boundary conditions yield $y(0) = C_1 = 0,\ \ y(\pi) = -C_1 = 0$, so $y = C_2 \sin x$.

34. The general solution is $y = C_1 \cos x + C_2 \sin x$. The boundary conditions yield $y(0) = C_1 = 0,\ \ y(\pi) = -C_1 = 1$, which is a contradiction. No solution.

35. The general solution is $y = C_1 \cos x + C_2 \sin x$. The boundary conditions yield $y'(0) = C_2 = 0,\ \ y'\left(\frac{1}{2}\right) = -C_1 = 2$, so $y = -2\cos x$.

36. The auxiliary equation is $m^2 - 1 = 0$, so $m = \pm 1$. The general solution is $y = C_1e^x + C_2e^{-x}$. The boundary conditions yield $y(0) = C_1 + C_2 = 1,\ \ y(1) = C_1e + C_2e^{-1} = -1$, or $C_1 = \dfrac{-1 - e^{-1}}{e - e^{-1}}$ and $C_2 = \dfrac{e+1}{e - e^{-1}}$, so $y = \left(\dfrac{-1 - e^{-1}}{e - e^{-1}}\right)e^x + \left(\dfrac{e+1}{e - e^{-1}}\right)e^{-x}$.

37. The auxiliary equation is $m^2 - 2m + 2 = 0$, so $m = 1 \pm i$. The general solution is $y = e^x\,(C_1 \cos x + C_2 \sin x)$. The boundary conditions yield $y(0) = C_1 = 1$ and $y(\pi) = -e^{\pi}C_1 = -1$, which is a contradiction. No solution.

38. The general solution is $y = e^x (C_1 \cos x + C_2 \sin x)$. The boundary conditions yield $y(0) = C_1 = 1$ and $y(\pi/2) = C_2 e^{\pi/2} = 1$, so $y = e^x \left(\cos x + e^{-\pi/2} \sin x\right)$

39. The auxiliary equation is $m^2 - 4m + 4 = 0$, so $m = 2$ is a repeated root. The general solution is $y = C_1 e^{2x} + C_2 x e^{2x}$. The boundary conditions yield $y(0) = C_1 = 0$ and $y(1) = C_2 e^2 = 1$, so $y = xe^{-2} e^{2x} = xe^{2(x-1)}$.

40. The general solution is $y = C_1 e^{2x} + C_2 x e^{2x}$. The boundary conditions yield $y'(0) = 2C_1 + C_2 = 1$ and $y(1) = (C_1 + C_2)\, e^2 = 2$, or $C_1 = 1 - 2e^{-2}$ and $C_2 = -1 + 4e^{-2}$, so $y = (1 - 2e^{-2})e^{2x} + (-1 + 4e^{-2})xe^{2x}$.

41. Assuming a solution of the form $y = e^{mx}$ we obtain the auxiliary equation $m^3 - 9m^2 + 25m - 17 = 0$. Since $y_1 = e^x$ is a solution we know that $m_1 = 1$ is a root of the auxiliary equation. The equation can then be written as $(m-1)(m^2 - 8m + 17) = 0$. The roots of this equation are 1 and $4 \pm i$. The general solution of the differential equation is $y = C_1 e^x + e^{4x}(C_2 \cos x + C_3 \sin x)$.

42. Assuming a solution of the form $y = e^{mx}$ we obtain the auxiliary equation $m^3 + 6m^2 + m - 34 = 0$. Since $y_1 = e^{-4x} \cos x$ is a solution, we know that $m_1 = -4 + i$ is a root of the auxiliary equation. Using the fact that complex roots of real polynomial equations occur in conjugate pairs we have that $m_2 = -4 - i$ is also a root. Thus $[m - (-4 + i)][m - (-4 - i)] = m^2 + 8m + 17$ is a factor of the auxiliary equation and we can write it as $m^3 + 6m^2 + m - 34 = (m^2 + 8m + 17)(m - 2) = 0$. The general solution of the differential equation is $y = C_1 e^{2x} + e^{-4x}(C_1 \cos x + C_2 \sin x)$.

43. $y' = me^{mx}, \;\; y'' = m^2 e^{mx}, \;\; y''' = m^3 e^{mx}; \;\; m^3 e^{mx} - 4m^2 e^{mx} - 5me^{mx} = 0 \Longrightarrow (m^3 - 4m^2 - 5m)e^{mx} = 0 \Longrightarrow m^3 - 4m^2 - 5m = 0 \Longrightarrow m(m-5)(m+1) = 0 \Longrightarrow m = 0, \;\; -1, \;\; 5; \;\; y = C_1 + C_2 e^{-x} + C_3 e^{5x}$

44. $y' = me^{mx}, \;\; y'' = m^2 e^{mx}, \;\; y''' = m^3 e^{mx}; \;\; m^3 e^{mx} + 3m^2 e^{mx} - 4me^{mx} - 12e^{mx} = 0 \Longrightarrow (m^3 + 3m^2 - 4m - 12)e^{mx} = 0 \Longrightarrow m^3 + 3m^2 - 4m - 12 = 0 \Longrightarrow m^2(m+3) - 4(m+3) = 0 \Longrightarrow (m^2 - 4)(m+3) = 0 \Longrightarrow m = -3, \;\; -2, \;\; 2; \;\; y = C_1 e^{-3x} + C_2 e^{-2x} + C_3 e^{2x}$

45. Case 1: $\lambda = -\alpha^2 < 0$
Auxiliary equation is $m^2 - \alpha^2 = 0$, so $m = \pm\alpha$ and general solution is $y = C_1 e^{\alpha x} + C_2 e^{-\alpha x}$. Boundary conditions yield $y(0) = C_1 + C_2 = 0$ and $y(1) = C_1 e^{\alpha} + C_2 e^{-\alpha} = 0$, or $C_1 = C_2 = 0$. So Case 1 yields no nonzero solutions.

    Case 2: $\lambda = 0$
Auxiliary equation is $m^2 = 0$, so $m = 0$ is a repeated root and general solution is $y = C_1 + C_2 x$. Boundary conditions yield $y(0) = C_1 = 0$ and $y(1) = C_2 = 0$. So Case 2 yields no nonzero solutions.

    Case 3: $\lambda = \alpha^2 > 0$
Auxiliary equation is $m^2 + \alpha^2 = 0$, so $m = \pm\alpha i$ and the general solution is $y = C_1 \cos \alpha x + C_2 \sin x$. Boundary conditions yield $y(0) = C_1 = 0$ and $y(1) = C_2 \sin \alpha = 0$. Hence, nonzero solutions exist only when $\sin \alpha = 0$, which implies $\alpha = \pm n\pi$ so that $\lambda = n^2\pi^2$ for $n = 1, 2, 3, \ldots$ ($n = 0$ is excluded since that would give $\lambda = 0$).

46. (a) If the earth has density ρ then $M = \rho\frac{4}{3}\pi R^3$ and $M_r = \rho\frac{4}{3}\pi r^3$, so that $M/M_r = R^3/r^3$ and $M_r = r^3M/R^3$. Then

$$F = -k\frac{M_r m}{r^2} = -k\frac{r^3 Mm/R^3}{r^2} = -k\frac{mM}{R^3}r.$$

(b) Since $a = d^2r/dt^2$,$F = ma = m\frac{d^2r}{dt^2} = -k\frac{mM}{R^3}r \Longrightarrow \frac{d^2r}{dt^2} + \frac{kM}{R^3}r = 0 \Longrightarrow \frac{d^2r}{dt^2} + \omega^2 r = 0 where \omega^2 = kM/R^3$. Since $kmM/R^2 = mg$ we have $\omega^2 = kM/R^3 = g/R.$

(c) The general solution of the differential equation in part (**b**) is $r(t) = c_1\cos\omega t + c_2\sin\omega t$. The initial conditions $r(0) = R$ and $r'(0) = 0$ imply $c_1 = R$ and $c_2 = 0$. Then $r(t) = R\cos\omega t$. The mass oscillates back and forth from one side of the earth to the other with a period of $T = 2\pi/\omega$. If we use R=3960 mi and g=32 ft/s^2, then $T \approx 5079$ s or 1.41 h.

47.

48.

16.3 Nonhomogeneous Linear Equations

1. $m^2 - 9 = 0 \Longrightarrow m = -3,\ \ 3;\ \ y_c = C_1e^{-3x} + C_2e^{3x};\ \ y_p = A,\ \ y_p' = y_p'' = 0;\ \ -9A = 54 \Longrightarrow A = -6;\ \ y_p = -6;\ \ y = C_1e^{-3x} + C_2e^{3x} - 6$

2. $2m^2 - 7m + 5 = 0 \Longrightarrow (2m-5)(m-1) = 0 \Longrightarrow m = 1,\ \ 5/2;\ \ y_c = C_1e^x + C_2e^{5x/2};\ \ y_p = A,\ \ y_p' = y_p'' = 0;\ \ 5A = -29 \Longrightarrow A = -29/5;\ \ y = C_1e^x + C_2e^{5x/2} - \frac{29}{5}$

3. $m^2 + 4m + 4 = 0 \Longrightarrow (m+2)^2 = 0 \Longrightarrow m = -2,\ \ -2;\ \ y_c = C_1e^{-2x} + C_2xe^{-2x};\ \ y_p = Ax + B,\ \ y_p' = A,\ \ y_p'' = 0;\ \ 4A + 4(Ax+B) = 2x + 6 \Longrightarrow 4Ax + 4(A+B) = 2x + 6$
Solving $4A = 2,\ \ 4A + 4B = 6$, we obtain $A = 1/2$ and $B = 1$. Thus, $y = C_1e^{-2x} + C_2xe^{-2x} + \frac{1}{2}x + 1$.

4. $m^2 - 2m + 1 = 0 \Longrightarrow (m-1)^2 = 0 \Longrightarrow m = 1,\ \ 1;\ \ y_c = C_1e^x + C_2xe^x;\ \ y_p = Ax^3 + Bx^2 + Cx + d,\ \ y_p' = 3Ax^2 + 2Bx + C,\ \ y_p'' = 6Ax + 2B$
$(6Ax + 2B) - 2(3Ax^2 + 2Bx + c) + (Ax^3 + Bx^2 + Cx + D) = x^3 + 4x$
$\Longrightarrow Ax^3 + (-6A + B)x^2 + (6A - 4B + C)x + (2B - 2C + D) = x^3 + 4x$
Solving $A = 1,\ \ -6A + B = 0,\ \ 6A - 4B + c = 4,\ \ 2B - 2C + D = 0$, we obtain $A = 1,\ \ B = 6, C = 22$, and $D = 32$. Thus, $y = C_1e^x + C_2xe^x + x^3 + 6x^2 + 22x + 32$.

5. $m^2 + 25 = 0 \Longrightarrow m = \pm 5i; y_c = C_1\cos 5x + C_2\sin 5x; y_p = A\sin x + B\cos x,\ \ y_p' = A\cos x - B\sin x,\ \ y_p'' = -A\sin x - B\cos x;\ \ -A\sin x - B\cos x + 25(A\sin x + B\cos x) = 6\sin x \Longrightarrow 24A\sin x + 24B\cos x = 6\sin x;\ \ A = 1/4,\ \ B = 0;\ \ y = C_1\cos 5x + C_2\sin 5x + \frac{1}{4}\sin x$

6. $m^2 - 4 = 0 \Longrightarrow m = -2,\ \ 2;\ \ y_c = C_1e^{-2x} + C_2e^{2x};\ \ y_p = Ae^{4x},\ \ y_p' = 4Ae^{4x},\ \ y_p'' = 16Ae^{4x} - 4Ae^{4x} = 7e^{4x} \Longrightarrow 12Ae^{4x} = 7e^{4x} \Longrightarrow A = 7/12;\ \ y = C_1e^{-2x} + C_2e^{2x} + \frac{7}{12}e^{4x}$

7. $m^2 - 2m - 3 = 0 \Longrightarrow (m-3)(m+1) = 0 \Longrightarrow m = -1,\ \ 3;\ \ y_c = C_1e^{-x} + C_2e^{3x}$
$y_p = Ae^{2x} + Bx^3 + Cx^2 + Dx + E,\ \ y_p' = 2Ae^{2x} + 3Bx^2 + 2Cx + D,\ \ y_p'' = 4Ae^{2x} + 6Bx + 2C$
$(4Ae^{2x} + 6Bx + 2C) - 2(2Ae^{2x} + 3Bx^2 + 2Cx + D) - 3(Ae^{2x} + Bx^3 + Cx^2 + Dx + E) = 4e^{2x} + 2x^3 \Longrightarrow -3Ae^{2x}3Bx^3 + (-6B-3C)x^2 + (6B-4C-3D)x + (2C-2D-3E) = 4e^{2x} + 2x^3$
Solving $-3A = 4,\ \ -3B = 2,\ \ -6B - 3C = 0,\ \ 6B - 4C - 3D = 0,\ \ 2C - 2D - 3E = 0$, we obtain $A = -4/3,\ \ B = -2/3,\ \ C = 4/3,\ \ D = -28/9$, and $E = 80/27$. Thus,

$$y = C_1e^{-x} + C_2e^{3x} - \frac{4}{3}e^{2x} - \frac{2}{3}x^3 + \frac{4}{3}x^2 - \frac{28}{9}x + \frac{80}{27}.$$

8. $m^2 + m + 1 = 0 \Longrightarrow m = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i;\ \ y_c = e^{-x/2}(C_1\cos\sqrt{3}x/2 + C_2\sin\sqrt{3}x/2)$
$y_p = Ax^2e^x + Bxe^x + Ce^x + D,\ \ y_p' = Ax^2e^x + (2A+B)xe^x + (B+C)e^x$
$y_p'' = Ax^2e^x + (4A+B)xe^x + (2A+2B+C)e^x$
$\left[Ax^2e^x + (4A+B)xe^x + (2A+2B+C)e^x\right] + \left[Ax^2e^x + (2A+B)xe^x + (B+C)e^x\right] + \left[Ax^2e^x + Bxe^x + Ce^x + D\right] = x^2e^x + 3 \Longrightarrow 3Ax^2e^x + (6A+3B)xe^x(2A+3B+3C)e^x + D = x^2e^x + 3$
Solving $3A = 1,\ \ 6A + 3B = 0,\ \ 2A + 3B + 3C = 0,\ \ D = 3$, we obtain $A = 1/3,\ \ B = -2/3,\ \ C = 4/9$, and $D = 3$. Thus,

$$y = e^{-x/2}(C_1\cos\sqrt{3}x/2 + C_2\sin\sqrt{3}x/2) + \frac{1}{3}x^2e^x - \frac{2}{3}xe^x + \frac{4}{9}e^x + 3.$$

9. $m^2 - 8m + 25 = 0 \Longrightarrow m = 4 \pm 3i;\ \ y_c = e^{4x}(C_1\cos 3x + C_2\sin 3x);\ \ y_p = Ae^{3x} + B\sin 2x + C\cos 2x,\ \ y_p' = 3Ae^{3x} + 2B\cos 2x - 2C\sin 2x,\ \ y_p'' = 9Ae^{3x} - 4B\sin 2x - 4C\cos 2x$
$(9Ae^{3x} - 4B\sin 2x - 4C\cos 2x) - 8(3Ae^{3x} + 2B\cos 2x - 2C\sin 2x) + 25(Ae^{3x} + B\sin 2x + C\cos 2x) = e^{3x} - 6\cos 2x \Longrightarrow 10Ae^{3x} + (21B + 16C)\sin 2x + (-16B + 21C)\cos 2x = e^{3x} - 6\cos 2x$
Solving $10A = 1,\ \ 21B + 16C = 0,\ \ -16B + 21C = -6$, we obtain $A = 1/10,\ \ B = 96/697$, and $C = -126/697$. Thus,

$$y = e^{4x}(C_1\cos 3x + C_2\sin 3x) + \frac{1}{10}e^{3x} + \frac{96}{697}\sin 2x - \frac{126}{697}\cos 2x.$$

10. $m^2 - 5m + 4 = 0 \Longrightarrow (m-1)(m-4) = 0 \Longrightarrow m = 1,\ \ 4;\ \ y_c = C_1e^x + C_2e^{4x}$
$y_p = A\sinh 3x + B\cosh 3x,\ \ y_p' = 3A\cosh 3x + 3B\sinh 3x,\ \ y_p'' = 9A\sinh 3x + 9B\cosh 3x$
$(9A\sinh 3x + 9B\cosh 3x) - 5(3A\cosh 3x + 3B\sinh 3x) + 4(A\sinh 3x + B\cosh 3x) = 2\sinh 3x \Longrightarrow (13A - 15B)\sinh 3x + (-15A + 13B)\cosh 3x = 2\sinh 3x$
Solving $13A - 15B = 2,\ \ -15A + 13B = 0$, we obtain $A = -13/28$ and $B = -15/28$. Thus,

$$y + C_1e^x + C_2e^{4x} - \frac{13}{28}\sinh 3x - \frac{15}{28}\cosh 3x.$$

11. $m^2 - 64 = 0 \Longrightarrow m = -8,\ \ 8;\ \ y_c = C_1e^{-8x} + C_2e^{8x};\ \ y_p = A,\ \ y_p' = y_p'' = 0$
$-64A = 16 \Longrightarrow A = -1/4;\ \ y = C_1e^{-8x} + C_2e^{8x} - \frac{1}{4},\ \ y' = -8C_1e^{-8x} + 8C_2e^{8x}.$
Using $y(0) = 1$ and $y'(0) = 0$ we obtain $C_1 + C_2 - \frac{1}{4} = 1,\ \ -8C_1 + 8C_2 = 0$, or $C_1 = C_2 = 5/8$.
Thus, $y = \frac{5}{8}e^{-8x} + \frac{5}{8}e^{8x} - \frac{1}{4}$.

12. $m^2+5m-6=0 \Longrightarrow (m+6)(m-1)=0 \Longrightarrow m=-6,\ \ 1;\ \ y_c=C_1e^{-6x}+C_2e^x;\ \ y_p=Ae^{2x},\ \ y_p'=2Ae^{2x},\ \ y_p''=4Ae^{2x};\ \ Ae^{2x}+5(2Ae^{2x})-6(Ae^{2x})=10e^{2x} \Longrightarrow 8Ae^{2x}=10e^{2x} \Longrightarrow A=5/4;\ \ y=C_1e^{-6x}+C_2e^x+\dfrac{5}{4}e^{2x},\ \ y'=-6C_1e^{-6x}+C_2e^x+\dfrac{5}{2}e^{2x}$.
Using $y(0)=1$ and $y'(0)=0$ we obtain $C_1+C_2+\dfrac{5}{4}=1,\ \ -6C_1+C_2+\dfrac{5}{2}=0$, or $C_1=\dfrac{9}{28}$ and $C_2=-\dfrac{4}{7}$. Thus, $y=\dfrac{9}{28}e^{-6x}-\dfrac{4}{7}e^x+\dfrac{5}{4}e^{2x}$.

13. $m^2+1=0 \Longrightarrow m=-i;\ \ i;\ \ y_c=C_1\cos x+C_2\sin x;\ \ W=\begin{vmatrix}\cos x & \sin x\\ -\sin x & \cos x\end{vmatrix}=1$
$u_1'=-\sin x\sec x=-\tan x,\ \ u_1=\ln|\cos x|;\ \ u_2'=\cos x\sec x=1, u_2=x$
$y_p=\cos x\ln|\cos x|+x\sin x;\ \ y=C_1\cos x+C_2\sin x+\cos x\ln|\cos x|+x\sin x$

14. $m^2+1=0 \Longrightarrow m=-i,\ \ i;\ \ y_c=C_1\cos x+C_2\sin x;\ \ W=\begin{vmatrix}\cos x & \sin x\\ -\sin x & \cos x\end{vmatrix}=1$
$u_1'=-\sin x\tan x=-\dfrac{\sin^2 x}{\cos x}=-\dfrac{1-\cos^2 x}{\cos x}=-\sec x+\cos x,\ \ u_1=-\ln|\sec x+\tan x|+\sin x$
$u_2'=\cos x\tan x=\sin x;\ \ u_2=-\cos x$
$y_p=-\cos x\ln|\sec x+\tan x|+\sin x\cos x-\sin x\cos x=-\cos x\ln|\sec x+\tan x|$
$y=C_1\cos x+C_2\sin x-\cos x\ln|\sec x+\tan x|$

15. $m^2+1=0 \Longrightarrow m=-i,\ \ i;\ \ y_c=C_1\cos x+C_2\sin x;\ \ W=\begin{vmatrix}\cos x & \sin x\\ -\sin x & \cos x\end{vmatrix}=1$
$u_1'=-\sin^2 x,\ \ u_1=-\dfrac{1}{2}x+\dfrac{1}{2}\sin x\cos x;\ \ u_2'=\sin x\cos x,\ \ u_2=\dfrac{1}{2}\sin^2 x$
$y_p=-\dfrac{1}{2}x\cos x+\dfrac{1}{2}\sin x\cos^2 x+\dfrac{1}{2}\sin^3 x$
$$\begin{aligned}y&=C_1\cos x+C_2\sin x-\frac{1}{2}x\cos x+\frac{1}{2}\sin x\cos^2 x+\frac{1}{2}\sin^3 x\\ &=C_1\cos x+C_2\sin x-\frac{1}{2}x\cos x+\frac{1}{2}\sin x(\cos^2 x+\sin^2 x)=C_1\cos x+C_3\sin x-\frac{1}{2}x\cos x\end{aligned}$$

16. $m^2+1=0 \Longrightarrow m=-i,\ \ i;\ \ y_c=C_1\cos x+C_2\sin x;\ \ W=\begin{vmatrix}\cos x & \sin x\\ -\sin x & \cos x\end{vmatrix}=1$
$u_1'=-\sin x\sec x\tan x=-\tan^2 x=1-\sec^2 x,\ \ u_1=x-\tan x;\ \ u_2'=\cos x\sec x\tan x=\tan x$
$u_2=-\ln|\cos x|;\ \ y_p=\cos x(\tan x-x)=x\cos x-\sin x-\sin x\ln|\cos x|$
$y=C_1\cos x+C_2\sin x+x\cos x-\sin x-\sin x\ln|\cos x|=C_1\cos x+C_3\sin x+x\cos x-\sin x\ln|\cos x|$

17. $m^2+1=0 \Longrightarrow m=-i,\ \ i;\ \ y_c=C_1\cos x+C_2\sin x;\ \ W=\begin{vmatrix}\cos x & \sin x\\ -\sin x & \cos x\end{vmatrix}=1$
$u_1'=-\sin x\cos^2 x\ \ u_1=\frac{1}{3}\cos^3 x;\ \ u_2'=\cos x\cos^2 x=\cos x-\cos x\sin^2 x,\ \ u_2=\sin x-\frac{1}{3}\sin^3 x$
$y_p=\frac{1}{3}\cos^4 x+\sin^2 x-\frac{1}{3}\sin^4 x=\sin^2 x+\frac{1}{3}(\cos^2 x-\sin^2 x)=\sin^2 x+\frac{1}{3}\cos 2x$
$$\begin{aligned}y&=C_1\cos x+C_2\sin x+\sin^2 x+\frac{1}{3}\cos 2x=C_1\cos x+C_2\sin x+\frac{1}{2}-\frac{1}{2}\cos 2x+\frac{1}{3}\cos 2x\\ &=C_1\cos x+C_2\sin x+\frac{1}{2}-\frac{1}{6}\cos 2x\end{aligned}$$

18. $m^2+1=0 \Longrightarrow m=-i,\ \ i;\ \ y_c=C_1\cos x+C_2\sin x;\ \ W=\begin{vmatrix}\cos x & \sin x\\ -\sin x & \cos x\end{vmatrix}=1$

$u_1' = -\sin x \sec^2 x = -\tan x \sec x, \quad u_1 = -\sec x; \quad u_2' = \cos x \sec^2 x = \sec x;$
$u_2 = \ln|\sec x + \tan x|$
$y_p = -\cos x \sec x + \sin x \ln|\sec x + \tan x| = -1 + \sin x \ln|\sec x + \tan x|$
$y = C_1 \cos x + C_2 \sin x - 1 + \sin x \ln|\sec x + \tan x|$

19. $m^2 - 1 = 0 \Longrightarrow m = -1, \ 1; \ \ y_c = C_1 e^{-x} + C_2 e^x; \ \ W = \begin{vmatrix} e^{-x} & e^x \\ -e^{-x} & e^x \end{vmatrix} = 2$

$$u_1' = \frac{1}{2} e^x \cosh x = -\frac{1}{4}(e^{2x} + 1), \quad u_1 = -\frac{1}{8} e^{2x} - \frac{1}{4} x;$$
$$u_2' = \frac{1}{2} e^{-x} \cosh x = \frac{1}{4}(1 + e^{-2x}) \quad u_2 = \frac{1}{4} x - \frac{1}{8} e^{-2x},$$
$$y_p = e^{-x}(-\frac{1}{8} e^{2x} - \frac{1}{4} x) + e^x(\frac{1}{4} x - \frac{1}{8} e^{-2x} = -\frac{1}{8} e^x - \frac{1}{4} x e^{-x} + \frac{1}{4} x e^x - \frac{1}{8} d^{-x}$$
$$= -\frac{1}{8} e^x - \frac{1}{8} e^{-x} + \frac{1}{2} x \sinh x$$
$$y = C_1 e^{-x} + C_2 e^x - \frac{1}{8} e^x - \frac{1}{8} e^{-x} + \frac{1}{2} x \sinh x = C_3 e^{-x} + C_4 e^x + \frac{1}{2} x \sinh x$$

20. $m^2 - 1 = 0 \Longrightarrow m = -1, \ 1; \ \ y_c = C_1 e^{-x} + C_2 e^x; \ \ W = \begin{vmatrix} e^{-x} & e^x \\ -e^{-x} & e^x \end{vmatrix} = 2$

$$u_1' = -\frac{1}{2} e^x \sinh 2x = -\frac{1}{4}(e^{3x} - e^{-x}), \quad u_1 = -\frac{1}{12} e^{3x} - \frac{1}{4} e^{-x}$$
$$u_2' = \frac{1}{2} d^{-x} \sinh 2x = \frac{1}{4}(e^x - e^{-3x}), \quad u_2 = \frac{1}{4} e^x + \frac{1}{12} e^{-3x}$$
$$y_p = e^{-x}(-\frac{1}{12} e^{3x} - \frac{1}{4} e^{-x}) + e^x(\frac{1}{4} e^x + \frac{1}{12} e^{-3x}) = -\frac{1}{12} e^{2x} - \frac{1}{4} e^{-2x} + \frac{1}{4} e^{2x} + \frac{1}{12} e^{-2x}$$
$$= \frac{1}{6} e^{2x} - \frac{1}{6} e^{-2x} = \frac{1}{3} \sinh 2x$$
$$y = C_1 e^{-x} + C_2 e^x + \frac{1}{3} \sinh 2x$$

21. $m^2 - 4 = 0 \Longrightarrow m = -2, \ 2; \ \ y_c = C_1 e^{-2x} + C_2 e^{2x}; \ \ W = \begin{vmatrix} e^{-2x} & e^{2x} \\ -2e^{-2x} & 2e^{2x} \end{vmatrix} = 4$

$$u_1' = -\frac{1}{4} e^{2x}(\frac{e^{2x}}{x}) = -\frac{1}{4} \frac{e^{4x}}{x}, \quad u_1 = -\frac{1}{4} \int_{x_0}^{x} \frac{e^{4t}}{t} dt; \quad u_2' = \frac{1}{4} e^{-2x} \left(\frac{e^{2x}}{x} = \frac{1}{4x} \right), \quad u_2 = \frac{1}{4} \ln|x|$$
$$y_p = -\frac{1}{4} e^{-2x} \int_{x_0}^{x} \frac{e^{4t}}{t} dt + \frac{1}{4} e^{2x} \ln|x|; \quad y = C_1 e^{-2x} + C_2 e^{2x} - \frac{1}{4} e^{2x} \int_{x_0}^{x} \frac{e^{4t}}{t} dt + \frac{1}{4} e^{2x} \ln|x|$$

22. $m^2 - 9 = 0 \Longrightarrow m = -3, \ 3; \ \ y_c = C_1 e^{-3x} + C_2 e^{3x}; \ \ W = \begin{vmatrix} e^{-3x} & e^{3x} \\ -3e^{-3x} & 3e^{3x} \end{vmatrix} = 6$

$$u_1' = -\frac{1}{6} e^{3x}(\frac{9x}{e^{3x}} = -\frac{3}{2} x, \quad u_1 = -\frac{3}{4} x^2; \quad u_2' = \frac{1}{6} e^{-3x}(\frac{9x}{e^{3x}}) = \frac{3}{2} x e^{-6x}$$

$$u_2 = \int \frac{3}{2} x e^{-6x} dx \quad \boxed{\text{Integration by parts}}$$
$$= -\frac{1}{4} x e^{-6x} - \frac{1}{24} e^{-6x}$$
$$y_p = -\frac{3}{4} x^2 e^{-3x} - \frac{1}{4} x e^{-3x} - \frac{1}{24} e^{-3x}; \quad y = C_3 e^{-3x} + C_2 e^{3x} - \frac{3}{4} x^2 e^{-3x} - \frac{1}{4} x e^{-3x}$$

23. $m^2+3m+2=0 \Longrightarrow (m+2)(m+1)=0 \Longrightarrow m=-2,\ \ -1;\ \ y_c=C_1e^{-2x}+C_2e^{-x}$

$$W=\begin{vmatrix} e^{-2x} & e^{-x} \\ -2e^{-2x} & -e^{-x} \end{vmatrix}=e^{-3x};\ \ u_1'=-\frac{1}{e^{-3x}}\frac{e^{-x}}{1+e^x}=-\frac{e^{2x}}{1+e^x}$$

$$u_1=-\int\frac{e^{2x}}{1+e^x}dx \quad \boxed{v=1+e^x,\ \ dv=e^xdx,\ \ e^x=v-1}$$

$$=-\int\frac{v-1}{v}dv=-v+\ln|v|=-1-e^x+\ln(1+e^x)$$

$$y_p=e^{-2x}[-1-e^x+\ln(1+e^x)]+e^{-x}\ln(1+e^x)=-e^{-2x}-e^{-x}+e^{-2x}\ln(1+e^x)$$
$$+e^{-x}\ln(1+e^x)$$

$$y=C_1e^{-2x}+C_2e^{-x}-e^{-2x}-e^{-x}+e^{-2x}\ln(1+e^x)+e^{-x}\ln(1+e^x)$$
$$=C_3e^{-2x}+C_4e^{-x}+e^{-2x}\ln(1+e^x)+e^{-x}\ln(1+e^x)$$

24. $m^2-3m+2=0 \Longrightarrow (m-1)(m-2)=0 \Longrightarrow m=1,\ \ 2;\ \ y_c=C_1e^x+C_2e^{2x};$

$$W=\begin{vmatrix} e^{x} & e^{2x} \\ e^{x} & 2e^{2x} \end{vmatrix}=e^{3x};\ \ u_1'=-\frac{1}{e^{3x}}e^{2x}\frac{e^{3x}}{1+e^x}=-\frac{e^{2x}}{1+e^x}$$

$$u_1=-\int\frac{e^{2x}}{1+e^x}dx \quad \boxed{v=1+e^x,\ \ dv=e^xdx,\ \ e^x=v-1}$$

$$=-\int\frac{v-1}{v}dv=-v+\ln|v|=-1-e^x+\ln(1+e^x)$$

$$u_2'=\frac{1}{e^{3x}}e^x\frac{e^{3x}}{1+e^x}=\frac{e^x}{1+e^x},\ \ u_2=\ln(1+e^x)$$

$$y_p=e^x[-1-e^x+\ln(1+e^x)]+e^{2x}\ln(1+e^x)=-e^x-e^{2x}+e^x\ln(1+e^x)+e^{2x}\ln(1+e^x)$$

$$y=C_1e^x+C_2e^{2x}-e^x-e^{2x}+e^x\ln(1+e^x)+e^{2x}\ln(1+e^x)$$
$$=C_3e^x+C_4e^{2x}+e^x\ln(1+e^x)+e^{2x}\ln(1+e^x)$$

25. $m^2+3m+2=0 \Longrightarrow (m+2)(m+1)=0 \Longrightarrow m=-2,\ \ -1;\ \ y_c=C_1e^{-2x}+C_2e^{-x}$

$$W=\begin{vmatrix} e^{-2x} & e^{-x} \\ -2e^{-2x} & -e^{-x} \end{vmatrix}=e^{-3x};\ \ u_1'=-\frac{1}{e^{-3x}}e^{-x}\sin e^x=-e^{2x}\sin e^x$$

$$u_1=-\int e^{2x}\sin e^x dx \quad \boxed{\text{Integration by parts}}$$

$$=e^x\cos e^x-\sin e^x$$

$$u_2'=\frac{1}{e^{-3x}}e^{-2x}\sin e^x=e^x\sin e^x,\ \ u_2=-\cos e^x$$

$$y_p=e^{-2x}(e^x\cos e^x-\sin e^x)+e^{-x}(-\cos e^x)=-e^{-2x}\sin e^x;\ \ y=C_1e^{-2x}+C_2e^{-x}-e^{-2x}\sin e^x$$

26. $m^2-2m+1=0 \Longrightarrow (m-1)^2=0 \Longrightarrow m=1,\ \ 1;\ \ y_c=C_1e^x+C_2xe^x;$

$$W=\begin{vmatrix} e^{x} & xe^{x} \\ e^{x} & xe^x+e^{x} \end{vmatrix}=e^{2x};\ \ u_1'=-\frac{1}{e^{2x}}xe^xe^x\tan^{-1}x=-x\tan^{-1}x$$

$$\begin{aligned}
u_1 &= -\int x\tan^{-1} x dx \quad \boxed{\text{Integration by parts}}\\
&= -\frac{1}{2}x^2\tan^{-1} x - \frac{1}{2}\tan^{-1} x + \frac{1}{2}x\\
u_2' &= \frac{1}{e^{2x}}e^x e^x \tan^{-1} x = \tan^{-1} x,\ \ u_2 = x\tan^{-1} x - \frac{1}{2}\ln(1+x^2)\\
y_p &= e^x\left[-\frac{1}{2}x^2\tan^{-1} x - \frac{1}{2}\tan^{-1} x + \frac{1}{2}x\right] + xe^x\left[x\tan^{-1} x - \frac{1}{2}\ln(1+x^2)\right]\\
&= \frac{1}{2}x^2e^x\tan^{-1} x - \frac{1}{2}e^x\tan^{-1} x + \frac{1}{2}xe^x - \frac{1}{2}xe^x\ln(1+x^2)\\
y &= C_1e^x + C_3xe^x + \frac{1}{2}x^2e^x\tan^{-1} x - \frac{1}{2}e^x\tan^{-1} x - \frac{1}{2}xe^x\ln(1+x^2)
\end{aligned}$$

27. $m^2 - 2m + 1 = 0 \Longrightarrow (m-1)^2 = 0 \Longrightarrow m = 1,\ \ 1;\ \ y_c = C_1e^x + C_2xe^x;$

$$W = \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix} = e^{2x};\ \ u_1' = -\frac{1}{e^{2x}}xe^x\frac{e^x}{1+x^2} = -\frac{x}{1+x^2},\ \ u_1 = -\frac{1}{2}\ln(1+x^2);$$

$$u_2' = \frac{1}{e^{2x}}e^x\frac{e^x}{1+x^2} = \frac{1}{1+x^2},\ \ u_2 = \tan^{-1} x$$

$$y_p = -\frac{1}{2}e^x\ln(1+x^2) + xe^x\tan^{-1} x;\ \ y = C_1e^x + C_2xe^x - \frac{1}{2}e^x\ln(1+x^2) + xe^x\tan^{-1} x$$

28. $m^2 - 2m + 2 = 0 \Longrightarrow m = 1 \pm i;\ \ y_c = e^x(C_1\cos x + C_2\sin x);$

$$W = \begin{vmatrix} e^x\cos x & e^x\sin x \\ -e^x\sin x + e^x\cos x & e^x\cos x + e^x\sin x \end{vmatrix} = e^{2x}$$

$$u_1' = -\frac{1}{e^{2x}}e^x\sin x\, e^x\sec x = -\tan x,\ \ u_1 = \ln|\cos x|;\ \ u_2' = \frac{1}{e^{2x}}e^x\sec x = 1, u_2 = x$$

$$y_p = e^x\cos x\ln|\cos x| + xe^x\sin x;\ \ y = e^x(C_1\cos x + C_2\sin x) + e^x\cos x\ln|\cos x| + xe^x\sin x$$

29. $m^2 + 2m + 1 = 0 \Longrightarrow (m+1)^2 = 0 \Longrightarrow m = -1,\ \ -1;\ \ y_c = C_1e^{-x} + C_2xe^{-x}$

$$W = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & -xe^{-x} + e^{-x} \end{vmatrix} = e^{-2x};\ \ u_1' = -\frac{1}{e^{-2x}}xe^{-x}e^{-x}\ln x = -x\ln x$$

$$\begin{aligned}
u_1 &= -\int x\ln x dx \quad \boxed{\text{Integration by parts}}\\
&= \frac{1}{4}x^2 - \frac{1}{2}x^2\ln x
\end{aligned}$$

$$u_2' = \frac{1}{e^{-2x}}e^{-x}e^{-x}\ln x = \ln x,\ \ u_2 = x\ln x - x$$

$$y_p = e^{-x}\left(\frac{1}{4}x^2 - \frac{1}{2}x^2\ln x\right) + xe^{-x}(x\ln x - x) = \frac{1}{2}x^2e^{-x}\ln x - \frac{3}{4}x^2e^{-x}$$

$$y = C_1e^{-x} + C_2xe^{-x} + \frac{1}{2}x^2e^{-x}\ln x - \frac{3}{4}x^2e^{-x}$$

30. $m^2 + 10m + 25 = 0 \Longrightarrow (m+5)^2 = 0 \Longrightarrow m = -5,\ \ -5;\ \ y_c = C_1e^{-5x} + C_2xe^{-5x}$

$$W = \begin{vmatrix} e^{-5x} & xe^{-5x} \\ -5e^{-5x} & -5xe^{-5x} + e^{-5x} \end{vmatrix} = e^{-10x};\ \ u_1' = \frac{1}{e^{-10x}}xe^{-5x}\frac{e^{-10x}}{x^2} = -\frac{e^{-5x}}{x}$$

$$u_1 = -\int \frac{e^{-5x}}{x}dx = -\int_{x_0}^{x}\frac{e^{-5t}}{t}dt;\ \ u_2' = \frac{1}{e^{-10x}}e^{-5x}\frac{e^{-10x}}{x^2} = \frac{e^{-5x}}{x^2},$$

$$u_2 = \int \frac{e^{-5x}}{x^2}dx = \int_{x_0}^{x}\frac{e^{-5t}}{t^2}dt;\ \ y_p = -e^{-5x}\int_{x_0}^{x}\frac{e^{-5x}}{t}dt + xe^{-5x}\int_{x_0}^{x}\frac{e^{-5t}}{t^2}dt$$

$$y = C_1e^{-5x} + C_2xe^{-5x} - e^{-5x}\int_{x_0}^{x}\frac{e^{-5t}}{t}dt + xe^{-5x}\int_{x_0}^{x}\frac{e^{-5t}}{t^2}dt$$

31. $4m^2 - 4m + 1 = 0 \Longrightarrow (2m-1)^2 = 0 \Longrightarrow m = 1/2,\ \ 1/2;\ \ y_c = C_1e^{x/2} + C_2xe^{x/2}$

$W = \begin{vmatrix} e^{x/2} & xe^{x/2} \\ \frac{1}{2}e^{x/2} & \frac{1}{2}xe^{x/2} + e^{x/2} \end{vmatrix} = e^x;\ \ u_1' = -\frac{1}{e^x}xe^{x/2}(2e^{-x} + \frac{1}{4}x) = -2xe^{-3x/2} - \frac{1}{4}x^2e^{-x/2}$

$u_1 = -2\int xe^{-3x/2}dx - \frac{1}{4}\int x^2e^{-x/2}dx \quad \boxed{\text{Integration by parts}}$

$= \frac{4}{3}xe^{-3x/2} + \frac{8}{9}e^{-3x/2} + \frac{1}{2}x^2e^{-x/2} + 2xe^{-x/2} + 4e^{-x/2}$

$u_2' = \frac{1}{e^x}e^{x/2}(2e^{-x} + \frac{1}{4}x) = 2e^{-3x/2} - \frac{1}{4}xe^{-x/2}$

$u_2 = 2\int e^{-3x/2}dx + \frac{1}{4}\int xe^{-x/2}dx \quad \boxed{\text{Integration by parts}}$

$= -\frac{4}{3}e^{-3x/2} - \frac{1}{2}xe^{-x/2} - e^{-x/2}$

$y_p = e^{x/2}(\frac{4}{3}xe^{-3x/2} + \frac{8}{9}e^{-3x/2} + \frac{1}{2}x^2e^{-x/2} + 2xe^{-x/2} + 4e^{-x/2})$

$+ xe^{x/2}(-\frac{4}{3}e^{-3x/2} - \frac{1}{2}xe^{-x/2} - e^{-x/2}) = \frac{8}{9}e^{-x} + x + 4$

$y = C_1e^{x/2} + C_2xe^{x/2} + \frac{8}{9}e^{-x} + x + 4$

32. $4m^2 - 4m + 1 = 0 \Longrightarrow (2m-1)^2 = 0 \Longrightarrow m = 1/2,\ \ 1/2;\ \ y_c = C_1e^{x/2} + C_2xe^{x/2}$

$W = \begin{vmatrix} e^{x/2} & xe^{x/2} \\ \frac{1}{2}e^{x/2} & \frac{1}{2}xe^{x/2} + e^{x/2} \end{vmatrix} = e^x;\quad u_1' = -\frac{1}{e^x}xe^{x/2}\frac{e^{x/2}}{4}\sqrt{1-x^2} = -\frac{x\sqrt{1-x^2}}{4}u_1' =$
$\frac{1}{12}(1-x^2)^{3/2};\ \ u_2' = \frac{1}{e^x}e^{x/2}\frac{e^{x/2}}{4}\sqrt{1-x^2} = \frac{\sqrt{1-x^2}}{4}$

$u_2 = \frac{1}{4}\int\sqrt{1-x^2}dx \quad \boxed{\text{Trig substitution}}$

$= \frac{1}{8}\sin^{-1}x + \frac{1}{8}x\sqrt{1-x^2}$

$y_p = \frac{1}{12}e^{x/2}(1-x^2)^{3/2} + \frac{1}{8}xe^{x/2}\sin^{-1}x + \frac{1}{8}x^2e^{x/2}\sqrt{1-x^2}$

$y = C_1e^{x/2} + C_2xe^{x/2} + \frac{1}{12}e^{x/2}(1-x^2)^{3/2} + \frac{1}{8}xe^{x/2}\sin^{-1}x + \frac{1}{8}x^2e^{x/2}\sqrt{1-x^2}$

33. $m^2 - 1 = 0 \Longrightarrow m = -1,\ \ 1;\ \ y_c = C_1e^{-x} + C_2e^x;\ \ W = \begin{vmatrix} e^{-x} & e^x \\ -e^{-x} & e^x \end{vmatrix} = 2$

$u_1' = \frac{1}{2}e^xxe^x = -\frac{1}{2}xe^{2x}$

$u_1 = -\frac{1}{2}\int xe^{2x}dx \quad \boxed{\text{Integration by parts}}$

$= \frac{1}{8}e^{2x} - \frac{1}{4}xe^{2x}$

$u_2' = \frac{1}{2}e^{-x}xe^x = \frac{1}{2}x,\ \ u_2 = \frac{1}{4}x^2;\ \ y_p = e^{-x}(\frac{1}{8}e^{2x} - \frac{1}{4}xe^{2x}) + e^x(\frac{1}{4}x^2) = \frac{1}{8}e^x - \frac{1}{4}xe^x + \frac{1}{4}x^2e^x$

$y = C_1e^{-x} + C_3e^{x} - \frac{1}{4}xe^{x} + \frac{1}{4}x^2e^{x}$; $y' = -C_1e^{-x} + C_3e^{x} - \frac{1}{4}e^{x} + \frac{1}{4}xe^{x} + \frac{1}{4}x^2e^{x}$

Using $y(0) = 1$ and $y'(0) = 0$ we have $C_1 + C_3 = 1$, $-C_1 + C_3 - \frac{1}{4} = 0$, or $C_1 = 3/8$ and $C_3 = 5/8$. Thus, $y = \frac{3}{8}e^{-x} + \frac{5}{8}e^{x} - \frac{1}{4}xe^{x} + \frac{1}{4}x^2e^{x}$.

34. $2m^2 + m - 1 = 0 \Longrightarrow (2m-1)(m+1) = 0 \Longrightarrow m = -1,\ 1/2$; $y_c = C_1e^{-x} + C_2e^{x/2}$

$W = \begin{vmatrix} e^{-x} & e^{x/2} \\ -e^{-x} & \frac{1}{2}e^{x/2} \end{vmatrix} = \frac{3}{2}e^{-x/2}$; $u_1' = -\frac{2}{3e^{-x/2}}e^{x/2}\frac{(x+1)}{2} = -\frac{1}{3}(xe^{x} + e^{x})$, $u_1 = -\frac{1}{3}xe^{x}$

$u_2' = \frac{2}{3e^{-x/2}}e^{-x}\frac{(x+1)}{2} = \frac{1}{3}e^{-x/2}(x+1)$

$u_2 = \frac{1}{3}\int e^{-x/2}(x+1)dx$ $\boxed{\text{Integration by parts}}$

$= -\frac{2}{3}xe^{-x/2} - 2e^{-x/2}$

$y_p = e^{-x}(-\frac{1}{3}xe^{x}) + e^{x/2}(-\frac{2}{3}xe^{-x/2} - 2e^{-x/2}) = -x - 2$

$y = C_1e^{-x} + C_2e^{x/2} - x - 2$; $y' = -C_1e^{-x} + \frac{1}{2}C_2e^{x/2} - 1$

Using $y(0) = 1$ and $y'(0) = 0$ we obtain $C_1 + C_2 - 2 = 1$, $-C_1 + \frac{1}{2}C_2 - 1 = 0$, or $C_1 = 1/3$ and $C_2 = 8/3$. Thus, $y = \frac{1}{3}e^{-x} + \frac{8}{3}e^{x/2} - x - 2$.

35. $y'' - \frac{1}{x}y' + \frac{1}{x^2}y = \frac{4}{x}\ln x$; $y_c = C_1x + C_2x\ln x$; $W = \begin{vmatrix} x & x\ln x \\ 1 & 1+\ln x \end{vmatrix} = x$

$u_1' = -\frac{1}{x}(x\ln x)(\frac{4}{x}\ln x) = -\frac{4}{x}(\ln x)^2$, $u_1 = -\frac{4}{3}(\ln x)^3$; $u_2' = \frac{1}{x}(x)(\frac{4}{x}\ln x) = \frac{4}{x}\ln x$,

$u_2 = 2(\ln x)^2$; $y_p = -\frac{4}{3}x(\ln x)^3 + 2x(\ln x)^3 = \frac{2}{3}x(\ln x)^3$; $y + C_1x + C_2x\ln x + \frac{2}{3}x(\ln x)^3$

36. $y'' - \frac{4}{x}y' + \frac{6}{x^2}y = \frac{1}{x^3}$; $y_c = C_1x^2 + C_2x^3$; $W = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = x^4$;

$u_1' = -\frac{1}{x^4}(x^3)\left(\frac{1}{x^3}\right) = -\frac{1}{x^4}$; $u_1 = \frac{1}{3x^3}$; $u_2' = \frac{1}{x^4}(x^2)\left(\frac{1}{x^3}\right) = \frac{1}{x^5}$, $u_2 = -\frac{1}{4x^4}$;

$y_p = x^2\left(\frac{1}{3x^3}\right) + x^3\left(-\frac{1}{4x^4}\right) = \frac{1}{12x}$

$y = C_1x^2 + C_2x^3 + \frac{1}{12x}$

37. Writing the differential equation in the form $d^2C/dx^2 - (1/\lambda^2)C = -C(\infty)/\lambda^2$ we see that the auxiliary equation is $m^2 - 1/\lambda^2 = 0$. Thus, $C_c = c_1e^{x/\lambda} + c_2e^{-x/\lambda}$. Using undetermined coefficients with $C_p = A$ we find that $A = C(\infty)$. Then $C(x) = c_1ex/\lambda + c_2e^{-x/\lambda} + C(\infty)$. Since $C(0) = c_1 + c_2 + C(\infty) = 0$ and $\lim_{x\to\infty} C(x) = C(\infty)$ we see that $c_1 = 0$ and $c_2 = -C(\infty)$. Thus, $C(x) = C(\infty)(1 - e^{-x/\lambda})$.

38. If y_c is the complementary function and y_p is a particular solution, we have

$$ay_c'' + by_c' + cy_c = 0 \quad \text{and} \quad ay_p'' + by_p' + cy_p = g(x).$$

Therefore, letting $y = y_c + y_p$, we have

$$\begin{aligned} ay'' = by' + cy' &= a(y_c + y_p)'' + b(y_c + y_p)' + c(y_c + y_p) \\ &= ay_c'' + ay_p'' + by_c' + by_p' + cy_c + cy_p \\ &= [ay_c'' + by_c' + cy_c] + [ay_p'' + by_p' + cy_p] \\ &= ay_p'' + by_p' + cy_p = g(x) \end{aligned}$$

39. (a) Substituting Ae^x in for y in the DE, we have $Ae^x + 2Ae^x - 3Ae^x = 10e^x$ or $0 = 10e^x$, which is a contradiction for any value of A.

    (b) Substituting Axe^x for y, we have

    $$A(x+2)e^x + A(2x+2)e^x - 3Axe^x = 10e^x.$$

    Equating coefficients of xe^x and coefficients of e^x, we get

    $$A + 2A - 3A = 0 \quad \text{and} \quad 2A + 2A = 10$$

    which gives $A = \dfrac{5}{2}$. Therefore, $y_p = \dfrac{5}{2}xe^x$.

    (c) The auxiliary equation is $m^2 + 2m - 3 = 0$, so $m = -3$ or $m = 1$. This gives $y_c = C_1e^{-3x} + C_2e^x$. Therefore, the general solution is

    $$y = y_c + y_p = C_1e^{-3x} + C_2e^x + \frac{5}{2}xe^x$$

40. The auxiliary equation is $m^2 - 1 = 0$, so $m = \pm 1$. This gives $y_c = C_1e^{-x} + C_2e^x$. We look for a particular solution of the form $y_p = Axe^x + B(x-2)e^{-x} - Axe^x - Bxe^x = e^{-x} - e^x$. Equating coefficients of xe^x, e^x, xe^{-x}, and e^{-x}, we get $A - A = 0$, $2A = -1$, $B - B = 0$, $-2B = 1$, which gives $A = -\dfrac{1}{2}$, $B = -\dfrac{1}{2}$. Therefore, $y_p = -\dfrac{1}{2}xe^x - \dfrac{1}{2}xe^{-x}$ and the general solution is

    $$y = y_c + y_p = C_1e^{-x} + C_2e^x - \frac{1}{2}xe^x - \frac{1}{2}xe^{-x}$$

16.4 Mathematical Models

1. A weight of 4 pounds is pushed up 3 feet above the equilibrium position. At $t = 0$ it is given an initial speed upward of 2 feet per second.

2. A mass of 2 pounds is pulled down 0.7 feet below the equilibrium position and held. At $t = 0$ it is released from rest.

3. Using $m = W/g = 8/32 = 1/4$, the initial value problem is $\dfrac{1}{4}x'' + x = 0$; $x(0) = \dfrac{1}{2}$, $x'(0) = \dfrac{3}{2}$. The auxiliary equation is $\dfrac{1}{4}m^2 + 1 = 0$, so $m = \pm 2i$ and $x = C_1 \cos 2t + C_2 \sin 2t$, $x' = -2C_1 \sin 2t + 2C_2 \cos 2t$. Using the initial condition, we obtain $C_1 = 1/2$ and $C_2 = \dfrac{3}{4}$. The equation of motion is $x(t) = \dfrac{1}{2}\cos 2t + \dfrac{3}{4}\sin 2t$.

4. From Hooke's law we have $24 = k(1/3)$, so $k = 72$. Using $m = W/g = 24/32 = 3/4$, the initial value problem is $\frac{3}{4}x'' + 72x = 0$; $x(0) = -3$, $x'(0) = 0$. The auxiliary equation is $\frac{3}{4}m^2 + 72 = 0$, so $m = \pm 4\sqrt{6}i$ and $x = C_1 \cos 4\sqrt{6}t + C_2 \sin 4\sqrt{6}t$, $x' = -4\sqrt{6}C_1 \sin 4\sqrt{6} + 4\sqrt{6}C_2 \cos 4\sqrt{6}t$. Using the initial conditions, we obtain $C_1 = -1/4$ and $C_2 = 0$. Thus, $x(t) = -\frac{1}{4}\cos 4\sqrt{6}t$.

5. From Hooke's law we have $400 = k(2)$, so $k = 200$. The initial value problem is $50x'' + 200x = 0$; $x(0) = 0$, $x'(0) = -1 =$. The auxiliary equation is $50m^2 + 200 = 0$, so $m = \pm 2i$ and $x = C_1 \cos 2x + C_2 \sin 2x$, $x' = -2C_1 \sin 2x + 2C_2 \cos 2x$. Using the initial conditions, we obtain $C_1 = 0$ and $C_2 = -5$. Thus, $x(1) = -5 \sin 2x$.

6. Using $m = W/g = 2/32 = 1/16$, the initial value problem is $\frac{1}{16}x'' + 4x = 0$; $x(0) = \frac{2}{3}$, $x'(0) = -\frac{4}{3}$. The auxiliary equation is $m^2/16 + 4 = 0$, so $m = \pm 8i$ and $x = C_1 \cos 8x + C_2 \sin 8x$, $x' = -8C_1 \sin 8x + 8C_2 \cos 8x$. Using the initial conditions, we obtain $C_1 = 2/3$ and $C_2 = -1/6$. Thus, $x(t) = \frac{2}{3}\cos 8t - \frac{1}{6}\sin 8t$.

7. A 2 pound weight is released from the equilibrium position with an upward speed of 1.5 ft/s. A damping force numerically equal to twice the instantaneous velocity acts on the system.

8. A 16 pound weight is released from 2 feet above the equilibrium position with a downward speed of 1 ft/s. A damping force numerically equal to the instantaneous velocity acts on the system.

9. Using $m = W/g = 4/32 = 1/8$, the initial value problem is $\frac{1}{8}x'' + x' + 2x = 0$; $x(0) = -1$, $x'(0) = 8$. The auxiliary equation is $m^2/8 + m + 2 = 0$ or $(m+4)^2 = 0$, so $m = -4$, -4 and $x = C_1e^{-4t} + C_2te^{-4t}$, $x' = (C_2 - 4C_1)e^{-4t} - 4C_2te^{-4t}$. Using the initial conditions, we obtain $C_1 = -1$ and $C_2 = 4$. Thus, $x(t) = -e^{-4t} + 4te^{-4t}$. Solving $x(t) = -e^{4t} + 4te^{-4t} = 0$, we see that the weight passes through the equilibrium position at $t = 1/4s$. To find the maximum displacement we solve $x'(t) = 8e^{-4t} - 16te^{-4t} = 0$. This gives $t = 1/2$. Since $x(1/2) = e^{-2} \approx 0.14$, the maximum displacement is approximately 0.14 feet below the equilibrium position at $t = 1/2s$.

10. From Hooke's law we have $40(980) = k(10)$, so $k = 3920$. The initial value problem is $40x'' + 560x' + 3920x = 0$; $x(0) = 0$, $x'(0) - 2$. The auxiliary equation is $40m^2 + 560m + 3920 = 0$ or $m^2 + 14m + 98 = 0$, so $m = -7 \pm 7i$ and $x = e^{-7t}(C_1 \cos 7t + C_2 \sin 7t)$,
$x' = -7(C_1 + C_2)e^{-7t} \sin 7t - 7(C_1 - C_2)e^{-7t} \cos 7t$. Using the initial conditions, we obtain $C_1 = 0$ and $C_2 = 2/7$. Thus, $x(t) = \frac{2}{7}e^{-7t} \sin 7t$.

11. From Hooke's law we have $10 = k(7 - 5)$, so $k = 5$. Using $m = W/g = 8/32 = 1/4$, the initial value problem is $\frac{1}{4}x'' + x' + 5x = 0$; $x(0) = \frac{1}{2}$; $x'(0) = 1$. The auxiliary equation is $m^2/4 + m + 5 = 0$ or $m^2 + 4m + 20 = 0$, so $m = -2 \pm 4i$. Thus, $x = e^{-2t}(C_1 \cos 4t + C_2 \sin 4t)$ and $x' = -2(C_1 - 2C_2)e^{-2t} \cos 4t - 2(2C_1 + C_2)e^{-2t} \sin 4t$. Using the initial conditions, we obtain $\frac{1}{2} = C_1$ and $1 = -2(\frac{1}{2} - 2C_2)$, so $C_1 = 1/2$ and $C_2 = 1/2$. Therefore $x(t) = \frac{1}{2}e^{-2t}(\cos 4t + \sin 4t)$.

12. From Hooke's law we have $24 = k(4)$, so $k = 6$. Using $m = W/g = 24/32 = 3/4$, the initial value problem is $\frac{3}{4}x'' + \beta x' + 6x = 0$; $x(0) = 0$, $x'(0) = -2$. The auxiliary equation is $\frac{3}{4}m^2 + \beta m + 6 = 0$. Using the quadratic formula, $m = (-\beta \pm \sqrt{\beta^2 - 18}/(3/2)$. When $\beta > \sqrt{18} = 3\sqrt{2}$, we have $m_1 = -\frac{2}{3}\beta + \frac{2}{3}\sqrt{\beta^2 - 18}$ and $m_2 = -\frac{2}{3}\beta - \frac{2}{3}\sqrt{\beta^2 - 18}$. Thus,

$$\begin{aligned} x(t) &= C_1 e^{-2\beta t/3 + 2t\sqrt{\beta^2-18}/3} + C_2 e^{-2\beta t/3 - 2t\sqrt{\beta^2-18}/3} \\ &= e^{-2\beta t/3}\left[C_3 \cosh\left(\frac{2}{3}\sqrt{\beta^2 - 18}\right) t + C_4 \sinh\left(\frac{2}{3}\sqrt{\beta^2-18}\right) t\right] \end{aligned}$$

see Example 5 in Section 16.2.
From $x(0) = 0$ we obtain $C_3 = 0$ so that $x(t) = C_4 e^{-2\beta t/3}\sinh(\frac{2}{3}\sqrt{\beta^2 - 18t})$. The velocity is

$$x'(t) = \frac{2}{3}\sqrt{\beta^2 - 18}C_4 e^{-2\beta t/3}\cosh(\frac{2}{3}\sqrt{\beta^2-18t}) - \frac{2\beta}{3}C_4 e^{-2\beta t/3}\sinh(\frac{2}{3}\sqrt{\beta^2-18t}).$$

From $x'(0) = -2$ we obtain $-2 = \frac{2}{3}\sqrt{\beta^2 - 18}C_4$ or $C_4 = -3/\sqrt{\beta^2 - 18}$. Therefore,

$$x(t) = \frac{-3}{\sqrt{\beta^2-18}}e^{-2\beta t/3}\sinh(\frac{2}{3}\sqrt{\beta^2 - 18t}).$$

13. From Hooke's law we have $10 = k(2)$, so $k = 5$. Using $m = W/g = 10/32 = 5/16$, the differential equation is $\frac{5}{16}x'' + \beta x' + 5 = 0$. The auxiliary is $\frac{5}{16}m^2 + \beta m + 5 = 0$ Using the quadratic formula, $m = (-\beta \pm \sqrt{\beta^2 - 25/4})/(5/8)$. For $\beta > 0$ the motion is

(a) overdamped when $\beta^2 - 25/4 > 0$ or $\beta > 5/2$

(b) critically damped when $\beta^2 - 25/4 = 0$ or $\beta = 5/2$

(c) underdamped when $\beta^2 - 25/4 < 0$ or $\beta < 5/2$.

14. Since $W = mg = 1(32) = 32$, we have from Hooke's law $32 = k(2)$, so $k = 16$. The initial value problem is $x'' + 8x' + 16x = 8\sin 4t$; $x(0) = x'(0) = 0$. The auxiliary equation is $m^2 + 8m + 16 = (m+4)^2 = 0$ so $m = -4, \quad -4,$ and $x_c = C_1 e^{-4t} + C_2 t e^{-4t}$. Using $x_p = A\sin 4t + B\cos 4t$ we find $A = 0$ and $B = -1/4$. Thus,

$$x(t) = C_1 e^{-4t} + C_2 t e^{-4t} - \frac{1}{4}\cos 4t \text{ and } x'(t) = -4C_1 e^{-4t} - 4C_2 t e^{-4t} + C_2 e^{-4t} + \sin 4t.$$

Using the initial conditions, we obtain $0 = C_1 - \frac{1}{4}$ and $0 = -4C_1 + C_2$. Thus, $C_1 = 1/4$ and $C_2 = 4C_1 = 1$. Therefore $x(t) = \frac{1}{4}e^{-4t} + te^{-4t} - \frac{1}{4}\cos 4t$.

15. The initial value problem is $x'' + 8x' + 16x = e^{-t}\sin 4t$; $x(0) = x'(0) = 0$. Using $x_p = Ae^{-t}\sin 4t + Be^{-t}\cos 4t$ we find $A = -7/625$ and $B = -24/625$. Thus,

$$x(t) = C_1 e^{-4t} + C_2 te^{-4t} - \frac{7}{625}e^{-t}\sin 4t - \frac{24}{625}e^{-t}\cos 4t,$$

$$x'(t) = -4C_1 e^{-4t} - 4C_2 te^{-4t} + C_2 e^{-4t} - \frac{28}{625}e^{-t}\cos 4t + \frac{7}{625}e^{-t}\sin 4t + \frac{96}{625}e^{-t}\sin 4t + \frac{24}{625}e^{-t}\cos 4t.$$

Using the initial conditions, we obtain $C_1 = 24/625$ and $C_2 = 100/625$. Thus,

$$x(t) = \frac{24}{625}e^{-4t} + \frac{100}{625}te^{-4t} - \frac{7}{625}e^{-t}\sin 4t - \frac{24}{626}e^{-t}\cos 4t.$$

As $t \longrightarrow \infty$, $e^{-t} \longrightarrow 0$ and $x(t) \longrightarrow 0$.

16. A 32 pound weight is pulled 2 feet below the equilibrium position and held. At time $t = 0$ an external force equal to $5\sin 3t$ is applied to the system. The auxiliary equation is $m^2 + 9 = 0$, so $m = \pm 3i$ and $x_c = C_1\cos 3t + C_2 \sin 3t$. Using variation of parameters $x_p = -\frac{5}{6}t\cos 3t + \frac{5}{18}\sin 3t$, so

$$x(t) = C_1\cos 3t + C_3\sin 3t - \frac{5}{6}t\cos 3t$$

$$x'(t) = -3C_1\sin 3t + 3C_2\cos 3t + \frac{5}{2}t\sin 3t - \frac{5}{6}\cos 3t.$$

Using the initial conditions, we obtain $C_1 = 2$ and $C_2 = 5/18$. Thus, $x(t) = 2\cos 3t + \frac{5}{18}\sin 3t - \frac{5}{6}t\cos 3t$. The spring-mass system is in pure resonance.

17. The DE describing charge is $.05q'' + 2q' + 100q = 0$. The auxiliary equation is $0.5m^2 + 2m + 100 = 0$, so $m = -20 \pm 40i$. The general solution is $q = e^{-20t}(C_1\cos 40t + C_2\sin 40t)$. The initial conditions yield $q(0) = C_1 = 5$ and $i(0) = q'(0) = -20C_1 + 40C_2 = 0$, which gives $C_1 = 5$ and $C_2 = \frac{5}{2}$. Therefore $q(t) = e^{-20t}\left(5\cos 40t + \frac{5}{2}\sin 40t\right)$, and $q(0.01) = 4.568C$. $q(t) = 0$ when $5\cos 40t + \frac{5}{2}\sin 40t = 0$ which first occurs at $t = 0.0509$ s.

18. The DE describing charge is $\frac{1}{4}q'' + 20q' + 300q = 0$. The auxiliary equation is $\frac{1}{4}m^2 + 20m + 300 = 0$, so $m = -20$ or $m = -60$. The general solution is $q(t) = C_1e^{-20t} + C_2e^{-60t}$. The initial conditions yield $q(0) = C_1 + C_2 = 4$ and $i(0) = q'(0) = -20C_1 - 60C_2 = 0$, which give $C_1 = 6$ and $C_2 = -2$. Therefore, $q(t) = 6e^{-20t} - 2e^{-60t}$. The charge is never equal to zero.

19. The DE is $\frac{5}{3}q'' + 10q' + 30q = 300$. The auxiliary equation is $\frac{5}{3}m^2 + 10m + 30 = 0$. so $m = -3 \pm 3i$. This gives $q_c = e^{-3t}(C_1\cos 3t + C_2\sin 3t)$. Assume a particular solution of the form $q_p = A$. Substituting into the DE, we have $30A = 300$ so that $A = 10$ and therefore $q_p = 10$. Thus, the general solution is $q = q_c + q_p = e^{-3t}(C_1\cos 3t + C_2\sin 3t) + 10$. The initial conditions yield $q(0) = C_1 + 10 = 0$ and $i(0) = q'(0) = -3(C_1 - C_2) = 0$, which gives $C_1 = -10$ and $C_2 = -10$. Therefore, $q(t) = e^{-3t}(-10\cos 2t - 10\sin 3t) + 10 - 10 - 10e^{-3t}(\cos 3t + \sin 3t)$, $i(t) = q'(t) = 60e^{-3t}\sin 3t$. The charge $q(t)$ attains a maximum of 10.432 C at $t = \frac{\pi}{3}$.

20. The DE is $q''+100q'+2500q=30$. The auxiliary equation is $m^2+100m+250000$, so $m=-50$ is a repeated root. This gives $q_c = C_1e^{-50t}+C_2te^{-50t}$. Assume a particular solution of the form $q_p = A$. Substituting into the DE, we have $2500A = 30$ so that $A = \frac{3}{250}$ and therfore $q_p = \frac{3}{250}$. The general solution is $q = q_c+q_p = C_1e^{-50t}+C_2te^{-50t}+\frac{3}{250}$. The general solution is $q = q_c + q_p = C_1e^{-50t}+C_2te^{-50t}+\frac{3}{250}$. The initial conditions yield $q(0) = C_1 = 0$ and $i(0) = q'(0) = -50C_! + C_2 = 2$ which give $C_1 = 0$ and $C_2 = 2$. Therefore, $q(t) = 2te^{-50t}+\frac{3}{250}$ and $i(t) = q'(t) = (2-100t)e^{-50t}$. The charge $q(t)$ attains a maximum of 0.0267 C at $t = \frac{1}{50}$ s.

21.

16.5 Power Series Solutions

1. $$\underbrace{\sum_{n=2}^{\infty} n(n-1)c_nx^{n-2}}_{k=n-2}+\sum_{n=0}^{\infty} c_nx^n = \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k+\sum_{k=0}^{\infty}c_kx^k$$
$$= \sum_{k=0}^{\infty}[(k+2)(k+1)c_{k+2}+c_k]x^k = 0$$
$(k+2)(k+1)c_{k+2}+c_k = 0;\quad c_{k+2} = -\dfrac{c_k}{(k+2)(k+1)},\quad k=0,1,2,\ldots$
$c_2 = -\dfrac{c_0}{2} = -\dfrac{c_0}{2!},\quad c_3 = -\dfrac{c_1}{3\cdot 2} = -\dfrac{c_1}{3!},\quad c_4 = -\dfrac{c_2}{4\cos 3} = \dfrac{c_0}{4\cdot 3\cdot 2!} = \dfrac{c_0}{4!},$
$c_5 = -\dfrac{c_3}{5\cdot 4} = \dfrac{c_1}{5\cdot 4\cdot 3!} = \dfrac{c_1}{5!},\quad c_6 = -\dfrac{c_4}{6\cdot 5} = -\dfrac{c_0}{6\cdot 5\cdot 4!} = -\dfrac{c_0}{6!},$
$c_7 = -\dfrac{c_5}{7\cdot 6} = -\dfrac{c_1}{7\cdot 6\cdot 5!} = -\dfrac{c_1}{7!}$
$$y = c_0\left[1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+\cdots\right]+c_1\left[x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\frac{1}{7!}x^7+\cdots\right]$$
$$= c_0\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}x^{2n}+c_1\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}x^{2n+1}$$

2. $$\underbrace{\sum_{n=2}^{\infty} n(n-1)c_nx^{n-2}}_{k=n-2}-\sum_{n=0}^{\infty} c_nx^n = \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k-\sum_{k=0}^{\infty}c_kx^k$$
$$= \sum_{k=0}^{\infty}[(k+2)(k+1)c_{k+2}-c_k]x^k = 0$$
$(k+2)(k+1)c_{k+2}-c_k = 0;\quad c_{k+2} = \dfrac{c_k}{(k+2)(k+1)},\quad k=0,1,2,\ldots;\quad c_2 = \dfrac{c_0}{2!},$
$c_3 = \dfrac{c_1}{3\cdot 2} = \dfrac{c_1}{3!},\quad c_4 = \dfrac{c_2}{4\cdot 3} = \dfrac{c_0}{4!},\quad c_5 = \dfrac{c_3}{5\cdot 4} = \dfrac{c_1}{5!},\quad c_6 = \dfrac{c_4}{6\cdot 5} = \dfrac{c_0}{6!},\quad c_7 = \dfrac{c_5}{7\cdot 6} = \dfrac{c_1}{7!}$
$$y = c_0\left[1+\frac{1}{2!}x^2+\frac{1}{4!}x^4+\frac{1}{6!}x^6+\cdots\right]+c_1\left[x+\frac{1}{3!}x^3+\frac{1}{5!}x^5+\frac{1}{7!}x^7+\cdots\right]$$
$$= c_0\sum_{n=0}^{\infty}\frac{1}{(2n)!}x^{2n}+c_1\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}x^{2n+1}$$

3. $$\underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} - \underbrace{\sum_{n=1}^{\infty} nc_n x^{n-1}}_{k=n-1} = \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - \sum_{k=0}^{\infty}(k+1)c_{k+1}x^k$$
$$= \sum_{k=0}^{\infty}[(k+2)(k+1)c_{k+2} - (k+1)c_{k+1}]x^k = 0$$
$(k+2)(k+1)c_{k+2} - (k+1)c_{k+1} = 0;\;\; c_{k+2} = \dfrac{c_{k+1}}{(k+2)},\;\; k = 0, 1, 2, \ldots;\;\; c_2 = \dfrac{c_1}{2} = \dfrac{c_1}{2!},$
$c_3 = \dfrac{c_2}{3} = \dfrac{c_1}{3!},\;\; c_4 = \dfrac{c_3}{4} = \dfrac{c_1}{4!},$
$y = c_0 + c_1\left[x + \dfrac{1}{2!}x^2 + \dfrac{1}{3!}x^3 + \cdots\right] = c_0 + c_1\sum_{n=1}^{\infty}\dfrac{1}{n!}x^n$

4. $$\underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} - \underbrace{\sum_{n=1}^{\infty} nc_n x^{n-1}}_{k=n-1} = 2\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k + \sum_{k=0}^{\infty}(k+1)c_{k+1}x^k$$
$$= \sum_{k=0}^{\infty}[2(k+2)(k+1)c_{k+2} - (k+1)c_{k+1}]x^k = 0$$
$2(k+2)(k+1)c_{k+2} + (k+1)c_{k+1} = 0;\;\; c_{k+2} = -\dfrac{c_{k+1}}{2(k+2)},\;\; k = 0, 1, 2, \ldots;$
$c_2 = -\dfrac{c_1}{2\cdot 2} = -\dfrac{c_1}{2\cdot 2!},\;\; c_3 = -\dfrac{c_2}{2\cdot 3} = \dfrac{c_1}{2^2\cdot 3!},\;\; c_4 = -\dfrac{c_3}{2\cdot 4} = -\dfrac{c_1}{2^3\cdot 4!},$
$y = c_0 + c_1\left[x - \dfrac{1}{2\cdot 2!}x^2 + \dfrac{1}{2^2\cdot 3!}x^3 - \dfrac{1}{2^3\cdot 4!}x^4 + \cdots\right]$

5. $$\underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} - \underbrace{x\sum_{n=0}^{\infty} c_n x^{n}}_{k=n+1} = \sum_{k=1}^{\infty}(k+2)(k+1)c_{k+2}x^k - \sum_{k=1}^{\infty}c_{k-1}x^k$$
$$= 2c_2 + \sum_{k=0}^{\infty}[(k+2)(k+1)c_{k+2} - c_{k-1}]x^k = 0$$
$c_2 = 0;\;\; (k+2)(k+1)c_{k+2} - c_{k-1} = 0;\;\; c_{k+2} = \dfrac{c_{k-1}}{(k+2)(k+1)},\;\; k = 1, 2, 3, \ldots;$
$c_3 = \dfrac{c_0}{3\cdot 2},\;\; c_5 = \dfrac{c_2}{5\cdot 4} = 0,\;\; c_6 = \dfrac{c_3}{6\cdot 5} = \dfrac{c_0}{6\cdot 5\cdot 3\cdot 2},\;\; c_7 = \dfrac{c_4}{7\cdot 6} = \dfrac{c_1}{7\cdot 6\cdot 4\cdot 3}$
$c_9 = \dfrac{c_5}{8\cdot 7} = 0,\;\; c_9 = \dfrac{c_6}{9\cdot 8} = c_9 = \dfrac{c_0}{9\cdot 8\cdot 6\cdot 5\cdot 3\cdot 2},\;\; c_{10} = \dfrac{c_7}{10\cdot 9} = \dfrac{c_1}{10\cdot 9\cdot 7\cdot 6\cdot 4\cdot 3}$
$y = c_0\left[1 + \dfrac{1}{3\cdot 2}x^3 + \dfrac{1}{6\cdot 5\cdot 3\cdot 2}x^6 + \dfrac{1}{9\cdot 8\cdot 6\cdot 5\cdot 3\cdot 2}x^9 + \cdots\right]$
$\quad + c_1\left[x + \dfrac{1}{4\cdot 3}x^4 + \dfrac{1}{7\cdot 6\cdot 4\cdot 3}x^7 + \dfrac{1}{10\cdot 9\cdot 7\cdot 6\cdot 4\cdot 3}x^{10} + \cdots\right]$

6. $$\underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} + \underbrace{x^2 \sum_{n=0}^{\infty} c_n x^n}_{k=n+2} = \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k + \sum_{k=2}^{\infty} c_{k-2}x^k$$

$$= 2c_2 + +c_3 x + \sum_{k=2}^{\infty}[(k+2)(k+1)c_{k+2} + c_{k-2}]x^k = 0$$

$c_2 = c_3 = 0$; $(k+2)(k+1)c_{k+2} + c_{k-2} = 0$; $c_{k+2} = -\dfrac{c_{k-2}}{(k+2)(k+1)}$, $k = 2, 3, 4, \ldots$;

$c_4 = -\dfrac{c_0}{4 \cdot 3}$, $c_5 = -\dfrac{c_1}{5 \cdot 4}$, $c_6 = -\dfrac{c_2}{7 \cdot 6} = 0$, $c_7 = -\dfrac{c_3}{7 \cdot 6}$, $c_8 = -\dfrac{c_4}{8 \cdot 7} = \dfrac{c_0}{8 \cdot 7 \cdot 4 \cdot 3}$

$c_9 = -\dfrac{c_5}{9 \cdot 8} = \dfrac{c_1}{9 \cdot 8 \cdot 5 \cdot 4}$, $c_{10} = -\dfrac{c_6}{10 \cdot 9} = 0$, $c_{11} - \dfrac{c_7}{11 \cdot 10} = 0$,

$c_{12} = -\dfrac{c_8}{12 \cdot 11} = -\dfrac{c_0}{12 \cdot 11 \cdot 8 \cdot 7 \cdot 4 \cdot 3}$, $c_{13} = -\dfrac{c_9}{13 \cdot 12} = -\dfrac{c_1}{13 \cdot 12 \cdot 9 \cdot 8 \cdot 5 \cdot 4}$,

$y = c_0 \left[1 - \dfrac{1}{4 \cdot 3}x^4 + \dfrac{1}{8 \cdot 7 \cdot 4 \cdot 3}x^8 - \cdots\right] c_1 \left[x - \dfrac{1}{5 \cdot 4}x^5 + \dfrac{1}{9 \cdot 8 \cdot 5 \cdot 4}x^9 - \cdots\right]$

7. $$\underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} - 2x \sum_{n=1}^{\infty} c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n$$

$$= \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - \sum_{k=1}^{\infty} kc_k x^k + \sum_{k=0}^{\infty} c_k x^k$$

$$= c_0 + 2c_2 + \sum_{k=1}^{\infty}[(k+2)(k+1)c_{k+2} - (2k-1)c_k]x^k = 0$$

$c_0 + 2c_2 = 0$; $(k+2)(k+1)c_{k+2} - (2k-1)c_k = 0$; $c_2 = -\dfrac{c_0}{2}$

$c_{k+2} = \dfrac{(2k-1)c_k}{(k+2)(k+1)}$, $k = 1, 2, 3, \ldots$; $c_3 = \dfrac{c_1}{3 \cdot 2} = \dfrac{c_1}{3!}$, $c_4 = \dfrac{3c_2}{4 \cdot 3} = -\dfrac{3c_0}{4!}$,

$c_5 = \dfrac{5c_3}{5 \cdot 4} = \dfrac{5c_1}{5!}$, $c_6 = \dfrac{7c_4}{6 \cdot 5} = \dfrac{7 \cdot 3c_0}{6!}$, $c_7 = \dfrac{9c_5}{7 \cdot 6} = \dfrac{9 \cdot 5c_1}{7!}$

$y = c_0 \left[1 - \dfrac{1}{2!}x^2 - \dfrac{3}{4!}x^4 - \dfrac{7 \cdot 3}{6!}x^6 - \cdots\right] + c_1 \left[x + \dfrac{1}{3!}x^3 + \dfrac{5}{5!}x^5 + \dfrac{9 \cdot 5}{7!}x^7 + \cdots\right]$

8. $$\underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} - x \sum_{n=1}^{\infty} nc_n x^{n-1} + 2\sum_{n=0}^{\infty} c_n x^n$$

$$= \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - \sum_{k=1}^{\infty} kc_k x^k + 2\sum_{k=0}^{\infty} c_k x^k$$

$$= 2c_0 + 2c_2 + \sum_{k=1}^{\infty}[(k+2)(k+1)c_{k+2} - (k-2)c_k]x^k = 0$$

$2c_0 + 2c_2 = 0$; $(k+2)(k+1)c_{k+2} - (k-2)c_k = 0$; $c_2 = -c_0$

$c_{k+2} = \dfrac{(k-2)c_k}{(k+2)(k+1)}$, $k = 1, 2, 3, \ldots$; $c_3 = -\dfrac{c_1}{3 \cdot 2} = -\dfrac{c_1}{3!}$, $c_4 = 0$, $c_5 = \dfrac{c_3}{5 \cdot 4} = -\dfrac{c_1}{5!}$,

$c_6 = c_8 = c_{10} = 0,\quad c_7 = \dfrac{3c_5}{7\cdot 6} = -\dfrac{3c_1}{7!},\quad c_9 = \dfrac{5c_7}{9\cdot 8} = -\dfrac{5\cdot 3c_1}{9!}$

$y = c_0(1-x^2) + c_1\left[1 - \dfrac{1}{3!}x^3 - \dfrac{1}{5!}x^5 - \dfrac{3}{7!}x^7 - \dfrac{5\cdot 3}{9!}x^9 + \cdots\right]$

9. $\underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} + \underbrace{x^2\sum_{n=1}^{\infty} nc_n x^{n-1}}_{k=n+1} + \underbrace{x\sum_{n=0}^{\infty} c_n x^n}_{k=n+1}$

$$= \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k + \sum_{k=2}^{\infty}(k-1)c_{k-1}x^k + \sum_{k=1}^{\infty} c_{k-1}x^k$$

$$= 2c_2 + (6c_3 + c_0)x + \sum_{k=2}^{\infty}[(k+2)(k+1)c_{k+2} + kc_{k-1}]x^k = 0$$

$2c_2 = 0,\quad 6c_3 + c_0 = 0\quad (k+2)(k+1)c_{k+2} + kc_{k-1} = 0;\quad c_2 = 0,\quad c_3 = -\dfrac{c_0}{3\cdot 2}$

$c_{k+2} = -\dfrac{kc_{k-1}}{(k+2)(k+1)},\quad k = 2,3,4,\ldots;\quad c_4 = -\dfrac{2c_1}{4\cdot 3},\quad c_5 = 0,\quad c_6 = -\dfrac{4c_3}{6\cdot 5} = \dfrac{4c_0}{6\cdot 5\cdot 3\cdot 2}$

$c_7 = -\dfrac{5c_4}{7\cdot 6} = \dfrac{5\cdot 2c_1}{7\cdot 6\cdot 4\cdot 3},\quad c_8 = c_{11} = c_{14} = \cdots = 0,\quad c_9 = -\dfrac{7c_6}{9\cdot 8} = -\dfrac{7\cdot 4c_0}{9\cdot 8\cdot 6\cdot 5\cdot 3\cdot 2}$

$c_{10} = -\dfrac{8c_7}{10\cdot 9} = -\dfrac{8\cdot 5\cdot 2c_1}{10\cdot 9\cdot 7\cdot 6\cdot 4\cdot 3}$

$y = c_0\left[1 - \dfrac{1}{3!}x^3 + \dfrac{4^2}{6!}x^6 - \dfrac{7^2\cdot 4^2}{9!}x^9 \cdots\right] + c_1\left[x - \dfrac{2^2}{4}x^4 + \dfrac{5^2\cdot 2^2}{7!}x^7 - \dfrac{8^2\cdot 5^2\cdot 2^2}{10!}x^{10} + \cdots\right]$

10. $\underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} + 2x\sum_{n=1}^{\infty} nc_n x^{n-1} + 2\sum_{n=0}^{\infty} c_n x^n$

$$= \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k + 2\sum_{k=1}^{\infty} kc_k x^k + 2\sum_{k=0}^{\infty} c_k x^k$$

$$= 2c_2 + 2c_0 + \sum_{k=1}^{\infty}[(k+2)(k+1)c_{k+2} + 2(k+1)c_k]x^k = 0$$

$2c_0 + 2c_2 = 0;\quad (k+2)(k+1)c_{k+2} + 2(k+1)c_k = 0;\quad c_2 = -c_0$

$c_{k+2} = -\dfrac{c_k}{k+2},\quad k = 1,2,3,\ldots;\quad c_3 = -\dfrac{2c_1}{3},\quad c_4 = -\dfrac{2c_2}{4} = \dfrac{2c_0}{4},\quad c_5 = -\dfrac{2c_3}{5} = \dfrac{2^2c_1}{5\cdot 3},$

$c_6 = -\dfrac{2c_4}{6} = -\dfrac{2^2c_0}{6\cdot 4},\quad c_7 = -\dfrac{2c_5}{7} = -\dfrac{2^3c_1}{7\cdot 5\cdot 3},\quad c_8 = -\dfrac{2c_6}{8} = \dfrac{2^3c_0}{8\cdot 6\cdot 4}$

$y = c_0\left[1 - x^2 + \dfrac{2}{4}x^4 - \dfrac{2^2}{6\cdot 4}x^6 + \dfrac{2^3}{8\cdot 6\cdot 4}x^8 + \cdots\right] + c_1\left[x - \dfrac{2}{3}x^3 + \dfrac{2^2}{5\cdot 3}x^5 - \dfrac{2^3}{7\cdot 5\cdot 3}x^7 + \cdots\right]$

11. $$(x-1)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} nc_n x^{n-1}$$

$$= \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-1}}_{k=n-1} - \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} + \underbrace{\sum_{n=2}^{\infty} nc_n x^{n-1}}_{k=n-1}$$

$$= \sum_{k=1}^{\infty}(k+1)kc_{k+1}x^k - \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k + \sum_{k=0}^{\infty}(k+1)c_{k+1}x^k$$

$$= c_1 - 2c_2 + \sum_{k=1}^{\infty}[(k+1)kc_{k+1} - (k+2)(k+1)c_{k+2} + (k+1)c_{k+1}]x^k = 0$$

$c_1 - 2c_2 = 0;\;\; (k+1)kc_{k+1} - (k+2)(k+1)c_{k+2} + (k+1)c_{k+1} = 0;\;\; c_2 = \dfrac{c_1}{2}$

$c_{k+2} = \dfrac{(k+1)c_{k+1}}{k+2},\;\; k = 1, 2, 3, \ldots;\;\; c_3 = \dfrac{2c_2}{3} = \dfrac{c_1}{3},\;\; c_4 = \dfrac{3c_3}{4} = \dfrac{c_1}{4}$

$y = c_0 + c_1\left[x + \dfrac{1}{2}x^2 + \dfrac{1}{3}x^3 + \dfrac{1}{4}x^4 + \cdots\right] = c_0 + c_1\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}x^n$

12. $$(x+2)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x\sum_{n=1}^{\infty} nc_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n$$

$$= \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-1}}_{k=n-1} + 2\underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} + \sum_{n=1}^{\infty} nc_n x^n - \sum_{n=0}^{\infty} c_n x^n$$

$$= \sum_{k=1}^{\infty}(k+1)kc_{k+1}x^k + 2\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k + \sum_{k=1}^{\infty}kc_k x^k - \sum_{k=0}^{\infty}c_k x^k$$

$$= 4c_2 - c_0 + \sum_{k=1}^{\infty}[(k+1)kc_{k+1} + 2(k+2)(k+1)c_{k+2} + (k-1)c_k]x^k = 0$$

$4c_2 - c_0 = 0;\;\; (k+1)kc_{k+1} + 2(k+2)(k+1)c_{k+2} + (k-1)c_k = 0;\;\; c_2 = \dfrac{c_0}{4}$

$c_{k+2} = -\dfrac{(k+1)kc_{k+1} + (k-1)c_k}{2(k+2)(k+1)} = -\dfrac{kc_{k+1}}{2(k+2)} - \dfrac{(k-1)c_k}{2(k+2)(k+1)},\;\; k = 1, 2, 3, \ldots$

$c_3 = -\dfrac{c_2}{2\cdot 3} = -\dfrac{c_0}{2\cdot 3\cdot 4},\;\; c_4 = -\dfrac{2c_3}{2\cdot 4} - \dfrac{c_2}{2\cdot 4\cdot 3} = \dfrac{c_0}{2\cdot 3\cdot 4^2} - \dfrac{c_0}{2\cdot 3\cdot 4^2} = 0$

$c_5 = 0 - \dfrac{2c_3}{2\cdot 5\cdot 4} = \dfrac{c_9}{5\cdot 4^2\cdot 3\cdot 2},\;\; c_6 = -\dfrac{4c_5}{2\cdot 6} - 0 = -\dfrac{c_0}{6\cdot 5\cdot 4\cdot 3\cdot 2^2}$

$y = c_0\left[1 + \dfrac{1}{4}x^2 - \dfrac{1}{4\cdot 3\cdot 2}x^3 + \dfrac{1}{5\cdot 4^2\cdot 3\cdot 2}x^5 - \cdots\right] + c_1 x$

13. $$(x^2-1)\sum_{n=2}^{\infty}n(n-1)c_nx^{n-2}+4x\sum_{n=1}^{\infty}nc_nx^{n-1}+2\sum_{n=0}^{\infty}c_nx^n$$
$$=\sum_{n=2}^{\infty}n(n-1)c_nx^n-\underbrace{\sum_{n=2}^{\infty}n(n-1)c_nx^{n-2}}_{k=n-2}+4\sum_{n=1}^{\infty}nc_nx^n+2\sum_{n=0}^{\infty}c_nx^n$$
$$=\sum_{k=2}^{\infty}k(k-1)c_kx^k-\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k+4\sum_{k=1}^{\infty}kc_kx^k+2\sum_{k=0}^{\infty}c_kx^k$$
$$=(2c_0-2c_2)+(2c_1+4c_1-6c_3)x+\sum_{k=2}^{\infty}[k(k-1)c_k-(k+2)(k+1)c_{k+2}+4kc_k+2c_k]x^k$$
$$=0$$
$2c_0-2c_2=0$; $6c_1-6c_3=0$; $(k+2)(k+1)c_k-(k+2)(k+1)c_{k+2}=0$; $c_2=c_0$, $c_3=c_1$;
$c_{k+2}=c_k$, $k=2,3,4,\ldots$; $c_4=c_2=c_0$, $c_5=c_3=c_1$, $c_6=c_4=c_0$, $c_7=c_5=c_1$
$y=c_0\left[1+x^2+x^4+\cdots\right]+c_1[x+x^3+x^5+\cdots]=c_0\sum_{n=0}^{\infty}x^{2n}+c_1\sum_{n=0}^{\infty}x^{2n+1}$

14. $$(x^2+1)\sum_{n=2}^{\infty}n(n-1)c_nx^{n-2}-6\sum_{n=0}^{\infty}c_nx^n$$
$$=\sum_{n=2}^{\infty}n(n-1)c_nx^n+\underbrace{\sum_{n=2}^{\infty}n(n-1)c_nx^{n-2}}_{k=n-2}-6\sum_{n=0}^{\infty}nc_nx^n$$
$$=\sum_{k=2}^{\infty}k(k-1)c_kx^k+\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k-6\sum_{k=0}^{\infty}c_kx^k$$
$$=(2c_2-6c_0)+(6c_3-6c_1)x+\sum_{k=2}^{\infty}[k(k-1)c_k+(k+2)(k+1)c_{k+2}-6c_k]x^k=0$$
$2c_2-6c_0=0$; $6c_3-6c_1=0$; $(k-3)(k+2)c_k+(k+2)(k+1)c_{k+2}=0$; $c_2=3c_0$, $c_3=c_1$;
$c_{k+2}=-\dfrac{(k-3)c_k}{k+1}$, $k=2,3,4,\ldots$; $c_4=-\dfrac{-c_2}{3}=c_0$, $c_5=0$, $c_6=-\dfrac{c_4}{5}=-\dfrac{c_0}{5}$
$c_7=c_9=c_{11}=\cdots=0$, $c_8=-\dfrac{3c_6}{7}=\dfrac{3c_0}{7\cdot 5}$, $c_{10}=-\dfrac{5c_8}{9}=-\dfrac{5\cdot 3c_0}{9\cdot 7\cdot 5}$
$y=c_0\left[1+3x^2+x^4-\dfrac{3}{5\cdot 3}x^6+\cdots\right]+c_1(x+x^3)$

15. $(x^2+2)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + 3x\sum_{n=1}^{\infty} nc_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n$

$$= \sum_{n=2}^{\infty} n(n-1)c_n x^n + 2\underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} + 3\sum_{n=1}^{\infty} nc_n x^n - \sum_{n=0}^{\infty} c_n x^n$$

$$= \sum_{k=2}^{\infty} k(k-1)c_k x^k + 2\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k + 3\sum_{k=1}^{\infty} kc_k x^k - \sum_{k=0}^{\infty} c_k x^k$$

$$= (4c_2 - c_0) + (12c_3 + 3c_1 - c_1)x + \sum_{k=2}^{\infty}[k(k-1)c_k + 2(k+2)(k+1)c_{k+2} + 3kc_k - c_k]x^k$$

$$= 0$$

$4c_2 - c_0 = 0$; $12c_3 + 2c_1 = 0$; $2(k+2)(k+1)c_{k+2} + (k^2+2k-1)c_k = 0$; $c_2 = \dfrac{c_0}{4}$, $c_3 = -\dfrac{c_1}{6}$;
$c_{k+2} = -\dfrac{(k^2+2k-1)c_k}{2(k+2)(k+1)}$, $k = 2, 3, 4, \ldots$; $c_4 = -\dfrac{7c_2}{2\cdot 4\cdot 3} = -\dfrac{7}{4\cdot 4!}c_0$, $c_5 = -\dfrac{14c_3}{2\cdot 5\cdot 4} =$
$\dfrac{14}{2\cdot 5!}c_1$
$c_6 = -\dfrac{23c_4}{2\cdot 6\cdot 5} = \dfrac{23\cdot 7}{2^3\cdot 6!}c_0$, $c_7 = -\dfrac{34c_5}{2\cdot 7\cdot 6} = -\dfrac{34\cdot 14}{4\cdot 7!}c_1 y = c_0\left[1 + \dfrac{1}{4}x^2 - \dfrac{7}{4\cdot 4!}x^4 + \dfrac{23\cdot 7}{8\cdot 6!}x^6 - \cdots\right] +$
$c_1\left[x - \dfrac{1}{6}x^3 + \dfrac{14}{2\cdot 5!}x^5 - \dfrac{34\cdot 14}{4\cdot 7!}x^7 + \cdots\right]$

16. $(x^2-1)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x\sum_{n=0}^{\infty} nc_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n$

$$= \sum_{n=2}^{\infty} n(n-1)c_n x^n - \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} + \sum_{n=1}^{\infty} nc_n x^n - \sum_{n=0}^{\infty} c_n x^n$$

$$= \sum_{k=2}^{\infty} k(k-1)c_k x^k - \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k + \sum_{k=1}^{\infty} kc_k x^k - \sum_{k=0}^{\infty} c_k x^k$$

$$= -(2c_2 + c_0) + (c_1 - 6c_3 - c_1)x + \sum_{k=2}^{\infty}[k(k-1)c_k - (k+2)(k+1)c_{k+2} + kc_k - c_k]x^k$$

$$= 0$$

$2c_2 + c_0 = 0$; $-6c_3 = 0$; $(k+1)(k-1)c_k - (k+2)(k+1)c_{k+2} = 0$; $c_2 = -\dfrac{c_0}{2}$, $c_3 = 0$;
$c_{k+2} = \dfrac{(k-1)c_k}{k+2}$, $k = 2, 3, 4, \ldots$; $c_4 = \dfrac{c_2}{4} = -\dfrac{c_0}{4\cdot 2}$, $c_5 = c_7 = c_9 = \cdots = 0$, $c_6 = \dfrac{3c_4}{6} =$
$-\dfrac{c_0}{4\cdot 2^2}$
$y = c_0\left[1 - \dfrac{1}{2}x^2 - \dfrac{1}{8}x^4 - \dfrac{1}{16}x^6 - \cdots\right] + c_1 x$

17. $\displaystyle\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - (x+1)\sum_{n=1}^{\infty} nc_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n$

$$= \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} - \sum_{n=1}^{\infty} nc_n x^n - \underbrace{\sum_{n=1}^{\infty} nc_n x^{n-1}}_{k=n-1} - \sum_{n=0}^{\infty} c_n x^n$$

$$= \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - \sum_{k=1}^{\infty} kc_k x^k - \sum_{k=0}^{\infty}(k+1)c_{k+1}x^k - \sum_{k=0}^{\infty} c_k x^k$$

$$= 2c_2 - c_1 - c_0 + \sum_{k=1}^{\infty}[(k+2)(k+1)c_{k+2} - kc_k - (k+1)c_{k+1} - c_k]x^k = 0$$

$2c_2 - c_1 - c_0 = 0;\quad (k+2)(k+1)c_{k+2} - (k+1)c_{k+1} - (k+1)c_k = 0;\quad c_2 = \dfrac{c_0 + c_1}{2}$

$c_{k+2} = \dfrac{c_k + c_{k+1}}{k+2},\quad k = 1, 2, 3, \ldots;\quad c_3 = \dfrac{c_1 + c_2}{3} = \dfrac{c_1 + c_0/2 + c_1/2}{3} = \dfrac{c_0 + 3c_1}{6}$

$c_4 = \dfrac{c_2 + c_3}{4} = \dfrac{c_0/2 + c_1/2 + c_0/6 + c_1/2}{4} = \dfrac{2c_0 + 3c_1}{12}$

$c_5 = \dfrac{c_3 + c_4}{5} = \dfrac{c_0/6 + c_1/2 + c_0/6 + c_1/4}{5} = \dfrac{4c_0 + 9c_1}{60}$

$y = c_0\left[1 + \dfrac{1}{2}x^2 + \dfrac{1}{6}x^3 + \dfrac{1}{6}x^4 + \cdots\right] + c_1\left[x + \dfrac{1}{2}x^2 + \dfrac{1}{2}x^3 + \dfrac{1}{4}x^4 + \cdots\right]$

18. $\displaystyle\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - x\sum_{n=1}^{\infty} nc_n x^{n-1} - (x+2)\sum_{n=0}^{\infty} c_n x^n$

$$= \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} - \sum_{n=1}^{\infty} nc_n x^n - \underbrace{\sum_{n=0}^{\infty} c_n x^{n+1}}_{k=n+1} - 2\sum_{n=0}^{\infty} c_n x^n$$

$$= \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - \sum_{k=1}^{\infty} kc_k x^k - \sum_{k=1}^{\infty} c_{k-1}x^k - 2\sum_{k=0}^{\infty} c_k x^k$$

$$= 2c_2 - 2c_0 + \sum_{k=1}^{\infty}[(k+2)(k+1)c_{k+2} - kc_k - c_{k-1} - 2c_k]x^k = 0$$

$2c_2 - 2c_0 = 0;\quad (k+2)(k+1)c_{k+2} - (k+2)c_k - c_{k-1} = 0;\quad c_2 = 0$

$c_{k+2} = \dfrac{c_k}{k+1} + + \dfrac{c_{k-1}}{(k+2)(k+1)},\quad k = 1, 2, 3, \ldots;\quad c_3 = \dfrac{c_1}{2} + \dfrac{c_0}{3 \cdot 2} = \dfrac{c_0}{3!} + \dfrac{c_1}{2}$

$c_4 = \dfrac{c_2}{3} + \dfrac{c_1}{4 \cdot 3} = \dfrac{2c_0}{3!} + \dfrac{2c_1}{4!},\quad c_5 = \dfrac{c_3}{4} + \dfrac{c_2}{5 \cdot 4} = \dfrac{c_1}{4 \cdot 2} + \dfrac{c_0}{4!} + \dfrac{c_0}{5 \cdot 4} = \dfrac{11c_0}{5!} + \dfrac{3c_1}{4!}$

$c_6 = \dfrac{c_4}{5} + \dfrac{c_3}{6 \cdot 5} = \dfrac{2c_0}{5 \cdot 3!} + \dfrac{2c_1}{5!} + \dfrac{c_1}{6 \cdot 5 \cdot 2} + \dfrac{c_0}{6 \cdot 5 \cdot 3!} = \dfrac{52c_0}{6!} + \dfrac{4c_1}{5!}$

$y = c_0\left[1 + x^2 + \dfrac{1}{3!}x^3 + \dfrac{2}{3!}x^4 + \dfrac{11}{5!}x^5 + \cdots\right] + c_1\left[x + \dfrac{1}{2}x^3 + \dfrac{2}{4!}x^4 + \dfrac{3}{4!}x^5 + \cdots\right]$

19. $(x-1)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - x\sum_{n=1}^{\infty} nc_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n$

$$= \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-1}}_{k=n-1} - \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} - \sum_{n=1}^{\infty} nc_n x^n + \sum_{n=0}^{\infty} c_n x^n$$

$$= \sum_{k=1}^{\infty}(k+1)kc_{k+1}x^k - \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - \sum_{k=1}^{\infty} kc_k x^k + \sum_{k=0}^{\infty} c_k x^k$$

$$= c_0 - 2c_2 + \sum_{k=1}^{\infty}[(k+1)kc_{k+1} - (k+2)(k+1)c_{k+2} - kc_k + c_k]x^k = 0$$

$c_0 - 2c_2 = 0;\;\; (k+1)kc_{k+1} - (k+2)(k+1)c_{k+2} - (k-1)c_k = 0;\;\; c_2 = \dfrac{1}{2}c_0$

$c_{k+2} = \dfrac{(k+1)kc_{k+1} - (k-1)c_k}{(k+2)(k+1)},\;\; k = 1, 2, 3, \ldots;\;\; c_3 = \dfrac{2c_2}{3\cdot 2} = \dfrac{c_0}{3\cdot 2}$

$c_4 = \dfrac{3\cdot 2c_3 - c_2}{4\cdot 3} = \dfrac{c_0 - c_0/2}{4\cdot 3} = \dfrac{c_0}{4\cdot 3\cdot 2}$

$y = c_0\left[1 + \dfrac{1}{2}x^2 + \dfrac{1}{3\cdot 2}x^3 + \dfrac{1}{4\cdot 3\cdot 2}x^4 + \cdots\right] + c_1 x;\;\; y' = c_0\left[x + \dfrac{1}{2}x^2 + \cdots\right] + c_1$

Using the initial conditions, we obtain $-2 = y(0) = c_0$ and $6 = y'(0) = c_1$. The solution is

$$y = -2\left[1 + \frac{1}{2}x^2 + \frac{1}{3\cdot 2}x^3 + \frac{1}{4\cdot 3\cdot 2}x^4 + \cdots\right] + 6x = -2 + 6x - x^2 - \frac{1}{3}x^3 - \frac{1}{4\cdot 3}x^4 + \cdots$$

20. $(x-1)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 2x\sum_{n=1}^{\infty} nc_n x^{n-1} + 8\sum_{n=0}^{\infty} c_n x^n$

$$= \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} - 2\sum_{n=1}^{\infty} nc_n x^n + 8\sum_{n=0}^{\infty} c_n x^n$$

$$= \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - 2\sum_{k=1}^{\infty} kc_k x^k + 8\sum_{k=0}^{\infty} c_k x^k$$

$$= 2c_2 + 8c_0 + \sum_{k=1}^{\infty}[(k+2)(k+1)c_{k+2} - 2kc_k + 8c_k]x^k = 0$$

$2c_2 + 8c_0 = 0;\;\; (k+2)(k+1)c_{k+2} - 2(k-4)c_k = 0;\;\; c_2 = -4c_0$

$c_{k+2} = \dfrac{2(k-4)c_k}{(k+2)(k+1)},\;\; k = 1, 2, 3, \ldots;\;\; c_3 = \dfrac{-2\cdot 3c_1}{3\cdot 2} = -c_1,\;\; c_4 = \dfrac{-2\cdot 2c_2}{4\cdot 3} = \dfrac{4}{3}c_0$

$c_5 = \dfrac{-2\cdot 1c_3}{5\cdot 4} = \dfrac{1}{5\cdot 2}c_1,\;\; c_6 = \dfrac{2\cdot 0c_4}{6\cdot 5} = 0,\;\; c_8 = c_{10} = c_{12} = \cdots = 0$

$y = c_0\left[1 - 4x^2 + \dfrac{4}{3}x^4\right] + c_1\left[x - x^3 + \dfrac{1}{10}x^5 + \cdots\right]$

$y' = c_0\left[-8x + \dfrac{16}{3}x^3\right] + c_1\left[1 - 3x^2 + \dfrac{1}{2}x^4 + \cdots\right]$

Using the initial conditions, we obtain $3 = y(0) = c_0$ and $0 = y'(0) = c_1$. The solution is

$$y = 3\left[1 - 4x^2 + \frac{4}{3}x^4\right] = 3 - 12x^2 + 4x^4$$

Chapter 16 in Review

A. True/False

1. True

2. True. We know a general solution is $y = Ae^x + Be^{-x}$. Now

$$\begin{aligned} C_1 \cosh x + C_2 \sinh x &= C_1\left(\frac{e^x + e^{-x}}{2}\right) + C_2\left(\frac{e^x - e^{-x}}{2}\right) \\ &= \left(\frac{C_1}{2} + \frac{C_2}{2}\right)e^x + \left(\frac{C_1}{2} - \frac{C_2}{2}\right)e^{-x}. \end{aligned}$$

   By varying C_1 and C_2, we see that the two equations are different forms of the same general solution.

3. False. y_2 is a constant multiple of y_1. Specifically, $y_2 = 0 \cdot y_1$.

4. False. Plugging $y_p = A$ into the DE gives $0 = 10$, a contradiction.

5. True. Any constant function solves the DE.

6. False. $P_y = 2x$ while $Q_x = -2x$.

7. True

8. True

B. Fill in the Blanks

1. By inspection, the constant function $y = 0$ solves the DE.

2. The auxiliary equation is $m^2 - m = 0$, so $m = 0$ or $m = 1$. The general solution is $y = C_1 + C_2e^x$. Boundary conditions yield $y(0) = C_1 + C_2 = 1$ and $y(1) = C_1 + C_2e = 0$, which give $C_1 = \dfrac{e}{e-1}$ and $C_2 = \dfrac{-1}{e-1}$. Therefore, $y = \dfrac{e}{e-1} - \left(\dfrac{1}{e-1}\right)e^x$.

3. $10 = k(2.5) \Longrightarrow k = 4$ lb/ft;
   $32 = 4x \Longrightarrow x = 8$ ft

4. We have a repeated root $m = -7$. Therefore, $y = C_1e^{-7x} + C_2xe^{-7x}$.

5. $y_p = Ax^2 + Bx + C + Dxe^{2x} + Ee^{2x}$

C. Exercises

1. $P_y = -6xy^2 \sin y^3 = Q_x$, and the equation is exact.
$f_x = 2x\cos y^3$, $f = x^2 \cos y^3 + g(y)$, $f_y = -3x^2y^2 \sin y^3 + g'(y) = -1 - 3x^2y^2 \sin y^3$,
$g'(y) = -1$, $g(y) = -y$, $f = x^2 \cos^3 y - y$.
Therefore, the solution is $x^2 \cos y^3 - y = C$.

2. $P_y = 6y^2 = Q_x$, and the equation is exact. $f_x = 3x^2 + 2y^3$, $f = x^3 + 2xy^3 + g(y)$, $f_y = 6xy^2 + g'(y) = 6xy^2 + y^2$,
$g'(y) = y^2$, $g(y) = \dfrac{y^3}{3}$, $f = x^3 + 2xy^3 + \dfrac{y^3}{3}$.
Therefore, the solution is $x^3 + 2xy^3 + \dfrac{y^3}{3} = C$.

3. $P_y = -2xy^{-5} = Q_x$, and the equation is exact.
$f_x = \frac{1}{2}xy^{-4}$, $f = \frac{1}{4}x^2y^{-4} + g(y)$, $f_y = -x^2y^{-5} + g'(y) = 3y^{-3} - x^2y^{-5}$
$g'(y) = 3y^{-3}$, $g(y) = -\frac{3}{2}y^{-2}$, $f = \frac{1}{4}x^2y^{-4} - \frac{3}{2}y^{-2}$.
Therefore, the general solution is $\frac{1}{4}x^2y^{-4} - \frac{3}{2}y^{-2} = C$. Since $y(1) = 1$, we have $\frac{1}{4}(1)(1) - \frac{3}{2}(1) = C$ or $C = -\frac{5}{4}$. Thus, the solution is $\frac{1}{4}x^2y^{-4} - \frac{3}{2}y^{-2} = -\frac{5}{4}$.

4. $P_y = 2x + \sin x = Q_x$ and the equation is exact.
$f_x = y^2 + y\sin x$, $f = xy^2 - y\cos x + g(y)$, $f_y = 2xy - \cos x + g'(y) = 2xy - \cos x - \frac{1}{1+y^2}$,
$g'(y) = \dfrac{1}{1+y^2}$, $g(y) = \tan^{-1}(y)$, $f = xy^2 - y\cos x + \tan^{-1}(y)$. Therefore, the general solution is $xy^2 - y\cos x + \tan^{-1}(y) = C$. Since $y(0) = 1$, we have $-1 + \frac{\pi}{4} = C$. Thus, the solution is $xy^2 - y\cos x + \tan^{-1}(y) = \dfrac{\pi}{4} - 1$

5. $m^2 - 2m - 2 = 0 \Longrightarrow m = 1 \pm \sqrt{3}$; $y = C_1e^{(1-\sqrt{3})x} + C_2e^{(1+\sqrt{3})x}$

6. $m^2 - 8 = 0 \Longrightarrow m = \pm 2\sqrt{2}$; $y = C_1e^{-2\sqrt{2}x} + C_2e^{2\sqrt{2}x}$

7. $m^2 - 3m - 10 = 0 \Longrightarrow (m-5)(m+2) = 0 \Longrightarrow m = -2,\ 5$; $y = C_1e^{-2x} + C_2e^{5x}$

8. $4m^2 + 20m + 25 = 0 \Longrightarrow (2m+5)^2 = 0 \Longrightarrow m = -5/2,\ -5/2$; $y = C_1e^{-5x/2} + C_2xe^{-5x/2}$

9. $9m^2 + 1 = 0 \Longrightarrow m = \pm\dfrac{1}{3}i$; $y = C_1 \cos\dfrac{x}{3} + C_2 \sin\dfrac{x}{3}$

10. $2m^2 - 5m = 0 \Longrightarrow m(2m-5) = 0 \Longrightarrow m = 0,\ 5/2$; $y = C_1 + C_2e^{5x/2}$

11. Letting $y = ux$ we have

$$(x + uxe^u)dx - xe^u(udx + xdu) = 0 \Longrightarrow dx - xe^u du = 0 \Longrightarrow \frac{dx}{x} - e^u du = 0$$
$$\Longrightarrow \ln|x| - e^u = C_1 \Longrightarrow \ln|x| - e^{y/x} = C_1.$$

Using $y(1) = 0$ we find $C_1 = -1$. The solution of the initial-value problem is $\ln|x| = e^{y/x} - 1$.

12. The auxiliary equation is $m = m^2 + 4m + 4 = 0$, so $m = -2$ is a repeated root. The general solution is $y = C_1e^{-2x} + C_2xe^{-2x}$. Initial conditions yield $y(0) = C_1 = -2$ and $y'(0) = -2C_1 + C_2 = 0$ which give $C_1 = -2$ and $C_2 = -4$. The solution is $y = -2e^{-2x} - 4xe^{-2x}$.

13. $m^2 - m - 12 = 0 \Longrightarrow (m-4)(m+3) = 0 \Longrightarrow m = -3,\ \ 4;\ \ y_c = C_1e^{-3x} + C_2e^{4x}$
$y_p = Axe^{2x} + Be^{2x},\ \ y_p' = 2Axe^{2x} + (A+2B)e^{2x};\ \ y_p'' = 4Axe^{2x} + 4(A+B)e^{2x}$
$[4Axe^{2x} + 4(A+B)e^{2x}] - [2Axe^{2x} + (A+2B)e^{2x}] - 12[Axe^{2x} + Be^{2x}]$
$= -10Axe^{2x} + (3A - 10B)e^{2x} = xe^{2x} + e^x$
Solving $-10A = 1,\ \ 3A - 10B = 1$ we obtain $A = -1/10$ and $B = -13/100$. Thus,

$$y = C_1e^{-3x} + C_2e^{4x} - \frac{1}{10}xe^{2x} - \frac{13}{100}e^{2x}.$$

14. The auxiliary equation is $m^2 + 4 = 0$, so $m = \pm 2i$. Therefore, $y_c = C_1\cos 2x + C_2\sin 2x$. Assume a particular solution of the form $y_p = Ax^2 + Bx + C$. Substituting into the DE, we have

$$2A + Ax^2 + Bx + C = 16x^2.$$

Equating coefficients, we get $2A + C = 0,\ \ B = 0$, and $A = 16$. This gives $C = -32$. Therefore, $y_p = 16x^2 - 32$. The general solution is $y = y_c + y_p = C_1\cos 2x + C_2\sin 2x + 16x^2 - 32$.

15. $m^2 - 2m + 2 = 0 \Longrightarrow m = 1 \pm i;\ \ y_c = e^x(C_1\cos x + C_2\sin x)$
$W = \begin{vmatrix} e^x\cos x & e^x\sin x \\ -e^x\sin x + e^x\cos x & e^x\cos x + e^x\sin x \end{vmatrix} = e^{2x}$
$u' = -\dfrac{1}{e^{2x}}e^x\sin x\, e^x\tan x = -\dfrac{\sin^2 x}{\cos x} = \dfrac{\cos^2 x - 1}{\cos x} = \cos x - \sec x,\ \ u = \sin x - \ln|\sec x + \tan x|$
$v' = \dfrac{1}{e^{2x}}e^x\cos x\, e^x\tan x = \sin x,\ \ v = -\cos x$
$y_p = e^x\cos x(\sin x - \ln|\sec x + \tan x|) - e^x\sin x\cos x = -e^x\cos x\ln|\sec x + \tan x|$
$y = e^x(C_1\cos x + C_2\sin x) - e^x\cos x\ln|\sec x + \tan x|$

16. $m^2 - 1 = 0 \Longrightarrow m = -1,\ \ 1;\ \ y_c = C_1e^{-x} + C_2e^x;\ \ W = \begin{vmatrix} e^{-x} & e^x \\ -e^{-x} & e^x \end{vmatrix} = 2$
$u' = -\dfrac{1}{2}e^x\dfrac{2e^x}{e^x + e^{-x}} = -\dfrac{e^{2x}}{e^x + e^{-x}} = -\dfrac{e^{3x}}{e^{2x} + 1}$
$u = -\displaystyle\int \frac{e^{3x}}{e^{2x}+1}dx \quad \boxed{t = e^x,\ \ dt = e^x dx}$
$= -\displaystyle\int \frac{t^2}{t^2+1}dt = -\int\left(1 - \frac{1}{t^2+1}\right)dt = \tan^{-1} t - t = \tan^{-1} e^x - e^x$
$v' = \dfrac{1}{2}e^{-x}\dfrac{2e^x}{e^x + e^{-x}} = \dfrac{1}{e^x + e^{-x}} = \dfrac{e^x}{e^{2x}+1}$
$v = \displaystyle\int \frac{e^x}{e^{2x}+1}dx \quad \boxed{t = e^x,\ \ dt = e^x dx}$
$= \displaystyle\int \frac{dt}{t^2+1} = \tan^{-1} t = \tan^{-1} e^x$
$y_p = e^{-x}(\tan^{-1} e^x - e^x) + e^x\tan^{-1} e^x = (e^x + e^{-x})\tan^{-1} e^x - 1$
$y = C_1e^{-x} + C_2e^x + (e^x + e^{-x})\tan^{-1} e^x - 1$

17. $m^2 + 1 = 0 \Longrightarrow m = \pm i;\ \ y_c = C_1\cos x + C_2\sin x;\ \ W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$
$u' = -\sin x\sec^3 x = -\tan x\sec^2 x,\ \ u = -\dfrac{1}{2}\sec^2 x;\ \ v' = \cos x\sec^3 x = \sec^2 x,\ \ v = \tan x$

$$y_p = -\frac{1}{2}\cos x \sec^2 x + \sin x \tan x = \sin x \tan x - \frac{1}{2}\sec x = \frac{\sin^2 x}{\cos x} - \frac{1}{2\cos x}$$
$$= \frac{2\sin^2 x - 1}{2\cos x} = \frac{\sin^2 x - \cos^2 x}{2\cos x} = \frac{1}{2}\sin x \tan x - \frac{1}{2}\cos x$$
$$y = C_3\cos x + C_2\sin x + \frac{1}{2}\sin x\tan x, \quad y' = -C_3\sin x + C_2\cos x + \frac{1}{2}\sin x\sec^2 x + \sin x$$

Using the initial conditions, we obtain $C_3 = 1$ and $C_2 = 1/2$. Thus,

$$y = \cos x + \frac{1}{2}\sin x + \frac{1}{2}\sin x\tan x = \frac{2\cos^2 x}{2\cos x} + \frac{\sin^2 x}{2\cos x} + \frac{1}{2}\sin x$$
$$= \frac{\cos^2 x + 1}{2\cos x} + \frac{1}{2}\sin x = \frac{1}{2}(\sin x + \cos x + \sec x).$$

18. The auxiliary equation is $m^2+2m+2=0$, so $m=-1\pm i$. Therefore, $y_c = e^{-x}\left(C_1\cos x + C_2\sin x\right)$. Assume a particular solution of the form $y_p = A$. Substituting this into the DE, we have $2A = 1$, or $A = \frac{1}{2}$. Therefore, the general solution is $y = y_c + y_p = e^{-x}\left(C_1\cos x + C_2\sin x\right) + \frac{1}{2}$. The initial conditions yield $y(0) = C_1 + \frac{1}{2} = 0$ and $y'(0) = -C_1 + C_2 = 1$ which give $C_1 = -\frac{1}{2}$ and $C_2 = \frac{1}{2}$. Thus, the solution is $y = e^{-x}\left(-\frac{1}{2}\cos x + \frac{1}{2}\sin x\right) + \frac{1}{2}$.

19. $$\underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} + \underbrace{x\sum_{n=0}^{\infty} c_n x^n}_{k=n+1} = \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k + \sum_{k=1}^{\infty} c_{k-1}x^k$$
$$= 2c_2 + \sum_{k=1}^{\infty}[(k+2)(k+1)c_{k+2} + c_{k-1}]x^k = 0$$
$$c_2 = 0; \quad (k+2)(k+1)c_{k+2} + c_{k-1} = 0; \quad c_{k+2} = -\frac{c_{k-1}}{(k+2)(k+1)}, \quad k = 1, 2, 3, \ldots$$
$$c_3 = -\frac{c_0}{3\cdot 2}, \quad c_4 = -\frac{c_1}{4\cdot 3}, \quad c_5 = 0, \quad c_6 = -\frac{c_3}{6\cdot 5} = \frac{c_0}{6\cdot 5\cdot 3\cdot 2},$$
$$c_7 = -\frac{c_4}{7\cdot 6} = \frac{c_1}{7\cdot 6\cdot 4\cdot 3}$$
$$c_8 = 0. \quad c_9 = -\frac{c_6}{9\cdot 8} = -\frac{c_0}{9\cdot 8\cdot 6\cdot 5\cdot 3\cdot 2}, \quad c_{10} = -\frac{c_7}{10\cdot 9} = -\frac{c_1}{10\cdot 9\cdot 7\cdot 6\cdot 4\cdot 3}$$
$$y = c_0\left[1 - \frac{1}{3\cdot 2}x^3 + \frac{1}{6\cdot 5\cdot 3\cdot 2}x^6 - \frac{1}{9\cdot 8\cdot 6\cdot 5\cdot 3\cdot 2}x^9 + \cdots\right]$$
$$+ c_1\left[x - \frac{1}{4\cdot 3}x^4 + \frac{1}{7\cdot 6\cdot 4\cdot 3}x^7 - \frac{1}{10\cdot 9\cdot 7\cdot 6\cdot 4\cdot 3}x^{10} + \cdots\right]$$

20. $(x-1)\sum_{n=2}^{\infty} n(n-1)c_nx^{n-2} + 3\sum_{n=1}^{\infty} c_nx^n$

$$= \underbrace{\sum_{n=2}^{\infty}(n)(n-1)c_nx^{n-1}}_{k=n-1} - \underbrace{\sum_{n=2}^{\infty} n(n-1)c_nx^{n-2}}_{k=n-2} + 3\sum_{n=0}^{\infty} c_nx^n$$

$$= \sum_{k=1}^{\infty}(k+1)kc_{k+1}x^k - \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k + 3\sum_{k=0}^{\infty} c_kx^k$$

$$= 3c_0 - 2c_2 + \sum_{k=1}^{\infty}[(k+1)kc_{k+1} - (k+2)(k+1)c_{k+2} + 3c_k]x^k = 0$$

$3c_0 - 2c_2 = 0;\;\; (k+1)kc_{k+1} - (k+2)(k+1)c_{k+2} + 3c_k = 0;\;\; c_2 = \dfrac{3c_0}{2};$

$c_{k+2} = \dfrac{kc_{k+1}}{k+2} + \dfrac{3c_k}{(k+2)(k+1)},\;\; k = 1, 2, 3, \ldots;\;\; c_3 = \dfrac{c_2}{3} + \dfrac{3c_1}{3\cdot 2} = \dfrac{c_0}{2} + \dfrac{c_1}{2},\;\; c_4 = \dfrac{2c_3}{4} + \dfrac{3c_2}{4\cdot 3} =$
$\dfrac{c_0}{4} + \dfrac{c_1}{4} + \dfrac{3c_0}{8} = \dfrac{5c_0}{8} + \dfrac{c_1}{4},$

$c_5 = \dfrac{3c_4}{5} + \dfrac{3c_3}{5\cdot 4} = \dfrac{3c_0}{8} + \dfrac{3c_1}{20} + \dfrac{3c_0}{40} + \dfrac{3c_1}{40} = \dfrac{9c_0}{20} + \dfrac{9c_1}{40}$

$y = c_0\left[1 + \dfrac{3}{2}x^3 + \dfrac{1}{2}x^3 + \dfrac{5}{8}x^4 + \cdots\right] + c_1\left[x + \dfrac{1}{2}x^3 + \dfrac{1}{4}x^4 + \cdots\right]$

21. The differential equation is $mx'' + 4x' + 2x = 0$. The solutions of the auxiliary equation are

$$\frac{1}{2m}(-4 \pm \sqrt{16-8m}) = \frac{1}{m}(-2 \pm \sqrt{4-2m}).$$

The motion will be non-oscillatory when $4 - 2m \geq 0$ or $0 < m \leq 2$.

22. Substituting $x_p = \alpha A$ into the differential equation we obtain $\omega^2\alpha A = A$, so $\alpha = 1/\omega^2$ and $x_p = A/\omega^2$.

23. Using $m = W/g = 4/32 = 1/8$, the inital value problem is $\dfrac{1}{8}x'' + x' + 3x = e^{-t};\;\; x(0) = 2,$ $x'(0) = 0$. The auxiliary equation is $m^2/8 + m + 3 = 0$. Using the quadratic formula, $m = -4 \pm 2\sqrt{2}i$. Thus, $x_c = e^{-4t}(C_1\cos 2\sqrt{2}t + C_2\sin 2\sqrt{2}t)$. Using $x_p = Ae^{-t}$, we find $A = 8/17$. Thus,

$$x(t) = e^{-4t}(C_1\cos 2\sqrt{2}t + C_2\sin 2\sqrt{2}t) + \frac{8}{17}e^{-t}$$

$$\text{and}\quad x'(t) = e^{-4t}[(2\sqrt{2}C_2 - 4C_1)\cos 2\sqrt{2}t - (2\sqrt{2}C_1 - 4C_2)\sin 2\sqrt{2}t] - \frac{8}{17}e^{-t}.$$

Using the initial conditions, we obtain $2 = C_1 + 8/17$ and $0 = 2\sqrt{2}C_2 - 4C_1 - 8/17$. Then $C_1 = 26/17$ and $C_2 = 28\sqrt{2}/17$ and

$$x(t) = e^{-4t}\left(\frac{26}{17}\cos 2\sqrt{2}t + \frac{28}{17}\sqrt{2}\sin 2\sqrt{2}t\right) + \frac{8}{17}e^{-t}.$$

24. (a) From $k_1 = 2W$ and $k_2 = 4W$ we find $1/k = 1/2W + 1/4W = 3/4W$. Then $k = 4W/3 = 4mg/3$. The differential equation $mx'' + kx = 0$ then becomes $x'' + (4g/3)x = 0$. The solution is $x(t) = C_1 \cos 2\sqrt{g/3}t + C_2 \sin 2\sqrt{g/3}t$. The initial conditions $x(0) = 1$ and $x'(0) = 2/3$ imply $C_1 = 1$ and $C_2 = 1/\sqrt{3g}$.

    (b) To find the maximum speed of the weight we compute

$$x'(t) = 2\sqrt{\frac{g}{3}}\sin 2\sqrt{\frac{g}{3}} + \frac{2}{3}\cos 2\sqrt{\frac{g}{3}}t \quad \text{and} \quad |x'(t)| = \sqrt{4\frac{g}{3} + \frac{4}{9}} = \frac{2}{3}\sqrt{3g+1}.$$

25. The auxiliary equation is $m^2/4 + m + 1 = 0$ or $(m+2)^2 = 0$, so $m = -2,\ -2$ and $x(t) = C_1e^{-2t} + C_2te^{-2t}$ and $x'(t) = -2C_1e^{-2t} - 2C_2te^{-2t} + C_2e^{-2t}$. Using the initial conditions, we obtain $4 = C_1$ and $2 = -2C_1 + C_2$. Thus, $C_1 = 4$ and $C_2 = 10$. Therefore $x(t) = 4e^{-2t} + 10te^{-2t}$ and $x'(t) = 2e^{-2t} - 20te^{-2t}$. Setting $x'(t) = 0$ we obtain the critical point $t = 1/10$. The maximum vertical displacement is $x(1/10) = 5e^{-0.2} \approx 4.0937$.